Arbeitsgemeinschaft mit aktuellem Thema:
Homotopy of moduli spaces
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Organizers:

Michael Weiss	Søren Galatius
Department of Mathematics	Department of Math. Sciences
University of Aberdeen	University of Aarhus
Aberdeen AB24 3UE	Ny Munkegade, 8000 Aarhus
U.K.	DENMARK
m.weiss@maths.abdn.ac.uk	galatius@imf.au.dk

Introduction:

The moduli spaces in the title are moduli spaces of compact Riemann surfaces. For the purpose of calculating their rational (co–)homology, one may replace them by classifying spaces of certain mapping class groups $\Gamma_{g,b}$, that is, groups of isotopy classes of orientation preserving diffeomorphisms from a smooth, compact, connected and oriented surface to itself. The Mumford conjecture [32] postulates that the rational cohomology ring of $B\Gamma_{g,b}$ for "large" genus g is a polynomial ring on generators κ_1, κ_2, κ_3, ..., where the degree of κ_i is $2i$. A specific goal of the A.G. is to go over the proof of Mumford's conjecture as given in [24]. This proof relies on methods from homotopy theory and differential topology. For an overview without the technicalities, see also [44]. Following are some vocabulary, some of the recent history of the subject, and a few words on how the talks hang together.

Let $F_{g,b}$ denote a connected, compact, oriented smooth surface of genus g with b boundary circles (and no "punctures"). The topological group of smooth orientation preserving automorphisms of $F_{g,b}$ which restrict to the identity on $\partial F_{g,b}$ will be denoted by $\mathrm{Diff}(F_{g,b};\partial)$. The mapping class group of $F_{g,b}$ is

$$\Gamma_{g,b} = \pi_0 \mathrm{Diff}(F_{g,b};\partial)\,.$$

A fundamental result of Earle, Eells and Schatz [7], [8] states that the identity component of $\mathrm{Diff}(F_{g,b};\partial)$ is contractible when $g>1$ or $b>0$. Consequently $B\Gamma_{g,b}$ is then a classifying space for certain surface bundles.

Concatenation of surface automorphisms determines, for each g and h, homomorphisms $\Gamma_{g,2}\times\Gamma_{h,2}\longrightarrow\Gamma_{g+h,2}$. (This requires orientations on the boundary circles; we omit the details.) Hence the disjoint union of the spaces $B\Gamma_{g,2}$ for $g\geq 0$ becomes a topological monoid. The loop space of its classifying space,

$$\Omega B\big(\coprod_g B\Gamma_{g,2}\big),$$

is known as the *group completion* of $\coprod B\Gamma_{g,2}$. As is often the case, compare [1], the group completion process can be replaced by a Quillen plus construction. Namely, we have

$$\Omega B\big(\coprod_g B\Gamma_{g,2}\big)\ \simeq\ \mathbb{Z}\times B\Gamma^+_{\infty,2}\,.$$

Here $\Gamma_{\infty,2}$ is the direct limit of the groups $\Gamma_{g,2}$ under certain (obvious) stabilization homomorphisms $\Gamma_{g,2}\to\Gamma_{g+1,2}$. It is a perfect group. Hence the plus construction can be applied to its classifying space, giving a simply connected space $B\Gamma^+_{\infty,2}$ with the same integral homology as $B\Gamma_{\infty,2}$.

A pointed space X is an *infinite loop space* if there exist pointed spaces B^nX and pointed homotopy equivalences $B^nX\to\Omega(B^{n+1}X)$ for $n\geq 0$, with $B^0X=X$. Homotopy types of infinite loop spaces are in a bijective correspondence with generalized cohomology theories whose coefficient groups in dimensions <0 vanish. — The following was proved by Tillmann [40], around 1996, and was quite unexpected at the time.

Theorem 0.1 $\Omega B\big(\coprod_g B\Gamma_{g,2}\big)$ *is an infinite loop space.*

Following [23], we now turn to an old idea from bordism theory, the Thom–Pontryagin construction (in order to make a qualified guess as to "which" infinite loop space $\Omega B\big(\coprod B\Gamma_{g,2}\big)$ might be). Suppose given a sequence of spaces and inclusion maps

$$Z_0\to Z_1\to Z_2\to\cdots$$

and, for each k, a real vector bundle J_k on Z_k of fiber dimension i, and identifications $J_{k+1}|Z_k \cong J_k$. Then the *Thom spaces* $\mathrm{Th}\,(J_k)$, obtained by adding a point "at infinity" to the total space of each J_k, form a *spectrum*; i.e., they are related by canonical embeddings $S^1 \wedge \mathrm{Th}\,(J_k) \to \mathrm{Th}\,(J_k)$ respecting the base points. In this way they determine, for each $m \in \mathbb{Z}$, an infinite loop space $\mathrm{colim}_k\ \Omega^{m+k}\mathrm{Th}\,(J_k)$.

