Mathematisches Forschungsinstitut Oberwolfach

Report No. 17/2004

Arbeitsgemeinschaft mit aktuellem Thema: The Goodwillie Calculus of Functors

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March 28th – April 3rd, 2004

Introduction by the Organisers

The purpose of this meeting was to introduce the participants to calculus of functors, a theory aimed at "approximating" functors in algebra and topology. The focus of the talks was on two related branches of the theory, homotopy and manifold calculus.

The organizers, Thomas Goodwillie and Randy McCarthy, scheduled 16 talks. The first was given by Goodwillie who introduced the main ideas and outlined the plan for the rest of the meeting. In talks 2—5, speakers explained the most important terminology, techniques, and results used in both versions of the theory. In talks 6—8, participants learned about manifold calculus and how it applies to spaces of knots (talk 8), while talks 9—15 dealt with homotopy calculus. In the last four of those lectures, some applications, elaborations on results established in previous talks, and different versions of certain ideas and proofs encountered so far were given. Goodwillie explored some connections between the two versions of calculus of functors (as they apply to the embedding and identity functors) and gave the concluding remarks in the last talk.

It should be noted that a third version of the theory, orthogonal calculus, was not discussed due to time constraints, but Goodwillie explained some of its most salient features in an extra evening session. Other evening events took place as well, such as Rainer Vogt's elaboration of an aspect of his talk and informal research reports given by some participants.

Arbeitsgemeinschaft on Arbeitsgemeinschaft mit aktuellem Thema: The Goodwillie Calculus of Functors

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Abstracts

1. Introduction

Thomas Goodwillie

This workshop is about two related sets of ideas: the homotopy calculus and the manifold calculus. Each of them is a method of describing spaces (or other objects) up to weak homotopy equivalence by making heavy use of categories, functors, and naturality. In a typical application of the method, one gains information about a space by viewing the space as a special value of a suitable functor, analyzes the functor using "calculus", and then specializes. Thus the principal objects of study become some rather broad category of functors. A constant theme is the systematic approximation of these functors by functors of much more special kinds.

The homotopy calculus deals with homotopy functors from, for example, the category of topological spaces to itself. Here a *homotopy functor* is a functor that preserves (weak) equivalences. An optional additional axiom is that the functor is continuous on morphisms, in the sense that for any finite complex K and space X the canonical map from $K \times F(X)$ to $F(K \times X)$ is continuous. Another option is the *limit axiom*: the functor preserves filtered homotopy colimits. The main sources for the general theory are [4][5][6].

The manifold calculus deals with contravariant functors from the partially ordered set of open subsets of a fixed smooth manifold M to, for example, the category of spaces. Again the functors must satisfy a kind of homotopy invariance; roughly speaking, if $U \supseteq V$ is a collar then the map $F(U) \to F(V)$ is an equivalence. The main sources for the general theory are [24] [11].

Because of time constraints we have not planned any talks about a third theory, the orthogonal calculus, which deals with functors, continuous on morphisms, from the category of finite-dimensional real Hilbert spaces and isometric linear injections to the category of spaces. See [25].

The central idea in the homotopy calculus is approximation of functors by "linear" functors, just as in the ordinary differential calculus the central idea is the approximation of functions by linear functions. Linearity means the following. Call the homotopy functor F *excisive* if it takes homotopy pushout squares to homotopy pullback squares and call it *reduced* if the unique map $F(*) \to *$ is a weak equivalence. Call it *linear* if it is both excisive and reduced. A typical linear functor from based spaces to based spaces will, up to natural equivalence, have the form $L(X) = \Omega^\infty(C \wedge X)$, at least on finite CW complexes X. Here C is some spectrum, which can be called the coefficient of the linear functor.

There is a standard process, which is sometimes called stabilization and here is called linearization, for turning a reduced functor F into a linear functor L. Roughly speaking, there is a natural map from $F(X)$ to $\Omega F(\Sigma X)$ and one iterates this to make the stabilization, the homotopy colimit of $\Omega^k F(\Sigma^k X)$ as k goes to infinity. If the functor is continuous on morphisms and is reduced in the strong sense that $F(*) \simeq *$, then one literally has such a map F to $\Omega F \Sigma$. In general we prefer not to assume this much. Instead we apply F to the pushout diagram that makes a suspension from two cones and consider the map from $F(X)$ to the homotopy limit of the other three spaces in the resulting square diagram

$$\begin{array}{ccc} F(X) & \rightarrow & F(CX) \\ \downarrow & & \downarrow \\ F(CX) & \rightarrow & F(\Sigma X) \end{array}$$

(This homotopy limit is equivalent to $\Omega F(\Sigma X)$ if F is reduced.) If F is linear then L is (equivalent to) F, and in general L is the universal example (in an appropriate up-to-homotopy sense) of a linear functor under L. The coefficient of L is called the derivative of F at the one point space.

More generally the derivative $\partial_y F(Y)$ of F at the space Y and basepoint y can be defined as the coefficient of the stabilization of the functor

$$Z \mapsto \text{hofiber}(F(Y \vee_y Z) \rightarrow F(Y))$$

from based spaces to based spaces.

There is another useful generalization. The excision condition concerns the behavior of a functor on two-dimensional cubical diagrams. We call a functor n-excisive if it satisfies a cerain condition involving $(n+1)$-dimensional cubical diagrams, so that 1-excisive means excisive. It turns out that again for any F there is a universal n-excisive functor under F. We call it P_nF and think of it as the nth Taylor polynomial of F. There are maps $P_nF \rightarrow P_{n-1}F$, and F maps into the limit of this "Taylor tower".

The nth layer of the tower, meaning the homotopy fiber of $P_nF \rightarrow P_{n-1}F$, is analogous to a homogeneous polynomial; it is an n-excisive functor whose $(n-1)$-excisive approximation is trivial. Such things turn out always to have the form $\Omega^\infty(C \wedge X^{\wedge n})_{h\Sigma_n}$, at least on finite CW complexes X. Here the coefficient C is a spectrum with an action of the symmetric group Σ_n, and it is called the nth derivative of F (at $*$).

Most functors encountered in practice are not n-excisive for any n, but are stably n-excisive. F is called stably 1-excisive if for a homotopy pushout square

$$\begin{array}{ccc} X & \rightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \rightarrow & X_{12} \end{array}$$

the functor always yields a square diagram such that the map from the first space $F(X)$ to the homotopy limit of the other three is $k_1 + k_2 - c_1$ connected, where k_i is the connectivity of the map $X \rightarrow X_i$ and c_1 is a constant depending only on F. F is called stably n-excisive if it satisfies a similar condition involving $(n+1)$-dimensional cubes. If F is stably n-excisive for all n and the associated sequence of constants c_n has slope ρ, then the functor is called ρ-analytic.

If F is ρ-analytic then for ρ-connected spaces X the canonical map from $F(X)$ to $P_nF(X)$ has a connectivity that tends to infinity with n. ("The Taylor series converges to the function" within a "radius" determined by ρ.)

If F is ρ-analytic and $\partial_y F(Y) \simeq *$ for all (Y, y) then F is locally constant: any $(\rho - 1)$-connected map $X \to Y$ of spaces, or at least of finite complexes, induces an equivalence $F(X) \to F(Y)$. This can be proved using Taylor towers. It was proved in [5] by a more direct method.

So much for the homotopy calculus. We now turn more briefly to the manifold calculus. The most important example is the functor $\mathrm{Emb}(-, N)$ which takes an open set U of M to the space of smooth embeddings of U in another manifold N.

Here again there is a notion of n-excisive functor, and there is a way of building a universal n-excisive functor $T_n F$ under F. It can be defined in a few words: $(T_n F)(U)$ is the homotopy limit of $F(V)$ over all open sets V in U that are tubular neighborhoods of sets having at most n elements. Once again, if F satisfies a kind of analyticity (stable excision) condition then the resulting tower converges for a large class of objects. Again there is a classification theorem for homogeneous functors (n-excisive functors with trivial $(n-1)$-excisive part). The functor $\mathrm{Emb}(-, N)$ is sufficiently analytic that these methods give very strong information about the space of embeddings of M in N if the codimension $\dim(N)$-$\dim(M)$ is at least three. In fact, in some useful but complicated sense the homotopy type of $\mathrm{Emb}(M, N)$ is determined by the family of spaces $\mathrm{Emb}(U, N)$, where U ranges through those open sets of M that are tubular neighborhoods of finite sets.

The talks at this workshop will deal mostly with the general results mentioned above and some generalizations. Of course important examples will be introduced, but we will not venture very far into serious applications of the theory, such as applications of homotopy calculus to algebraic K-theory and to classical homotopy theory.

Homotopy calculus and manifold calculus can be presented as separate and parallel subjects, but in fact the former had its genesis in the latter and there is an ongoing interplay between the two. This will be the subject of the final talk.

2. Cubical diagrams and n-th order excision

Konstantin Salikhov

Let $\mathcal{T}$ be the be the category of based spaces and $\mathcal{C}$ be a small category.

Definition ([5]). A diagram of spaces is a functor $\mathcal{X} : \mathcal{C} \to \mathcal{T}$. If $\mathcal{C}$ is the category $P(\underline{n})$ of subsets of the n-element set $\underline{n} = \{1, \ldots, n\}$ with the morphisms given by inclusions, we call such a diagram *cubical*, or simply *n-cube*.

For any diagram $\mathcal{X}$ of spaces we can talk about its homotopy limit [1]. Since a cubical diagram has the initial object, $\mathrm{holim}_{P(\underline{n})}(\mathcal{X})$ will not give us anything new. Instead, consider $\mathrm{holim}_{P_0(\underline{n})}(\mathcal{X})$ over the subcategory $P_0(\underline{n})$ of non-empty subsets of $\underline{n}$.

Definition ([5]). A cube $\mathcal{X}$ is called *homotopy cartesian*, or just *cartesian*, if the natural map $\mathcal{X}(\emptyset) \to \mathrm{holim}_{P_0(\underline{n})}(\mathcal{X})$ is a weak equivalence. By $\mathrm{tfiber}(\mathcal{X})$ we denote the homotopy fiber of this map. If this map is k-connected, the cube $\mathcal{X}$ is called *k-cartesian.*

There are dual notions of *co-cartesian* and *k-co-cartesian* cubes, and also of tcofiber($\mathcal{X}$). Note that we can look at an $(n+1)$-cube $\mathcal{X}$ as a componentwise map $\mathcal{X}_1 \to \mathcal{X}_2$ between two n-cubes. Then tfiber($\mathcal{X}$) $\simeq$ hofiber(tfiber($\mathcal{X}_1$) $\to$ tfiber($\mathcal{X}_2$)).

Proposition ([5]). For any map $\mathcal{X}_1 \to \mathcal{X}_2$ of n-cubes and the corresponding $(n+1)$-cube $\mathcal{X}$
(i) If $\mathcal{X}$ and $\mathcal{X}_2$ are k-cartesian, then $\mathcal{X}_1$ is k-cartesian.
(ii) If $\mathcal{X}_1$ is k-cartesian and $\mathcal{X}_2$ is $(k+1)$-cartesian, then $\mathcal{X}$ is k-cartesian.

There is, of cause, the corresponding dual proposition for k-co-cartesian cubes. Note that we can use all the above constructions to define diagrams of spectra (tfiber of a diagram is understood as the spectrum of tfiber's). Such cubes of spectra will be heavily used in the next sections. If $\mathcal{X} : P(\underline{n}) \to \mathcal{S}p$ is an n-cube of spectra, then $\mathcal{X}$ is k-cartesian iff it is $(k+n-1)$-co-cartesian.

We say that a cube $\mathcal{X}$ is strongly co-cartesian, if all of its faces of dimension at least two are co-cartesian.

Definition ([5]). A functor $F : \mathcal{T} \to \mathcal{T}$ is called *n-excisive* if for any strongly co-cartesian $(n+1)$-cube $\mathcal{X}$, the composition cube $F \circ \mathcal{X}$ is cartesian.

The basic example of a 1-excisive functor is $L(X) = \Omega^\infty(C \wedge X)$, where C is a spectrum. We can also talk about a functor of several variables $M : \mathcal{T}^r \to \mathcal{T}$. We say that M is $(n_1, \ldots, n_r)$-excisive if it is n_i-excisive in i-th variable, with the other variables fixed.

