Arbeitsgemeinschaft

Ergodic Theory and Heegner Points

Oberwolfach 1.4-7.4. 2001

The following assertion is a special case of Marina Ratner's theorem on the Raghunathan conjecture.

Theorem. *Let p be a prime and let us consider a finite product*

$$G = \prod_{i=1}^{n} PGL_2(\mathbf{Q}_p).$$

Let $\Gamma = \prod_{i=1}^{n} \Gamma_i \subset G$ be a product of a family $\{\Gamma_i\}_{i=1,\ldots n}$ of noncommensurable, discrete, cocompact subgroups of $PGL_2(\mathbf{Q}_p)$. Let $\Delta \tilde{\to} PGL_2(\mathbf{Q}_p) \subset G$ be the diagonal subgroup. Then the orbit of Δ on $\Gamma \backslash G$ is dense.

This theorem plays a crucial role in the proof of Mazur's conjecture on Heegner points, by Cornut and Vatsal. This proof is at the center of the second half of this "Arbeitsgemeinschaft". The goal of the first half will be to get some understanding of the ideas behind Ratner's proof of Raghunathan's conjecture.

Unfortunately, we had some problems to find an organizer for the first half of the programme, so I (G.H.) tried to do it myself. I found it very difficult to read Ratner's papers, and I believe that it is almost impossible to give an outline of the proof in the most general versions of those theorems, which imply the above result as its special case. In the paper [Ra1] we find a proof of a theorem whose proof can be explained in 3 lectures or so. This proof contains some of the fundamental ideas of M. Ratner.

Last minute change After a first version of this programme was finished, Raghunathan pointed out that the paper [M-T2] contains a complete proof of Rathner's general theorem, and is much more readable. Since there is some urgency to get the programme out, I indicate two alternative possibilities for lectures 7–10. Which one will be chosen will depend on the volunteers for one or the other solution.

PART I: RATNER'S WORK

Lectures 1–2: *Statement of the main Results and applications, following [R5].*
This should be two lectures on Monday morning explaing the Raghunathan conjectures and Ratners's results. The Theorems 1-4 should be formulated. Some applications should also be discussed here, e.g., Oppenheim's conjecture and the applications to rigidity results: [Ra4] Section 6, see also [Gh] 4.1 and 4.2.

Theorems S1 and S2 (p. 168 of [R5]) will both be needed in the second part (in a special case) and should therefore also be dicussed.

Lecture 3: *Hedlund's theorem.*

The theorem of Hedlund says that, for any cocompact discrete subgroup $\Gamma \subset SL_2(\mathbf{R})$, every orbit of the group $U(t) = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbf{R} \right\}$ is dense. The proof is not so hard, it is outlined in [Gh, 2.2]. This in itself would be a short talk, but one should include a discussion of the non-cocompact case, which is more interesting, and helpful for studying Ratner's papers (see [Gh] 3.1, 3.2, 3.3). At this point, Dani's theorem in [Da1] should at least be mentioned. If one wants to say something about the proof of Dani's theorem, the whole lecture may be even cover two talks.

Lecture 4: *Ergodic properties of unipotent actions*

[Ra1], section 2, Theorem 2.1 and 2.2.

Lecture 5: *The (R)-property.*

This is a crucial point and perhaps the most important idea in Ratner's papers (it always comes back like an ergodic flow). Ghys makes some effort to explain this idea in [Gh] 3.1 and 3.2. (I think property (H) and property (R) are closely related.)

The person giving this talk ought to communicate with the speaker for lecture 3.

Lecture 6: *Proof of theorem 4.1 and theorem 1 in [Ra1].*

Good luck! — The ideas are nicely explained on the second half of p. 477.

Note that the proof of the theorem for solvable groups is much easier than that of the general case. The next step would be to discuss the proof of Theorem 1 (Main Theorem) in [Ra2]. I think that these proofs simplify considerably when specialized to the case we are interested in:

Thus we want to apply the Main Theorem to the case $\Gamma \backslash G$ and the unipotent flow given by the group

$$U(t) = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbf{Q}_p \right\} \subset \Delta.$$

We have several advantages over the general case:

i) The quotient is compact, hence the closures of orbits are compact as well. Therefore assertions like theorem 2.1 and 2.2 from lecture 4 become obvious.

ii) If we succeed in finding a diagonal element $a \in T(\mathbf{Q}_p) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid t \in \mathbf{Q}_p^\times \right\} \subset \Delta$, where a is not a unit, in the group Λ, then we can use the same arguments as in Lecture 3 to show that the two unipotent radicals

$$U_+, U_- \subset G = \prod_{i=1}^{n} PGL_2(\mathbf{Q}_p)$$

are contained in Λ, and this is certainly enough.

