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Miniworkshop: Finite Elements and Layer-Adapted Meshes

November 3rd – November 7th, 2003

This week was organized by H.-G. Roos (Dresden) and M. Stynes (Cork). The subject deals with the numerical solution of problems that change their character if one or more parameters approach a critical value. Although several books on the subject have appeared some years ago (1996), in practice the numerical solution of PDEs with significantly varying behaviour in different parts of the domain of definition is still a real challenge. During the workshop the main topic of discussion was the application of the finite element method on a priori or a posteriori defined layer-adapted meshes.
During the week an extremely interesting exchange of ideas took place. The number of participants at a mini-workshop allows each one to present his/her actual results and moreover permits a thorough discussion. The subjects discussed concentrated on: several aspects of a posteriori error estimates and adaptive methods, parameter-uniform convergence for problems with more complicated layer structure, aspects of discretization methods with special emphasize on the discontinuous Galerkin finite element method, and several approaches to stabilization on anisotropic meshes.
Many people made progress in ongoing cooperative projects and also some new collaborations were started. It became apparent that some of the scientific questions raised during the meeting still require much serious research. During one evening session, each participant briefly presented an open problem whose solution is of interest to him/her. All the participants expressed their enjoyment of the mini-workshop to the organizers and found the special form of a mini-workshop conducive to furthering their scientific research.
Consequently, the organizers hope to organize a further meeting on this or an extended topic a few years from now.

Abstracts

Reliable a posteriori error control for non-conforming finite element approximation of Stokes flow

Willy Dörfler

(joint work with M. Ainsworth)

We derive computable a posteriori error estimates for the lowest order non-conforming Crouzeix–Raviart element in case of approximation of incompressible Stokes flow. The estimator provides an explicit upper bound that is free of any unknown constants. In addition, it is shown that the estimator provides an equivalent lower bound on the error up to a generic constant.

Local a posteriori error estimation for reaction-diffusion equations on anisotropic FEM meshes

Sergej Grosman

Singularly perturbed reaction-diffusion problems exhibit in general solutions with anisotropic features, e.g. strong boundary and/or interior layers. This anisotropy is reflected in the discretization by using meshes with anisotropic elements. The quality of the numerical solution rests on the robustness of the a posteriori error estimator with respect to both the perturbation parameters of the problem and the anisotropy of the mesh. // The simplest local error estimator from the implementation point of view is the so-called hierarchical error estimator. The robustness proof is mainly based on two facts: the saturation assumption and the strengthened Cauchy-Schwarz inequality. The proofs of both these facts are given in the present work as well as the concluding proof of the robustness of the estimator. A numerical example confirms the theory.

Adjoint consistent approximation of linear functionals using adaptive finite element methods

Kathryn Harriman

(joint work with E. Süli and D. Gavaghan)

In many areas of application the quantity of interest is not the solution of a partial differential equation but a linear functional of its solution. For such problems, what is needed is an adaptive finite element algorithm to give a numerical approximation of the functional to within a given tolerance. We describe the discontinuous Galerkin finite element method (DGFEM) for a model convection diffusion problem. We give an a priori error bound for the computed functional which shows that it is important to use a symmetric interior penalty Galerkin method (rather than nonsymmetric) and a weak form of the functional which leads to an adjoint consistent dual problem. This leads to optimal convergence rates. We also state an a posteriori bound on the error in the computed functional which could be used as the basis for an hp-adaptive finite element method.

On a two-dimensional discontinuous Galerkin discretisation with embedded Dirichlet boundary condition

Piet W. Hemker

In this talk we introduced a discretisation of Discontinuous Galerkin (DG) type for solving two-dimensional second order elliptic PDEs on a regular rectangular grid, while the boundary value problem has a curved Dirichlet boundary.
According to the same principles that underlie DG-methods, we adapt the discretisation in the cell in which the (embedded) Dirichlet boundary cannot follow the gridlines of the orthogonal grid.
The DG-discretisation aims at a high order of accuracy. We discretize by using tensor products of cubic polynomials. By construction, such a DG discretisation is fourth order consistent, both in the interior and at the boundaries. By experiments we show fourth order convergence in the presence of a curved Dirichlet boundary. Stability is proved for the one-dimensional Poisson equation with an embedded boundary condition.
To illustrate the possibilities of our DG-discretisation, we solve a convection dominated boundary value problem on a regular rectangular grid with a circular embedded boundary condition [J Comp and Appl Math 76 (1996) 277 – 285]. We show how accurately the boundary layer with a complex structure can be captured by means of piece-wise cubic polynomials. The example shows that the embedded boundary treatment can be very effective.

