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Mini-Workshop: Quantum Topology in Dimension Three

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Quantum topology grew out of the work of Jones, Witten, Reshetikhin and Turaev in the 80s and early 90s. The major task facing quantum topologists in the new millennium is to show that it can be used as a tool for addressing classical topological problems. The most obvious testing ground is knot theory and topology of 3-manifolds. This seems particularly interesting and tempting in the light of the recent announcement of a proof of the Geometrization conjecture for 3-dimensional manifolds.

The Mini-workshop brought together experts with different ideas and background in 3-dimensional quantum topology like gauge theory, TQFT, category theory, representation theory, moduli spaces and skein theory. This has been the stimulus to numerous discussions in groups and exchange of ideas during the week in Oberwolfach (and beyond). In particular the discussions and talks have shown that the ideas of quantum topology are central for understanding the role of topology in relation with its origins in algebra, analysis and geometry. Of course the new role of quantum topology in dimension 3 in the possible aftermath of a proof of the Geometrization Conjecture has been a topic of many discussions. The famous Volume conjecture, which directly relates geometry and quantum structure in dimension 3, now appears to be of even more crucial importance.

The strength of quantum topology in relating topology with other areas of mathematics became apparent in particular in Andersen's lecture, where the use of gauge theory allows to understand how quantum invariants detect the unknot, or in Blanchet's lecture relating skein theory and contact homology, or in Sikora's survey lecture relating 3-manifold topology and number theory.

The 15 participants had 13 formal lectures, a survey lecture and a problem session. The survey lecture has been given by Adam Sikora with title: "Primes and knots". The results of the problem session is a list of 43 problems, which is contained in this report as "Open Problems".

The organizers plan to publish proceedings from the workshop as a special issue of the Journal of Knot Theory and its Ramifications.

Abstracts

Gauge theory approach to Reshitikin-Turaev TQFT

Jorgen E. Andersen

In the talk I first recalled the gauge theory construction of quantum representations of the mapping class groups via the geometric quantization of moduli space of flat connections on compact surface. I then discussed recent joint work with Prof. K. Ueno, where we have constructed the Reshetikhin-Turaev TQFT's from the sheaf of vacua constructed by A. Tsuchiya, K. Ueno and Y. Yamada. Using the known relation between the sheaf of vacua construction and the geometric quantization of the moduli spaces, we can therefore use the gauge theory approach to study the Reshetikhin-Turaev TQFT. Using this approach I have recently proved:

1. The Asymptotic Faithfulness property of the quantum representations of the mapping class groups.
2. Results on the Growth Rate of the quantum invariant for a certain class of 3-manifolds.

A consequence of 2. is that the HOMFLY knot polynomial of all cables of a knot is an unknot detector.

Geometric construction of spinors in orthogonal modular categories

Anna Beliakova

I give a geometric construction of orthogonal modular categories. I consider the category of tangles modulo the kernel of the Kontsevich integral combined with an orthogonal weight system. This category admit a natural $\mathbb{Z}_2$-grading with the 0-graded part equivalent to the category of tangles modulo the Kauffman skein relations. The last category was studied by C. Blanchet and me earlier. I use our previous results in order to give a semi-simple decomposition of the whole $\mathbb{Z}_2$-graded category for the two choices of parameter corresponding to the odd and even orthogonal cases. This further leads to the two series of orthogonal modular categories. I show that they admit refinements and compute refined Verlinde formulas.

Spinors in Quantum Topology

Christian Blanchet

We consider the link invariant associated with the spinor representation of the odd orthogonal Lie algebra $so(2n+1)$. Using Kontsevich integral and the spinor weight system we establish skein relations for this invariant.

A similarity between the obtained relations and Ng's definition of contact homology of a knot is pointed out. Following this ideas we define the Clifford module of a knot and the Clifford chords algebra on n strings. The definitions are skein theoretic. Generators are embedded chords ending on the knot or strings; relations are essentially Kauffman relations on chords together with a Clifford relation. Our definition leads to a series of questions:

- Is the Clifford module of a knot finite dimensional ?
- Is it computable ?
- What is the relation between the Clifford chord algebra and the spinor centralizer algebra ?
- Is there a graded Clifford homology of knots ?

