This is the author's translation of the introduction of "Differenzierbare Strukturen auf Orbiträumen".

Differentiable Structures on Orbit Spaces

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Diplomarbeit

Bonn, April 1999

Abstract

Let a compact Lie group H act differentiably on a differentiable manifold M. Denote the coarse slice diagram of this action by Δ. Assume M and Δ tacitly being endowed with tubular systems.

Then a linearisation of Δ — if one exists — determines a differentiable structure on the orbit space $H\backslash M$. For M compact, isotopic linearisations yield isotopic differentiable structures. For Δ abelian, especially for H abelian, the statements "$H\backslash M$ is a topological manifold", "there is a differentiable structure on $H\backslash M$" and "Δ is linearisable" are even equivalent.

Two examples are examined geometrically: $\mathbb{C}P^2$, devided by complex conjugation, is diffeomorphic to S^4. SU(3), devided by $T_{\max} \times T_{\max}$, is homeomorphic to S^4. More generally, for K a compact Lie group the orbit space $T_{\max}\backslash K/T_{\max}$ is a manifold if and only if K is locally isomorphic to $\mathrm{U}(1)^k \times \mathrm{SU}(2)^l \times \mathrm{SU}(3)^m$ for suitalbe $k, l, m \geqslant 0$.

Mathematics Subject Classification (1991):
57 S 15 Compact Lie groups of differentiable transformations

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Introduction

A differentiable manifold M with a fixed differentiable action of a compact Lie group H is called a differentiable H-manifold.[1] While the dimension is the only local information of an ordinary manifold, to each point of an H-manifold there corresponds a slice representation. Globally we can — on the one hand — stratify M by means of the slice types. On the other hand we can divide M by H. It seems natural to ask for stuctures that the quotient carries or may carry.

The sheaf of invariant differentiable functions on the H-manifold induces a sheaf on the quotient. Moreover the quotient inherits the stratification[2] and hence is triangulable.[3] It is even a topological manifold (with boundary) after having removed a suitable closed set of codimension at least three.[4] The question if the whole quotient be a topological manifold is local and — in the abelian case — has an answer in terms of representation theory.[5] But under which circumstances does a differentiable structure on $H\backslash M$ exist, and to what extent is it compatible with the structures on the quotient mentioned above?

The quotient map is a differentiable fibre bundle in a canonical way only if all the isotropy groups are conjugate to each other. The so called special H-manifolds are next as regards complexity. For them Jänich constructs a differentiable structure on the quotient. According to Bredon a slightly modified structure is even canonical. Now for arbitrary H-manifolds the question seems to have not yet been treated in the literature.

The present paper therefore deals in its first chapter with the construction of a differentiable structure on the quotient of an arbitrary H-manifold by formulating a sufficient condition for existence, fixing the choices to be made on the way and investigating the influence of the choices on the structure thus constructed.

First we define various kinds of objects: An equivariant manifold refines the notion of an H-manifold by letting H lie normally in a compact Lie group G still acting on M. H works as "dividing group" whereas G works as symmetry group: The orbits, the slice representations and the strata are taken with respect to H. They as well as the

[1] cf. e. g. [Hirzebruch/Mayer], [Jänich] or [Bredon]
[2] cf. [Lellmann]
[3] cf. [Verona], p. 128
[4] cf. [Bredon], p. 187
[5] cf. [Böhm] and chapter II

tubular systems and the quotient structures to be considered later are invariant under the action of G. The symmetry group of ${}_H\backslash M$ is the quotient $G/_H$.

Equivariant vector spaces are faithful orthogonal representations with no trivial summand. Their isomorphy classes are called slice types.[6] Stable sets of slice types are called slice diagrams. We measure the complexity of a slice diagram by its length.[7] The most important examples of a slice diagram is the set of slice types of an equivariant manifold.

Equivariant vector bundles are differentiable fibre bundles with equivariant vector spaces as fibres. These are exactly the normal bundles of the strata of our equivariant manifolds.

The layer next to the objects are the tubular systems: These are compatible families of tubes of the respective strata, with the compatibility defined recursively. Tubular systems always exist and each two tubular systems of a fixed object are isotopic.

