Quotients of compact Lie group actions

Given a differentiable action ρ of a compact Lie group H on a manifold M.

Is the orbit space ${}_H\backslash^M$ again a manifold?

In general : No!

${}_H\backslash^M$ is a stratified space.

$\exists \mathcal{E} : M \longrightarrow ({}_H\backslash^M, \mathcal{E})$ is a diff. submersion

$\Longleftrightarrow$ All orbits are principal conditions? top / diff? boundary?

point x isotropy representation $\rho_x : H_x \to GL(T_x M)$

orbit Hx slice type $H_x \to GL(N_x Hx)$

stratum X normal type $H_x / \ker \hookrightarrow GL(N_x X)$

slice diagram $\Delta_J(M)$ normal diagram $\Delta(M)$

${}_H\backslash^M$ is a top. mfd $\Leftrightarrow$ the quotients of the lowest slice types are

A linearisation of $\Delta(M)$ induces a diff. structure on ${}_H\backslash^M$.

Let $\Delta(M)$ be abelian. Then ${}_H\backslash^M$ top. mfd $\Leftrightarrow$ ${}_H\backslash^M$ diff. mfd $\Leftrightarrow$ $\Delta(M) \subset \Delta^{ab}$:= generated by prime normal types of codimension 1 or 0

Let G be a compact Lie group and T a maximal torus of G.

$${}_T\backslash^{G}/_T \text{ is a mfd} \Leftrightarrow G \underset{\text{locally}}{\cong} (S^1)^k \times SU(2)^l \times SU(3)^m$$

In this case $${}_T\backslash^{G}/_T \approx \pi_0(G) \times I^l \times (S^4)^m$$

$${}_T\backslash^{SU(3)}/_T \approx S^4$$

Kreck & Kastenholz

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elementary geometric proof

equivariantly with respect to an orthogonal action of

$$(\mathfrak{S}_3^2 \rtimes {}^t(\;)) \rtimes \overline{(\;)}$$

contains $Q\backslash SO(3)/Q \approx S^3$

suffices to show $SU(3)/(T \times T) \rtimes \overline{(\;)} \approx D^4$

$$SU(3) \xrightarrow{\alpha} M(3\times 3, \mathbb{R})$$

$$A \longmapsto \left(a_{ij}\overline{a_{ij}}\right)_{ij}$$

image lies in affine subspace $\{B \mid \sum_i b_{ij} = 1, \sum_j b_{ij} = 1\}$

... in nonnegative cone $\{B \mid b_{ij} \geq 0\}$

... in semi-algebraic set $\{B \mid p(B) \leq 0\}$

where $p\begin{pmatrix} a & x & \cdot \\ b & y & \cdot \\ c & z & \cdot \end{pmatrix} = a^2x^2 + b^2y^2 + c^2z^2$

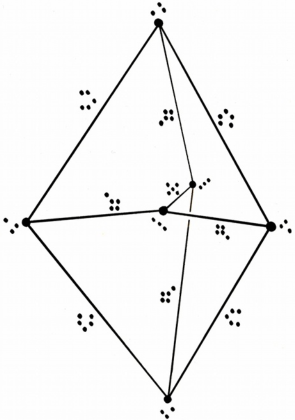
$$-2(bycz + czax + axby)$$

$$SU(3)/_{(T\times T)\rtimes \Pi} \hookrightarrow M(3\times 3, \mathbb{R})$$

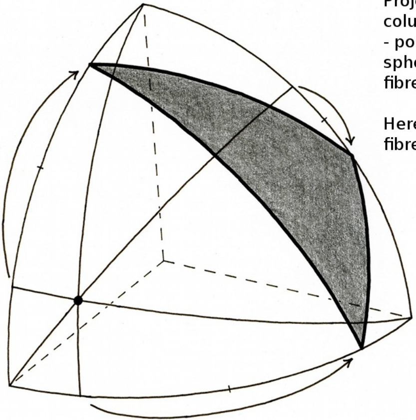
image <u>is</u> the intersection of the above-mentioned sets

image is "strongly" star-shaped with resp. to $\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