Let M be a smooth closed m–manifold equipped with a map $u : M \to Z_k$, a smooth embedding $e : M \to \mathbb{R}^{m+k}$ and an identification of $u^*(J_k)$ with the normal bundle of e. The Thom–Pontryagin construction associates with these data a certain map $S^{m+k} \to \mathrm{Th}\,(J_k)$, in other words a point in $\Omega^{m+k}\mathrm{Th}\,(J_k)$, which we can also view as a point in $\mathrm{colim}_k\ \Omega^{m+k}\mathrm{Th}\,(U_k)$.

Let now $J_k = L_k^\perp$ be the canonical vector bundle of fiber dimension k on the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^{2+k})$ of oriented 2–dimensional linear subspaces in $\mathbb{R}^{2+k}$. A closed smooth oriented surface F can be embedded in $\mathbb{R}^{2+k}$ for some k. Then there is a canonical (Gauss) map $u : F \to \mathrm{Gr}_2(\mathbb{R}^{2+k})$ and an identification of $u^*L_k^\perp$ with the normal bundle. By Thom–Pontryagin, these data determine a point $\alpha_\infty(F)$ in $\mathrm{colim}_k\ \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$. The construction works equally well for families, i.e., a bundle $E \to B$ of oriented closed surfaces determines a map

$$\alpha_{\infty,E} : B \longrightarrow \mathrm{colim}_k\ \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)\,.$$

We can specialize to the case $B = \coprod_g B\Gamma_{g,2}$. This comes with a universal bundle of compact connected surfaces where each fiber has an incoming and an outgoing boundary circle; gluing these two circles together, for each fiber, we obtain a bundle of closed oriented surfaces $E \to \coprod B\Gamma_{g,2}$ and hence, as above, a map from $\coprod B\Gamma_{g,2}$ to $\mathrm{colim}_k\ \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$. This factors through the group completion, giving finally

$$\alpha_\infty : \Omega B\big(\coprod_g B\Gamma_{g,2}\big) \longrightarrow \mathrm{colim}_k\ \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)\,.$$

0.2 Madsen's integral Mumford conjecture [23], now a theorem [24]: *The map α_∞ is a homotopy equivalence.*

Madsen and Tillmann showed in [23] that α_∞ respects the infinite loop space structures, with Tillmann's infinite loop space structure on the source. The Mumford classes κ_i can be defined, in a slightly roundabout way, as $\alpha_\infty^*(\bar\kappa_i)$ for certain classes $\bar\kappa_i$ in the integral cohomology of the base point

component of $\operatorname{colim}_k \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$. It is a trivial matter to identify the *rational* cohomology of the base point component of $\operatorname{colim}_k \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$ with the polynomial ring $\mathbb{Q}[\bar{\kappa}_1, \bar{\kappa}_2, \dots]$. In this way, Madsen's integral Mumford conjecture implies the original Mumford conjecture.
The calculation of the cohomology of $\operatorname{colim}_k \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$ with coefficients in a finite field is more of a challenge, but has been carried out quite recently by Galatius [10].

* * *

The proof of 0.2 given in [24] relies on several established theorems and theories. The most important among these are the homological stability theorem of Harer [16], [19], [20] and a theorem about the cohomology of spaces of smooth functions with "moderate singularities" due to Vassiliev [42], [41]. At a more basic level, submersion theory [33], [15] and immersion theory [38], [17], [15] contribute to the proof. Of these, immersion theory, submersion theory and Harer's theorem will be used as "black boxes" in the A.G. But Vassiliev's theorem will be treated in detail because it is less well known and Vassiliev's proof of it is not easy to read.
From a strategic perspective, the proof of 0.2 given in [24] revolves around a commutative diagram of spaces and maps

$$\begin{tikzcd}
 & {|\mathcal{W}|} \arrow[r] \arrow[d, "\simeq"] & {|\mathcal{W}_{\mathrm{loc}}|} \arrow[d, "\simeq"] \\
{|h\mathcal{V}|} \arrow[r] & {|h\mathcal{W}|} \arrow[r] & {|h\mathcal{W}_{\mathrm{loc}}|}
\end{tikzcd}$$