Proposition ([5]). If $M : \mathcal{T}^r \to \mathcal{T}$ is $(n_1, \ldots, n_r)$-excisive, then the composition $M \circ \Delta$ with the diagonal map $\Delta : \mathcal{T} \to \mathcal{T}^r$ is $(n_1 + \cdots + n_r)$-excisive.

Definition ([6]). For a functor $F : \mathcal{T} \to \mathcal{T}$ we define the n^{th} cross-effect $cr_n(F) : \mathcal{T}^n \to \mathcal{T}$ by $cr_n(F)(X_1, \ldots, X_n) = \text{tfiber}(S \mapsto F(\vee_{i \notin S} X_i))$ where $S \subset \underline{n}$.

Proposition ([6]). If $F : \mathcal{T} \to \mathcal{T}$ is n-excisive, then $cr_n(F) : \mathcal{T}^n \to \mathcal{T}$ is 1-excisive in each variable.

Let us remind that in the manifold calculus we study contravariant functors from the category of open subsets of a smooth manifold M without boundary to spaces. Here the notion of n-excisiveness looks like

Definition ([24]). A cofunctor F is called *polynomial of* deg $\leq n$, if for any open subset $V \subset M$, and pairwise disjoint closed subsets $A_1, A_2, \ldots, A_{n+1} \subset V$, the $(n+1)$-cube $S \mapsto F(V \setminus \cup_{i \in S} A_i)$, with $S \subset \underline{n+1}$, is cartesian.

The basic example of a cofunctor of deg ≤ 1 is $F(V) = \Gamma(p, V)$, the space of sections of a fibration $p : E \to M$. In particular, $F(V) = \{\text{Immersions of } V \text{ in } N\}$ for some fixed manifold N, is a cofunctor of deg ≤ 1. An example of a cofunctor of deg $\leq n$ can be constructed in the same way. In $M^{(n)} = M \times M \times \cdots \times M$ (product of n copies of M) consider the fat diagonal Δ^n, consisting of all n-tuples $(x_1, \ldots, x_n)$ with $x_i = x_j$ for some $i \neq j$. Denote by $\binom{M}{n}$ the orbit space $(M^{(n)} - \Delta^n)/\Sigma_n$. If $p : E \to \binom{M}{n}$ is a fibration, then $F(V) = \Gamma(p, \binom{V}{n})$ is a cofunctor of deg $\leq n$.

3. Analyticity and homotopy excision

Rainer Vogt

We say a homotopy functor $F : \mathcal{C} \to \mathcal{D}$ is in $E_n(c, \kappa)$ if for any strongly cocartesian cube $\chi : \mathcal{P}(S) \to \mathcal{C}$, $|S| = n+1$, such that $\chi(\emptyset) \to \chi(s)$ is k_s-connected with $k_s \geq \kappa$, the cube $F(\chi)$ is $(-c + \Sigma_{s \in S} k_s)$-cartesian.

Definition A homotopy functor $F : \mathcal{C} \to \mathcal{D}$ is called

- *stably n-excisive* if $F \in E_n(c, \kappa)$ for some c and κ
- *ρ-analytic* if there is an integer q such that $F \in E_n(n\rho - q, \rho + 1)$ for all $n \geq 1$.

The aim of this talk is to prove the following analyticity results.

Theorem 1:

(1) The identify functor $\mathcal{T}op \to \mathcal{T}op$ is 1-analytic.
(2) Let K be a finite CW-complex. Then the functors

$$\begin{array}{ll} \mathcal{T}op \longrightarrow \text{Spectra}, & X \mapsto \Sigma^\infty(Map(K, X)_+) \\ \mathcal{T}op \longrightarrow \mathcal{T}op, & X \mapsto \Omega^\infty\Sigma^\infty(Map(K, X)_+) \end{array}$$

are $(\dim K)$-analytic.

Part (1) is consequence of homotopy excision:

Theorem 2 (Ellis-Steiner): Let $\chi : \mathcal{P}(S) \to \mathcal{T}op$ be a strongly cocartesian S-cube, $|S| \geq 1$, such that $\chi(\emptyset) \to \chi(s)$ is k_s-connected. Then χ is k-cartesian with $k = (1 - |S| + \Sigma k_s)$. Moreover, $\pi_k(\text{fiber}(a\chi))$ has an algebraic presentation in terms of the groups $\pi_{k_s}(\text{fiber}(\chi(\emptyset) \to \chi(s)))$. Here fiber$(a\chi)$ is the total fiber of the cube χ.

This result has an Eckmann-Hilton dual.

Theorem 3: Let $\chi : \mathcal{P}(S) \to \mathcal{T}op$ be a strongly cartesian S-cube, $|S| \geq 1$, and let $F_s = \text{fiber } (g_s : \chi(S - s) \to \chi(S))$. Then fiber$(b\chi)$ is the join of the F_s, $s \in S$. Here $b\chi : \text{hocolim}\chi|\mathcal{P}_1(S) \to \chi(S)$ is the canonical map, where $\mathcal{P}_1(S) \subset \mathcal{P}(S)$ is the full subcategory of proper subsets of S.

Corollary: If $\chi : \mathcal{P}(S) \to \mathcal{T}op$ is a strongly cartesian S-cube, $|S| = n \geq 1$, and $\chi(S-s) \to \chi(S)$ is k_s-connected, then χ is at least $(n-1+\Sigma k_s)$-cocartesian.

For the proof of Part (2) of Theorem 1 we need partial refinements of Theorems 2 and 3.

Let $\chi : \mathcal{P}(S) \to \mathcal{C}$ be an S-cube, $|S| \geq 1$. For $\emptyset \neq T \subset S$ let $\partial^T \chi$ be the T-subcube with initial vertex $\chi(\emptyset)$ and terminal vertex $\chi(T)$, and let $\partial_{S-T}\chi$ be the T-subcube with initial vertex $\chi(S-T)$ and terminal vertex $\chi(S)$.

Theorem 2*: Let $\chi : \mathcal{P}(S) \to \mathcal{T}op$ be an S-cube, $|S| = n \geq 1$. Suppose that

(i) for all T, $\emptyset \neq T \subset S$, the T-cube $\partial^T \chi$ is $k(T)$-cocartesian

(ii) $k(U) \leq k(T)$ for all $U \subset T$.

Then χ is k-cartesian with

$$k = \min\{1 - n + \Sigma_\alpha k(T_\alpha)\}$$

over all partitions $\{T_\alpha\}$ of S into non-empty subsets.

Theorem 3*: Let $\chi : \mathbb{P}(S) \to \mathcal{T}op$ be an S-cube, $|S| = n \geq 1$. Suppose that

(i) for all T, $\emptyset \neq T \subset S$, the T-cube $\partial_{S-T}\chi$ is $k(T)$-cartesian

(ii) $k(U) \subset k(T)$ for all $U \subset T$.

Then χ is k-cocartesian with

$$k = \min\{n - 1 + \Sigma_k k(T)_\alpha)\}$$

over all partitions $\{T_\alpha\}$ of S into non-empty subsets.

We prove Theorem 1 in detail using Theorems 2, 3, 2*, 3*. We then indicate how those can be proved.

4. Disjunction and excision for spaces of embeddings

Tibor Macko

In this talk we consider good contravariant functors $F \colon \mathcal{O}(M) \to$ Spaces from the partially ordered set $\mathcal{O}(M)$ of open subsets of a fixed smooth mani-fold M to the category of spaces. Typical example of such a functor is $U \mapsto \mathrm{Emb}(U, N)$, where N is some fixed smooth manifold. We define analytic functors in this setting. Analyticity of a given functor F has a consequence that the Taylor tower of F converges to F within the radius of convergence.

Definition 1 ([11]). A good contravariant functor $F \colon \mathcal{O}(M) \to$ Spaces is ρ-*analytic* with *excess* c if for all $k \geq 1$ the $\underline{k+1}$-cube

$$S \mapsto F(V_S)$$

is $(c + \sum_{i=0}^{k}(\rho - q_i))$-cartesian, whenever $S \mapsto V_S$ is a $\underline{k+1}$-cube such that

$$V_i = V_\emptyset \cup \{\text{handles of index } \leq q_i\},$$

and $\rho > q_i$ for all i. For $S \subseteq \underline{k+1}$ the symbol V_S denotes $V_S = \cup_{i \in S} V_i$.

In the rest of the talk we concentrate on the functor $U \mapsto \mathrm{Emb}(U,N)$. The main result is:

Theorem 2 ([9],[3],[7],[8])**.** *In the situation as described above the* $\underline{k+1}$*-cube*

$$S \mapsto \mathrm{Emb}(V_S, N)$$

is $((3-n)+\sum_{i=0}^{k}(n-q_i-2))$ *cartesian, where* $n = \dim(N)$*. Hence the functor* $U \mapsto \mathrm{Emb}(U,N)$ *is* $(n-2)$*-analytic with excess* $(3-n)$*.*

A consequence is that the Taylor tower converges to $\mathrm{Emb}(M,N)$ whenever $\dim(M) \leq n-3$.

There is a weaker statement which can be proved using much less sophisticated methods than are those used to prove the previous statement.

Theorem 3 ([8])**.** *In the situation as described above the* $\underline{k+1}$*-cube*

$$S \mapsto \mathrm{Emb}(V_S, N)$$

is $((3-n)+\sum_{i=0}^{k}(n-2q_i-2))$ *cartesian, where* $n = \dim(N)$*.*

A consequence of this statement is the convergence of the Taylor tower to $\mathrm{Emb}(M,N)$ whenever $2\ \dim(M) \leq n-3$.

The proof of the weaker statement consists of converting the excision statement into the multiple disjunction statement which is proved using general position arguments and the generalized higher Blakers-Massey theorem (see [8]).

The proof of the stronger statement is much more difficult. We sketch a proof of a slightly weaker version presented in [8]. It first converts the statement from Theorem 2 to a statement about the cubes of spaces of diffeomorphisms. After that the main ingredients are the following:

- the proof of a corresponding statement about the cubes of spaces of homotopy equivalences given in [7],
- a multiple disjunction lemma for smooth concordance embeddings which is a content of Goodwillie's thesis [3], and
- an application of two theorems from surgery theory given in [8].

5. n-excisive functors

Andrew Blumberg

In this talk we develop and make precise the slogan "n-excisive functors are determined by their restrictions to sets with n elements or fewer, and the restriction can be arbitrary". Let $\mathcal{F}_n$ denote the category of finite pointed sets with $\leq n$ elements. In the setting of the homotopy calculus, we prove the following theorem:

Theorem 1 Let F and G be n-excisive functors from based spaces to spectra, and $t : F \to G$ a natural transformation. Then $t : F(X) \to G(X)$ is a weak equivalence for all finite dimensional CW-complexes X provided that $t : F(U) \to G(U)$ is a weak equivalence for $U \in \mathrm{obj}(\mathcal{F}_n)$. (Note that in the presence of a limit axiom for F and G, this result can be extended to all CW-complexes).

Working in the manifold calculus, we replace $\mathcal{F}_n$ by the category $\mathcal{O}_n(M)$ which consists of all open subsets of M which are diffeomorphic to a disjoint union of $\leq n$ disks (copies of $\mathbb{R}^m$, where m is the dimension of M). We have the analogous theorem:

Theorem 2 Let F and G be "good" cofunctors from $O(M)$ to spaces which are polynomial of degree $\leq n$, and $t : F \to G$ a natural transformation. Then $t(X) : F(X) \to G(X)$ is a weak equivalence for all X provided that $t : F(U) \to G(U)$ is a weak equivalence for $U \in \mathrm{obj}(\mathcal{O}_n(M))$.

The proofs of theorems 1 and 2 are essentially similar and use induction over cell attachment and handlebody decompositions respectively.

With these results in hand, we now proceed to describe stronger characterizations of n-excisive (and polynomial of degree $\leq n$) functors. Returning to the homotopy calculus, given a functor F from $\mathcal{F}_n$ to spectra we can prolong this via homotopical left Kan extension to a functor LF from finite CW-complexes to spectra.

Definition For a functor F from $\mathcal{F}_n$ to spectra, define $LF(X)$ to be the geometric realization of the simplicial set with k-simplices given as

$$\coprod_{x_0, x_1, \ldots, x_n} F(x_0) \wedge (\mathrm{hom}(x_0, x_1) \times \mathrm{hom}(x_1, x_2) \times \ldots \times \mathrm{hom}(x_{n-1}, x_n) \times X^{x_n})_+$$

As one would expect, for $\underline{\mathrm{m}} \in \mathcal{F}_n$, there is a weak equivalence $LF(\underline{\mathrm{m}}) \simeq F(m)$.