Invariant measures, Reidemeister torsion and Turaev-Viro invariants

Charles Frohman

This is a preliminary lecture about a computation of the total Reidemeister torsion of the $SU(2)$-character variety of an oriented 3 dimensional homology sphere that satisfies strong rigidity, that is $H^1(M, ad\rho) = 0$ for all irreducible representations. We roughly follow the outline of Witten's computation of the volume of the moduli space of semistable bundles of rank 2, degree 0 and fixed determinant over a nonsingular algebraic curve. We find that the total Reideimeister torsion is the Heat kernel regularization of a formula that resembles the formula for the Turaev-Viro invariant of a three manifold, except that all the quantized quantities are replaced by their corresponding classical quantities.

Integral TQFT and application

Patrick Gilmer

We discuss conditions which allow one to reduce the ring of coefficients of a TQFT if one restricts the associated cobordism category slightly. We give applications to the integrality of Turaev-Viro polynomials, and more refined strong shift equivalence class invariants of knots.

Deformation of homotopy into isotopy in oriented 3-manifolds

Uwe Kaiser

It is shown that deformation quantization in skein theory of oriented 3-manifolds can be induced from a *topological* deformation quantization of the fundamental 2-groupoid of the space of immersions of circles in M. The structure of skein module and its relations with string topology homomorphisms appear through representations of the groupoid structure into modules generated by objects. The deformation of the fundamental 2-groupoid is defined by the singularity stratification, the quantization by passage to isotopy classes. Several explicit properties and computations of skein modules are proved. Moreover, local systems on the space of immersions are important for the understanding of HOMFLY oriented and framed skein theory.

A tangled tale

Louis H. Kauffman

(joint work with Sofia Lambropoulou, Heather Dye)

We describe two projects involving tangles. The first is joint work with Sofia Lambropoulou and produces infinitely many examples of unknot diagrams (of increasing complexity) that must be made more complex by Reidemeister moves before they can be simplified. The second is joint work with Heather Dye where we give examples of pairs of links, differing by one crossing switch, whose Jones polynomials are equal to the Jones polynomial of the unlink. These examples can be used to produce instances of virtual links whose non-classicality is difficult to detect.

On the coloured Jones-polynomial of twist knots

Gregor Masbaum

Habiro has shown using quantum groups that the coloured Jones polynomial of a knot has a so-called cyclotomic expansion. This expansion is useful for studying the volume conjecture. It is also an important tool for constructing Habiro's universal sl_2 invariant of integral homology 3-spheres. Explicit computations of this expansion, however, existed only for a few knots.

In this talk I will explain how to use skein theory to compute the cyclotomic expansion for twist knots. This is an infinite series of knots including the trefoil and the figure eight knot, and my formulas generalize previously known formulas of Habiro and Le for those two knots.

Khovanov homology of links in I-bundles over surfaces

Jozef H. Przytycki

(joint work with Marta M. Asaeda, Adam S. Sikora)

Recently, Khovanov defined for any link L in $\mathbb{R}^3$ graded homology groups whose polynomial Euler characteristic is the Jones polynomial of L.

Unfortunately, Khovanov's construction does not extend in a straightforward way to links in I-bundles over surfaces (except the homology with $\mathbb{Z}_2$ coefficients only). Therefore the goal of this paper is to provide a nontrivial generalization of his method leading to homology invariants of links in such manifolds with arbitrary rings of coefficients.

We prove the basic properties of our homology groups including Viro's exact sequence for Khovanov homology groups. We also introduce cohomology groups of any link L and relate them via duality theorem to the homology groups of the mirror image of L.

Oriented Quantum Algebras

David Radford

In this talk a general introduction and discussion of the theory of oriented quantum algebras and its applications in 3-dimensional quantum topology has been given.

