Arbeitsgemeinschaft mit aktuellem Thema:

Minimal Surfaces

Mathematisches Forschungsinstitut Oberwolfach

October 4th - 9th, 2009

Organizers:

William Meeks III
Mathematics Department
University of Massachusetts
Amherst, MA 01003
USA
profmeeks@gmail.com

Matthias Weber
Mathematics Department
Indiana University
Bloomington, IN 47405
USA
matweber@indiana.edu

Introduction:

The theory of Minimal Surfaces has developed rapidly in the past 10 years. There are many factors that have contributed to this development:

- Sophisticated construction methods ([11, 28, 30]) have been developed and have supplied us with a wealth of examples which have provided intuition and spawned conjectures.

- Deep curvature estimates by Colding and Minicozzi ([3]) give control on the local and global behavior of minimal surfaces in an unprecedented way.

- Much progress has been made in classifying minimal surfaces of finite topology or low genus in $\mathbb{R}^3$ or in other flat 3-manifolds. For instance, all properly embedded minimal surfaces of genus 0 in $\mathbb{R}^3$, even those with an infinite number of ends, are now known ([24, 20, 21]).

- There are still numerous difficult but easy to state open conjectures, like the genus-g helicoid conjecture: *There exists a unique complete embedded minimal surface with one end and genus g for each $g \in \mathbb{N}$* or the related Hoffman-Meeks conjecture: *A finite topology surface with* *genus g and $n \geq 2$ ends embeds minimally in $\mathbb{R}^3$ with a complete metric if and only if $n \leq g + 2$.*

- Sophisticated tools from 3-manifold theory have been applied and generalized to understand the geometry and topological properties of properly embedded minimal surfaces in $\mathbb{R}^3$.
- Minimal surfaces have had important applications in topology and play a prominent role in the larger context of Geometric Analysis.

This workshop will lead us to some recent highlights of the theory. For a general reference to the subject of classical minimal surface theory and for most of the topics covered in the first 14 talks below, please see the recent survey [17].

Talks:

1. **Plateau Solutions and Conjugate Plateau Constructions**
Outline the basic existence results for the solution of Plateau's problem. Apply this solution to construct a minimal surface by first finding its conjugate surface. Example: Scherk surfaces. Mention the Jenkins-Serrin theorem and its application (see 2.6.1 in [12], see also [13, 14]).

2. **A Quasiperiodic Minimal Surface**
This talk will explain the construction of a quasiperiodic minimal surface due to Mazet and Traizet ([15]). This is the first example of an embedded minimal surface without a quotient of finite genus. The techniques are based on the conjugate Plateau method. Knowledge of the doubly periodic surfaces of Karcher-Meeks-Rosenberg ([12, 23]) will be helpful.

3. **Gluing Constructions for Minimal Surfaces**
The purpose of this talks is to present Traizet's regeneration technique, which allows the construction of minimal surfaces of any genus in various contexts by regenerating from nodal limits using the implicit function theorem. There are many incarnations of this method, surveyed in [28]. A possible goal for this talk would be to explain the Minimal Surfaces a la Riemann according to [29], but the applicants can feel free to suggest another application of the method.

4. **The Classification of Doubly Periodic Minimal Tori With Parallel Ends**
   This talk will prove the following theorem of Pérez, Rodríguez, and Traizet [27]: Any embedded, doubly periodic minimal surface of genus one and parallel ends belongs to the 3-dimensional moduli space constructed by Karcher and Meeks-Rosenberg.

5. **Osserman's Theorem**
   Following [16], this talk will give an elementary proof of Osserman's theorem: A complete minimal surface of finite total curvature is conformally equivalent to a compact Riemann surface with finitely many points removed.

6. **Curvature Estimates for Stable Minimal Surfaces**
   One of the main tools in minimal surface theory is the existence of curvature estimates for compact stable minimal surfaces in terms of the intrinsic distance from the boundary of the surface, see [17]. The speakers for this and the following talk should work together.

7. **Barrier Constructions**
   When the curvature estimates from the previous talk are used in conjunction with a barrier construction by Meeks and Yau [26] and the maximum principle one can obtain basic theoretical results like the strong halfspace theorem [10]. See [17] for a discussion of curvature estimates and these applications and [22] for applications of curvature estimates for minimal and constant mean curvature surfaces in 3-manifolds.