Now, given a functor F from based spaces to spectra, we can restrict down to a functor with domain $\mathcal{F}_n$ and then use the homotopical left Kan extension to prolong back up to a functor with domain finite CW-complexes. Call the functor associated in this fashion to F, LF_n. There is a natural transformation $l : LF_n \to F$.

We establish the following theorem:

Theorem 3 If F is an n-excisive functor, the natural transformation $l : LF_n \to F$ is a weak equivalence.

In light of the first theorem we proved above and the observation that homotopical left Kan extensions are indeed prolongations up to homotopy, it suffices to show:

Proposition Given any functor G from $\mathcal{F}_n$ to spectra, LG is n-excisive.

This proposition is proved by analyzing the simplicial set LG and in particular observing that homotopy cofibers commute with realization.

In the manifold setting essentially the same situation occurs. Specifically, given any cofunctor from $O(M)$ to spaces, we can restrict to a cofunctor from $O_k(M)$ to spaces. Such a cofunctor can be prolonged back up to a cofunctor with domain $O(M)$ using homotopical right Kan extension, and there is a natural transformation $r : F \to RF_n$.

Definition Given a cofunctor F from $O_n(M)$ to spectra, define $RF(X)$ to be $\operatorname{holim}_{U \in O_k(X)} F(U)$.

(Note that the definition of the homotopical left Kan extension could also have been written in such a form as an appropriate homotopy colimit).

Finally, we state the following theorem:

Theorem 4 If F is an "good" cofunctor which is polynomial of degree $\leq n$, the natural transformation $r : F \to RF_n$ is a weak equivalence.

Once again, this theorem follows immediately from theorem 2 above and the following proposition (which is proved in the next talk):

Proposition Given any cofunctor G from $O_k(M)$ to spectra, RG is a n-degree polynomial cofunctor.

References for the material discussed in this talk are Goodwillie's "Calculus III : Taylor series" and Weiss' "Embeddings from the point of view of immersion theory: Part 1".

6. Weiss' Taylor tower in the manifold case, part I

Ben Wieland

In this talk, we construct polynomial approximations of (good) contravariant functors from the partially ordered set $\mathcal{O}(M)$ of open sets of a manifold M to the category of spaces. Everything is taken from the papers of Weiss and Goodwillie [23, 24, 11]. From the previous talk, we know that polynomials of degree n are determined by their values on the subposet $\mathcal{O}_n(M)$ of disjoint unions of at most n disks, so a natural candidate is the homotopy Kan extension. Since the property of being a polynomial functor involves a limit, the right Kan extension is the correct choice.

Definition. For a good functor $F\colon \mathcal{O}(M) \to \text{Spaces}$, the *nth Taylor polynomial* T_nF is the homotopy right Kan extension of the restriction of F to $\mathcal{O}(M)$. Since

Kan extensions are pointwise limits, this is given by $T_nF(U) = \operatorname*{holim}_{V\in\mathcal{O}_n(U)} F(U)$. They come with maps $F \to \ldots \to T_nF \to T_{n-1}F \to \ldots$

Lemma. T_nF is n-excisive, or a polynomial of degree n.

Theorem. $F \to T_nF$ is the universal map from F to a polynomial of degree n. An interpretation is that any natural transformation $F \to G$ to a polynomial of degree n canonically factors (in the homotopy category) as $F \to T_nF \to T_nG \xleftarrow{\sim} G$.

The theorem follows from the lemma by formal properties of the Kan extension. In particular, since $\mathcal{O}_n(M)$ is a full subcategory of $\mathcal{O}(M)$ the Kan extension is truly an extension: if we restrict back to the subcategory, the values do not change (up to equivalence). Thus the Kan extension of a polynomial agrees with it on $\mathcal{O}_n(M)$ and both are polynomial, so they are equivalent. Thus we may, through the Kan extension, identify the (homotopy) category of polynomial functors of degree n with the (homotopy) category of functors on $\mathcal{O}_n(M)$. Then, the universal property of the Kan extension assures us that the Kan extension of the restriction of a functor is the universal approximation (on the right) of the functor by a polynomial.

Remark. Weiss's notion of a *good* functor [23, 2.2] [24, 1.1] involves two parts: that the functor is *locally constant* in that it takes isotopy equivalences to homotopy equivalences and a mild sheaf condition that its value on an increasing union is the homotopy limit. The first part is all that is needed to define the polynomial approximation, since that requires only that the restriction to $\mathcal{O}_n(M)$ be locally constant. The second condition is needed only if we wish our polynomial approximations, which are good, to tell us about the values that the original, nongood functor takes on infinite manifolds. But if we only care about its values on manifolds with finite handle decompositions, we might as well replace it with its universal good approximation. This may be constructed as the Kan extension of its restriction to the full subcategory $\mathcal{O}'(M)$ of manifolds with finite handle decompositions. Finally, I should warn that if we drop the mild sheaf condition of goodness, the polynomial concept becomes ambiguous. Two definitions of polynomials, as (almost) locally constant homotopy sheaves for a particular Grothendieck topology and the right Kan extensions of locally constant functors on $\mathcal{O}_n(M)$ automatically satisfy the mild sheaf condition, but the definition involving cubes does not.

An *analytic* functor is one that is stably n-excisive for all n. Such a functor is well-approximated by its Taylor polynomials. We see this by Mayer-Vietoris induction, which shows how a polynomial is determined by its values on $\mathcal{O}_n(M)$. The analytic functor is only approximately determined, in that each $n+1$-cube which for the polynomial is cartesian is for the analytic functor only highly connected. Thus each cube tells us that analytic functor is uniformly approximated by the polynomial. The connectivity of the approximation depends on the degree of the polynomial and on the handle dimension of the domain. With the handle dimension inside a radius of convergence, the connectivity grows linearly with the degree n of approximation. Good approximations are made on manifolds with low handle dimension, which is quite different from the homotopy calculus, where good approximations are made on spaces with high connectivity. A relation between the two notions is that an inclusion of manifolds having low handle dimension is equivalent to the inclusion of their complements having high connectivity.

Definition. A good functor F on $\mathcal{O}(M)$ is *ρ-analytic with excess c* if for all $U \subset M$ and pairwise disjoint closed subsets $A_i \subset U$ with dimension q_i, for $i = 1 \dots r$, the r-cube $S \mapsto F(U \backslash \bigcup_{i \in S} A_i)$ is $c + \sum \rho - q_i$-connected. One might require this only for $q_i < \rho$.

Theorem. If F is ρ-analytic with excess c and $U \subset M$ has handle dimension $q < \rho$, then $F(U) \to T_n F(U)$ is $c + n(\rho - q)$-connected. In particular, $F(U) \xrightarrow{\sim} \operatorname{holim} T_n F(U)$

It remains to prove the lemma that $T_n F$ is n-excisive. This is proved by replacing the category $\mathcal{O}_n(V)$ with a slightly smaller category $\epsilon\mathcal{O}_n(V)$ of those open sets U subordinate to the open cover ϵ, in the sense that each component (which is a disk) U must be contained in one of the open sets of ϵ. Restricting to this smaller category does not lose information because our functors preserve isotopy equivalences and every collection of big disks is isotopic to a collection of small subdisks. Moreover, restricting a functor $\epsilon\mathcal{O}_n(M)$ and right homotopy Kan extending back to $\mathcal{O}_n(M)$ preserves the property of sending isotopy equivalences to homotopy equivalences, so this does not change the functor, up to equivalence. Since Kan extension is compatible with composition $\epsilon\mathcal{O}_n(M) \to \mathcal{O}_n(M) \to \mathcal{O}(M)$, this shows that we may define $T_n F$ using either $O_n(M)$ or $\epsilon O_n(M)$.

The key to polynomial properties is usually a pigeonhole argument. Here we wish to make our n disks completely miss one of the $n+1$ closed sets used to test the polynomial property. Given an open set U and $n+1$ disjoint closed sets A_i, we wish to show that the cube $S \mapsto T_n F(U_S)$ is cartesian, where $U_S = U \backslash A_S$ and $A_S = \bigcup_i A_i$. We choose our cover ϵ so that no open set in ϵ touches more than one of the A_i. For example, we could choose the sets $M \backslash A_{\{j \neq i\}}$. Then an open set in $\epsilon\mathcal{O}_n(V)$ having only n components, each of which may hit only one of the A_i, must have all of them miss some A_i. Thus the category $\epsilon\mathcal{O}_n(V)$ is the union of the categories $\epsilon\mathcal{O}_n(V \backslash A_i)$ and in fact the colimit of the cube of their intersections. Since these subcategories are ideals, not only is their union the whole category, but the union of their nerves is the nerve of the whole category. This enables us to pull the colimit out of the indexing set and make it a limit of spaces. All together:

$$\operatorname*{holim}_{S \neq \emptyset} T_n F(V_S) = \operatorname*{holim}_{S} \operatorname*{holim}_{\epsilon\mathcal{O}_n(V_S)} F = \operatorname*{holim}_{\operatorname{hocolim}_S \epsilon\mathcal{O}_n(V_S)} F$$
$$\simeq \operatorname*{holim}_{\operatorname{colim}_S \epsilon\mathcal{O}_n(V_S)} F = \operatorname*{holim}_{\epsilon\mathcal{O}_n(V)} F = T_n F(V)$$

The one equivalence which is not a homeomorphism is the step that uses that the subcategories are ideals.

7. Weiss' Taylor tower in the manifold case, part II

Brian Munson

We continue a discussion of the Taylor tower of a good cofunctor from the poset $\mathcal{O}(M)$ of open subsets of a smooth manifold M to the category of spaces. The main point of this talk was to classify the k^{th} layer of the Taylor tower for a good cofunctor F, defined as the cofunctor $E_k(V) = \text{hofiber}(T_kF(V) \to T_{k-1}F(V))$, which makes sense if we choose a basepoint in $T_{k-1}F(M)$. This functor has the property that it is k-excisive (because it is the homotopy fiber of two functors which are k-excisive), and its $(k-1)$st Taylor approximation is contractible. With this in mind, we call a cofunctor $E : \mathcal{O}(M) \to$ Spaces *homogeneous of degree* k if E is k-excisive and $T_{k-1}E(V) \simeq *$ for all $V \in \mathcal{O}(M)$.

Interesting examples are the cofunctors $\text{Emb}(-, N)$, $\text{Imm}(-, N)$, $\text{Map}(-, X)$, and $\Gamma(p, \binom{-}{k})$, where $p : Z \to \binom{M}{k}$ is some fibration over the space of unordered configurations of k points in M. They are, respectively, the functors which assign to each open subset of V in M the space of embeddings of V in a smooth manifold N, the corresponding space of immersions, the space of maps of V into any space X, and the space of sections of the fibration p. Except for the space of embeddings, all of these cofunctors are l-excisive for some l. Moreover, the space of sections example is very general, as we can think of $\text{Map}(V, X)$ as the space of sections of a trivial fibration over V with fiber X, and $\text{Imm}(V, N)$ as a section of a bundle over V whose fiber is a Stiefel manifold. This suggests the example $\Gamma(p, \binom{V}{k})$ deserves further study.

A related example is the functor $V \mapsto \Gamma(p, \partial\binom{V}{k})$, the space of sections "near infinity". One can more properly define this as the homotopy colimit over all neighborhoods Q of the fat diagonal in V^k/Σ_k of $\Gamma(p, \binom{V}{k} \cap Q)$. It turns out that this functor is $(k-1)$ excisive, and is the $(k-1)$st Taylor approximation to the k-excisive functor $\Gamma(p, \binom{V}{k})$. Hence if we pick a basepoint in $\Gamma(p, \partial\binom{M}{k})$, and define $\Gamma^c(p, \binom{V}{k}) = \text{hofiber}(\Gamma(p, \binom{V}{k}) \to \Gamma(p, \partial\binom{V}{k}))$, then this is homogeneous of degree k, as observed above. The classification theorem for homogeneous cofunctors says that this is the only example. More precisely,

Theorem: Given $E : \mathcal{O}(M) \to$ Spaces a homogeneous cofunctor of degree k, there exists a fibration $p : Z \to \binom{M}{k}$ and an equivalence $E(V) \to \Gamma^c(p, \binom{V}{k})$ for all $V \in \mathcal{O}(M)$, natural in V.

