Arbeitsgemeinschaft mit aktuellem Thema

Operaden und ihre Anwendungen

Oberwolfach 08.10. - 14.10.2000

The basic idea behind the notion of an operad is the idea of substitution: given r functions $f_1, \dots, f_r$ in $n_1, \dots, n_r$ variables and given a function g in r variables the substitution yields a function h of $n = n_1 + \dots + n_r$ variables

$$h(x_1, \dots, x_n) = g(f_1(x_1, \dots, x_{n_1}), \dots, f_r(x_j, \dots, x_n))$$

with $j = 1 + \sum_{i=1}^{n-1} n_i$. The arguments x_i and the values of the functions are all in some X, which might be something like a vector space or a topological space, and the functions should be morphisms in the related category. An operad is intended to organize a system of such functions and substitutions.

The definition of an operad $\mathcal{P}$ in some symmetric monoidal category $\mathcal{C}$ is now as follows:

The operad $\mathcal{P}$ consists of a sequence $\mathcal{P}(0), \mathcal{P}(1), \dots$ of objects and for every r-tuple $\overline{n} = (n_1, \dots, n_r)$ of natural numbers $n_1, \dots, n_r \geq 0$ there is a structure morphism

$$\gamma : \mathcal{P}(r) \otimes \mathcal{P}(n_1) \otimes \dots \otimes \mathcal{P}(n_r) \longrightarrow \mathcal{P}(n_1 + \dots + n_r).$$

Here the $\otimes$ denotes the symmetric monoidal structure in $\mathcal{C}$. These structure maps are associative and there is a unit element in $\mathcal{P}(1)$. Usually the n-th term of an operad $\mathcal{P}(n)$ carries an action of the symmetric group on n letters.

The master example of an operad is the endomorphism operad $\mathcal{E}nd(X)$ of an object $X \in \mathrm{Ob}(\mathcal{C})$: For every n one considers the morphisms from the n-fold product of X to X and defines

$$\mathcal{E}nd(X)(n) := \mathrm{Mor}(X^{\otimes n}, X).$$

Operads are important because of their representations: An operad $\mathcal{P}$ may act on some object $X \in \mathrm{Ob}(\mathcal{C})$ via maps

$$\rho : \mathcal{P}(n) \otimes X^{\otimes n} \longrightarrow X,$$

which are compatible with the unit in the operad and which are associative and equivariant, then X is called a $\mathcal{P}$-algebra. $\mathcal{P}(2)$ encodes binary operations on X and $\mathcal{P}(3)$ is a parameter space for ternary operations. Thus a property like having an associative product on X or more general a system of operations and relations, i.e. something like a theory can be expressed by the means of an operad. Every operad yields a theory in the sense of Lawvere (see [SSM], p. 8) but not all theories come from operads.

Historically the theory of operads began with the investigations of iterated loop spaces in the sixties. There have been some predecessors, mainly algebraic ones. Lazard's idea of formal group laws is one of those: To look for all power series $G(x, y)$ in two variables x and y over a commutative ring R such that the substitution

$$(f_1(x), f_2(x)) \mapsto G(f_1(x), f_2(x))$$

yields a group structure on the ring of power series $R[[x]]$ is similar to the idea of an operad acting on $R[[x]]$ and it connects the concepts of operads to the idea of deformations.

Due to Adams and MacLane there are the notions of PROPs and PACTs which were invented in order to deal with commutativity "up to higher homotopies". Boardman's PROP of linear isometries was a key tool for understanding infinite loop spaces. J. Stasheff constructed the family of associahedra $\{K_n\}$ – now known as Stasheff polytopes – in order to characterize spaces of the homotopy type of a loop space. In 1973, Boardman and Vogt published their book on "Homotopy invariant algebraic structures on topological spaces" [BV]. J.P. May introduced the term "operad" for these kind of structures in [Ma].

The operad which is used to study H-spaces and iterated loop spaces, i.e. "topological groups up to homotopy", is the so-called little cubes operad $\mathcal{C}^n, (n \leq \infty)$, in $\mathbb{R}^n$. It can be used to identify n-fold loop spaces. This operad acts on every n-fold loop space and a space can be unmasked to be an iterated loop space if it allows such an action.

In general there are two main questions associated to a given operad $\mathcal{P}$ in a symmetric monoidal category $\mathcal{C}$:

1. Which objects $X \in \mathrm{Ob}(\mathcal{C})$ possess an action of the operad $\mathcal{P}$?

   For example, a connected space X carries an action of the operad of little n-cubes iff it is an n-fold loop space.