We are, however, interested mainly in the third layer made up by the quotient structures. A quotient structure of an equivariant manifold is a structure of a differentiable manifold on its quotient, subject to certain recursive compatibility conditions. Analogously we would like the quotient of an equivariant vector space or vector bundle to be again a euclidean vector space and a riemannian vector bundle respectively. the quotient structures of the linear objects "equivariant vector space", "slice type" and "slice diagram" are called linearisations as well. Any tubular system or quotient structure of an equivariant manifold is defined with respect to a tubular system and a quotient structure of the slice diagram of that manifold.

Theorem I 7.7. *Let M be an equivariant manifold, φ a tubular system of its slice diagram Δ and let ψ be a tubular system of M with respect to φ. Then there is one and only one quotient structure of (M, ψ) lying over any given linearisation of (Δ, φ).*

This theorem steps from "local" to "global": (The existence of) a linearisiation of the local datum "slice diagram" yields (the existence of) a global differentiable structure on the orbit space. The latter is characterised by the property that the quotients of the tubes of the strata are tubes of the quotients of the strata.

By passing to isotopy classes one can get rid of the tubular systems in the case of compact objects:

Theorem I 8.11.

(1) Linearisability is a well defined property of slice diagrams without refering to tubular systems.

(2) An isotopy class of a linearisation of a slice diagram of a compact equivariant manifold determines an isotopy class of differentiable structures on the respective orbit space.

[6] The kind of slice types and diagrams I use are coarser than those of Jänich.

[7] also called depth or height

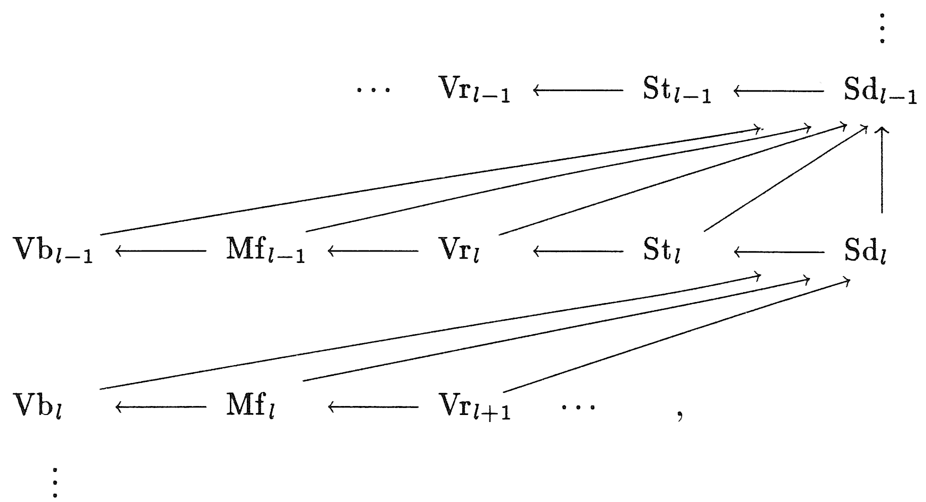
Now what do we know about existence and uniqueness of linearisations? An equivariant vector space is linearisable iff its unit sphere possesses a quotient structure, such that this quotient becomes equivariantly diffeomorphic to the standard sphere with an orthogonal action. Then by conical extension the quotient of the equivariant vector space is itself a euclidean vector space with an orthogonal action.

In spite of this rather strong condition, linearisability is not just a sporadic phenomenon. In fact, there is an infinity of linearisable slice types of any given length greater than one.

In 5.10 we explicitely list all linearisations of slice types of length two. In IV 2.8 we give a linearisation of a non abelian slice type of length four.

In this context at least two questions remain: Does topological "manifoldness" of the quotient imply the linearisability of an equivariant vector space? And: Can an equivariant vector space have two or more non isotopic linearisations?

Technically the first chapter is ruled by recursive definitions according to the following scheme:



where Vr_l denotes the equivariant vector spaces of length l, for example. Defining quotient structures for Vr_l, say, requires quotient structures for equivariant manifolds of length $l-1$ in addition to quotient structures for slice diagrams of length $l-1$. One actually proceeds from Mf_{l-1} to Vr_l by conical extension as described earlier.