One constructs the fibration from the cofunctor E by restricting E to a suitable subcategory $\mathcal{I}^{(k)}(M)$ of $\mathcal{O}(M)$ whose geometric realization is the space of unordered configurations of k points in M, and on which E takes all morphisms to homotopy equivalences. The subcategory $\mathcal{I}^{(k)}(M)$ has as its objects those open subsets of M which are diffeomorphic to a disjoint union of k balls, with morphisms the inclusions which are isotopy equivalences. Then the homotopy colimit of E over this subcategory quasifibers over its realization, and this produces the desired fibration. The fiber over a given $V \in \mathcal{I}^{(k)}(M)$ is just $E(V)$, and if one thinks of making E by taking the k^{th} layer of the Taylor tower of some good cofunctor F, then one can go further to obtain a description of the fibers in terms of F alone. It is through this description that one can see the symmetric group Σ_k acting, which is an important observation if we are trying to make an analogy between derivatives in the homotopy calculus (spectra with Σ_k action built from the functor in question), and in the manifold case (the total fiber of a k-cube built from the cofunctor in question).

The reference for this talk is sections 7 and 8 of Michael Weiss' paper titled "Embeddings from the point of view of immersion theory, Part I", published in Geometry and Topology.

8. Spaces of long knots from the calculus of embeddings viewpoint

Pascal Lambrechts

In this talk we give an application of Goodwillie-Weiss embedding calculus to the space of knots. Another summary of these ideas can be find in [10, Section 5.1] and there is also a discussion around this theme in [11, Section 5]

The space of long knots. Set $I = [0,1]$ and let M be a smooth manifold with non empty boundary. Fix two unit tangent vectors $\alpha \in STM$ (resp. $\beta \in STM$) located on ∂M and pointing inward (resp. outward). A *lonk knot* in M is a smooth embedding $f\colon I \hookrightarrow M$ such that $df(0) = \alpha$, $df(1) = \beta$, and f is transverse to ∂M. We assume also that $\|f'(t)\| = 1$ for each $t \in I$, i.e. the module of the speed is constant. We denote by $Emb(I, M; \partial)$ the space of long knots in M; sometimes we will replace I by some open subset $V \subset I$ containing ∂I.

When $M = \mathbb{R}^{n-1} \times [0,1]$, this space of long knots is closely related to the usual space of knots $Emb(S^1, S^n)$. Indeed it is fairly easy to check that $Emb(I, \mathbb{R}^{n-1} \times [0,1]; \partial)$ is homotopy equivalent to the fibre of the map

$$Emb(S^1, S^n) \to V_2(\mathbb{R}^{n+1}) = SO(n+1)/SO(n-1)\ ,\ g \mapsto (g(1), g'(1)/\|g'(1)\|)$$

Since we are considering embeddings with boundary conditions, the general framework for the Goodwillie-Weiss tower has to be slightly adapted (see [24, Section 10].) Here $\mathcal{O}$ is the poset of open subsets of I *that contain* $\partial I = \{0,1\}$. We define the contravariant functor

$$F\colon \mathcal{O} \to Spaces\,,\ V \mapsto F(V) = Emb(V, M; \partial).$$

This functor is good (in the sense of [24, Definition 1.1]) and we can study its Taylor tower. For $k \geq 1$ let $\mathcal{O}(k)$ be the subposet of elements of $\mathcal{O}$ consisting of a disjoint union of a collar of ∂I with at most k open balls inside I. Following [24, p.84], the k-th stage of the Taylor tower of F is given by, for $U \in \mathcal{O}$,

$$T_k F(U) = \mathrm{holim}_{V \in \mathcal{O}(k), V \subset U} F(V).$$

Fix $k \geq 1$ and set $\underline{k} = \{0, 1, \cdots, k\}$.
Fix $k+1$ disjoint subintervals $A_0, A_1, \cdots, A_k \subset \mathrm{int}(I)$. For $S \subset \underline{k}$ define

$$E_S = Emb(I \setminus \cup_{s \in S} A_s, M; \partial).$$

If $S' \subset S$ we have an obvious restriction map $\rho_{S',S} \colon E_{S'} \to E_S$. This defines a diagram of spaces, $S \mapsto E_S$, indexed by the poset $\mathcal{P}_0(\underline{k})$ of non empty subsets $S \subset \underline{k}$. We have the following

Proposition 1. $(T_k F)(I) \simeq \mathrm{holim}_{\emptyset \neq S \subset \underline{k}} E_S$.

The proof of Proposition 1 is easy: just use the fact that $T_k F$ is k-excisive, that $I \setminus \cup_{s \in S} A_s \in \mathcal{O}(k)$, and that the restrictions to $\mathcal{O}(k)$ of $T_k F$ and F are naturally homotopy equivalent.

Configuration spaces. We relate now the spaces E_S to certain configuration spaces. Elements of STM are denoted by $\xi = (x, v)$ with $x \in M$ and $v \in T_x M$ with $\|v\| = 1$. Define the configuration space $C'_q(M; \partial)$ consisting of $(q+2)$-tuples

$$(\xi_0 = (x_0, v_0), \xi_1 = (x_1, v_1), \cdots, \xi_{q+1} = (x_{q+1}, v_{q+1})) \in (STM)^{q+2}$$

such that $\xi_0 = \alpha$, $\xi_{q+1} = \beta$, and $x_i \neq x_j$ for $i \neq j$.

Let $S \subset \underline{k}$ be a subset of cardinality $q+1$. Then $I \setminus \cup_{s \in S} A_s$ is a disjoint union of a collar about ∂I with q disjoint open subintervals $J_1, \cdots, J_q$. Let t_i be the middle point of J_i. The following proposition is not difficult to prove, the key argument being the fact that the space of *free* (i.e. without boundary conditions) embeddings of I in M is homotopy equivalent to STM.

Proposition 2. Let $q = |S| - 1 \geq 0$. We have a homotopy equivalence

$$\phi \colon E_S \xrightarrow{\simeq} C'_q(M; \partial)\ ,\ f \mapsto (df(0), df(t_1), \cdots, df(t_q), df(1))\,.$$

The latter proposition suggests that in the homotopy limit of Proposition 1 we could replace the spaces E_S by these configuration spaces. For this we would also need maps corresponding to the restriction maps $\rho_{S',S}$. This can only be done after replacing the configuration spaces $C'_q(M, \partial)$ by a suitable compactification $C'_q[M, \partial]$, à la Fulton-MacPherson. The intuitive idea is that elements of $C'_q[M, \partial]$ consist of "virtual" configurations

$$(\xi_0 = (x_0, v_0), \xi_1 = (x_1, v_1), \cdots, \xi_{q+1} = (x_{q+1}, v_{q+1}))$$

where x_i and x_j *may be equal*, in wich case some extra data serves to distinguish these two points infinitesimally. A precise definition of $C'_q[M, \partial]$ is given in [20, Definitions 4.1 and 4.12]. It turns out that this compactification has the same homotopy type as the configuration space itself. We can define also *doubling maps* d_i, for $0 \leq i \leq q$,

$$d_i \colon C'_{q-1}[M; \partial] \to C'_q[M; \partial],\ (\xi_0, \cdots, \xi_i = (x_i, v_i), \cdots, \xi_q) \mapsto (\xi_0, \cdots, \xi_i, \xi'_i, \cdots, \xi_q)$$

where $\xi'_i = (x'_i, v_i)$ with $x_i = x'_i$ but "infinitesimally" $x'_i - x_i = v_i$ (see [20, Definitions 4.1 and 4.12].) The following generalization of Proposition 2 express the fact that these doubling maps correspond to the restriction maps on E_S:

Proposition 3 ([20, Proposition 5.14])**.** The diagrams of spaces

$$\{E_S, \text{restriction maps}\}_{S\in\mathcal{P}_0(\underline{k})}$$

and

$$\{C'_{|S|-1}[M;\partial], \text{composite of doubling maps}\}_{S\in\mathcal{P}_0(\underline{k})}$$

are homotopy equivalent.

Combining this with Proposition 1 we get the following

Corollary 4 ([20, Lemma 5.18])**.** $(T_kF)(I) \simeq \operatorname{holim}_{\emptyset\neq S\subset\underline{k}} C'_{|S|-1}[M;\partial]$.

A cosimplicial space. One can also define forgetting maps, s_j, for $1 \le j \le q$,

$$s_j\colon C'_q[M,\partial] \to C'_{q-1}[M,\partial]\ ,\ (\xi_0,\cdots,\xi_i,\cdots,\xi_q) \mapsto (\xi_0,\cdots,\hat{\xi}_i,\cdots,\xi_q).$$

A natural guess would be that this gives a cosimplicial space with d_i as cofaces and s_j as codegeneracies. This is not true because certain cosimplicial identities are not satisfied. However it is possible to replace $C'_q[M,\partial]$ by a homotopy equivalent quotient $C'_q\langle[M,\partial]\rangle$ for which the induced map satisfy the cosimplicial identities. We get then a cosimplicial space ([20, Definition 6.1]):

$$X_\bullet := \{C'_q\langle[M,\partial]\rangle, d_i, s_j\}_{q\ge 0}.$$

To such a cosimplicial space one can associate its partial k-th totalisation, $\mathrm{Tot}^k X_\bullet$, which is also homotopy equivalent to the homotopy limit of the cofaces maps on the k-coskeleton. Therefore we get

Proposition 5. $T_kF(I) \simeq \mathrm{Tot}^k(X_\bullet)$.

If we work in codimension at least 3 the Goodwillie-Klein excision result implies that the Taylor tower converges to the embedding space. In our case this yields to the following

Theorem 6 ([20, Theorem 6.2])**.** If $\dim M \ge 4$ then $Emb(I, M; \partial) \simeq \mathrm{Tot}(X_\bullet)$.

The latter theorem is very effective because Bousfield-Kan have constructed spectral sequences to compute the homotopy groups or cohomology of the totalization of a cosimplicial space and because configuration spaces are fairly well understood. For example Scanell and Sinha have used Theorem 6 to compute certain homotopy groups of the space of long knots in $\mathbf{R}^n$ ([19]). Also, using the Kontsevich's theorem on the formality of configuration space in $\mathbf{R}^n$, we have proved that the rational cohomology Bousfield-Kan spectral sequence for computing $\mathrm{Tot}(X_\bullet)$ when $M = \mathbf{R}^{n-1}\times I$, $n \ge 4$, collapses at the E_2-term. This determines completely the rational homotopy type of $Emb(I, \mathbf{R}^{n-1}\times I; \partial)$ (see [17].)

A difficult open question is whether Theorem 6 still holds under the weaker hypothesis $\dim M = 3$. This is of course of great interest since when $M = \mathbf{R}^2 \times I$ the space of long knots is closely related to the space of usual knots $Emb(S^1, S^3)$. This question is in fact very much related to the conjecture that the Vassiliev invariants separate all knots. Indeed Ismar Volic has proved that a certain algebraization of the cohomology Bousfield-Kan spectral sequence contains in its E_2-term the Vassiliev invariants. More precisely $E_2^{2p,-2p}$ is isomorphic to Vassiliev invariants of order p. Note that $E_2^{2p,-2p}$ is strongly related to the cohomology of the $2p$-th layer of the Taylor tower for this embedding space. We refer the reader for more details to the very interesting work [21].

9. and 10. Goodwillie's Taylor tower in the homotopy case, parts I and II

Gerald Gaudens and Christian Ausoni

We explain the construction and the basic properties of the Taylor tower of a homotopy functor, as developed in [Goodwillie, Calculus III]. This amounts to the first three sections of the paper.