rescaling along the rays yields $SU(3)/\ldots \approx D^4$

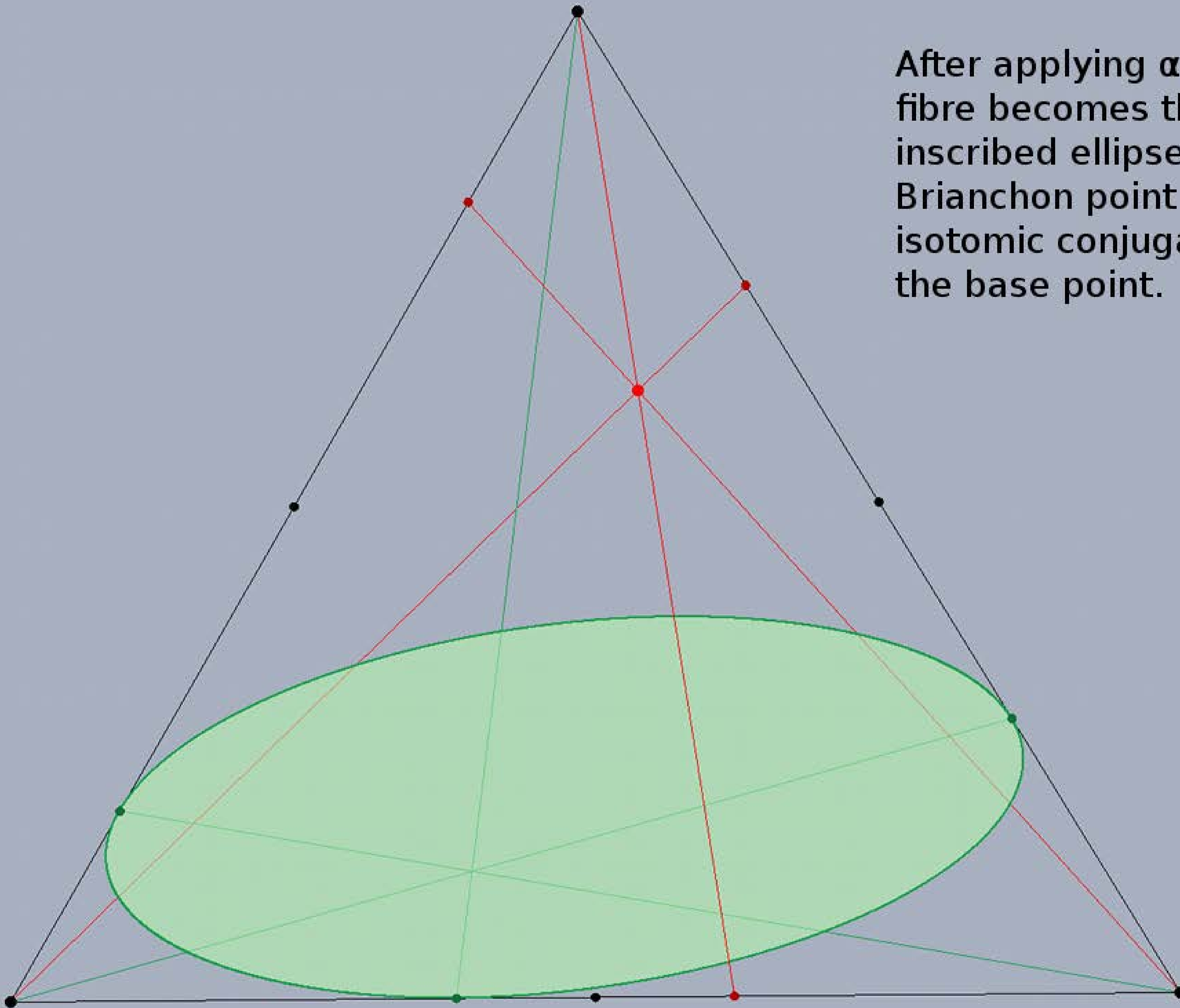


skeleton of non-principal strata



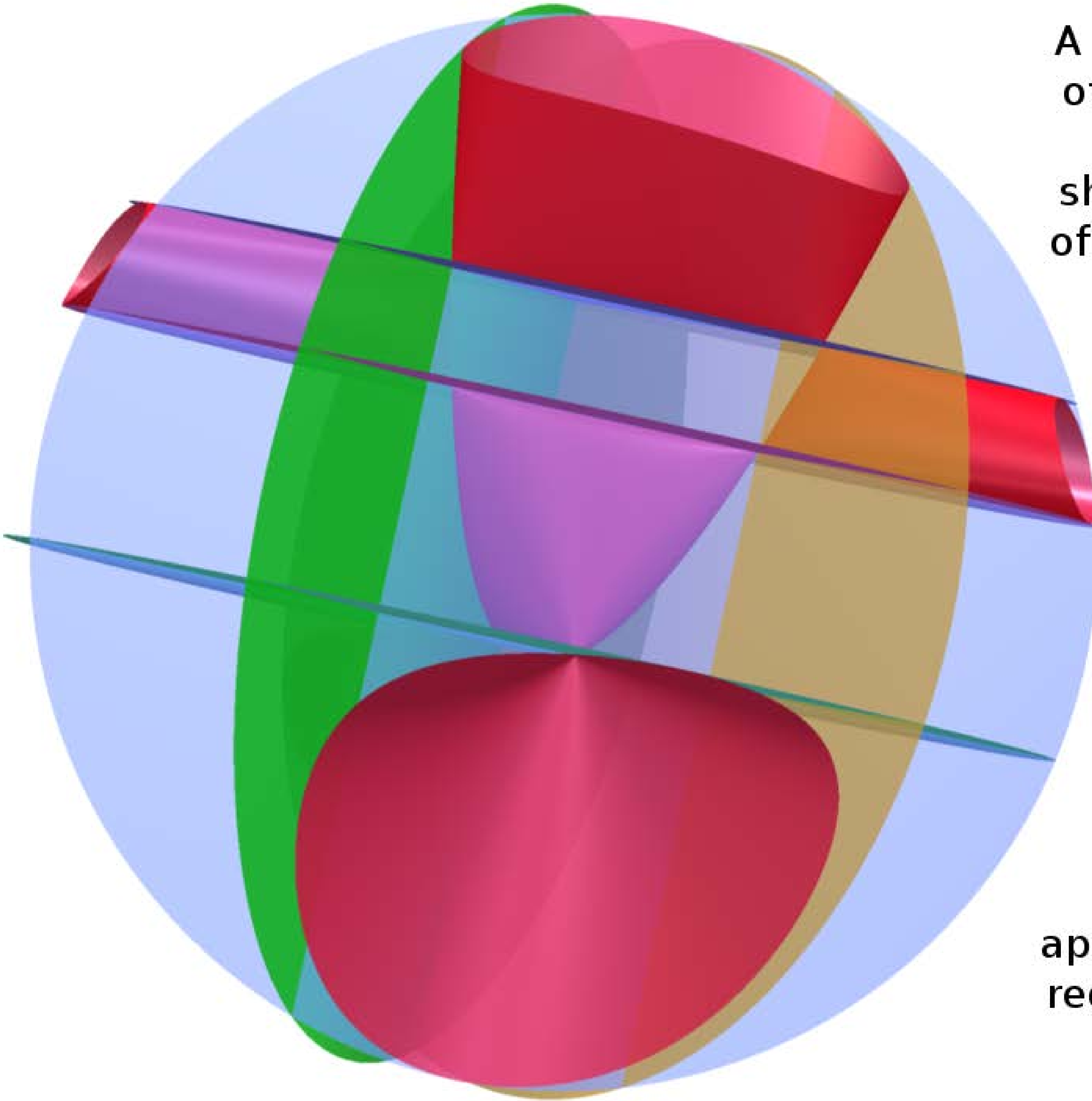
Projecting onto the first column vector yields - possibly degenerate - spherical triangles as fibres.

Here we construct the fibre from the base point.