where the lower row is a homotopy fiber sequence. The spaces in the lower row are hybrids between geometry and homotopy theory; those in the upper row are pure geometry. The main topic of talk 5 is a homotopy equivalence $|h\mathcal{V}| \simeq \operatorname{colim}_k \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$. Talk 8 is about Vassiliev's theorem [42], [41], which in talk 9 leads to the homotopy equivalence $|\mathcal{W}| \simeq |h\mathcal{W}|$. The right–hand column of the diagram, and the fact that the lower row is a homotopy fiber sequence, are established in talk 10. Talks 12, 13 and 14 use the geometric definitions of $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$ to obtain homotopy colimit decompositions of these spaces. In the case of $|\mathcal{W}|$, which is the more important of the two, the pieces in the decomposition are classifying spaces for certain surface bundles equipped with certain surgery data. The decompositions are improved in talks 15 and 16 in such a way that all non–connected surfaces disappear from the picture. They are finally used in talk 17 to establish, with the help of Harer's theorem [16], a *homology fiber sequence*

$$\mathbb{Z} \times B\Gamma_{\infty,2} \longrightarrow |\mathcal{W}| \longrightarrow |\mathcal{W}_{\mathrm{loc}}|,$$

and this completes the proof of 0.2. The remaining talks 1, 2, 3, 6, 7, 4 and 11 are more self–contained and supply background information or necessary abstract notions to be used in later talks.

Talks:

1. **Mapping class groups and homological stability**
   This should be an introduction listing the basic results from the geometric theory of surfaces which we need. It should mention the theorem of Earle–Eells–Schatz [7], [8], Harer–Ivanov homological stability [16], [19], [20] and the fact that (most) mapping class groups are perfect [34], probably all without proof. But then the Quillen plus construction, the group completion process applied to the topological monoid

   $$\coprod_g B\Gamma_{g,1+1}$$

   and the homotopy equivalence of $\Omega B\big(\coprod B\Gamma_{g,1+1}\big)$ with $\mathbb{Z} \times B\Gamma^+_{\infty,1+1}$ should be explained in some detail. For the last part, state and use the group completion theorem. Despite (or because of) its informal style, [1] is an excellent reference for these topics.

2. **Tillmann's theorem**
   The goal is to prove 0.1 from the introduction above. This is cultural, i.e., it doesn't feed into the proof of Mumford's conjecture, but it is enormously important as a motivation. The talk should contain a definition of Tillmann's symmetric monoidal category $\mathcal{Y}_b$ following [40] or [23] and explain why the group completion of the classifying space $B\mathcal{Y}_b$ is homotopy equivalent to $\mathbb{Z} \times B\Gamma^+_{\infty,2}$. The fact that (group completions of) classifying spaces of symmetric monoidal categories are infinite loop spaces [37], [1], can be used as a black box — or try to sketch a proof.
   See also appendix C in [24] for some useful general facts related to the definition of $\mathcal{Y}_b$ and Tillmann's theorem.

3. **Thom–Pontryagin construction and MMM classes**
   The first major topic here should be the Thom–Pontryagin construction [39] and the related description of homotopy groups of Thom spectra as bordism groups. See also chapter 7 of [18]. The construction of the Madsen–Tillmann map α_∞ should be obtained as a special (but parametrized) case, first as a map from $\coprod_g B\Gamma_{g,2}$ to $\mathrm{colim}_k\ \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$. See section 1.2 of [23] or follow the sketch above. To factor it through the group completion $\Omega B(\coprod_g B\Gamma_{g,2})$, replace the latter by $\mathbb{Z}\times B\Gamma^+_{\infty,2}$. See the description of talk 1.
   The second major topic should be the construction of the Mumford–Miller–Morita characteristic classes for surface bundles as done by Miller [25] and Morita [29], [30] or as sketched in [44]. Then explain how these two things, the Madsen–Tillmann map and the MMM classes, are related — sketch in [44].
   It is a good idea to stick to *closed* surfaces mostly in both of these topics; where surfaces of type $F_{g,2}$ appear, the two boundary circles can be glued together.
   If there is still time left, a few words on the "Thom spectrum" made from the spaces $\mathrm{Th}\,(L_k^\perp)$ for $k\geq 0$ would be useful. This could lead to a quick calculation of the rational cohomology of $\mathrm{colim}_k\ \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$.