Let Y be a fixed topological space. By $\mathcal{U}_Y$ we denote the category of spaces over Y and by $\mathcal{T}_Y$ we denote the category of sectionned spaces over Y. Let $\mathcal{S}p$ be the category of spectra. When $Y = *$, we simply suppress it from the notation. Let $\mathcal{C} \in \{\mathcal{U}_Y, \mathcal{T}_Y\}$ and $\mathcal{D} \in \{\mathcal{T}, \mathcal{S}p\}$. We consider homotopy functors

$$F : \mathcal{C} \longrightarrow \mathcal{D},$$

namely functors which preserve weak homotopy equivalences. The aim is to study such a functor F by means of polynomial approximations. More precisely, we explain how to construct a natural tower of functors under F, called *the Taylor tower of F*, as diplayed below :

$$\begin{tikzcd}
 & \vdots \arrow[d, "q_{n+2}F"] \\
 & P_{n+1}F \arrow[d, "q_{n+1}F"] \\
F \arrow[ur, "p_{n+1}F"] \arrow[r, "p_nF"] \arrow[ddr, "p_0F"'] & P_nF \arrow[d, "q_nF"] \\
 & \vdots \arrow[d, "q_1F"] \\
 & P_0F \simeq F(Y)
\end{tikzcd}$$

This Taylor tower is characterized by the following universal property. For all $n \in \mathbb{N}$,

$(\mathrm{i})_n$ P_nF is *n-excisive* (it takes strongly cocartesian $(n+1)$-cubes to cartesian cubes), and

$(\mathrm{ii})_n$ $P_nF : F \longrightarrow P_nF$ is initial as a map from F to an n-excisive functors, in the homotopy category of homotopy functors.

Although properties $(\mathrm{i})_n$ and $(\mathrm{ii})_n$ always hold (see abstract for talk XV), we prove them only for *stably n-excisive functors* (see abstract for talk III). Furthermore, we show that if F is ρ-analytic and X is $(\rho+1)-$connected then the natural map

$$F(X) \longrightarrow \mathrm{holim}_{n\in\mathbb{N}} P_nF(X)$$

is a weak homotopy equivalence. See talk III for the definition of a ρ-analytic functor. A space X is $(\rho+1)-$connected if the structural map $X \to Y$ is.

We break down the study of the Taylor tower of a functor F by introducing the layers

$$D_nF = \mathrm{hofiber}(q_n : P_nF \longrightarrow P_{n-1}F).$$

For technical reasons, we need to assume that $\mathcal{C}$ is *a pointed category*, so we will restrict ourselves to functors $\mathcal{T}_Y \to \mathcal{D}$. It follows from the definition that D_nF is n-homogeneous : for $n \geq 1$, we call a functor H *n-homogeneous* if

- H is n-excisive,
- H is n-reduced : $P_{n-1}H \sim *$.

Let us denote by $\mathcal{H}_n(\mathcal{T}_Y, \mathcal{D})$ the category of n-homogeneous functors $\mathcal{T}_Y \to \mathcal{D}$. In order to classify n-homogeneous functors, we introduce *symmetric n-multilinear* functors. A functor

$$L : \mathcal{T}_Y^n \longrightarrow \mathcal{D}$$

is n-multilinear if L is 1-homogeneous in each variable. Such a functor L is *symmetric* provided that for any permutation of n elements $\sigma \in \Sigma_n$ there exists a natural isomorphism

$$L(\sigma) : L(X_1, \ldots, X_n) \longrightarrow L(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$$

such that $L(\sigma\sigma') = L(\sigma')L(\sigma)$. We denote by $\mathcal{L}_n(\mathcal{T}_Y, \mathcal{D})$ the category of symmetric multilinear functors $\mathcal{T}_Y^n \to \mathcal{D}$.

The next aim is to construct a diagram

$$\begin{tikzcd}
\mathcal{H}_n(\mathcal{T}_Y, \mathcal{T}) \arrow[r, shift left, "B^\infty"] \arrow[d, "\mathrm{cr}_n"'] & \mathcal{H}_n(\mathcal{T}_Y, \mathcal{S}p) \arrow[l, shift left, "\Omega^\infty"] \arrow[d, shift left, "\mathrm{cr}_n"] \\
\mathcal{L}_n(\mathcal{T}_Y, \mathcal{T}) \arrow[r, shift left, "B^\infty"] & \mathcal{L}_n(\mathcal{T}_Y, \mathcal{S}p) \arrow[l, shift left, "\Omega^\infty"] \arrow[u, shift left, "\Delta_n"]
\end{tikzcd}$$

The m^{th} cross-effect of a homotopy functor $F : \mathcal{T}_Y \to \mathcal{D}$ is the symmetric m-multifunctor defined by

$$\mathrm{cr}_m F(X_1, \ldots, X_m) = \text{total homotopy fiber of } F(S(X_1, \ldots, X_m)) \; .$$

Here $S(X_1, \ldots, X_n)$ is the n-cube

$$S \longmapsto \bigvee_{i \in \underline{n} \setminus S} X_i$$

with the obvious projection maps (the one point sum $\vee$ is meant as 'sum over Y'). If F is n-homogeneous, then $\mathrm{cr}_n F$ is not only symmetric but also *multilinear*. On the other hand, one can define a functor Δ_n from $\mathcal{L}_n(\mathcal{T}_Y, \mathcal{S}p)$ to $\mathcal{H}_n(\mathcal{T}_Y, \mathcal{S}p)$

$$\Delta_n L(X) = F(X, \ldots, X)_{h\Sigma_n} \ .$$

One can also build natural deloopings B^∞ of homogeneous functors $\mathcal{T}_Y \longrightarrow \mathcal{T}$ and symmetric multilinear functors $\mathcal{T}_Y^n \longrightarrow \mathcal{T}$.

We show that all (bottom, top, and left) pairs of functors in the diagram induce equivalences at the level of homotopy categories, hence cr_n is also an equivalence on the left hand side.

It follows from talk II that for any $L \in \mathcal{L}_n(\mathcal{T}_Y, \mathcal{S}p)$ and for n-tuple $(X_1, \ldots, X_n)$ of finite complexes, one has a natural equivariant weak homotopy equivalence

$$L(X_1, \ldots, X_n) \simeq L(S^0, \ldots, S^0) \wedge X_1 \wedge \ldots \wedge X_n \ .$$

Here $L(S^0, \ldots, S^0)$ is a Σ_n equivariant spectrum in the naive sense, making the right hand side a symmetric multilinear functor. Thus this equivalence extends to an equivariant weak homotopy equivalence of functors in presence of suitable limit axioms.

To sum up, for any homotopy functor from $\mathcal{C}$ to $\mathcal{D}$, one can describe (at least on finite complexes) $D_n F$ as

$$D_n F(X) \simeq (C_n \wedge X^{\wedge n})_{h\Sigma_n}$$

where C_n is some fixed spectrum with Σ_n action. This spectrum C_n is called the n^{th} derivative of F at S^0. One should compare this with the fact that in (ordinary) calculus, the n^{th}-homogeneous part of the Taylor expansion of a C^∞ function f is

$$(P_n f - P_{n-1} f)(x) = \frac{f^{(n)}(0)}{n!} \cdot x^n \ .$$

11. The derivatives of the identity functor

Carl-Friedrich Bödigheimer

For a functor $F : \mathrm{TOP}_0 \to \mathrm{TOP}_0$ we studied the layers

$$D_n F : \mathrm{hfiber}(\mathrm{q_n F} : \mathrm{P_n F} \to \mathrm{P_{n-1} F})$$

and their n-th cross-effect, the n-th derivative

$$D^{(n)} F := \mathrm{cr_n}(\mathrm{D_n F}),$$

which is a functor of n variables. Such a functor can be multilinearized to

$$\mathrm{MultLin(G)(X_1, \ldots, X_n) := hocolim_{k_1, \ldots, k_n} \Omega^{k_1 + \ldots + k_n} G(\Sigma^{k_1} X_1, \ldots, \Sigma^{k_n} X_n).}$$

We used the following easily proved facts :

(1) $$\mathrm{cr_n(D_nF) \simeq cr_n(P_nF)}$$

(2) $$D^{(n)}F \quad \text{is symmetric}$$

(3) $$D_nF(X) \simeq D^{(n)}F(X,\ldots,X)_{h\Sigma_n}$$

(4) $$\mathrm{MultiLin(cr_nF) \simeq D^{(n)}F}$$

The following proposition is together with the last equivalence above the important tool to recognize the n-th derivative of a functor.

Proposition 1
Let G, G' be two reduced, symmetric functors of n variables and let $T : G \to G'$ be a Σ_n-equivariant natural transformation. If there exists a constant c such that for any k and all k-connected spaces $X_1, \ldots, X_n$ the map

$$T_{X_1,\ldots,X_n} : G(X_1,\ldots,X_n) \to G'(X_1,\ldots,X_n)$$

is $((n+1)k - c)$-connected, then

$$\mathrm{MultiLin(T) : MultiLin(G) \to MultiLin(G')}$$

is an equivalence.

The strategy to identify $D^{(n)}F$ is (1) to find a reduced, symmetric functor M_nF of n veriables, (2) a Σ_n-equivariant transformation $T : cr_nF \to M_nF$, which satisfies the connectivity condition of the Proposition.

Following the work of Brenda Johnson [13] we did (a) describe a space $\bar{C}_n$ with a Σ_n-action, and (b) define the functor

$$M_nF(X_1,\ldots,X_n) := \mathrm{map_0(\bar{C}_n, \bigwedge_{i=1}^{n} F(X_i))},$$

and a Σ_n-equivariant transformation $T : \mathrm{cr_nF \to M_nF}$. Furthermore, we determined the homotopy type of $\bar{C}_n$ to be a bouquet of $(n-1)!$ spheres of dimension $n-1$.

To give some more details for (a) note that a point in $\mathrm{cr_nF(X_1,\ldots,X_n)}$ is a collection of maps $\Phi_U : I^U \to F(X_U)$ for each $U \subset n$, where $X_U = \bigwedge_{i\notin U} X_i$. These maps satisfy certain boundary conditions and coherence conditions. This turns out to be the same as a map from $\bar{C}$ to $\bigwedge_{i=1}^n F(X_i)$. We describe the space $\bar{C}$ as a subspace of an n-fold configuration space, and we showed in several steps that $\bar{C} \simeq \bigvee^{(n-1)!} \mathbb{S}^{n-1}$ Thus

$$M_nF(X_1,\ldots,X_n) \simeq \prod^{(n-1)!} \Omega^{n-1} \bigwedge_{i=1}^{n} F(X_i).$$

The transformation T is straightforward.

Only now do we spezialize to $F = \mathrm{Id}$ being the identity functor and prove the connectivity condition of the proposition. We used here fact that Id and $\Omega\Sigma$ have the same derivatives. The main ingredient is the

Hilton-Milnor-Theorem

$$\Omega\Sigma(X_1 \vee \ldots \vee X_n) \simeq \prod_{\alpha=1}^{\infty} \Omega\Sigma \bigwedge_{j\in w_\alpha} X_j,$$

where w_α denotes the α-th basic word in a (Hall) basis of the free non-associative algebra on the symbols $x_1, \ldots, x_n$.

The cross-effect, defined as the total fiber of a cube consisting of the projection maps, is now seen to be a subproduct, consisting of all the factors w_α containing each symbol $x_1, \ldots, x_n$ at least once.

Proposition 2
If $X_1, \ldots, X_n$ are k-connected, then

$$\pi_m(\prod^{(n-1)!} X_1 \wedge \ldots \wedge X_n) \simeq \pi_m(\mathrm{cr_n}\Omega\Sigma(\mathrm{X_1, \ldots, X_n}))$$

is an isomorphism for $0 \le m \le (n+1)(k+1) - 1$.

The final result is now

$$\mathrm{Id}^{(\mathrm{n})} \simeq \bigvee^{(\mathrm{n}-1)!} \Omega^\infty\Sigma^\infty(\mathbb{S}^{1-\mathrm{n}})$$

.

12. The Taylor towers of $X \mapsto \Sigma^\infty Map_*(K, X)$ and $X \mapsto \Sigma^\infty\Omega^\infty X$

Hal Sadofsky

We wish to describe Goodwillie's Taylor tower for the functor from based spaces to spectra given by

$$X \mapsto F(X) = \Sigma^\infty Map_*(K, X).$$

Here K is a CW-complex with finitely many cells, and Map_* is the space of basepoint preserving maps.

Our main source is Arone's paper *A generalization of Snaith-type filtration* (Trans. Amer. Math. Soc. **351** (1999), no. 3, 1123–1150). Other sources are Goodwillie's *Calculus II* (K-Theory **5** 1991/92), no. 4, 295-332), *Calculus III* (Geom. Topol. **7**, (2003), 645-711), and Ahearn-Kuhn's *Product and other fine structure in polynomial resolutions of mapping spaces* (Algebr. Geom. Topol. **2** (2002), 591–647).

