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Model Theory and Complex Analytic Geometry

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Abstract: The aim of the workshop was to discuss the connections between model theory and complex analytic geometry, a particularly fascinating point of interaction, where model-theoretic methods can both serve to extend the scope of classical results, and establish new ones.

Introduction by the Organisers

The workshop consisted of 2 tutorials of four 1-hours talks each, 10 1-hours talks, and 5 half-hour talks. The tutorials were given by Ya'acov Peterzil and Sergei Strachenko on *Complex analytic geometry, an o-minimal viewpoint*, and by Boris Zil'ber and Alex Wilkie on *Pseudoanalytic structures and Hrushovski's construction.*

For many years there were two main lines of research in model theory:

- the abstract study of mathematical structures and theories, this line of the subject is often referred to as *stability theory* or *classification theory*;

- applications to the study of definability in concrete mathematical structures (like algebraically closed fields, the real field or the field of p-adic numbers).

At first, only the most basic tools from the general theory were needed in applications, but, over the last ten years, some of the most sophisticated ideas from stability theory have played an important role in applications, most notably Hrushovski's proof of the Mordell-Lang Conjecture for function fields. At the same time, these applications have given us new examples of stable structures which have led to new insights in the general theory. We shall briefly describe some of the recent work.

Compact Complex Spaces. Zil'ber showed that a compact complex space equipped with all analytic relations is an ω-stable structure with quantifier elimination. He and Hrushovski showed that any strongly minimal set definable in these structures is either locally modular or closely related to the field of complex numbers. This type of dichotomy is the fundamental insight in many of the modern applications of model theory. Pillay began the systematic model theoretic study of these structures and was able to show that many interesting model theoretic phenomena arise naturally in this context. For example, simple non-algebraic tori are exactly the locally modular groups. In addition to giving us new examples of locally modular strongly minimal sets, this result led Pillay to a model theoretic methods to extend Falting's theorem to a proof of Mordell-Lang Conjecture for complex tori. Pillay, in collaboration with Scanlon and Kowalski, have carried on a detailed model theoretic analysis of the groups definable in compact complex spaces, their results extend and generalize Fujiki's work on meromorphic groups. A highlight of this work is Pillay and Scanlon's proof that any meromorphic group is an extension of a complex torus by a linear algebraic group, a generalization of Chevalley's theorem for algebraic groups. Recently Pillay was able to show how results of Campana and Fujiki on cycle spaces leads to a relatively easy proof of the dichotomy theorem for strongly minimal sets. With this as a model he and Ziegler were able to find new proofs of the dichotomy theorem in several other important settings (differential fields, difference fields of characteristic 0) that greatly simplify and offer new insights to some applications of model theory to diophantine geometry. In model theory one often needs to not only understand the structures we are studying but also their nonstandard extensions. While these extensions have no classical analogs, problems about nonstandard extensions often give rise to interesting classical problems about uniformity. An important recent result in this direction is Moosa's proof of the nonstandard Riemann Existence Theorem.

Quasi-analytic structures. Zil'ber originally conjectured that the dichotomy property was true for all strongly minimal sets. Hrushovski refuted this by giving a very combinatorial construction of a counterexample. Zil'ber's current research program is designed to show that the type of examples constructed by Hrushovski actually arise naturally. The first major success of this program was recently completed by Koiran, building on work of Wilkie, who showed that one could construct analytic functions f such that the structure $(\mathbb{C}, +, \cdot, f)$ is isomorphic to an expansion built by a Hrushovski construction. The most intriguing part of this program is Zil'ber's work on pseudoexponentiation. Zil'ber has shown that a Hrushovski style construction can be used to expand the complex field by adding an homomorphism from the additive to multiplicative group with very good model theoretic properties. The proof uses a wide array of ingredients including some diophantine geometry of intersections of varieties in algebraic tori developed by Zil'ber and Shelah's very abstract work on the classification theory of excellent classes. The most remarkable part of this program is Zil'ber's conjecture that the structure he has built is actually the complex field with the usual exponential function. An outright proof of this is unlikely, as it would require strong forms of Schanuel's conjecture, but, if true, this would give us a much better understanding of the model theory of this structure. For example, one could prove that any subset of $\mathbb{C}$ definable using exponentiation is either countable or co-countable, and show that there are many automorphism. These are two difficult open problems.

A related question is whether one can obtain model theoretically interesting new structures on the complex numbers by adding sets defined in tame expansions of the real field. Marker showed that it was impossible to add any new real algebraic structure and Peterzil and Starchenko recently generalized this to show that one cannot add any o-minimal structure. It is still interesting to ask of o-minimal structures have any interesting ω-stable reducts.