4. **Classifying spaces: what they classify**
   The classifying space of a group G obviously classifies principal G–bundles. But what does the classifying space of a category [36] classify ? There is an answer in thm. 4.1.3 of [24]. What it does is simply to generalize and formalize Steenrod's view of principal bundles as 1–cocycles (which you get from local trivializations of the bundle). It is written in a sheaf theoretic language introduced in section 2.2 of [24]. Ideally therefore, in this talk the sheaf theoretic language of [24, §2.2] should be introduced; the construction of a representing space for certain sheaves [24, app. A] should be explained; and finally thm. 4.1.3 in of [24] should be stated and proved. That may be too much for one talk. Make compromises as necessary, but don't sacrifice the statement of 4.1.3.
   There is a different and probably a more elegant answer to the same question, *what does the classifying space of a category classify*, in [28]. It would be interesting to learn about that, too, if there is time.

5. **First desingularization procedure**
The main task here is to define the space $|h\mathcal{V}|$ mentioned earlier, and to prove that it is homotopy equivalent to $\mathrm{colim}_k\, \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$. This proof uses the *submersion theory h–principle* [33], [15] which should be presented as a black box in this talk. (It should nevertheless be presented carefully, because it is the first of three h–principles which we will encounter in the talks. Try to emphasize the general philosophy — integrability up to homotopy. See also [13], [9]. The next three talks are about another h–principle, due to Vassiliev.)
The proof of $|h\mathcal{V}| \simeq \mathrm{colim}_k\, \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$, and the definition of $|h\mathcal{V}|$, are written in a a sheaf theoretic language, section 2.2 and appendix A of [24], which you have to familiarize yourself with. Note that the sheaf theoretic language should already have been introduced in talk 4, so you needn't worry about that.
The definition of $|h\mathcal{V}|$ in section 2.1 of the current version of [24] mentions certain boundary data (item iii of dfn 2.1.5). These are replaced by *empty boundary* in [44]. Both versions are correct, but the version with empty boundaries is recommended. The homotopy equivalence between $|h\mathcal{V}|$ and $\mathrm{colim}_k\, \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$ is in section 3.2, especially thm 3.2.4 of [24]. The proof given there needs some weeding out, because it proves something more general (we'll get there later). All you need is a map from $|h\mathcal{V}|$ to $\mathrm{colim}_k\, \Omega^{2+k}\mathrm{Th}\,(L_k^\perp)$ and a proof that it induces isomorphisms on homotopy groups. The latter relies strongly on Thom–Pontryagin theory (talk 3) and submersion theory, as mentioned before. Finally, don't forget to explain briefly what the Madsen–Tillmann map α_∞ looks like with $|h\mathcal{V}|$ as the target.

6. **Interpolation theory on manifolds**
This is fairly self–contained. The main reference is [11], unless you find a better one. Some knowledge of French and a sense of humor are prerequisites. Some knowledge of algebraic geometry, especially "Hilbert varieties", could be an advantage but is not a prerequisite.
Let M be a closed manifold, $Z = C^\infty(M, \mathbb{R}^n)$ the vector space of smooth functions from M to $\mathbb{R}^n$, and R the $\mathbb{R}$–algebra of smooth functions from M to $\mathbb{R}$. Then Z is an R–module. Interpolation theory on M, from an abstract/modern point of view, is about R–submodules Y of Z for which $\dim_{\mathbb{R}}(Z/Y) < \infty$. In the talk, such submodules should be "classified" (with the help of Nakayama's lemma). From the

classification it should be deduced that, for fixed $k \geq 0$, the set of all R–submodules Y of Z for which $\dim_{\mathbb{R}}(Z/Y) \leq k$ forms a *compact* Hausdorff space (with a fairly obvious topology). It should be explained how this fact leads to the existence of "many" *finite dimensional* linear or affine subspaces of the $\mathbb{R}$–vector space Z with good interpolation properties. For the definition of *good interpolation properties*, you can use "condition $(\mathbf{c_{2,r}})$" in dfn A.2 of [44]. We also need to know that condition $(\mathbf{c_{2,r}})$ is an "open" condition. That is, if D satisfies $(\mathbf{c_{2,r}})$ and D_1 is another finite dimensional affine subspace of Z sufficiently close to D, then D_1 satisfies $(\mathbf{c_{2,r}})$.