By Calculus II, the functor $F(X)$ is analytic, and the Taylor tower converges if the connectivity of X is at least the dimension of K. We describe the k-th excisive approximation $P_k(X)$, and the kth level, $D_k(X)$.

Arone's main theorem is that

$$P_kF(X) \simeq \mathrm{Nat}_{\mathcal{M}_k}(K^\wedge, X^\wedge). \tag{5}$$

Here $\mathcal{M}_k$ is the category with objects the integers $\underline{0}$ through $\underline{k}$ where 0 is the empty set and $\underline{j} = \{1, \dots, j\}$ and with morphisms surjections. Note that there is a natural map

$$\mathrm{Nat}_{\mathcal{M}_k}(K^\wedge, X^\wedge) \to \mathrm{Nat}_{\mathcal{M}_{k-1}}(K^\wedge, X^\wedge)$$

by restricting the functors on $\mathcal{M}_k$ to $\mathcal{M}_{k-1}$. Thus the functors $\mathrm{Nat}_{\mathcal{M}_k}(K^\wedge, X^\wedge)$ fit together into a tower.

If Y is a pointed space, $Y^\wedge : \mathcal{M}_k \to \underline{\mathrm{Spectra}}$ is the functor that associates $\underline{j}$ to the j-fold smash product of Y, and associates a surjection $\underline{j} \to \underline{i}$ to the evident "diagonal" map $X^{\wedge i} \to X^{\wedge j}$.

First, assuming (5), we follow the argument from Arone's paper to describe the right hand side explicitly so that we can determine the kth layer combinatorially. To be precise,

$$D_kF(X) = \mathrm{Map}_{\mathrm{Spectra}}(K^{(k)}, Y^{\wedge k})^{\Sigma_k}.$$

Then we sketch the proof of (5) in the special case $K = S^n$ using the argument provided in Ahearn-Kuhn. This relies on three main ideas.

(1) We can reduce to considering $X = \Sigma^n Y$ for Y connected. This is true by Calculus II, or by an ad-hoc argument involving induction over cells.
(2) In the case $X = \Sigma^n Y$, the classical combinatorial model for $\Omega^n \Sigma^n Y$ has a filtration which splits on applying Σ^∞. The pieces of the splitting are homogeneous functors, so this gives the Taylor tower in that case.
(3) We map the tower determined by 2. above to the tower described by using the functors in the right hand side of (5). We check that we get a weak equivalence at the kth level by assuming we have such an equivalence at the $k-1$st level, and calculating the effect of the kth cross effect. This involves calculating the kth cross effect on both sides, and the effect on the map.

Finally, following (for example) Ahearn-Kuhn we can calculate the Taylor tower of $\Sigma^\infty \Omega^\infty X$ for X taking values in spectra by taking the hocolim of the Taylor towers for

$$\Omega^n \underline{X}_n \to \Omega^{n+1}(\Sigma \underline{X}_n) \to \Omega^{n+1} \underline{X}_{n+1} \to \dots$$

An example conclusion from this is

$$D_k(\Sigma^\infty \Omega^\infty)(X) = E(\Sigma_k)_+ \wedge_{\Sigma_k} X^{\wedge k}.$$

13. Homotopy calculus via cotriples

Andrew Mauer-Oats

The purpose of this talk is to explain the relationship between the $(n+1)$ cross effect and the n-excisive approximation for a functor from spaces to spectra, stated carefully as Theorem 3 below.

As a consequence of Theorems 3 and 4, the cross-effect cotriple can be used to study the n-excisive approximation to a functor F anywhere inside the radius of convergence (which is to say, anywhere that the calculus of functors itself gives information about F). One advantage of this approach is that to define cross effects requires only finite coproducts in the source category and homotopy fibers in the target category; not even "connectivity" is needed, hence the construction can be applied to functors with a wide variety of source and target categories.

Definition. ($[n]$, C_n): Let $[n]$ denote the set $\{0, 1, 2, \ldots, n\}$ with basepoint 0. Let C_n denote the full subcategory of pointed spaces generated by the objects $\{[0], \ldots, [n]\}$.

For any continuous functor F, there is a categorical map

$$|F(X.)| \to F(|X.|)$$

from the realization of F applied dimensionwise to a simplicial space to F applied to the realization (that is from, hocolim$F() \to F$(hocolim)).

To avoid any confusion, when we use the word "equivalence", it means "weak homotopy equivalence." The symbol "$\simeq$" denotes an equivalence.

Definition. A functor commutes with realizations if the above map is a weak equivalence.

Theorem 1. (Goodwillie) If $F \simeq P_{n+1}F$ and $cr_{n+1}F \simeq 0$, then $F \simeq P_nF$.

Theorem 2. If F commutes with realizations and $cr_{n+1}F \simeq 0$, then $F \simeq P_nF$.

Examples:

- The integral homology functor, $F(X) = \mathbf{HZ} \wedge X$ commutes with realizations.
- The functor to Eilenberg-MacLane spectra given by $\pi_2 F(X) := H_2(X; \mathbf{Z})$ does not commute with realizations. On S^3 it is zero, but applied dimensionwise to S^2 wedged with a discrete model for S^1 it produces $K(\mathbf{Z}, 3)$.

Definition (L_nF). Given a functor F from spaces to spectra, let L_nF denote the homotopy invariant left Kan extension of F along the inclusion of C_n into the category of spaces. One formula for $L_nF(X)$ involves the realization of a simplicial space that in dimension $[n]$ is:

$$\bigvee_{A_0, \ldots, A_n \in C_n} F(A_0) \times \left(\mathrm{Map}_{C_n}(A_0, A_1) \times \cdots \times \mathrm{Map}_{Top}(A_n, X)\right)_+$$

Notice that $\mathrm{Map}_{Top}(A_n, X)$ is a product of copies of X because A_n is a finite set of points $[i] = \{0, \ldots, i\} \in C_n$.

Facts about left Kan extensions of functors from spectra to spectra:

(1) If $F \simeq P_nF$, then $L_nF \to F$ is an equivalence. (This was shown in an earlier talk.)
(2) L_nF commutes with realizations for all n. (Using the above definition, because finite products commute with realizations.)
(3) $L_\infty F \simeq F$ if and only if F commutes with realizations.
(4) If $cr_{n+1}F \simeq 0$, then $L_nF \to L_{n+1}F$ is an equivalence.

The last fact is a little more involved: one way to show it is by arguing that $cr_{n+1}L_{n+1}F$ is excisive in each variable, and hence is determined by smashing with the spectrum $cr_{n+1}L_{n+1}F([1],\dots,[1])$, which is the same as $cr_{n+1}F([1],\dots,[1])$, and hence is contractible.

We then use these facts to prove Theorem 2.

Theorem 3. Let F be a functor from spaces to spectra that commutes with realizations. Then there is a homotopy fiber sequence

$$cr^*_{n+1}F \to F \to P_nF,$$

where the fiber is a simplicial spectrum built from the iterated cross effects (which form a cotriple).

The argument for Theorem 3 is to show that Theorem 2 produces an equivalence on the homotopy fibers of a cube:

$$\begin{tikzcd} cr^*_{n+1}F \arrow[r] \arrow[d] & F \arrow[d] \\ P_n(cr^*_{n+1}F) \arrow[r] & P_nF \end{tikzcd}$$

Theorem 4. Let F be an r-analytic functor from spaces to spectra. The functor F commutes with realizations of dimensionwise r-connected simplicial spaces.

Analyticity means that the Taylor tower converges on r-connected spaces, so this follows by approximating F by some large P_NF and then using the facts that $L_NF \simeq P_NF$ and L_NF commutes with realizations.

14. Linear functors of spaces over a space

Morten Brun

The aim of this talk was to make sense of the right hand column of the following table, that is to explain some versions of the chain rule in functor calculus. Our main references are the papers “Calculus III” of Goodwillie and “A chain rule in the calculus of homotopy functors” of Klein and Rognes, both published in Geometry and Topology. However our chain rule is slightly different from the one of Klein and Rognes.

M smooth manifold	$\mathcal{U}_M$ for a space M
$F\colon M \to N$ smooth map	$F\colon \mathcal{U}_M \to \mathcal{U}_N$ homotopy functor
$X \in M$ point in M	$X \in \mathcal{U}_M$ space over M
$T_X M$ vector space of 1-jets of smooth maps $\alpha\colon \mathbb{R} \to M$ with $\alpha(0) = X$	$T_X\mathcal{U}_M$ category of excisive functors $L\colon \mathcal{T} \to \mathcal{T}_M$ with $L(*) = X$
$T_X F\colon T_X M \to T_Y N$ $T_X F([\alpha]) = [F\alpha]$	$T_X F\colon T_X\mathcal{U}_M \to T_Y\mathcal{U}_N$ $T_X F(L) = P_1(FL)$
Chain rule: $T_X(GF) = (T_Y G)(T_X F)$	Chain rule: $T_X(GF) \simeq (T_Y G)(T_X F)$.
$(T_X F)_{ij} =< T_X F(e_i), e_j >$	$\partial_x^y F =$ $\{n \mapsto \mathrm{hofib}_y(F(S^n \cup_x X) \to F(X))\}$
$(T_X GF)_{ij} = \sum_k (T_Y G)_{ik}(T_X F)_{kj}$	$\partial_x^z(GF) = \int_{y\in Y} \partial_y^z G \wedge \partial_x^y F dy$.

1. Analytic functors

Given a space M, $\mathcal{U}_M$ denotes the category of spaces over M, and $\mathcal{T}_M$ denotes the category of spaces over and under M. In case M is a one-point space, we denote these categories $\mathcal{U}$ and $\mathcal{T}$ respectively.

We consider analytic functors $F\colon \mathcal{U}_M \to \mathcal{U}_N$ and $G\colon \mathcal{U}_N \to \mathcal{U}_Q$. Note that the composite functor GF also is analytic.

Given a space X over M we let $J^n_X\mathcal{U}_M$ denote the category of analytic n-excisive functors $L\colon \mathcal{T} \to \mathcal{T}_X$ with $L(*) = X$, and let $J^n_X F\colon J^n_X\mathcal{U}_M \to J^n_Y\mathcal{U}_N$ denote the obvious functor taking L to the n-excisive approximation $P_n(FL)$ of FL. (Working with homotopy limits and -colimits in $\mathcal{T}_Y$ we can assure that $P_n(FL)$ is an object if $J^n_Y\mathcal{U}_N$.)

Proposition. *The natural map $P_n(GFL) \to P_n(GP_n(FL))$ is a weak equivalence for every $L \in J^n_X\mathcal{U}_M$.*

As a consequence we obtain a **first chain rule:** $J^n_X(GF) \simeq (J^n_Y G)(J^n_X F)$.

2. Systems of spectra

From now on we require homotopy functors to be *finitary* and *continuous.* Working with homotopy limits and -colimits in $\mathcal{T}_Y$, with $Y = F(X)$, the definition of $T_X F$ in the above table makes perfect sense!

We want to explain our **second chain rule:** $T_X(GF) \simeq (T_Y G)(T_X F)$. We prefer to switch to parametrized spectra in order to explain this chain rule.

Remark. A linear functor $L\colon \mathcal{T} \to \mathcal{T}_X$ naturally takes values in infinite fiberwise loop spaces. The reason is the same as for $X = *$: since the square

$$\begin{array}{ccc} L(S^0) & \longrightarrow & L(I) \\ \downarrow & & \downarrow \\ L(I) & \longrightarrow & L(S^1) \end{array}$$

is cartesian, there is a weak equivalence $L(S^0) \simeq \Omega_X L(S^1)$ with the convention $\Omega_X Z = \mathrm{map}_{\mathcal{T}_X}(S^1, Z^f)$, where Z^f is a fibrant replacement of Z and $\mathrm{map}_{\mathcal{T}_X}$ is the internal hom-object in $\mathcal{T}_X$. (We must require X to be weak Hausdorff.)

Definition. A *system of spectra on* X consists of objects $\{E_n\}$ of $\mathcal{T}_X$ and maps $S^1 \wedge E_n \to E_{n+1}$. This category is denoted $\mathrm{Sp}(X)$.