In a slightly different direction, Miller and Speissegger have shown that the logarithmic spiral is d-minimal and Zil'ber believes this can be used to obtain some natural models of the theory of bi-colored field first built by Poizat using a Hrushovski construction.

Workshop: Model Theory and Complex Analytic Geometry

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Abstracts

Tamm's theorem on the boundary

Matthias Aschenbrenner
(joint work with C. Miller)

Let k be a subfield of $\mathbb{R}$ and $f\colon A \to \mathbb{R}$, $A \subseteq \mathbb{R}^{m+n}$, be a function definable in the o-minimal expansion $\mathbb{R}^k_{an}$ of the real field by restricted analytic functions and power functions with exponents in k. We consider the following question:

Question *Is the set*

$$\big\{(s,x) \in \mathbb{R}^{m+n} : x \in \operatorname{bd}(A_s) \text{ and } f(s,-) \text{ extends analytically at } x\big\}$$

definable (in $\mathbb{R}^k_{an}$)?

Here, $\operatorname{bd}(A_s)$ denotes the boundary of

$$A_s := \big\{x \in \mathbb{R}^n : (s,x) \in A\big\}$$

and $f(s,-)$ denotes the function:

$$\begin{array}{rcc} f(s,-)\colon A_s & \to & \mathbb{R} \\ x & \mapsto & f(s,x) \end{array}$$

for $s \in \mathbb{R}^m$. This question has a positive answer if "$x \in \operatorname{bd}(A_s)$" is replaced by "$x \in \operatorname{interior}(A_s)$" and *extends analytically* by *is analytic*, by a theorem of Tamm (for $k = \mathbb{Q}$, $m = 0$) and van den Dries-Miller (arbitrary k and m), and for $m = 0$ (Bierstone, Piękosz). We give a positive answer, provided that the singularities of $\operatorname{bd}(A_s)$ are *nice* in a way explained in the talk.

The decomposition of projective varieties according to the three pure geometries. Arithmetic and hyperbolicity aspects

Frederic Campana

We intrinsically describe any complex projective variety X by fibrations with fibres of one of the three fundamental pure geometries of algebraic geometry.

We first define 3 classes of projective complex varieties X in arbitray dimension n, which generalise the usual trichotomy for curves (g=0,1, or at least 2) as follows: The first (resp. second; resp. third) class is the class of manifolds with $\kappa_+ = -\infty$ (resp. $\kappa = 0$; resp. $\kappa = n = dim$), where κ is the so-called Kodaira dimension. The condition $\kappa_+(X) = -\infty$ means that $\kappa(Y) = -\infty$, for any variety Y such that there is a surjective meromorphic map from X to Y. The property $\kappa = n$ is usually called *being of general type*, following B. Moishezon.

We shall decompose intrinsically any X in two steps as a tower of fibrations, each one of these having (orbifold) fibres in one of the above three classes. We next define X to be special if it has no surjective meromorphic maps with (orbifold) base of general type. The orbifold structure on the base is given by the divisor (*possibly empty*) of multiple fibres of the map

The first decomposition theorem is as follows:

Theorem 1. *For any X, there is a unique fibration $c_X : X \to C(X)$, called the* **core** *of X, such that:*

1. its (general) fibres are special.

2. Its (orbifold) base is of general type.

This fibration is functorial, birational and invariant under finite étale covers.

Thus , the core *decomposes* any X into two parts: its *special* part (the fibres), and the *general type* one (the orbifold base). This *splitting* should also occur conjecturally at the arithmetic and hyperbolicity levels, giving a very simple description of the rational points on X and of the Kobayashi pseudometric of X.

The second decomposition is the following:

Theorem 2. *For any X as above, we have that $c_X = (J^\circ)^n$, where r (resp. J) are intrinsically defined fibrations with orbifold fibres having $k_+ = -\infty$ (res. $\kappa = 0$).*

In particular, X is special if and only if it is a tower of fibrations with orbifold fibres in the first two classes of the 3 pure geometries.

The above decomposition theorems are similar to the ones for complex Lie algebras L: first, Levi-Maltsev's theorem yields a decomposition of L as solvable-by-semi-simple Lie algebras, and then the derived series decomposes the solvable part as a tower of abelian ones).

Are real numbers nicer (less nice) than p-adic numbers?

Raf Cluckers

Constructible functions We present a project which consists in extending some results by Lion, Rolin, and Comte on integration of subanalytic functions, see [LR], [LRC], to a more general framework. We compare with analogue p-adic results, recently proven by Denef, Cluckers, and Loeser. This is one example of a theory in which the p-adics lie much ahead of the real numbers, and even more, the presented project has its origins in the p-adic theory.