7. **Advanced transversality**
   The main reference is chapter 4 of [12]. Ideally, the goal would be to prove lemma A.3 in [44], which also uses results from talk 6. That might however be far too ambitious, not least because lemma A.3 refers to a definition A.1 which is only sketched.
   An extremely conscentious but realistic speaker might want to read the whole of appendix A in [44] to get an idea of what lemma A.3 is used for, then make definition A.1 explicit, then use many different kinds of transversality theorems to prove lemma A.3 (modulo the results from talk 6), then isolate the transversality theorems used and finally construct the talk around a selection of those transversality theorems.
   Under all circumstances the talk should present early on, and carefully, what Golubitsky and Guillemin call the basic transversality result, lemma 4.6 in [12]. Note the remark following it. The versatility of the lemma should then be illustrated by some examples, and preferably these examples should be related to lemma A.3 in [44].

8. **Spaces of functions with moderate singularities**
   This will be a very difficult talk. It should first of all *state* Vassiliev's theorem, that is, Thm. 0 part A) of [42] or equivalently III.1.1 of [41]; see [2] for details on semi–algebraic sets. Then it should attempt to take us through the proof. This is outlined in the first 5 pages of [42] and, for the case of a *closed* manifold, the appendix to [44]. You may of course try to find more details in [41]. Suggestion: the closed manifold case is hard enough.
   The sole purpose of the preceding two talks is to give strong evidence for statements such as lemma A.3 in [44], or Thm. 1.4 of [42], or

theorem 4.2.1 of chapter III in [41]. This means that you should use such statements without proof.

9. **Second desingularization procedure**
   The main tasks here are (a) to define the spaces $|h\mathcal{W}|$ and $|\mathcal{W}|$ mentioned earlier, (b) to prove that $|\mathcal{W}| \simeq |h\mathcal{W}|$ and (c), to show that $|h\mathcal{W}|$ is homotopy equivalent to an infinite loop space associated with a certain Thom spectrum. For part (a), see chapter 2 of [24]. For part (b), see chapter 4 of [24]. For part (c), the reference is section 3.2 of [24]; this is very similar to the content of talk 5 and should therefore not have priority.

10. **Localization**
    In this talk the spaces $|\mathcal{W}_{\mathrm{loc}}|$ and $|h\mathcal{W}_{\mathrm{loc}}|$ will be introduced; see chapter 2 of [24]. It will then be shown that $|\mathcal{W}_{\mathrm{loc}}| \simeq |h\mathcal{W}_{\mathrm{loc}}|$, following section 3.4 of [24]. Finally the homotopy fiber sequence
    $$|h\mathcal{V}| \to |h\mathcal{W}| \to |h\mathcal{W}_{\mathrm{loc}}|$$
    will be established. This involves identifying $|h\mathcal{W}_{\mathrm{loc}}|$ with an infinite loop space associated to a certain spectrum, following section 3.3 of [24]. Then it remains to arrange this spectrum in a cofiber sequence with the two spectra defined in talks 9 and 5, respectively. This is section 3.1 of [24].

11. **Homotopy colimits**
    This talk should define the homotopy colimit of a functor F from a small category to the category of spaces, and explain how this definition "repairs" the fact the the categorical colimit is not homotopy invariant; cf. example D.1.1 in [24]. In appendix D of [24], the homotopy colimit of F is defined as the categorical colimit of a "resolution" of F, so that the homotopy colimit appears as something like a total derived functor of the categorical colimit. The talk should be specially addressed to the non–topologists in the audience — most topologists under 60 know the homotopy colimit concept inside out.
    The bible for homotopy colimits (and homotopy limits) is [4], but the resolution point of view is not stressed enough there. It is also written in an extremely simplicial language which doesn't appeal to everybody. Therefore [6] is also recommended. "Facts" about homotopy colimits

which we need are C.1.3, D.1.2, D.1.3, D.1.5 in [24]. Among these, D.1.5 should have the highest priority; this relies on the definitions and results of talk 4.
If there is time left (unlikely), you could mention a certain spectral sequence converging to $H_*(\text{hocolim } F)$, with second page given by the "higher" colimits of $H_* \circ F$. These higher colimits are the left derived functors of the functor "colimit" applied to graded–abelian–group–valued functors. The details can be found somewhere in [4].