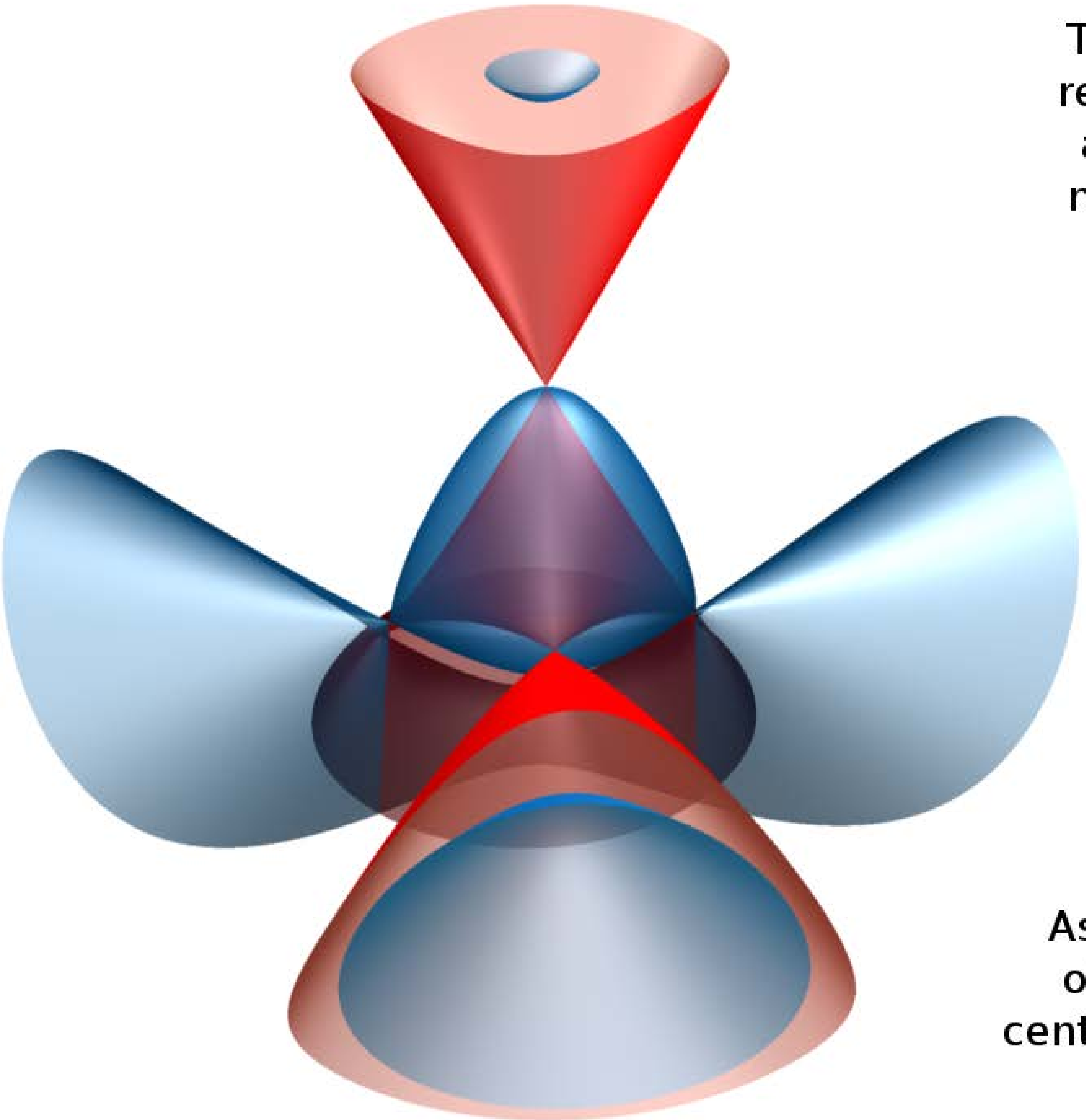
After applying α the fibre becomes the inscribed ellipse whose Brianchon point is the isotomic conjugate of the base point.

A 3-dimensional slice of th 4-dimensional affine subspace showing the zero set of the polynomial p.



A 3-dimensional slice of the 4-dimensional affine subspace showing the zero set of the polynomial p in red and the coordinate hyperplanes.

The image of α appears as the central red "tetrahedron" and its interior.



The zero set of p in red once again, and a certain "cage of non-transversality" in blue.

As before, the image of α appears as the central red "tetrahedron" and its interior.