The symmetry group comes into play when passing from Vr_l to Vb_l: The symmetry group of the equivariant vector space is at the same time the structure group of the equivariant vector bundle. By virtue of this group tubular systems and quotient structures of the fibre are turned into corresponding objects of the total space. The recursive nature of both tubular systems and quotient structures appears here in the guise of nesting of bundles, and their compatibility with the nesting is crucial for the strata to fit together.

Passing from Vb_k, $k \leqslant l$, to Mf_l involves the tubular systems: An equivariant manifold is covered by neighbourhoods of its strata. As tubes provide diffeomorphisms between neighbourhoods in the normal bundles and neighbourhoods in the manifold, and as the compatibility relations are carefully chosen, the differentiable structure on the quotient of the manifold can be furnished from the differentiable structures on the quotients of the normal bundles.

Equivariant vector spaces decompose into prim factors. In view of the second chapter we improve the definitions of tubular systems and quotient structures to match with this decompositon. The results of the first chapter remain true when using the new definitions instead of the old.

Chapter II deals with objects whose slice diagrams are abelian, i. e. each isotropy group modulo the kernel of the slice representation is abelian. In this case the linearisable slice types can be given explicitely, and we can show that there is no difference between orbit spaces that are topological manifolds and orbit spaces that are differentiable ones.

For abelian slice types we define a real and a 2-primary codimension respectively. The prime slice types of codimension 0 are the slice types of the standard representations of S^1 and $\mathbb{Z}_2$. In codimension 1 one obtains two series $(\sigma_a)_a$ and $(\tau^n)_n$. The finite products of abelian slice types of codimension 0 or 1 form a slice diagram Δ^{ab}, the meaning of which becomes clear by the following

Theorem II 5.11. *For an equivariant manifold with abelian slice diagram the following statements are equivalent:*

(1) The orbit space is a topological manifold (with boundary).
(2) The slice diagram is contained in Δ^{ab}.
(3) The slice diagram is linearisable.
(4) The orbit space admits a structure of a differentiable manifold (with smooth boundary).

The boundary is empty iff all the prime slice types have codimension one.

The equivalence of (1) and (2) translates a topological property of the equivariant manifold into representation theory. The proof follows from [Böhm] by a close analysis of the occuring representations. (3) implies (4) by virtue of the first chapter. (2) implies (3) by constructing a linearisation of Δ^{ab}; the following diagram gives the idea for linearising the slice type σ_a:

$$
\begin{array}{ccccc}
G(a)\backslash\mathbb{C}^n & & \xrightarrow{\quad f_a \quad} & & \mathbb{C}\times\mathbb{R}^{n-1} \\
\downarrow & & & & \downarrow \\
T^n\backslash\mathbb{C}^n & \cong & \mathbb{R}_{\geqslant 0}{}^n & \xrightarrow{\;g^n\;} & \mathbb{R}_{\geqslant 0}\times\mathbb{R}^{n-1}
\end{array}
\qquad
\begin{array}{ccccc}
G(a)\cdot z & & \longmapsto & & \omega_a\left(\frac{z}{|z|}\right)\cdot g^n(|z|) \\
\downarrow & & & & \downarrow \\
T^n\cdot z & \longmapsto & |z| & \longmapsto & g^n(|z|)
\end{array}
$$

$\mathbb{C}^n$ modulo the torus is the cubical corner $\mathbb{R}_{\geqslant 0}{}^n$, which is turned into the half space $\mathbb{R}_{\geqslant 0}\times\mathbb{R}^{n-1}$ by a smoothing g^n. Apart from the vector $|z|$ of absolute values the orbit $G(a)\cdot z$ possesses the angle $\omega_a\left(\frac{z}{|z|}\right)$ as an additional invariant. This angle fits well into the "complexification" $\mathbb{C}\times\mathbb{R}^{n-1}$ of the half space. To sum up f_a is a homeomorphism between the quotient $G(a)\backslash\mathbb{C}^n$ and a euclidean vector space.