8. **Universal Superharmonic Functions and their Application to the Conformal Type of Proper Minimal Surfaces in $\mathbb{R}^3$**
   This talk is very self-contained and would cover some of the basic definitions like *parabolic* Riemann surface with boundary (full harmonic measure on its boundary) and the "opposite" kind of Riemann surface called *hyperbolic*, as well as the main tool for studying this property which is the notion of a *universal superharmonic function*. One result that should be presented is the proof that a properly immersed minimal surface in a closed half space in $\mathbb{R}^3$ is parabolic. Another basic application of universal superharmonic functions that could be presented is: A properly immersed minimal surface in $\mathbb{R}^3$ with compact boundary and

contained in a slab has quadratic area growth. We refer the applicants to [6, 17] for references to the material of this talk.

9. **Colding-Minicozzi Theory: 1-Sided Curvature Estimates**
This one and the following two talks will be an introduction into Colding-Minicozzi theory with the goal to prove that any complete embedded minimal disk in Euclidean space is proper. We recommend that all participants read the surveys [1, 3, 4, 17]. This first talk will explain the 1-sided curvature estimates of Colding-Minicozzi as in Corollary 0.4 from [2].

10. **Colding-Minicozzi Theory: Chord-Arc Properties for Minimal Disks**
The content of this talk is to explain that a certain weak chord-arc estimate for embedded compact minimal disks in $\mathbb{R}^3$. By chord-arc estimate we refer to an estimate that relates intrinsic and extrinsic distance in the disk up to some universal constant. A key consequence of this result is that there is a small universal constant $\delta > 0$ such that for any compact embedded minimal disk $D \subset \mathbb{R}^3$ and for any point $p \in D$ with intrinsic distance $d = d_D(p, \partial D)$ from the boundary of D, then the component C_p of $\mathbb{B}_{\mathbb{R}^3}(p, \delta d)$ containing p is a disk disjoint from ∂D. This talk should focus only on the weak chord-arc case where ∂D lies on the unit sphere and passes through the origin. See Proposition 2.1 in Section 2 in [5] . For a somewhat different approach to proving this kind of chord-arc result in 3-manifolds see [25].

11. **Colding-Minicozzi Theory: Properness of Complete Embedded Minimal Surfaces with Finite Topology**
This talk is a continuation of the previous talk, but where the embedded compact disk D does not have its boundary on the boundary of a sphere. A simple consequence of this result and previous classical work is that a complete embedded minimal surface $\Sigma \subset \mathbb{R}^3$ of finite topology and compact boundary is always properly embedded in $\mathbb{R}^3$. Since many theoretical results are known only for properly embedded minimal surfaces, this type of result allows us to obtain the same result after replacing the stronger hypothesis of "proper" by the weaker one of "complete" when the surface has finite topology. See Section 3 in [5] and for a somewhat different approach see [25].

12. **Rescaling Arguments and Minimal Laminations**
This talk explains three essential tools in the subject of minimal surfaces in 3-manifolds. The first tool consists of rescaling arguments which analyze the behavior of a minimal surface M very near a point $p \in M$ where some geometric function (like the absolute Gaussian curvature function or the injectivity radius function) of the surface is extremely small or large, by homothetically rescaling coordinates/metric of the ambient space centered at p so that on the new scaled surface the function attain an almost maximal or almost minimum value of 1 at p. Natural limits of such rescaled surfaces frequently yield minimal laminations in $\mathbb{R}^3$ when the sequence of rescaled surfaces fails to have local area bounds. The second tool is the Minimal Lamination Closure Theorem in [25]: The closure of complete embedded minimal surface $M \subset N$ in a Riemannian 3-manifold N has the structure of a minimal lamination if and only if its injectivity function is bounded away from 0 on each compact domain in N. The third tool is the Stable Limit Leaf Theorem of Meeks, Pérez, and Ros [22], pages 25–30: The limit leaves of a codimension one minimal lamination of a Riemannian manifold are stable. The speaker should introduce minimal laminations and foliations of a Riemannian 3-manifold, explain the statement of the Minimal Lamination Closure Theorem, present the (short) proof of the Stable Limit Leaf Theorem for a minimal lamination of a 3-manifold, and give one example of a rescaling or blow-up argument.

