Arbeitsgemeinschaft

“A_∞-structures and mirror symmetry”

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HOW TO APPLY

Not every participant will give a talk. However, in order to apply you have to choose two talks (ordered by preference) which you feel you could give. Suggestions for other subjects to be covered are also possible. Please write an e-mail to **both** organizers. Practical questions should be addressed to the Institute at Oberwolfach, admin@mfo.de

PROGRAM OUTLINE

A_∞ structures were invented by Stasheff for uses in homotopy theory. Subsequently, they were applied to homological algebra (deformation theory) by many authors. In symplectic geometry, the Floer homology groups associated to Lagrangian submanifolds carry A_∞-structures (this observation has only recently been made rigorous). Kontsevitch used this A_∞ structure in his homological mirror conjecture, which says that it should coincide with a similar structure arising from sheaf cohomology in algebraic geometry. There is a lot of evidence for this conjecture:

- the case of elliptic curves is completely understood, and the case of higher-dimensional abelian varieties at least partially.

- relations between the automorphism groups of the two structures have been found experimentally.

- it appears to be compatible with the Strominger-Yau-Zaslow differential geometric approach to mirror symmetry.

- there is a simpler analogue for Fano manifolds, which can be proved in some cases.

Finally, physicists have discovered that A_∞-structures naturally arise in connection with open string theory and D-branes.

LIST OF AVAILABLE TALKS, WITH REFERENCES

(1) A_∞-algebras: A_∞-algebras, A_∞-homomorphisms, and quasi-isomorphisms [1a,1b]. Geometric picture as noncommutative vector fields. Dg-models (bar construction) [1a] and minimal models (homological perturbation Lemma) [1c,1d,1e]. Possibly, relation with deformation theory [1f]. Example: the A_∞-structure on the cohomology of a topological space [1d].

1a. Getzler-Jones, A_∞ algebra and cyclic bar comlex, Illinois J. Math. 34 (1990), 256
1b. Keller, math.RA/9910179
1c. Kontsevich-Soibelman, math.SG/0011041 (published in "Mirror Symmetry and Symplectic Geometry", World scientific),
1d. Merkulov, math.SG/9809172, published in Internat. Math. Res. Notices (1999) 153.
1e. Gugenheim and Stasheff, on perturbations and A_∞-structures,
1f. Fukaya, Deformation theory, homological algebra, and mirror symmetry, preprint (this and other preprints by Fukaya can be obtained from http://www.kusm.kyoto-u.ac.jp/˜fukaya/)

(2) A_∞-categories: A_∞-categories, A_∞-functors and their natural transformations. Example: the directed Fukaya category associated to an ordered collection of curves on a surface [2a]. Various notions of identity morphisms (strict identity, identity on cohomology, homotopy identity [2b]) and their equivalence. A_∞-pre-categories [1c]. Example: again curves on a surface.

2a. Gautschi, Robbin, Salamon, de Silva, Combinatorial Floer homology (unfinished but we can get hold of some version)
2b. Fukaya-Oh-Ohta-Ono, Lagrangian intersection Floer theory - Anomaly and Obstruction, preprint

(Opt) Rational homotopy theory (optional): Rational homotopy theory uses homological and homotopical algebra of differential graded algebra. It is a prototype of homological and homotopical algebra of A_∞ algebra and catetory.

o. Griffiths Morgan, Rational homotopy theory and differential forms, Birkhauser Prog. in Math. 16.

(3) Obstructions: A_∞-algebras with a zero-order term [1a]. Trivial example: $\Omega^*(Hom(E,E))$ for a vector bundle E with a non-flat connection. Filtered version [2b,1f]. The Maurer-Cartan (MC) equation [1f] in such an A_∞-algebra. "Linearized homology" associated to solutions of MC (the parallel story in the L_∞-context is explained in [3a]). Example: Chekhanov's invariant of Legendrian knots [3b] and the problem of "augmentations".

3a. Alexandrov, Kontsevich, Schwarz, Zaboronsky, The Geometry of the Master equation and TQFT, hep-th/9502010
3b. Chekhanov, Differential algebra of Legendrian links, math.GT/9706005 (see also preprints by Ng and collaborators)

(4) Derived categories: Classical derived categories (of abelian categories, such as coherent sheaves or modules over an algebra) [4c]. Example: the derived category of coherent sheaves on projective space according to Beilinson [4b]. Derived categories of A_∞-categories, via twisted complexes [4a,1b]. The relation with the classical construction.

4a. Kontsevich, homological algebra of mirror symmetry, ICM Proceedings Zurich, 1994
4b. Beilinson, Coherent sheaves on P^n and problems of linear algebra. Funct. Anal. Appl. 12 (1978), 214. See also Bernstein-Gelfand-Gelfand paper immediately preceding this one
4c. Gelfand-Manin, homological algebra, Springer. Or: Weibel, Introduction to homological algebra, Cambridge Univ. Press

(5) Morse theory: Stasheff polytopes [5a, 5b] as moduli spaces of trees [5c]. Morse homology [5e] and the cup product on it [5d]. A_∞-pre-category arising from Morse theory [5d]. Equivalence between Morse theory and differential forms [1c].