That the family $(f_a)_a$ and its 2-primary analogon define in fact a linearisation of Δ^{ab}, imposes strong conditions on the sequence $(g^n)_n$ of smoothings and on the tubular systems of the slice types involved. To this end we join cubical corner an half space of each dimension by a family of corners whose angle varies. Then we construct by recursion a coherent choice of tubular systems for these families of corners, the complex standard vector spaces and the complexified half spaces. Finally we construct a sequence of smoothings matching as well with each other as with the tubular systems.

Chapter III exhibits an elaborate example: the complex projective plane with "dividing group" complex conjugation and symmetry group SO(3). The orbit space is isomorphic to the four dimensional sphere, as is mentioned in [Arnold I] and proved in [Massey], [Kuiper] and [Arnold II], the authors being independent of each other.

Massey considers an action of the dihedral group on $S^2\times S^2$. The quotients with respect to the various subgroups form a diagram of coverings — partially branched. Thereby he finds one of the quotient spaces being homeomorphic to $\mathbb{C}P^2/(\bar{\ })$ as well as to S^4.

Kuiper starts with a veronese-like embedding of $\mathbb{C}P^2$ into the selfadjoint mappings of $\mathbb{C}^3$. Taking the averidge over the $(\bar{\ })$-action leads to an embedding of the quotient $\mathbb{C}P^2/(\bar{\ })$. Algebro-geometric reasoning proves the image of the embedding to be S^4. According to Kuiper the isomorphy in question is piecewise linear and therefore even differentiable. Arnold claims diffeomorphy on the basis of this map, too, while his arguments are more geometric in nature.

The next theorem states in what way the differentiable structures of Kuiper and Arnold are intrinsic. Moreover it shows the equivariance of the homeomorphy as conjectured by Massey.

Theorem III 2.7. *The equivariant manifold $\mathbb{C}P^2$ possesses one and only one isotopy class of quotient structures, and with respect to this the quotient $\mathbb{C}P^2/(\bar{\ })$ is* SO(3)*-equivariantly diffeomorphic to the four dimensional standard sphere.*

The set of oriented, normed ellipses that are centered in the origin of a three dimensional euclidean vector space V is a model of the complex projective plane $P(V_{\mathbb{C}})$. In this model many geometric properties, caused by $V_{\mathbb{C}}$ being the complexification of a euclidean vector space, become visable, e. g. does complex conjugation correspond with the change of orientation. The set of unoriented ellipses is therefore a model of the orbit space $P(V_{\mathbb{C}})/(\bar{\ })$. The isomorphy of $\mathbb{C}P^2$ to the second symmetric power of $\mathbb{C}P^1$ used by Massey is constructed by means of the ellipse model and elemental geometry.

Chapter IV treats the example that marked the starting point for investigating into the general quotients of chapter I and the special quotient of chapter III as well: On SU(3) there acts the square of the maximal torus T of diagonal matrices by $\big((x,y)\,,\,g\big) \mapsto xgy^{-1}$. The quotient $T\backslash \mathrm{SU}(3)/T$ is homeomorphic to S^4.

The general setting is examined by the next theorem, it's first part being an application of chapter II:

Theorem IV 1.1. *Let T be a maximal torus of a compact connected Lie group K. Then the quotient $T\backslash K/T$ is a manifold — possibly with boundary — iff K is locally isomorphic to $\mathrm{U}(1)^k \times \mathrm{SU}(2)^l \times \mathrm{SU}(3)^m$ for suitable $k, l, m \geqslant 0$. In this case the quotient is homeomorphic to $I^l \times (S^4)^m$.*

I consider the normaliser of $T \times_{\mathbb{Z}_3} T$ within the isometries of SU(3) to be the symmetry group of the equivariant manifold SU(3). Then the quotient symmetry group is finite and has exactly one faithful orthogonal representation of dimension five.

Conjecture. *The quotient $T\backslash \mathrm{SU}(3)/T$ has a distinguished isotopy class of differentiable structures, and with respect to this it is equivariantly diffeomorphic to the four dimensional standard sphere with an orthogonal action.*

Approaching this conjecture we prove:

(1) The quotient $T\backslash \mathrm{SU}(3)/T$ possesses a distinguished isotopy class of differentiable structures.
(2) The quotient is homeomorphic to S^4.
(3) The quotient possesses an equivariant topological embedding into $\mathbb{R}^5$ by invariant homogeneous polynomials of degree two and three on SU(3).