Multigrid iteration for the solution of the discontinuous Galerkin discretisation of elliptic PDEs

Piet W. Hemker

We discussed the discretisation of elliptic PDEs by the Baumann-Oden and by the symmetric DG method, and the solution of the resulting discrete operators by multigrid iteration. For our purpose we introduce a convenient hierarchical basis for the approximating space. For the one- and the two-dimensional linear second order elliptic equation, we study the convergence of MG-iteration for cell- and point-wise block-relaxation strategies by means of Fourier analysis.
We show that point-wise block partitioning gives much better results than the classical cell-wise partitioning. Independent of the mesh size, for Poisson's equation, with block-Gauss-Seidel and symmetric block-Gauss-Seidel smoothing, simple MG-cycles yield a convergence rate of 0.4 - 0.6 per iteration sweep for both DG-methods studied.
In contrast to the higher-order methods, the second-order DG-methods are unstable if no interior penalty is applied. We show for the 2nd-order methods how convergence depends on the additional penalty parameter. Like with the higher-order methods, we find that point-wise block-relaxations give much better results than the classical cell-wise relaxations. Both for the Baumann-Oden and for the symmetric DG method, with a sufficient interior penalty, the block relaxation methods studied (Jacobi, Gauss-Seidel and symmetric Gauss-Seidel) all make excellent smoothing procedures in a classical multigrid setting.

Optimal uniform convergence analysis for a singularly perturbed quasilinear problem and superconvergence analysis

JICHUN LI

The standard conforming finite element methods on a highly nonuniform rectangular meshes are considered for solving the quasilinear singular perturbation problem $-\varepsilon^2(u_{xx} + u_{yy}) + f(x, y; u) = 0$. By using a special interpolation operator and the integral identity technique, optimal uniform convergence rates of $O(N^{-(k+1)})$ in the L^2-norm are obtained for all $k-$th ($k \geq 1$) order conforming tensor-product finite elements, where N is the number of intervals in both x- and y-directions.

Layer-adapted meshes for one-dimensional reaction-convection-diffusion problems

TORSTEN LINSS

We study convergence properties of an upwinded finite element method for the solution of linear one-dimensional reaction-convection-diffusion problems on arbitrary meshes. We derive conditions that are sufficient for (almost) first-order convergence in the L_∞ norm, uniformly in the diffusion parameter, of the method. These conditions are easy to check and enable one to immediately deduce the rate of convergence. The key ingredients of our analysis are sharp bounds on the $W^{1,1}$ norm of the discrete GREEN's function associated with the discretization.

Stabilized finite element methods on anisotropic meshes

GERT LUBE

The motivation of the research stems from the numerical simulation of turbulence models for incompressible, non-isothermal flows. Linear(ized) advection-diffusion-reaction problems and saddle-point problems of Oseen-type appear as auxiliary problems. The solution of these subproblems is performed using stabilized finite element methods together with anisotropic meshes in boundary layers.
We review recent results by S. MICHELETTI ET.AL. in SINUM 41 (2003) 3, 1131-1162 on anisotropic interpolation estimates with the piecewise linear Lagrange and Clement interpolation operators. These results are applied to the advection-diffusion problem and to the Stokes system. In the lecture we show how the analysis can be extended to the advection-diffusion-reaction problem and to the Oseen system. Moreover, we discuss how previous results of APEL/LUBE can be used to extend the analysis to higher-order schemes and to the three-dimensional case.

Regularity of a singularly perturbed reaction-diffusion equation in curvilinear polygons

JENS MARKUS MELENK

We analyze the regularity properties of a singularly perturbed model problem of elliptic-elliptic type in a polygon. The regularity results capture the key properties of the solution, namely, boundary layers and corner singularities. Since the regularity results are derived with a view to an application of the *hp*-version of the finite element method (FEM), bounds on the derivatives of arbitrary order are provided, where the differentiation order enters explicitly in the bounds.
In the second part of the talk, mesh design issues are discussed. Design principles for the *hp*-FEM and the *h*-FEM are contrasted.

Optimal error estimates for FEM on Bakhvalov meshes

Hans-Görg Roos

Let us consider standard linear finite elements for a convection-diffusion problem with an exponential layer. Then, it is well-known that the error of the finite element approximation u^N using N subintervals for the mesh satisfies in the $\varepsilon-$ weighted H^1-Norm on a Shishkin-type mesh

$$\|u - u^N\|_\varepsilon \leq CN^{-1} \max|\varphi'|.$$

Further, there are meshes such that the mesh characterizing function φ has the property $\max|\varphi'| \leq CN^{-1}$ leading to an optimal error estimate. For a Bakhvalov mesh, however, it was up to now open whether or not the estimate

$$\|u - u^N\|_\varepsilon \leq CN^{-1}$$

holds true. Our proof of that conjecture uses Clement interpolants instead of standard nodal interpolants used so far.