5a. Stasheff, Homotopy associativity of H-spaces I, Trans. Amer. Math. Soc. 108 (1963), 275
5b. Stasheff, The prehistory of operads, In: Operads, Proceedings of the Renaissance conference, Contemp. Math. 202 (1997)
5c. Fukaya, Morse homotopy and its quantization, In: Geometry and Topology, International press, 1997
5d. Fukaya-Oh, Zero-loop open string on cotangent bundle and Morse homotopy, Asian. J. Math. 1 (1997)
5e. Schwarz, Morse homology, Birkhaeuser

(6) Introduction to Floer theory: Floer (co)homology as the Morse theory of the action functional. Its definition in the exact [6a] and monotone [6b,6e] case. Basic properties. Gradings [6c] and the use of *spin*-structures to introduce signs [2b]. The first appearance of obstructions [6d].

6a. Floer, Morse theory for Lagrangian intersections, J. Differential Geom. 28 (1988), 513
6b. Oh, Floer cohomology of Lagrangian intersections and pseudo-holomorphic discs I, Comm. Pure Appl. Math. 46 (1993), 949
6c. Seidel, Graded Lagrangian submanifolds, Bull. Soc. Math. France 128 (2000), 193
6d. Oh, Addendum to [5b], same issue p. 1299

6e. Oh, Floer cohomology, spectral sequence, and the Maslov class of Lagrangian embeddings, Internat. Math. Res. Notices (1996), 305

(7) The Fukaya category for exact symplectic manifolds: Stasheff polyhedra as real Deligne-Mumford moduli spaces [3c]. Fukaya category in the exact (and maybe also monotone) case as A_∞-pre-category [5c,7b,7c], or as honest A_∞-category. Cotangent bundles, and the equivalence between Morse and Floer theory [3d] (there is also a version for tori discussed in [1c]). Directed Fukaya categories [7a].

7a. Seidel, Vanishing cycles and mutation, math.SG/0007115
7b. Fukaya, Floer homology, A_∞-categories and topological field theory, in: Geometry and Physics, Aarhus 1995
7c. Fukaya, Morse homotopy, A_∞-categories, and Floer homologies, in: Proceedings of GARC Workshop on Geometry and Topology, Seoul National University, 1993

(8) Floer cohomology in the general context: Obstructions to Floer cohomology, and the corresponding deformation theory (in a framework similar to that established by talk 3). This follows [2b], a survey is [8a].

8a. Ohta, Obstruction to and deformation of Lagrangian intersection Floer theory, In: Mirror symmetry and Symplectic geometry, World Scientific, 2001.

(9) Virtual fundamental chains and Kuranishi structures: This talk, based on [9a,2b] or an equivalent approach, should be given by someone who is working in pseudo-holomorphic curve theory. If this applies to you, you will already know roughly what the contents should be!

9a. Fukaya, Ono, Arnold conjecture and Gromov-Witten invariant, Topology 38 (1999), 933

(10) Fukaya categories in general: Sequel to talk 8, which now describes how to define Fukaya A_∞-categories for general symplectic manifolds. Based on [10a,10b]

10a. Fukaya, Floer homology and mirror symmetry II, preprint
10b. Fukaya, Deformation theory, homological algebra, and mirror symmetry

(11) Introduction to homological mirror symmetry: Formulation of the conjecture according to [4a]. Some implications: monodromy maps and automorphisms of derived categories [11a,11b]. Extensions of vector bundles and connected sum of Lagrangian submanifolds (optional, [11c]).

11a. Aspinwall, Some Navigation Rules for D-Brane Monodromy, hep-th/0102198
11b. Seidel-Thomas, braid group actions on derived categories of coherent sheaves, math.AG/0001043 (see also Horja's papers, like math.AG/0103231)
11c. Thomas, moment maps, monodromy, and mirror manifolds, math.DG/0104196

(12) Tori: Various approximate forms of the mirror conjecture have been proved for elliptic curves and abelian varieties.

12a. Polishchuk, Zaslov, Categorical mirror symmetry: the elliptic curve, math.AG/9801119. See also math.AG/0001048, math.AG/0003076
12b. Fukaya, mirror symmetry of abelian variety and multi-theta function, preprint

(13) The Strominger-Yau-Zaslow picture: [13a] suggests a direct way of going from Lagrangian submanifolds to holomorphic vector bundles. This "Legendre transform" [13b,13c] gives the correct results for tori and abelian varieties. [1c] extends it to an equivalence of A_∞-pre-categories.

13a. Strominger-Yau-Zaslow, mirror symmetry is T-duality, hep-th/9606040
13b. Leung, mirror symmetry without corrections, math.DG/0009235
13c. Leung, Yau, and Zaslow, from Special Lagrangian to HYM via Fourier-Mukai transform, math.DG/0005118

(14) Physics: The role of A_∞-structures in open string theories. This is supposed to be an introductory talk. References are tentative, the speaker should presumably know enough physics to find his own way around.

14a. Lazaroiu, String field theory and brane superpotentials, hep-th/0107162
14b. Gaberdiel, Zwiebach, Tensor constructions of open string theories I, hep-th/9705038

(15) Homological mirror symmetry for Fano varieties, and Picard-Lefschetz theory: Symplectic Picard-Lefschetz theory, outlined in [7a] with details in [15a]. Distinguished basis of vanishing cycles, Hurwitz moves [7a]. Invariance of the derived category of the directed Fukaya category (without proof, but maybe with some mention of mutations) [7a, plus manuscript to be made available to the speaker]. Mirror symmetry for CP^2 [15b].

15a. Seidel, A long exact sequence for symplectic Floer cohomology, math.SG/0105186
15b. Seidel, More about vanishing cycles and mutation, math.SG/0010032