   Like a group action on a space translates to a group action on the homology of that space, an operad $\mathcal{P}$ leads to actions on the homology of a $\mathcal{P}$-algebra.

2. How does a $\mathcal{P}$-action help to study the homology of X?

   For example the homology of an H-space is a Hopf-algebra and the homology of the operad of little cubes yields the so-called Dyer-Lashof operations on every iterated loop space. In the algebraic setting an algebra over a special kind of operads (called Koszul operads) bring their appropriate homology theory with them.

The first part of the Arbeitsgemeinschaft (talks 1-6) is mainly concerned with algebraic operads. After a general introduction mentioning the relationship of operads and polynomial functors our aim is a description of the important rôle which Koszul operads play in the algebraic setting. The important concept of "algebras up to homotopy" is presented in talk 6.

The origin of operad theory as a means to understand loop space structures is the subject of the talks 7 to 9. These operad structures are still a very important subject in nowadays research (see talks 14 and 15).

Applications of operads in various fields of mathematics and physics will be presented in the last part of the Arbeitsgemeinschaft. The relationships to other subjects are widely spread – from field theories to moduli spaces and the mapping class group, from vertex operator algebras to the Grothendieck-Teichmüller group. Besides this, we will deal with Deligne's famous formality conjecture and its proof.

Introductory literature

As a good source for a brief introduction to operads and for a survey of their history we suggest the first part of the (unpublished) book by S. Shnider, J. Stasheff and M. Markl. If you are interested in a preliminary version of this book please mail to `richter@math.uni-bonn.de` and it will be sent to you as an attachment in ps.gz format. In addition we recommend the two overview papers "Definitions: operads, algebras and modules" and "Operads, algebras, modules" by J.P. May in [LSV, pp. 1-7, 15-31].

Talks

1. **Monads and operads**
   In this talk the definition of operads as monoids in the category of symmetric objects as in [J] and [Sm], their associated Schur functors and the concept of monads ([ML, chapter VI]) are introduced. The monad associated to an operad is related to the classification of polynomial functors which is described in [Md, pp. 74-84].

2. **Important examples of operads**
   Besides the definition of algebras over operads important examples such as iterated loop spaces as algebras over the operad of little cubes (see [A] and [Ma] for a description) and some of the classical algebraic examples like associative and commutative algebras and Lie algebras [KM] will be considered. Conceptual examples such as free algebras and a first example of algebras up to explicit homotopies, namely A_∞-algebras [GJ2], should be mentioned.

3. **Koszul duality I – the case of associative algebras**
   The notion of Koszul duality for operads defined in [GiK] has its origin in Koszul duality for associative algebras. S. Priddy introduced this term in [P] to describe a certain kind of quadratic filtered algebras which admit a small resolution. For graded associative algebras an introduction to this concept is described in [L3].

4. **Koszul duality II**
   For quadratic operads such as the operads encoding associative, commutative or Lie algebras there is a natural notion of an operadic dual defined in [GiK]. Algebras over such operads possess a chain complex which defines a homology theory. If the encoding operad is Koszul, this complex is acyclic on free objects; hence it provides an appropriate theory for algebras over the operad. As additional literature we recommend [Man], [L1] and [SSM, 3.2-3.4]

5. **Algebraic homology theories**
   This talk should give an introduction to the classical algebraic homology theories such as Hochschild homology [L2, chapter 1], homology of Lie algebras [L2, chapter 10] and André-Quillen homology [Q], [L2, 3.5]. The complexes arising from Koszul theory for associative, commutative algebras and Lie algebras coincide with the defining complexes in these classical cases [GiK], [SSM, 3.4]. For general references on the classical theories see also [CE] and [W].

6. **Homotopy algebras**
   Starting from the notion of a minimal model in the rational homotopy theory of algebras ([GM, chapter IX] and [FHT]) the concept of homotopy algebras is introduced via the language of minimal operads [M1]. For a given differential graded operad the homotopy algebras over these operad are the algebras over the associated minimal model. In the case of quadratic Koszul algebras the minimal model coincides with the cobar construction on the quadratic dual of the operad. For further reference see also [M2] and [SSM]. The "classical" notions of A_∞-algebras in the sense of [GJ2] and homotopy Gerstenhaber algebras [GV2] should be mentioned.

7. **Loop spaces**
   H. Hopf had begun to study the homology and cohomology of Lie groups arriving at the notion of a Hopf algebra and H-spaces. The (based) loop spaces ΩX are the most important H-spaces. Sugawara gained important results on the identification of H-spaces and loop spaces in the fifties and sixties. The talk should address the problem how to identify a space to be a loop space and introduce Stasheff's non-symmetric operad and the notion of A_∞-spaces including a proof of the main result of [Sta].