A uniform analysis of non-symmetric and coercive linear operators

Giancarlo Sangalli

In this talk, I show how to construct, by means of the function space interpolation theory, a natural norm for a generic linear coercive and non-symmetric operator L. The natural norm allows for continuity and inf-sup conditions which holds independently of L. Particularly, I will consider the convection-diffusion-reaction operator, for which we obtain continuity and inf-sup conditions that are uniform with respect to the operator coefficients. In this case, the results give some insight for the analysis of the singular perturbed behaviour of the operator, occurring when the diffusivity coefficient is small. Furthermore, the analysis is preliminary to applying some recent numerical methodologies (such as least-squares and adaptive wavelet methods) to this class of operators, and more generally to analyzing the numerical methods within the classical framework.

About the optimality of the residual-based a-posteriori estimators

Giancarlo Sangalli

In this talk, I discuss how the natural norm, introduced in my previous talk, is useful for the a-posterior error analysis of numerical methods. In particular, considering a very simple 1-dimensional convection-diffusion-reaction model problem, I present an improvement of the residual-based estimates for SUPG-FEM proposed in by R. Verfürth [A posteriori error estimators for convection-diffusion equations. Numer. Math. 80 (1998), no. 4, 641–663].

Grid approximation of singularly perturbed elliptic convection-diffusion equations in unbounded domains

Grigory I. Shishkin

In the quarter plane $\{(x, y) : \ x, y \geq 0\}$, we consider the class of Dirichlet problems for a singularly perturbed elliptic convection-diffusion equation. The higher derivatives of the equation, and also the first derivative along the y-axis contain respectively the parameters ε_1 and ε_2, which takes arbitrary values from the half-open interval $(0, 1]$ and the segment $[-1, 1]$. For small values of the parameter ε_1, a boundary layer appears in a neighbourhood of the domain boundary. Depending on the ratio between the parameters ε_1 and ε_2, this layer may be regular, parabolic or hyperbolic. Besides a boundary-layer scale, controlled by the perturbation parameters, one can observe a resolution scale, which is specified by the "width" of the domain on which the problem is to be solved on a computer. It turns out that, for solutions of the boundary value problem and of a formal difference scheme (on meshes with an infinite number of nodes) considered on bounded subdomains (referred to as the resolution subdomains), the domains of essential dependence, i.e., such domains outside which the finite variation of the solution causes relatively small disturbances of the solution on the resolution subdomains, are bounded uniformly with respect to the vector-parameter $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2)$. Using the conception of the domain of essential solution dependence, we design a constructive finite difference scheme (on meshes with a finite number of the nodes) that converges $\bar{\varepsilon}$-uniformly on the (bounded) resolution subdomains. The technique given in the paper can be applied to the construction of parameter-uniform numerical methods, capable of actual computation, for other types of singularly perturbed problems on unbounded domains.

On conditioning of ε-uniformly convergent schemes for singularly perturbed convection-diffusion problems

Grigory I. Shishkin

For a numerical method to solve boundary value problems with layers in the L_∞-norm, it is necessary to use such grids whose step-size in a neighbourhood of the layer becomes arbitrarily small for small values of the perturbation parameter ε (ε is a coefficient multiplying the highest derivatives). The use of such grids, in general, may cause a high sensitivity of the numerical solution to the disturbance of the problem data and, in particular, to round-off errors. As a result, such a numerical method appears to be inapplicable for practical computations.

In this presentation we consider disturbances of numerical solutions generated by disturbances of the problem data when special piecewise uniform grids are used to solve convection-diffusion problems. It is shown that the condition number for the matrix associated with the discrete problem is not ε-uniformly bounded. However, the condition number for this discrete problem is ε-uniformly bounded and, moreover, is the same as the condition number for a regular problem on uniform grids. As a result, for classical finite difference approximations on special piecewise uniform grids (their solutions on such "correct" grids converge ε-uniformly), the disturbances of the data generate disturbances of the discrete solutions for singularly perturbed problems which are of the same order of magnitude as for regular problems.