Now, (ii) should be doable by following Ratner's arguments in [Ra2] 3 and translating them to the p-adic context:

Lectures 7–10: Specialize to our concrete situation and translate [Ra2] 3. – 4. into the p-adic context.

I think the point is the decomposition 3.6. In our special situation, this is not difficult to write down. Formulate the basic Lemma 3.1. If we now succeed in proving theorem 4.1, then we are through.

One might not need 4 lectures for the proof, and accordingyl part I may be done before the end of Wednesday morning. If this happens, part II may be slightly expanded, with the help of the experts present. This will be decided in Oberwolfach.

Alternative for Lectures 7–10: Discuss the proof given in [M-T2] and [M-T3]. Its general strategy is discussed in the Comptes Rendus note. No further subdivision of these lectures can be offered so far.

PART II: MAZUR'S CONJECTURE

Lecture 11: *The arithmetic of definite quaternion algebras and Gross curves.*

This talk presents algebraic and class field theoretic ingredients for the constructions of the subsequent talks.

(a) Imaginary quadratic fields, orders, ring class fields and their connection with CM elliptic curves, anticyclotomic extensions and characters. — [Cox], [Serre], and a little bit of [Mazur]. *Care should be taken to avoid repetition with lecture 14 where classical Heegner points will be defined; talk 10 only presents their natural fields of definition.*

(b) Definite quaternion algebras, maximal and Eichler orders, orientations, embeddings of K and conductors. — [Vign].

(c) Gross's curve attached to a definite quaternion algebra and its connected components, Hecke operators. "Gross points" (= analogues of Heegner points on Gross's curve) and their conductors, Galois action. — [Gross], [BD2], [V1].

(d) Normal form, tree structure, rewriting the set of connected components $\Gamma'\backslash T$, where Γ is the subgroup of elements of norm 1 in $R[1/p]$. The commensurator of Γ. Galois action on normal forms — [V1, sec. 3], [BD3].

Lecture 12: *Gross's formula, statement and beginning of proof of the main result in the definite case.*

Gross's formula is the analogue for Gross points of the Gross-Zagier theorem. The talk also states the basic non vanishing result for L-values in the "definite case" which will follow from Ratner's theorems. — References for almost everything treated here are [Gross], [BD2], and in particular [V1], [V2], esp. sections 4 and 5.

(a) The function $\psi : \mathrm{Cl}(B) \longrightarrow \mathbf{Z}$; the special value formula.

Theorem. *Let K be an imaginray quadratic field with prime discriminant D not dividing N, and let p be a prime number not dividing $2NDh_K$. Then $L(g, \chi, 1) \neq 0$, for almost all ring class characters χ of p-power conductor.*

(b) First reduction of the proof: write $\chi = \chi_t \cdot \chi_w$ (tame and wild part), and consider only χ's with fixed χ_t. — *The Galois conjugacy of the special values was proved by Shimura, see [Shim]. The proof does not have to be reproduced in this lecture.*

(c) "Regularized Heegner points" (in the terminology of [BD2]), theta elements. Define the theta element $\theta = \theta(\chi_t)$. — [V2].

(d) It suffices to show that: *for all $n >> 0$ there exists χ_w of conductor p^n such that $\chi_w(\theta) \neq 0$.*

(e) Evaluate $\sum \chi_w(\theta)$ the sum being taken over all χ_w of conductor p^n, and express the result in terms of the points $\tilde{P}_n^\tau, \tilde{P}_{n-1}^\tau$, where $\tilde{P}_n$ is a certain regularized Heegner point of conductor p^n, with predecessor $\tilde{P}_{n-1}$, and τ is in the tame group.

(f) From the above, obtain a criterion for the non-vanishing in terms of distribution of the vectors $(\tilde{P}_n^\tau)_{\tau \in G_1}$ and $(\tilde{P}_{n-1})_{\tau \in G_1}$.