12. **Morse functions, surgeries and long traces**
This is one of three talks covering chapter 5 of [24]. The overall main result of the chapter is a homotopy colimit decomposition of $|\mathcal{W}|$ into pieces which are classifying spaces for certain surface bundles, and a compatible decomposition of $|\mathcal{W}_{\text{loc}}|$.
The first part of this talk, related to section 5.2, should be a fairly self–contained and "classical" exposition of the relationship between Morse singularities and surgeries. See also the relevant passages of [26] and [27]. More precisely, the talk should give an answer to the following. Given a proper smooth function $f : N \to \mathbb{R}$ with a single critical value, say $0 \in \mathbb{R}$, but possibly several critical points, how can we reconstruct both N and f from the smooth manifold $M = f^{-1}(c)$ where $c < 0$, and suitable "surgery data" in M ? More generally, we will also need to deal with *families* of such pairs (N, f), parametrized by another smooth manifold X.
The second part of the talk, related to section 5.3 of [24], is an application of the first part. The goal is to obtain the upper square in diagram (5.1) of [24].
Chapter 5 of [24] is probably difficult to read on its own because of the sheaf–theoretic language. You will have to consult chapter 2 as well, especially 2.2. Note that talk 4 takes care of section 2.2.

13. **A homotopy colimit decomposition**
This is the second of three talks covering chapter 5 of [24]. (Therefore, read the description of talk 12 first.) It will not be easy to prepare since it relies on most of the previous talks in the list, both for facts and for motivation.
The main result to be covered in this talk is the middle square in diagram (5.1) of [24]. This gives a homotopy colimit decomposition of

the space $|\mathcal{W}|$, and a similar one for $|\mathcal{W}_{\mathrm{loc}}|$. More precisely, we decompose the improved models $|\mathcal{L}|$ and $|\mathcal{L}_{\mathrm{loc}}|$ introduced in the second part of the previous talk. The technique used to get these decompositions is a standard "recognition principle" for homotopy colimits based on the main result of talk 4. The main reference for this talk is section 5.4 of [24]. For the homotopy colimits and the recognition principle, read also appendix D in [24]. Note that lemma D.1.5 should have been mentioned in talk 11.

14. **A return to surface bundles**
    This is the third of three talks on chapter 5 of [24]. It establishes the lower square in diagram (5.1) of [24]. It is technical in nature and will not be easy to prepare. At the end of it, we will have a description of $|\mathcal{W}|$ in terms of classifying spaces of certain surface bundles with additional structure. (If you use the "empty–boundaries" definition of $|\mathcal{W}|$, as suggested in [44], then you should also use bundles of *closed* surfaces throughout.) The main references are sections 5.5 and 5.6 of [24], and perhaps 5.7 for a useful observation. Many of the definitions are in section 5.1. This talk relies strongly on talk 12; therefore, read the description of that talk, too.

15. **Surgery, I**
    This is the first of two talks covering chapter 6 of [24]. It should concentrate on sections 6.1 and 6.2.
    The goal of the two talks together is to improve the homotopy colimit decomposition of $|\mathcal{W}|$ obtained in the previous talks so that the pieces can be interpreted as classifying spaces for certain bundles of *connected* surfaces. The method consists, unsurprisingly, in making non–connected surfaces connected by doing surgery on embedded 0–spheres. This must be done in a well organised way — our homotopy colimit decomposition of $|\mathcal{W}|$ must survive in some form.

16. **Surgery, II**
    This is the second of two talks covering section 6 of [24]. It should cover sections 6.3 and 6.4. The speaker should also be aware of what is in sections 6.1 and 6.2. The description of talk 15 is also valid for this one here, except that this talk has even more categories and even fewer geometric arguments in it.

17. **Reduction to Harer's theorem and conclusion**
This talk should cover the rather short chapter 7 in [24], which finishes the proof of Madsen's integral Mumford conjecture, 0.2 above. Many ideas here are taken from the proof of Tillmann's theorem, talk 2. Section 7.1 produces versions of the main results of section 6 (see the description of talk 15) where all surfaces in sight have large genus, in addition to being connected. Thanks to Harer's stability theorem [16] which should be used as a black box, and some elementary theory of quasifibrations and homology fibrations, Appendix C.1 in [24], these results lead in section 7.2 to a *homology fiber sequence* relating the spaces $\mathbb{Z} \times B\Gamma_{\infty,2}$, $|\mathcal{W}|$ and $|\mathcal{W}_{\mathrm{loc}}|$. This completes the proof.

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Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`m.weiss@maths.abdn.ac.uk`

by AUGUST 1, 2003, at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.