Here $\wedge$ is the fiberwise smash-product in $\mathcal{T}_X$ and $S^1 = (S^1 \times X) \in \mathcal{T}_X$ and $F\colon \mathcal{T}_X \to \mathcal{T}_Y$ induces $\mathrm{Sp}(F)\colon \mathrm{Sp}(X) \to \mathrm{Sp}(Y)$ with $(\mathrm{Sp}(F)(E))_n = F(E_n)$. The structure map $S^1 \wedge F(E_n) \to F(S^1 \wedge E_n) \to F(E_{n+1})$ is constructed using the adjoint to the map $S^1 \to \mathrm{map}_{\mathcal{U}}(E_n, S^1 \times E_n) \to \mathrm{map}_{\mathcal{U}}(F(E_n), F(E_{n+1}))$.

Our **third chian rule:** $\mathrm{Sp}(GF) = \mathrm{Sp}(G)\mathrm{Sp}(F)$ follows directly from the definition of $\mathrm{Sp}(F)$.

Definition. A morphism $E \to E'$ of $\mathrm{Sp}(X)$ is a *stable equivalence* if the induced map $\mathrm{hocolim}_n \Omega_X^n E_{m+n} \to \mathrm{hocolim}_n \Omega_X^n E'_{m+n}$ is a weak equivalence for every $m \in \mathbb{Z}$.

Lemma. *For a homotopy functor* $F\colon \mathcal{T}_X \to \mathcal{T}_Y$*, the morphism* $\mathrm{Sp}(F)(E) \to \mathrm{Sp}(P_1F)(E)$ *is a stable eqivalence for every* $E \in \mathrm{Sp}(X)$.

There are functors $\alpha_X\colon T_X\mathcal{U}_M \leftrightarrows \mathrm{Sp}(X) : \beta_X$ with $\alpha_X(L) = \mathrm{Sp}(L)(\mathbb{S})$ and $(\beta_X(E))(Z) = \Omega_X^\infty(E \wedge Z)$ Here $\mathbb{S}$ denotes the sphere spectrum in $\mathrm{Sp}(*)$. Using the following proposition, the second chain rule is a consequence of the third chain rule.

Proposition. α_X *and* β_X *induce inverse isomorphisms on homotopy categories.*

3. PARAMETRIZED BROWN REPRESENTABILITY

Given a system of spectra E on A and $f\colon A \to B$, the system of spectra f_*E on B is given by

$$(f_*E)_n = \mathrm{colim}(B \leftarrow A \to E_n).$$

In calculus notation the homotopy fibre over $b \in B$ is

$$(f_*E)_b = \int_{a \in f^{-1}(b)} E_a da.$$

Theorem. *Let $p_Y^{X \times Y}: X \times Y \to Y$ is the projection. For every linear functor $L\colon \mathcal{T}_X \to \mathcal{T}_Y$ there exists a system of spectra ∂L on $X \times Y$ with a natural stable equivalence*

$$\eta\colon (p_Y^{X \times Y})_*(\partial L \wedge (E \times Y)) \to \mathrm{Sp}(L)(E)$$

for $E \in \mathrm{Sp}(X)$.

4. THE PARAMETRIZED CHAIN RULE

$F\colon \mathcal{T}_X \to \mathcal{T}_Y$ is a homotopy functor with $Y = F(X)$ and $x \in X$.

Definition. $\partial_x F = \mathrm{Sp}(F)(\mathbb{S} \cup_x X) \simeq \mathrm{Sp}(P_1 F)(\mathbb{S} \cup_x X)$, where $S^n \cup_x X \to X$ maps S^n to x. Further we let $\partial_x^y F$ denote the homotopy fiber at $y \in Y$ of the map $\partial_x F \to Y$.

Note that

$$\begin{aligned} (\partial P_1 F)_{x_0,y} &\simeq \int_{x \in X} (\partial P_1 F)_{x,y} \wedge (\mathbb{S} \cup_{x_0} X)_x dx \\ &= ((p_X^{X \times Y})_*(\partial(P_1 F) \wedge ((\mathbb{S} \cup_{x_0} X) \times Y)))_y \\ &\simeq (\mathrm{Sp}(P_1 F)(\mathbb{S} \cup_{x_0} X))_y = \partial_{x_0}^y F. \end{aligned}$$

Theorem. $\partial_x(GF) \simeq (p_Z^{Y \times Z})_*(\partial G \wedge (\partial_x F \times Z))$.

Taking homotopy fibres over $z \in Z$ the theorem gives a **fourth chain rule:**

$$\begin{aligned} \partial_x^z(GF) &\simeq (p_Z^{Y \times Z})_*(\partial G \wedge (\partial_x F \times Z))_z \\ &= \int_{y \in Y} (\partial_y^z G \wedge (\partial_x F \times Z)_{y,z} dy \\ &= \int_{y \in Y} \partial_y^z G \wedge \partial_x^y F dy. \end{aligned}$$

15. Calculus without estimates

Ben Walter

In previous talks (see talks 9-11), we constructed and analyzed the Taylor tower for a homotopy functor assuming that the functor satisfied certain stable n-excision hypotheses (these hypotheses ranged, depending on the property or construction under consideration, from being stably n-excisive for a single n, to being ρ-analytic – which implies stable n-excision for all n). The property of ρ-analyticity for a functor implies convergence of its Taylor tower (at all ρ-connected spaces) – which greatly simplifies many proofs. Yet, just as the formal Taylor series for a function can be written even when the series will not converge, the "formal" Taylor tower for a functor can be constructed using the methods of the previous talks even in the absence of ρ-analyticity, indeed even in the absence of stable n-excision for any n. However, without any stable n-excision, some work is required in order to justify the name "Taylor tower" for this formal object. In particular, three fundamental properties of the tower are no longer clear. These are:

Theorem 1. *If $L(X_1, \dots, X_n)$ is $(1, \dots, 1)$-multilinear, then $(L \circ \Delta)(X)$ is n-homogeneous.* [6, 3.1]

Theorem 2. *The multilinearization of the n^{th} cross effect of F is equivalent to the n^{th} cross effect of $P_n(F)$.* [6, 6.1]

Theorem 3. *$P_n(F)$ is n-excisive.* [6, 1.8]

Parts of these theorems have already been proven without stable n-excision. In particular, the excisive half of Theorem 1 is proven in [5, 3.4] using only standard properties of cartesian and cocartesian cubes – this proof was discussed in a previous talk. The proof of the reduced half of Theorem 1 – that L $(1, \dots, 1)$-reduced implies $L \circ \Delta$ is n-reduced – relies on a property of homotopy limits. If $D : \mathcal{I} \to \mathcal{C}$ is a diagram over the index category $\mathcal{I}$ and $\mathcal{J}$ is a full subcategory of $\mathcal{I}$, then we can consider the homotopy limit of the diagram $D|_{\mathcal{J}}$. It is a standard fact that a map $\operatorname{holim}_{\mathcal{I}} D \to \operatorname{holim}_{\mathcal{J}} D$ exists and is a weak equivalence if $\mathcal{J}$ is left cofinal in $\mathcal{I}$.

To show that $P_{n-1}(L \circ \Delta)$ is weakly contractible, we show that the map $t_{n-1} : (L \circ \Delta) \to T_{n-1}(L \circ \Delta)$ factors through a weakly contractible object. The map t_{n-1} can be (essentially) displayed as a map of the type discussed above for the diagram $D_X(U_1, \dots, U_n) = L(X * U_1, \dots, X * U_n)$ over the indexing categories $\mathcal{I} = \mathcal{P}(\underline{n})^n$, $\mathcal{J} = \mathit{Diagonal}(\mathcal{P}_0(\underline{n})^n)$. We factor the map t_{n-1} by factoring the restriction of categories map $\mathcal{I} \to \mathcal{J}$. Given some other full subcategory ε of $\mathcal{I}$, let $(\mathcal{J} \cup \varepsilon)$ denote the full subcategory of $\mathcal{I}$ generated by the objects of $\mathcal{J}$ and the objects of ε. For any such category, we will get maps

$$\operatorname*{holim}_{\mathcal{I}} D \to \operatorname*{holim}_{\mathcal{J} \cup \varepsilon} D \to \operatorname*{holim}_{\mathcal{J}} D$$

The theorem is proven by producing ε such that both $\operatorname{holim}_{\varepsilon} D$ is contractible, and also ε is left cofinal in $(\mathcal{J} \cup \varepsilon)$.

In Calculus III, Goodwillie uses $\varepsilon_{\mathcal{G}} = \{\vec{S} \in \mathcal{P}_0(\underline{n})^n \mid S_j = \{j\}, \text{ some } j\}$ – the largest ε for which his argument will work. The smallest ε which can be chosen is

$$\varepsilon_{\mathcal{W}} = \left\{ \vec{S} \in \mathcal{P}_0(\underline{n})^n \;\middle|\; S_j = \begin{cases} B & j \notin A \\ \{j\} & j \in A \end{cases}, \ \emptyset \neq A \subset B \subset \underline{n} \right\}.$$

Theorem 2 asserts $P_{1,\dots,1} cr_n F \sim cr_n P_n F$. To prove this, we note that the distinction between functors of one variable and functors of many variables can be blurred. To wit, there is a natural isomorphism between the categories $\mathcal{C}_{Y_1} \times \cdots \times \mathcal{C}_{Y_n}$ and $\mathcal{C}_{Y_1 \coprod \cdots \coprod Y_n}$. Thus, any functor of many variables $F : \mathcal{C}_{Y_1} \times \cdots \times \mathcal{C}_{Y_n} \to \mathcal{D}$ may be viewed, instead, as a functor of one variable $F : \mathcal{C}_{Y_1 \coprod \cdots \coprod Y_n} \to \mathcal{D}$. This allows us to consider the n^{th} cross effect of F, $cr_n F$, as a functor of only one variable – so we can take $P_n cr_n F$. Now the theorem may be proven in two steps, first we show $P_{1,\ldots,1} cr_n F \sim P_n cr_n F$ and then we show $P_n cr_n F \sim cr_n P_n F$. The first of these statements is shown by proving that $P_{1,\ldots,1} cr_n F$ is n-excisive and $P_n cr_n F$ is $(1,\ldots,1)$-excisive. The statement then follows from the universality of P_n and $P_{1,\ldots,1}$. The second statement follows from the fact that the homotopy limit and cross effect operations commute (because cross effects are homotopy fibers, a type of homotopy limit).

The proof of Theorem 3 again involves the indexing categories $\mathcal{I}$, $\mathcal{J}$, ε, and $(\mathcal{J} \cup \varepsilon)$ from the proof of Theorem 1. To show $P_{n-1}F$ is $(n-1)$-excisive, we prove that for $\mathcal{X}: \mathcal{P}(\underline{n}) \to \mathcal{C}_Y$ any strongly co-cartesian $(n-1)$-cube, the map of cubes $t_{n-1}: F(\mathcal{X}) \to T_{n-1}F(\mathcal{X})$ factors through a cartesian cube. Thus the colimit cube, $P_{n-1}F(\mathcal{X})$ is itself cartesian. The proof relies on the construction of a new cube, $\hat{\mathcal{X}}: \mathcal{P}(\underline{n}) \times \mathcal{P}(\underline{n})^n \to \mathcal{C}_Y$ satisfying three properties:

$$\mathcal{X}(T) \xrightarrow{\sim} \operatorname*{holim}_{T\times\mathcal{I}} \hat{\mathcal{X}} \tag{6}$$

$$\text{The cube } \operatorname*{holim}_{T\times\varepsilon} F(\hat{\mathcal{X}}) \text{ is cartesian} \tag{7}$$

$$\operatorname*{holim}_{T\times\mathcal{J}} \hat{\mathcal{X}} \to \operatorname*{holim}_{U\in\mathcal{P}_0(\underline{n})} \mathcal{X}(T) *_Y U \tag{8}$$

This will complete the proof, since property (2) guarantees that the factorization $\operatorname{holim}_{T\times\mathcal{I}} \hat{\mathcal{X}} \to \operatorname{holim}_{T\times(\mathcal{J}\cup\varepsilon)} \hat{\mathcal{X}} \to \operatorname{holim}_{T\times\mathcal{J}} \hat{\mathcal{X}}$ induces a factorization through a cartesian cube upon applying F, and properties (1) and (3) complete the induced factorization to one of $t_{n-1}: F(\mathcal{X}) \to T_{n-1}F(\mathcal{X})$. For the cube $\hat{\mathcal{X}}$, we take iterated unions along $\mathcal{X}(T)$ of joins with the sets U_s:

$$\hat{\mathcal{X}}(T, U_1, \ldots, U_n) = \bigcup_{\substack{\mathcal{X}(\mathrm{T}) \\ s=1}}^{n} \mathcal{X}(T) *_{\mathcal{X}(T\cup\{s\})} U_s$$

The map in property (1) is then the standard equivalence and the map in property (3) is the fold map. Property (2) follows from the fact that the cubes $\tilde{\mathcal{X}}_{\vec{U}}: T \to F(\hat{\mathcal{X}}(T, \vec{U}))$ are cartesian for all $\vec{U} \in \varepsilon$ – in fact, for each $\vec{U} \in \varepsilon$, the cube $\tilde{\mathcal{X}}_{\vec{U}}$ is merely a map between two identical subcubes.