Lecture 13: *Surjectivity of reduction.*

This is first talk where the application of Ratner's theorems occurs.

(a) The basic non-commensurability of the groups Γ and Γ_τ, for τ in the tame group. — This is done both in [V1] and in [Cor1, §3.5], see also [Cor2].

(b) Proof of the surjectivity of the reduction map (to the connected components, via Ratner and group theory. — See [V1], combined with [Cor1, §3.6], [Cor2, §6.3] (cf. [V1, §4.8], [V2, §5.10].

Remark. Contrary to what will be seen in the proof of Mazur's Conjecture, more than the simple surjectivity is required for the definite case: one wants to prescribe the images of $\tilde{P}_n^\tau$ and $\tilde{P}_{n-1}$ simultaneously. Furthermore, one needs to know that any desired pattern occurs at level p^n, for every $n >> 0$. To get this, one has to use Ratner's theorem on uniform distribution of 1-parameter subgroups.

Lecture 14: Heegner Points and Modular Curves.

This talk prepares for the second application of Ratner's theorems. — Care should be taken to avoid repetition with lecture 10.

(a) Elliptic curves over fields and their endomorphism rings, classical theory of complex multiplication in characteristic 0 (j-invariants and ring class fields, fields of definition for finite subgroups). Supersingular elliptic curves. — Any book on the arithmetic of elliptic curves, for instance by Silverman; for the classical CM-theory: [Cox], esp. chapter 2, and [Serre].

(b) The modular curve $X_0(N)$ over $\mathbf{C}$ and over $\mathbf{Z}[1/N]$, including its modular interpretation. CM points and Heegner Points: Fields of definition, Galois action. Description of the supersingular locus. Bijection between isomorphism classes of (N,N)-inclusions of left R-lattices in B (where R is a maximal order in the quaternion algebra B), and the set of supersingular points: $[J \subset I] \mapsto [E^I \to E^J]$. Since $\widehat{B}^*$ acts transitively on these inclusions (on the right), one finally obtains:

$$\begin{array}{ccc} \widehat{R'}^* \backslash \widehat{B}^*/B^* & \longrightarrow & X_0^{\mathrm{ss}}(N)(\mathbf{F}_{\ell^2}) \\ \hat{b} & \mapsto & [E^{I_0\hat{b}} \to E^{J_0\hat{b}}] \end{array}$$

where $J_0 \subset_{(N,N)} I_0$ is a fixed inclusion of left R-ideals, and $R' = O_r(J_0) \cap O_r(I_0)$ is the intersection of two maximal orders in specified order, i.e., an oriented Eichler order. — [Rohrlich], [KM], [Cor1, §§1 and 2], see also [Cor2].

Lecture 15: *Mazur's Conjecture for classical Heegner points.*

The second arithmetic application of Ratner's theorems. The source is [Cor1], or [Cor2].

(a) Statements of results: Mazur's Conjecture and various corollaries. — See introduction of [Cor2]; cf. [Mazur], [Bertolini], last section of [NS], and [Nek].

(b) Preparing the proof: indexing the family $\mathcal{L}$; finiteness of torsion over all ring class fields; it suffices to show that there are many points in the image, and then: that there are many points in the reduced image. — [Cor1], or [Cor2].

(c) The geometric part of the trace. — [Cor1], esp. [Cor1, §5.1]; see also [Cor2], esp. [Cor2, §9.2].

(d) The chaotic part of the trace: Description of the reduction map

$$RED : \mathcal{L} \longrightarrow X_0^{\mathrm{ss}}(N')^{\mathcal{R}}(\mathrm{F}_{\ell^2}).$$

The map one finally winds up with is the same as in lecture 13 above, and is therefore surjective. — [Cor1, §3], or [Cor2, §6.2].

Lecture 16 **and** 17: *The parity conjecture.*

This is Nekovář's corollary from the previous results.

(a) Euler systems over ring class fields — the main result of [BD1]. Possibly also Lemma 2.1 of [Nek] (summary of Iwasawa theory). — *Note. The resuts of Bertolini and Darmon carry a number of presumably unnecessary restrictive hypotheses on p. They have not been weeded out yet in the existing literature.*

(b) Proof of the parity conjecture for Selmer groups. — [Nek], and handwritten notes of a talk of Jan Nekovář's which can be sent to the speaker(s) volunteering for this lecture.

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