Dear Professor Harder,

The simpler proof of Ratner's theorem [7, 9] due to Margulis and Tomanov [5, 6] can be sketched in 3 lectures as follows:

Statement of Ratner's Theorem: Let G be an algebraic group over $K = \mathbf{Q}_p$ or $K = \mathbf{R}$, U be an algebraic unipotent subgroup of G, and μ be a U-ergodic invariant probability measure on G/Γ, where Γ any discrete subgroup of G. Let

$$\Sigma = \{g \in G : \text{action of } g \text{ preserves } \mu\}.$$

Then Σx is closed, and $\mu(\Sigma x) = 1$ for some $x \in G/\Gamma$.

1. Lecture I: Some observations and Basic Lemma
   (a) [5, Lemma 1]: Reduction to the case when μ is Zariski dense (i.e. the image in G/Γ of any proper algebraic subvariety of G has zero μ-measure); and U is a maximal unipotent subgroup of Σ. Also Σ contains no normal subgroup of G containing U.
   (b) [5, Lemma 2]: Ergodic theorem from Tempelman's book; and existace of generic set M with uniform convergence property and $\mu(M) > 1-\epsilon$.
   (c) [6, §5]: Construction of 'U-quasi-regular' rational maps $\phi : U \to N_G(U)$ associate to 'upper topological limits of orbits $Ug_n \cdot \mathbf{p}$' in a linear representation of G with $\mathrm{Stab}_G(\mathbf{p}) = U$, where the sequence $g_n \to e$ in G is such that $g_n M \cap M \neq \emptyset$.
   Moreover $\phi(U) \not\subset U$. This is acheived by using the Zariski density condition on μ to assume that $\{g_n\} \cap N_G(U) = \emptyset$.
   (d) [5, Basic Lemma]: Using the above ergodic theorem, one shows that $\phi(U) \subset \Sigma \cap N_G(U)$.
   Let F be the subgroup generated by $\phi(U)$ and U.

2. Lecture II: Relation between $\{g_n\}$ and F.
   (a) [6, Prop. 6.7]: F contains a unipotent subgroup strictly larger than U, if $\{g_n\}$ satisfies certain algebraic Condition-$(**)$.
   Therefore, by maximality of U, the Condition-$(**)$ fails to hold.
   (b) [5, Prop. 2]: F contains a one-dimensional K-split torus T.
   Fix $s \in T$ such that $T/<s>$ compact. Note that $G = \overline{W^-(s)Z(s)W^+(s)}$, where $W^+(s) = \{u \in G : s^{-n}us^n \overset{n\to\infty}{\to} e\}$, $Z(s) = Z_G(s)$, etc.
   (c) [5, Section 4]: Introduction of subgroups $U^+ = W^+(s) \cap U$, $\mathcal{I}(s) = \{g : U^+g \subset \overline{W^-(s)Z(s)U^+}\}$, $U^-(s) = W^-(s) \cap \mathcal{I}(s)$, and regular $\mathrm{Int}(s)$-invariant cross section V^- of W^-/U^-.
   Condition-$(**)$ $\Leftrightarrow$ V^--components of g_n's are at least as large as their U^--components.

(d) [5, Proposition 5, Lemma 4]: Failure of Condition-($**$) leads to existence of a subset Ψ of full μ-measure such that

$$(E) \qquad \Psi \cap W^-(s)x \subset U^-(s)x, \ \forall x \in \Psi.$$

3. Lecture III: Entropy formula and conditional measures.

(a) [5, Prop. 6]: Simplified version of a result of Ledrappier and Young [3], which compares the entropy of the action of s on G/Γ and the modulus of Int(s) on various subgroups normalized by s, and shows that in case of equality $U^- \subset \Sigma$.

(b) [5, Prop. 7]: In view of condition-(E), from the above result one concludes that, in fact, $W^-(s) \subset \Sigma$ and $W^+ \subset \Sigma$.

Since $W^-(s)$ and $W^+(s)$ generate a normal subgroup of G containing U, by our assumption (via the initial reduction), $\Sigma = G$.

It is better to explain the entire proof by restricting to the case of $G = PGL_2(K) \times PGL_2(K)$, and U being the diagonal embedding of a unipotent subgroup of $PGL_2(K)$. This case already reveals the main important techniques involved in the proof of the Ratner's general theorem.

The proof of Margulis and Tomanov specialized to the case of $G = SL_2(\mathbf{C})$, and U being the real one-dimensional group, can be found in [12]. The article [4], by Margulis, is written for algebraists, and it can be helpful for our purpose.

For most number theoretic applications, one usually needs to show that certain sequences of algebraically defined measures on G/Γ become equidistributed in the limit. In such a situation, sometimes one can easily show that the limiting distribution of such a sequence of measures is invariant under a unipotent subgroup U. But one cannot easily conclude that such a measure is U-ergodic. To analyse such a measure by decomposing it into U-ergodic components, and to use Ratner's description for such components, there are well developed [8, 2, 11, 9, 10] techniques. These techniques should be explained, if there is time. Some particular cases of these techniques are very relevant to the theorem stated in the beginning of 'Main Program description'.

The results of Dani and Margulis [1] on non-divergence of unipotent trajectories are required to show that certain sequences of algebraically defined measures have convergent subsequences. Only after this one can start applying Ratner's theorem. Therefore the Dani-Margulis' non-divergence results are essential complements of Ratner's theorem. They should also be explained, if there is time.

If I could be of any use in lecturing/preparation please let me know.

With best regards,

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