13. **Harmonic Function Theory on Minimal Surfaces**
See [19]. This talk demonstrates that if a properly immersed minimal surface Σ in $\mathbb{R}^3$ with finite topology and compact boundary intersects some plane transversely in a finite collection of immersed arcs and curves, then it is conformally a compact Riemann surface with boundary which has been punctured in a finite number of points, which means Σ is a parabolic Riemann surface. Knowing the conformal structure of a complete minimal surface $\Sigma \subset \mathbb{R}^3$ of finite topology is parabolic is extremely useful because the stereographic projection of the Gauss map is a meromorphic function on the underlying Riemann surface. For example, via an application of Picard's Theorem applied to the Gauss map, we see that a complete minimally immersed surface $\Sigma \subset \mathbb{R}^3$ with compact boundary has finite total curvature if and only if it has finite topology and there exist 2 non-parallel planes which intersect it

transversely in a finite number of immersed arcs and curves.

14. **The Uniqueness of the Helicoid**
This talk will prove that any complete, embedded, simply connected minimal surface in $\mathbb{R}^3$ is either the plane or the helicoid. This theorem is due to Meeks and Rosenberg when the surface is proper, a property which we understand holds by the work of Colding and Minicozzi given in Talk 11. We will follow a new proof of the uniqueness of the helicoid by Meeks and Perez [18] which is partly based on the results in Talk 13. Depending on the organization of the talk, the speaker could also cover the more general classification of the asymptotic behavior of complete minimal embeddings $\mathbb{S}^1 \times [0, \infty)$ into $\mathbb{R}^3$.

15. **Minimal Graphs in Nil**
This talk will develop the basics of the Abresch-Rosenberg Hopf differential and use it to prove the classification of entire minimal graphs in the Heisenberg group by Fernandez and Mira [9].

16. **The Halfspace Theorem in Nil**
This talk continues the discussion of the previous talk, presenting the work of Daniel and Hauswirth [8]: They construct horizontal minimal annuli in Nil, and use them prove a halfspace theorem. This in turn allows them to classify all complete minimal graphs in Nil.

17. **Minimal Surfaces in $M^2 \times \mathbb{R}$**
Collin and Rosenberg [7] construct entire minimal graphs in $\mathbb{H}^2 \times \mathbb{R}$ that are conformally equivalent to $\mathbb{C}$. The projection of these graphs yields a harmonic diffeomorphism from $\mathbb{C}$ to $\mathbb{H}$, thus contradicting a conjecture of R. Schoen.

References

[1] T. H. Colding and W. P. Minicozzi II. Minimal disks that are double spiral staircases. *Notices of the AMS*, 50(3):327–339, 2003. MR1954009, Zbl pre02115046.

[2] T. H. Colding and W. P. Minicozzi II. The space of embedded minimal surfaces of fixed genus in a 3-manifold IV; Locally simply-connected. *Ann. of Math.*, 160:573–615, 2004. MR2123933, Zbl 1076.53069.

[3] T. H. Colding and W. P. Minicozzi II. Embedded minimal disks. In *Global theory of minimal surfaces*, pages 405–438. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.

[4] T. H. Colding and W. P. Minicozzi II. Shapes of embedded minimal surfaces. *Proc. National Academy of Sciences*, 103:11106–11111, 2006. MR2242650, Zbl pre05272457.

[5] T. H. Colding and W. P. Minicozzi II. The Calabi-Yau conjectures for embedded surfaces. *Ann. of Math.*, 167:211–243, 2008. MR2373154, Zbl 1142.53012.

[6] P. Collin, R. Kusner, W. H. Meeks III, and H. Rosenberg. The geometry, conformal structure and topology of minimal surfaces with infinite topology. *J. Differential Geom.*, 67:377–393, 2004. MR2153082.

[7] P. Collin and H. Rosenberg. Construction of harmonic di?eomorphisms and minimal graphs. Preprint, 2007.

[8] B. Daniel and L. Hauswirth. Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group. *Proc. Lond. Math. Soc. (3)*, 98(2):445–470, 2009. MR2481955, Zbl pre05530823.

[9] I. Fernandez and P. Mira. Holomorphic quadratic differentials and the Berstein problem in Heisenberg space. To appear in Trans. Amer. Math. Soc., arxiv:0705.1436v2.

[10] D. Hoffman and W. H. Meeks III. The strong halfspace theorem for minimal surfaces. *Invent. Math.*, 101:373–377, 1990. MR1062966, Zbl 722.53054.