Skein calculus for SU_n-quantum invariants

Adam Sikora

The Kauffman bracket skein relations provide an important method of studying SU_2-quantum invariants of links and 3-manifolds. Among its many applications, it makes possible to relate the SU_2-quantum invariants to Khovanov homology, skein modules, the (noncommutative) A-polynomial, and the SL_2-character varieties. In this talk, we develop an analogous skein calculus for SU_n-quantum invariants for all n, which hopefully has equally broad applications. We show that our skein relations coincide with those of Kauffman for $n = 2$ and those of Kuperberg for n=3. Finally, we show that the SU_n-skein module of a 3-manifold M based on our skein relations is a q- deformation of the coordinate ring of the SL_n-character variety of $\pi_1(M)$.

Virtual strings

Vladimir G. Turaev

A virtual string is a scheme of self-intersections of a generic oriented closed curve on an oriented surface. More precisely, a virtual string of rank $m \geq 0$ is an oriented circle with $2m$ distinguished points partitioned into m ordered pairs. These m ordered pairs of points are called arrows of the virtual string.

A (generic oriented) closed curve on an oriented surface gives rise to an "underlying" virtual string whose arrows correspond to the self-crossings of the curve. The usual homotopy of curves on surfaces suggests a notion of homotopy for strings. The homotopy of curves in 3-manifolds with boundary suggests a notion of cobordism for strings. The main objective of the theory of virtual strings is a study (and eventually classification) of their homotopy classes and cobordism classes. To this end, we introduce algebraic invariants of virtual strings, specifically, a one-variable polynomial u and a so-called based matrix. We formulate obstructions to homotopy/cobordism of strings in terms of these invariants. This leads us to a purely algebraic study of analogues of homotopy and cobordism for skew-symmetric matrices.

One of our main results: if two strings are cobordant, then their u-polynomials are equal. In particular, if a string is slice, that is if it can be realized by a closed curve on the boundary of an orientable 3-manifold M that is contractible in M, then its u-polynomial is zero.

We introduce a natural Lie cobracket in the free abelian group generated by the homotopy classes of strings. Dually, the abelian group of $\mathbb{Z}$-valued homotopy invariants of strings becomes a Lie algebra. This Lie algebra is integrated into an infinite dimensional Lie group. This Lie group gives rise to further algebraic objects including a Hopf algebra structure on the (commutative) polynomial algebra generated by the homotopy classes of strings.

Virtual strings are closely related to virtual knots introduced by L. Kauffman. In particular, the term "virtual knots" suggested to us the term virtual strings. Virtual knots can be defined as equivalence classes of arrow diagrams which are just virtual strings whose arrows are provided with signs $+$ or $-$. Forgetting these signs, we obtain a map from the set of virtual knots into the set of homotopy classes of virtual strings. We give a more elaborate construction which associates with each virtual knot a polynomial expression in virtual strings with coefficients in the ring $\mathbb{Q}[z]$. This leads to an isomorphism between a "skein algebra" of virtual knots and a polynomial algebra generated by the homotopy classes of strings.

Kirby elements and Quantum invariants

Alexis Virelizier

We define the notion of a Kirby element of a ribbon category (not necessarily semisimple). Kirby elements lead to 3-manifolds invariants. In general, determining all the Kirby elements is a quite difficult problem. We characterize (in terms of the structure maps of some categorical Hopf algebra) a set of Kirby elements which is sufficiently large to recover the known quantum invariants. In the case of a category of representations of a ribbon Hopf algebra, all can be described and computed in purely algebraic terms (avoiding representations).