On robust numerical methods for flow problems with large reynolds number: Problems with parabolic and interior boundary layers

Grigory I. Shishkin

Boundary value problems for boundary layer equations often arise when flows of an incompressible fluid for the large Reynolds number are studied. These quasilinear equations are singularly perturbed with a perturbation parameter $\boldsymbol{\varepsilon} = \boldsymbol{Re^{-1}}$, where $\boldsymbol{Re}$ is the Reynolds number. Parabolic boundary and interior layers are typical for such problems. Standard numerical methods when applied to the problems bring to large errors in the solution. Much more larger errors appear for the computed friction drag where derivatives for the solution are involved. For the discrete derivatives of the solution, the errors can be many times larger than exact values of the derivatives, when the parameter $\boldsymbol{\varepsilon}$ is small

(***Re***-number is large). Similar difficulties are typical when heat/masstransfer processes for a flow with the large Reynolds (and/or Peclet) number are studied. Moreover, the number of iterations, that are required to solve the quasilinear equations, depends on the perturbation parameter. Thus, the development of numerical methods for which errors of the solution and scale derivatives (as the number of iterations) are independent of the perturbation parameter, i.e., robust methods, is an important problem.
Earlier it was shown that in the case of problems with parabolic layers, use of a technique based on condensing meshes is necessary for construction of numerical methods which converge $\boldsymbol{\varepsilon}$-uniformly (in the maximum norm).
To construct a robust numerical method, that is capable to resolve the parabolic boundary and interior layers, we consider an approach based on special meshes, condensing in the layer region. The efficiency of the technique used is shown in a few numerical experiments. We treat three examples: a flow problem over a flat plate, a laminar jet problem, and an oscillating pipe flow [H. Schlichting. Boundary-Layer Theory, 7th ed. McGraw Hill, New York (1979)].

Corner singularities and boundary layers in a simple convection-diffusion problem

Martin Stynes
(joint work with R. B. Kellogg)

A singularly perturbed convection-diffusion problem posed on the unit square is considered. Its solution may have exponential and parabolic boundary layers, and corner singularities may also be present. Pointwise bounds on the solution and its derivatives are derived. The dependence of these bounds on the small diffusion coefficient, on the regularity of the data, and on the compatibility of the data at the corners of the domain are all made explicit. The bounds are derived by decomposing the solution into a sum of solutions of elliptic boundary-value problems posed on half-planes, then analyzing these simpler problems.

The SDFEM for a convection-diffusion problem with a boundary layer: Optimal error analysis and enhancement of accuracy

Lutz Tobiska
(joint work with Martin Stynes)

The streamline diffusion finite element method (SDFEM) is applied to a convection-diffusion problem posed on the unit square, using a Shishkin rectangular mesh with piecewise bilinear trial functions. The hypotheses of the problem exclude interior layers but allow exponential boundary layers. An error bound is proved for $\|u^I - u^N\|_{SD}$, where u^I is the interpolant of the solution u, u^N is the SDFEM solution, and $\|\cdot\|_{SD}$ is the streamline-diffusion norm. This bound implies that $\|u - u^N\|_{L^2}$ is of optimal order, thereby settling an open question regarding the L^2-accuracy of the SDFEM on rectangular meshes. Furthermore, the bound shows that u^N is superclose to u^I, which allows the construction of a simple postprocessing that yields a more accurate solution. Enhancement of the rate of convergence by using a discrete streamline-diffusion norm is also discussed.

The discontinuous Galerkin finite element method for singularly perturbed problems

Helena Zarin
(joint work with H.-G. Roos)

A nonsymmetric discontinuous Galerkin finite element method with interior penalties is considered for two–dimensional singularly perturbed problems. On an anisotropic Shishkin mesh with bilinear elements we prove error estimates (uniformly in the perturbation parameter) in an integral norm associated with this method. On different types of inter-element edges we derive the values of discontinuity–penalization parameters. Numerical experiments support the theoretical results.

Polynomial preserving recovery and its application to a posteriori error estimates under anisotropic grids

Zhimin Zhang
(joint work with A. Naga)

A polynomial preserving gradient recovery method (PPR) is introduced and analyzed. The method maintains the simplicity, efficiency, and superconvergence property of the Zienkiewicz-Zhu patch recovery (SPR), and performs better than SPR under certain grids (such as the Chevron mesh), higher-order elements, as well as on boundaries.
The PPR is used in the ZZ estimator (instead of the SPR) to form a recovery type a posteriori error estimator. As the SPR, the PPR can be used to arbitrary triangulations and anisotropic grids. Numerical comparisons between the two methods are demonstrated. Some theoretical explanation are also provided. Under mildly structured meshes, including meshes generated by the Delaunay triangulation and some meshes with high aspect ratio, the ZZ estimator based on PPR is proven to be asymptotically exact.

Edited by Torsten Linß

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