[11] N. Kapouleas. Complete embedded minimal surfaces of finite total curvature. *J. Differential Geom.*, 47(1):95–169, 1997. MR1601434, Zbl 0936.53006.

[12] H. Karcher. Embedded minimal surfaces derived from Scherk's examples. *Manuscripta Math.*, 62:83–114, 1988. MR0958255, Zbl 658.53006.

[13] H. Karcher. Construction of minimal surfaces. *Surveys in Geometry*, pages 1–96, 1989. University of Tokyo, 1989, and Lecture Notes No. 12, SFB256, Bonn, 1989.

[14] H. Karcher. Introduction to conjugate Plateau constructions. In *Global theory of minimal surfaces*, pages 489–524. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.

[15] L. Mazet and M. Traizet. A quasi-periodic minimal surface. *Commentarii mat. Helv.*, 83:573–601, 2008.

[16] W. H. Meeks III. Proofs of some classical results in minimal surface theory. *Indiana J. of Math.*, 54(4):1031–1045, 2005. MR2164416, Zbl 1085.53010.

[17] W. H. Meeks III and J. Pérez. The classical theory of minimal surfaces. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[18] W. H. Meeks III and J. Pérez. Embedded minimal surfaces of finite topology. Work in progress.

[19] W. H. Meeks III and J. Pérez. Finite type annular ends for harmonic functions. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[20] W. H. Meeks III and J. Pérez. Properly embedded minimal planar domains with infinite topology are riemann minimal examples. To appear in Current Developments in Mathematics 2008 Conference Proceedings, Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[21] W. H. Meeks III, J. Pérez, and A. Ros. The embedded Calabi-Yau conjectures for finite genus. Preprint, available at `http://www.ugr.es/local/jperez/papers/papers.htm`.

[22] W. H. Meeks III, J. Pérez, and A. Ros. Stable constant mean curvature surfaces. In *Handbook of Geometrical Analysis*, volume 1, pages 301–380. International Press, edited by Lizhen Ji, Peter Li, Richard Schoen and Leon Simon, ISBN: 978-1-57146-130-8, 2008. MR2483369, Zbl 1154.53009.

[23] W. H. Meeks III and H. Rosenberg. The global theory of doubly periodic minimal surfaces. *Invent. Math.*, 97:351–379, 1989. MR1001845, Zbl 676.53068.

[24] W. H. Meeks III and H. Rosenberg. The uniqueness of the helicoid. *Ann. of Math.*, 161:723–754, 2005. MR2153399, Zbl 1102.53005.

[25] W. H. Meeks III and H. Rosenberg. The minimal lamination closure theorem. *Duke Math. Journal*, 133(3):467–497, 2006. MR2228460, Zbl 1098.53007.

[26] W. H. Meeks III and S. T. Yau. The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.*, 179:151–168, 1982. MR0645492, Zbl 0479.49026.

[27] J. Pérez, M. Rodríguez, and M. Traizet. The classification of doubly periodic minimal tori with parallel ends. *Journal of Diff. Geometry*, 69(3):523–577, 2005. MR2170278, Zbl pre05004289.

[28] M. Traizet. Construction of minimal surfaces by gluing Weierstrass Representations. In *Global theory of minimal surfaces*, pages 439–452. American Mathematical Society, Providence, RI, for the Clay Mathematics Institute, Cambridge, MA, edited by D. Hoffman, 2005. MR2167253, Zbl 1078.53002.

[29] M. Traizet. Adding handles to Riemann minimal examples. *J. Differential Geom.*, 79:243–275, 2008. MR2420019, Zbl pre05294962.

[30] M. Weber and M. Wolf. Teichmüller theory and handle addition for minimal surfaces. *Ann. of Math.*, 156:713–795, 2002. MR1954234, Zbl 1028.53009.

Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`minimal.oberwolfach@gmail.com`

by **August 15th, 2009** at the latest.

You should also indicate which talk you are willing to give:
First choice: talk number ...
Second choice: talk number ...
Third choice: talk number ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Schwarzwaldstrasse 9-11, 77709 Oberwolfach-Walke, Germany. The institute offers board and lodging free of charge to the participants. However, travel expenses cannot be covered. Further information on the conference and on how to arrive at the Oberwolfach conference center will be given out to the participants, shortly after the deadline for applications.