A geometric understanding of the quotient of SU(3) starts with the graph shown in figure 25. Its vertices constitute the deepest stratum of the quotient, its open edges make up the middle one, the main stratum is missing however. The symmetry group of the graph is — enlarged by a factor $\mathbb{Z}_2$ — exactly the symmetry group of the quotient.

The two-sided action of T on SU(3) yields the two-sided action of the maximal 2-primary torus Q on SO(3) by passing to the fixed points of the complex conjuga-

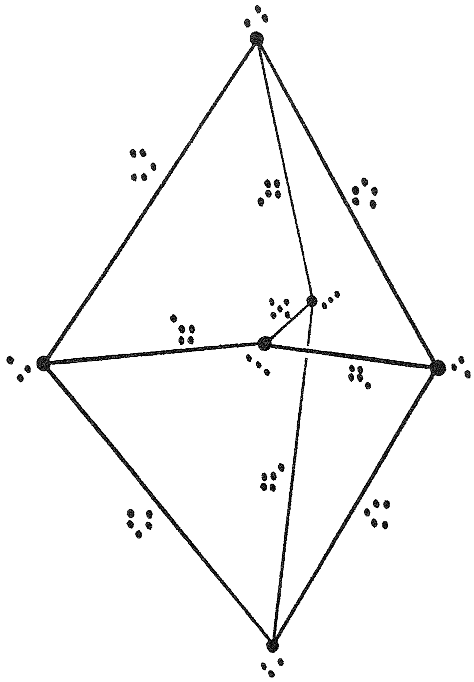


Figure 25: skeleton of $_T\backslash\mathrm{SU}(3)/_T$ and $_Q\backslash\mathrm{SO}(3)/_Q$ respectively

tion. Its quotient lies embedded in the quotient of SU(3) and has already the same graph formed of the lower strata. The quotient $_Q\backslash\mathrm{SO}(3)/_Q$ is equivariantly diffeomorphic to S^3. By the homeomorphy (2) $_Q\backslash\mathrm{SO}(3)/_Q$ becomes the equator of S^4; the embedding (3) maps $_Q\backslash\mathrm{SO}(3)/_Q$ into a hyperplane. A suitable triangulation of SO(3) makes the quotient map rather comprehensible, and the isomorphy with S^3 is explicitely constructed.

Utilising the graph made up by the lower strata Prof. Dr. Matthias Kreck first proved the homeomorphy of $_T\backslash\mathrm{SU}(3)/_T$ to S^4 by means of algebraic topology. Here we sketch an elementary proof with the help of complex conjugation on the quotient:

$$\begin{array}{rcl} f:\ (\bar{\ })\Big\backslash\Big(_T\backslash\mathrm{SU}(3)/_T\Big) & \longrightarrow & \mathbb{R}^4 \\ {[A]} & \longmapsto & (a_{ij}\overline{a_{ij}})_{i,j\leqslant 2} \end{array}$$

is a topological embedding: the orbit of a matrix is characterised by the absolute values of its entries. The image of f is homeomorphic to D^4, its boundary is just the image of $_Q\backslash\mathrm{SO}(3)/_Q$. As the latter is the fixed point set of complex conjugation, $_T\backslash\mathrm{SU}(3)/_T$ is homeomorphic to $D^4 \cup_{S^3} D^4$.

I like to thank Prof. Dr. Matthias Kreck for patiently supervising this project. He as well as Dr. Rainer Jung have always welcomed me in Mainz. I had the opportunity to discuss several questions with people at the Mathematisches Institut of the university of Bonn and at the Max-Plank-Institut für Mathematik. Tillmann Klingholz filled me with enthusiasm for TEX while sharply criticizing my style and my layout. Susanne Dahlmann worked persistently towards the clarity of my seminar talk. Dorthe Herrmann did the proof-reading of a large part of the manuscript and brought me back from the wrong track in some respects. I thank them all very much. By their listening, their advices and by some discussions when they were necessary, they took part in the completion of this work.