The usefulness of being able to construct "formal" Taylor towers with all of the important universal properties which a Taylor tower is expected to exhibit has been shown, in particular, by recent work of Nick Kuhn on the effects of localization with respect to Morava K-theories on Taylor towers. [15, 16]

16. Some relations between manifold calculus and homotopy calculus

Thomas Goodwillie

This talk is about relations between homotopy calculus and manifold calculus and about some central examples.

Example: The derivative of the identity functor ought to look like an identity matrix. It does. Recall that if F is a homotopy functor from spaces to spaces and $Y = F(X)$ then F gives a homotopy functor from spaces over and under X to spaces over and under Y. Recall from talk 14 that the linearization of the latter functor can be encoded in a spectrum object in the category of spaces over and under $X \times Y$. This is something like an X by Y matrix of partial derivatives whose (x, y) entry is the spectrum

$$k \mapsto hofiber_y(F(X \vee_x S^k) \to F(X)).$$

In the case when F is the identity (so $Y = X$) we obtain

$$k \mapsto hofiber_y(X \vee_x S^k \to X) \simeq (\Omega_x^y X)_+ \wedge S^k = \Sigma_+^\infty \Omega_x^y X$$

where $\Sigma_+^\infty Z$ is the suspension spectrum of Z_+ and $\Omega_x^y X$ is the space of paths from x to y in X, or in other words the homotopy fiber of the diagonal map $X \to X \times X$. It is reasonable to regard this matrix as a fibrant replacement for a sort of Kronecker delta object, a spectrum object on $X \times X$ whose fiber over (x, y) is the sphere spectrum or the point spectrum according as $x = y$ or not.

This description of the derivative of the identity might also be written as $i_* p^* S$ (taken in a suitable derived sense), where $i : X \to X \times X$ is the diagonal inclusion, $p : X \to *$, and S is the sphere spectrum.

Rhetorical question: If the first derivative of the identity is the identity matrix, why is the second derivative not zero? Answer: Some of the terminology of homotopy calculus works better for functors from spaces to spectra than for functors from spaces to spaces. Specifically, since "linearity" means taking pushout squares to pullback squares, the identity functor is not linear and the composition of two linear functors is not linear.

Attempted cryptic remark: Unlike the category of spectra, where pushouts are the same as pullbacks, the category of spaces may be thought of has having nonzero curvature.

Correction: After the talk Boekstedt asked about that remark. We discussed the matter at length and found more than one connection on the category of spaces, but none that was not flat. In fact curvature is the wrong thing to look for. There are in some sense exactly two tangent connections on the category of spaces (or should we say on any model category?). Both are flat and torsion-free. There is a map between them, so it is meaningful to subtract them. As is well-known in differential geometry, the difference between two connections is a 1-form with values in endomorphisms (whereas the curvature is a 2-form with values in endomorphisms). Thus there is a way of discussing the discrepancy between pushouts and pullbacks in the language of differential geometry, but it is a tensor field of a different type from what I had guessed.

Example: Let F_K be the functor $F_K(X) = \Sigma^\infty_+ Map(K, X)$ determined by a finite complex K. The n^{th} derivative of F_K at the one-point space was described in talk 12. Here is a formula for its first derivative at an arbitrary space X [4]:

$$\partial_x F_K(X) = \int^{\kappa \in K} \Sigma^\infty_+ Map((K, \kappa), (X, x)) d\kappa.$$

The left-hand side may be defined as the spectrum

$$k \mapsto fiber(F_K(X \vee_x S^k) \to F_K(X)).$$

The notation in the right-hand side is as in section 6 of [6]; the integral sign denotes cohomology spectrum of K with coefficients in a spectrum object whose fiber at k is $\Sigma^\infty Map((K, k), (X, x))_+$. Compact notation for the spectrum object on X would be $p_! e_* \pi^* S$, where p and π are projections and e is given by $e(k, f) = (k, f(k))$:

$$\begin{array}{ccc} K \times X & \leftarrow & K \times Map(K, X) \\ \downarrow & & \downarrow \\ X & & * \end{array}$$

Example (a fundamental object in manifold calculus): According to talk 7, the key to the k^{th} layer ($k > 1$) of Weiss' Taylor tower for embeddings of M in N is the following. For a set S of k points in N, let $\Phi_S(N)$ be the total homotopy fiber of the k-cubical diagram that associates to each subset $T \subset S$ the space of embeddings of T in N. If N is n-dimensional Euclidean space then it is easy to see that $\Phi_S(N)$ is equivalent to

$$cr_{k-1} I(S^{n-1}, \ldots, S^{n-1}),$$

the $(k-1)^{st}$ cross-effect of the identity evaluated at the $(n-1)$-sphere. Thus it is related to the $(k-1)^{st}$ layer of the Taylor tower of the identity in homotopy calculus.

Review of concordance theory: Let $Diff(N)$ be the space of all diffeomorphisms from the compact n-manifold N to itself fixing the boundary pointwise. Even when N is a disk, the homotopy type of $Diff(N)$ is not very well understood. $Diff(D^n)$ is contractible when $n < 4$, but $Diff(D^4)$ is a big mystery. For larger n $\pi_0 Diff(D^n)$ is a finite group (isomorphic to the Kervaire-Milnor group of exotic $(n+1)$-spheres), understood by surgery theory. Information beyond π_0 is gained mainly by downward induction on n, using spaces of concordances (or pseudoisotopies).

The space $C(N)$ of concordances of N is the space of diffeomorphisms from $N \times I$ to itself fixing $(N \times 0) \cup (\partial N \times I)$ pointwise. It fibers over $Diff(N)$ with fiber $Diff(N \times I)$ over the base point (and perhaps empty fibers over some components). There is a canonical map $C(N) \to C(N \times I)$. By [12] the map is approximately $(n/3)$-connected, and by [22] the *stable concordance space*, the homotopy colimit of $C(N \times I^k)$ as k tends to infinity, is closely related to Waldhausen's algebraic K-theory space $A(M)$. This has been used [2] to describe the rational homotopy groups of $\pi_j Diff(D^n) \otimes Q$ in the range roughly $j < n/3$.

The derivative of stable concordance [4]: Stable concordance can be made into a homotopy functor from spaces to spectra. Calculating its derivative amounts to finding stable-range descriptions of spaces of *concordance embeddings*. If H is a submanifold of N then a concordance embedding of H in N is an embedding of $H \times I$ in $N \times I$ that fixes $(H \times 0) \cup ((H \cap \partial N) \times I)$ pointwise and takes $H \times 1$ into $N \times 1$. The space of all such is $CEmb(H, N)$. The key case is when H is a tubular neighborhood of a p-disk with $\partial P \subset \partial N$. Using a few ideas from manifold calculus one finds that the space $CEmb(H, N)$ is $(n-p-3)$-connected and that it has a $(2n-2p-5)$-connected map to $\Omega^2 \Sigma^{n-p} \Omega N$. In the end this becomes a proof that the first derivative of stable concordance theory at (N, x) is $\Omega^2 \Sigma^\infty \Omega_x N$, and thus that $\partial_x A(X)$ is $\Sigma^\infty_+ \Omega_x N$. A neat way of expressing this is by saying that there is a certain map $A(X) \to (\Sigma^\infty_+ Map(S^1, X))^{hT}$, where T is the circle group, and that that map induces an equivalence from $\partial_x A(X)$ to

$$\partial_x (\Sigma^\infty_+ Map(S^1, X))^{hT} \simeq (\int^{\kappa \in S^1} \Sigma^\infty_+ Map((S^1, \kappa), (X, x)) d\kappa)^{hT}.$$

References

[1] D. Bousfield and D. Kan, *Homotopy Limits, Completions and Localizations*, Lecture Notes in Mathematics 304, Springer-Verlag, 1972.

[2] T. Farrell and W.-c. Hsiang, *On the rational homotopy groups of the diffeomorphism groups of disks, spheres, and aspherical manifolds*, Alg. and Geom. Topology (AMS Proc, Symp. Pure Math.), Part 1 (1978), 325-337.

[3] T. Goodwillie, *A multiple disjunction lemma for smooth concordance embeddings*, Mem. AMS 86 (1990), no. 431, 1-317.

[4] T. Goodwillie, *Calculus I: The first derivative of pseudoisotopy theory*, K-Theory, 4 (1990), 1-27.

[5] T. Goodwillie, *Calculus II: Analytic Functors*, K-Theory, 5 (1992), 295-332.

[6] T. Goodwillie, *Calculus III: Taylor series*, Geometry and Topology, 7 (2003), 645-711.

[7] T. Goodwillie, *Excision estimates for spaces of homotopy equivalences*, preprint at: http://www.math.brown.edu/faculty/goodwillie.html

[8] T. Goodwillie, *Excision estimates for spaces of diffeomorphisms*, preprint at: http://www.math.brown.edu/faculty/goodwillie.html

[9] T. Goodwillie, J. Klein, *Excision estimates for spaces of Poincaré embeddings*, in preparation.

[10] T. Goodwillie, J. Klein, and M. Weiss, *Spaces of smooth embeddings, disjunction and surgery*, Surveys on surgery theory, Vol. 2, 221–284, Ann. of Math. Stud., 149, Princeton Univ. Press, Princeton, NJ, 2001.

[11] T. Goodwillie, M. Weiss, *Embeddings for the point of view of immersion theory: Part II*, Geometry and Topology, 3 (1999) 103-118.

[12] K. Igusa, *The stability theorem for smooth pseudoisotopies*, K-Theory, 2 (1988), 1-355.

[13] B.Johnson, *The derivatives of homotopy theory*, Transactions AMS, 347 (1995), 1295-1321.

[14] B. Johnson and R. McCarthy, *Deriving calculus with cotriples*, Trans. Amer. Math. Soc., 356 (2004), 757–803.

[15] N. Kuhn, *Localization of Andrè-Quillen-Goodwillie Towers and the Periodic Homology of Infinite Loopspaces*, preprint at http://hopf.math.purdue.edu/cgi-bin/generate?/Kuhn/Kuhn
[16] N. Kuhn, *Tate Cohomology and Periodic Localization of Polynomial Functors*, preprint at http://hopf.math.purdue.edu/cgi-bin/generate?/Kuhn/Tate
[17] P. Lambrechts and I. Volic, *The rational homotopy type of the space of long knots in euclidean space of codimension* ≥ 3., in preparation.
[18] A. Mauer-Oats, *Algebraic Goodwillie calculus: a cotriple formula for the remainder*, `http://arXiv.org/math.AT/0212095`.
[19] K. Scannell and D. Sinha, *A one-dimensional embedding complex.*, J. Pure Appl. Algebra 170 (2002), no. 1, 93–107.
[20] D. Sinha, *The topology of spaces of knots*, preprint at http://arXiv.org/math.AT/0202287.
[21] I. Volic *Finite type knot invariants and calculus of functors*, preprint at http://arXiv.org/math.AT/0401440.
[22] F. Waldhausen, *Algebraic K-theory of spaces, concordance, and stable homotopy*, Annals of Math. Studies 113 (1987), 392-417.
[23] M. Weiss, *Calculus of embeddings*, Bull. AMS 33 (1996), no. 2, 177–187.
[24] M. Weiss, *Embeddings for the point of view of immersion theory: Part I*, Geometry and Topology, 3 (1999) 76-101.
[25] M. Weiss, *Orthogonal Calculus*, Trans. Amer. Math. Soc. 347 (1995) 3743-3796, *Erratum*, Trans. Amer. Math. Soc., 350 (1998), 851-855.

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