Open Problems

(1) Suppose that a 3-manifold M has the same quantum invariants as $S^1 \times S^2$. Does that imply that it's fundamental group is $\mathbb{Z}$?
(2) **Quantum Poincare Conjecture:** M^3 is simply connected if and only if it has the same Reshetikhin-Turaev invariants associated with $U_q(sl(n, C))$ as the three-sphere.
(3) **Jones Conjecture:** Prove that the Jones polynomial detects the unknot. Equivalently prove that all nontrivial virtual knots with Jones polynomial equal to 1 are non-classical.
(4) Does the coloured Jones polynomial detect the unknot?
(5) Do the quantum $su(2)$-invariants detect the volume?
(6) Find an interpretation of the noncommutative A-ideal in terms of geometric representation theory.
(7) What is the relationship between the classical A-ideal and the noncommutative A-ideal? Is the complexity of the classical A-polynomial of a knot reflected in the minimum degree of linear recursion for computing the coloured Jones polynomial?
(8) Are the Reshetikhin-Turaev invariants finite-to-one?
(9) The Turaev-Viro invariant can be written as a sum over spinal surfaces in the manifold. Can you see aspects of the normal surface theory of the manifold reflected in its Turaev-Viro invariants, or Witten-Reshetikhin-Turaev invariants?
(10) Can you see the asymptotic behaviour of the coloured Jones polynomial of a knot reflected in a family of integrals over $SL_2\mathbb{C}$ representations of some object associated to the knot, that can be analyzed using classical techniques?
(11) Can quantum invariants detect whether +1-surgery on a prime knot can yield a connected sum of Poincar/'e homology spheres?
(12) Can quantum invariants detect whether surgery on a hyperbolic knot can yield a connected sum where one factor is a lens space?
(13) Beyond Reidemeister torsion, what aspects of the spectral theory of a hyperbolic 3-manifold are reflected in quantum invariants?
(14) Use quantum invariants to disprove that every knot can be reduced to a trivial knot by 4-twists.
(15) The (t_3, t_4)-move consists of replacing two parallel strands oriented in the same direction with 3 twists and two parallel oppositely oriented strands with 4 twists. Use quantum invariants to disprove the conjecture that every knot can be reduced to a trivial one using this move.
(16) Define Khovanov homology for 3-manifolds.
(17) Define quantum Khovanov homology for all quantum knot invariants arising from classical groups.
(18) Is it true that order of every cyclic component of torsion of Khovanov homology is always a power of 2?
(19) What is the basis of the skein module of a cylinder over a closed surface for $SU(n)$-invariants? (The answer is known only for surfaces with boundary.)
(20) Give a description of a skein algebra of a surface for $SU(2)$ in terms of generators and relations.
(21) Compute the Kauffman bracket skein module for $S^1 \times P$, where P is a pair of pants (or in other words a thrice punctured sphere). Does this module have torsion?
(22) Compute skein modules of non-cabled knot complements.

(23) What determines the growth of $SU(n)$ quantum invariant of a knot?
(24) Does the 3-cabled Jones polynomial determine the unknot?
(25) How to compute the fundamental group of the virtual knot represented as a diagram on a surface?
(26) Formulate a definition of virtual 3-manifolds.
(27) Do the Reshetikhin-Turaev TQFT representations detect conjugacy classes of the mapping class group?
(28) Is there some relation between Khovanov homology and contact homology?
(29) Is contact homology finite dimensional?
(30) Compute contact homology from the Blanchet-Clifford module.
(31) Le M be a 3-manifold with boundary. Turaev-Viro ideal at level r (in the appropriate ring of cyclotomic integers) is generated by the Turaev-Viro invariants at level r of all closed 3-manifolds containing M. Reshetikhin-Turaev ideals are defined analogously using Reshetikhin-Turaev invariants. How are these two kinds of ideals related? *Remark:* It is easy to see that the Turaev-Viro ideal is included in the Reshetikhin-Turaev ideal.
(32) Find a finite set of generators for the Turaev-Viro ideals.
(33) Understand the explicit transform from the topological $*$-product and the Berezin-Toeplitz product on moduli spaces.
(34) Characterize polynomials which arise as Jones polynomials of knots.
(35) Do the finite dimensional quasitriangular Hopf algebras yield a complete set of knot invariants?
(36) Understand and compute the Henning's invariants of 3-manifolds.
(37) Study non-semisimple quantum invariants (that is quantum invariants constructed from ribbon categories C which cannot be computed by using a semisimple quotient of C. For example: for $U_q(sl_2)$, Reshetikhin-Turaev invariants are semisimple, HKR-invariant is not-semisimple.
(38) What is the analogy of quantum invariants in arithmetic topology?
(39) If a prime number is a knot, what is a crossing?
(40) Trefoil and figure 8 knot are the unknotting number 1 knots with diagrams where any crossing can be changed in order to turn the diagram into a diagram of the unknot. Are there any other knots with this property?
(41) Let K be a hyperbolic knot in the 3-sphere. We know that