8. **Iterated loop spaces I**
   The first truly important period for the notion of an operad came with the introduction of the "little n-disc operad" $\mathcal{C}^n$ of Boardman-Vogt [BV], May [Ma] and others, in order to study spaces of the homotopy type of iterated loop spaces $\Omega^n X$. The talk introduces this operad and its connection to configuration spaces and then proves the recognition principle. Furthermore, the Dyer-Lashof algebra and its action on the homology of a $\mathcal{C}^n$-space should be introduced, following [Co]; the Browder operation should be mentioned.

9. **Iterated loop spaces II**
   To continue the previous talk we introduce the "little ∞-disc operad" $\mathcal{C}^\infty$ and prove the recognition principle for E_∞-spaces, i.e. $\mathcal{C}^\infty$-algebras. One can emphasize the rôle of the Browder operation as a first obstacle to an E_∞-structure. In a second part one should introduce Boardman's "linear isometries operad" and describe its importance for the construction in [EKMM] of a symmetric monoidal category of spectra.

10. **Formality and the Deligne conjecture**
Starting with [GV1] one defines "formality" and the operad of chains belonging to the operad $\mathcal{C}^2$ of little discs and explains the statement of the Deligne conjecture on the structure of the Hochschild cochains. Following [K] one should sketch a proof of this conjecture, also using [T1], [T2], [KS] and [MS].

11. **Batalin-Vilkovisky algebras**
The $\mathbb{S}^1$-action on a double loop space and the Pontryagin product endow the homology of a double loop space with a structure of a Batalin-Vilkovisky (BV) algebra. The associated bracket is the Browder operation [Br]. Besides the definition of BV algebras the structure of a BV algebra on the cohomology of a topological conformal field theory (TCFT) in two dimensions should be explained along the lines of [G]. To this end the little framed disc operad is introduced as a variant of the little disc operad.

12. **Moduli spaces I**
A very interesting feature of the collection of moduli spaces $\mathfrak{M}^n_{g,r}$ of Riemann surfaces of genus g, with n punctures and r boundary curves is (a) that they form an operad if one boundary curve is specified, and (b) that they carry a $\mathcal{C}^2$-structure. The talk should define moduli spaces and should describe the operad structure. The mapping class group $\Gamma^n_{g,r}$ has to be introduced and the stability result for its homology has to be explained along the lines of [Ha]. A reference for the basic facts about the mapping class group is [Bi]

13. **Vanishing of the BV-structure for TCFTs**
This talk should give a more detailed discussion of the BV-structure for TCFTs. An introduction to TCFTs and the BRST-complex should be given and U. Tillmann's vanishing result [Ti1] for the BV-structure on the homology of the BRST-complex should be presented.

14. **Moduli spaces II**
One of the most striking applications of operads is U. Tillmann's result that the classifying space of the stable mapping class group $\Gamma^n_{\infty,1} = \mathrm{colim}_g \Gamma^n_{g,1}$ is an infinite loop space after group completion. The main point of the proof is that the operad of all moduli spaces has itself a $\mathcal{C}^\infty$-structure. The references are [Ti2] and [Ti3].

15. **Torsors and the Grothendieck-Teichmüller group**
The article by Kontsevich [K] which was already used in the talk on the formality theorem considers in its second part the action of the Grothendieck-Teichmüller group GT on an infinite dimensional manifold of deformations, namely on $\mathrm{Aut}(\mathrm{Chains}(\mathcal{C}^2))$. In this ambitious talk one might attempt to explain various conjectures made by Kontsevich.

16. **Vertex operator algebras**
Y.Z. Huang [Hu1], [Hu2] gave an operadic definition of topological vertex operator algebras. This should be explained, together with the Batalin-Vilkovisky structure on the homology. If possible, the motivation from physics should be given.

17. **Planar algebras and C^*-algebras**
Vaughan Jones introduces in [Jo] a "planar operad", melting the notion of the little 2-cubes operad from homotopy theory with tangles from knot theory. This operad is then used to produce subfactors of a type II_1-factor in a C^*-algebra. It is a first step towards a Galois theory for subfactors. The interesting example of bipartite graphs may be presented. Although this long article is difficult to survey it seems nevertheless to be worth while to attempt to understand its ideas. A further reference is [BJ].

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Participation

If you intend to participate, please send your name and address to

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until 14th of August 2000. We would like you to name three of the talks listed above which you are willing to give.