$$\frac{1}{2\pi}m(A(K)) = \sum_i Vol(\rho_i)$$

for certain representations $\rho_i : \Pi_1(S^3 - K) \to SL_2(C)$. Identify these representations.
(42) Does the Reshetikhin-Turaev $U_q(sl(n,C)$ quantum invariant of a knot K at the maximal level (say $S^k(V)$, for a representation V of SU(n)), evaluated at $e^{\frac{2\pi i}{k+n}}$, grow exponentially with rate $maxImCS(M_{SL(n,C)}(S^3 - K))$, where $M_{SL(n,C)}(S^3 - K)$ denotes moduli space of flat $SL(n,C)$ connections on $S^3 - K$.
(43) Let the triad labelled with a,b,c be evaluated as $\sqrt{-1}\epsilon_{abc}$, where ϵ_{abc} is equal to ± 1 (equal to the sign of the permutation abc) in the case when $abc \in Perm(123)$ and it is equal to 0 otherwise. Let G be a trivalent plane graph and define $[G] =$ Contraction of tensor sum corresponding to the above definition. Show that if G is a plane graph, isthmus free, then $[G] \neq 0$. *Remark:* this is equivalent to the Four Colour Theorem.

Edited by Uwe Kaiser

Participants

Prof. Dr. Jorgen E. Andersen
andersen@imf.au.dk
Matematisk Institut
Aarhus Universitet
Ny Munkegade
Universitetsparken
DK-8000 Aarhus C

Prof. Dr. Anna Beliakova
beliak@math.unibas.ch
Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 Basel

Prof. Dr. Christian Blanchet
christian.blanchet@univ-ubs.fr
Departement de Mathematiques
Universite de Bretagne-Sud
Centre Yves Coppens
BP 573
F-56017 Vannes

Prof. Dr. Charles Frohman
frohman@math.uiowa.edu
Dept. of Mathematics
University of Iowa
Iowa City, IA 52242-1466 – USA

Prof. Dr. Patrick Gilmer
gilmer@math.lsu.edu
Dept. of Mathematics
Louisiana State University
Baton Rouge, LA 70803-4918 – USA

Prof. Dr. Uwe Kaiser
kaiser@math.boisestate.edu
Department of Mathematics
Boise State University
1910 University Drive
Boise ID 83725-1555 – USA

Prof. Dr. Joanna Kania-Bartoszynska
kania@math.boisestate.edu
Department of Mathematics
Boise State University
1910 University Drive
Boise ID 83725-1555 – USA

Prof. Dr. Louis H. Kauffman
kauffman@uic.edu
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 South Morgan
Chicago, IL 60607-7045 – USA

Prof. Dr. Sofia Lambropoulou
sofia@math.ntua.gr
Department of Mathematics
National Technical University
of Athens
Zografou Campus
15780 Athens – Greece

Dr. Gregor Masbaum
masbaum@math.jussieu.fr
U. F. R. de Mathematiques
Case 7012
Universite de Paris VII
2, Place Jussieu
F-75251 Paris Cedex 05

Dr. Jozef H. Przytycki
przytyck@gwu.edu
Department of Mathematics
George Washington University
Washington, D. C. 20052 – USA

Prof. Dr. David Radford
radford@uic.edu
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045 – USA

Prof. Dr. Adam Sikora
asikora@buffalo.edu
School of Mathematics
Institute for Advanced Study
1 Einstein Drive
Princeton, NJ 08540 – USA

Dr. Alexis Virelizier
virelizi@math.univ-montp2.fr
Departement de Mathematiques
Universite Montpellier II
Place Eugene Bataillon
F-34095 Montpellier Cedex 5

Prof. Dr. Vladimir G. Turaev
turaev@math.u-strasbg.fr
Institut de Mathematiques
Universite Louis Pasteur
7, rue Rene Descartes
F-67084 Strasbourg Cedex