By a subanalytic set we will always mean a globally subanalytic subset $X \subset \mathbb{R}^n$.

For each subanalytic subset $X \subset \mathbb{R}^n$ let $\mathcal{C}(X)$ be the $\mathbb{R}$-algebra of functions on X generated by subanalytic functions $X \to \mathbb{R}$ and $\log(|f|)$ where $f : X \to \mathbb{R}^\times$ is subanalytic.

For a subanalytic set X, let $\mathcal{C}^{\leq d}(X)$ be the ideal of $\mathcal{C}(X)$ generated by the characteristic functions $\mathbf{1}_Z$ of subanalytic subsets $Z \subset X$ of dimension $\leq d$. Note that the support of a function in $\mathcal{C}(X)$ is in general not subanalytic, cf. the function

$(x, y) \mapsto x - \log(|y|)$ on $\mathbb{R} \times \mathbb{R}^{\times}$.

By $C^d(X)$ we denote the quotient

$$C^d(X) := \mathcal{C}^{\leq d}(X) / \mathcal{C}^{\leq d-1}(X).$$

Finally we set

$$C(X) := \bigoplus_{d \geq 0} C^d(X).$$

It is a module over $\mathcal{C}(X)$.

One may call all these functions *constructible functions*.

Suppose that $X \subset \mathbb{R}^n$ is a subanalytic set of dimension d. The set X contains a nonempty open submanifold $X' \subset \mathbb{R}^n$ such that $X \setminus X'$ has dimension $< d$. There is a canonical d-dimensional measure on X' coming from the submanifold structure and Euclidian structure on $\mathbb{R}^n$. We extend this measure to X by zero and denote it by μ_X. This measure allows us to define the subgroup $IC^d(X)$ of $C^d(X)$ for a subanalytic set X of dimension d, as the group consisting of all μ_d-integrable functions in $C^d(X)$. We define $IC^e(X)$ for general e as the subgroup of $C^e(X)$ consisting of the functions φ with support contained in a subanalytic subset $Z \subset X$ of dimension e and with $\varphi_{|Z} \in IC^e(Z)$. Finally, we define the graded group $IC(X)$ as $\oplus_r IC^r(X)$. Using the pullback of differential forms under analytic maps, it is possible to define the Jacobian of a subanalytic bijection $X \to Y$ and similarly, one can speak of Leray differential forms and so on.

By Sub we denote the category of (globally) subanalytic subsets $X \subset \mathbb{R}^n$ for $n > 0$, with subanalytic maps as morphisms. We can now state an abridged version of a general (work in progress) integration result which states unicity and existence of a certain integral operator. This integral operator is introduced as a push-forward operator of functions under subanalytic maps, inspired by integration in the fibers with a measure on the fibers essentially determined by the Leray-differential forms.

Theorem 1. *There exists a unique functor sending a subanalytic set X to the group $IC(X)$ such that a morphism $f : X \to Y$ in* Sub *is sent to a group morphism $f_! : IC(X) \to IC(Y)$ satisfying a list of axioms (one for the disjoint union, a projection formula, one for the projection of a* 1*-cell on the base, and one for the projection of a* 0*-cell on the base)*

The proof is based on the results by Lion, Rolin, and Comte, in [LR2], [LR] and [LRC], and functorial arguments.

Extensions of constructible functions Two possible extensions of the groups $C(X)$ could include functions of the form $x \in X \mapsto \exp(2\pi i f(x))$ or functions $x \in X \mapsto |f(x,t)|^s$ for all subanalytic functions $f : X \to \mathbb{R}$ and a formal complex variable s with $\mathrm{Re}(s) \geq 0$. In the p-adic case, for a subanalytic p-adic set X, one can built analogously groups $C(X)$ and their extensions with additive character or complex power. All these extensions are known, by recent work of Denef, Cluckers, and Loeser, [Denef1], [Denef2], [CLexp], [Ccell], to satisfy an analogue of the previous theorem. Even more nicely, in the p-adic case, when can work throughout with semialgebraic sets and functions, and still build a framework closed under integration.

However, in the real case, new transcendental functions come up when calculating parameterized integrals and the groups $C(X)$ extended with e.g. functions of the form $x \in X \mapsto \exp(2\pi i f(x))$ need to be extended by more transcendental functions in order to make them closed under integration as in the above theorem. During the lecture, I gave a suggestion of adding a small number of transcendental functions which might be sufficient.

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On the number of arithmetic steps needed to generate the greatest common divisor of two integers

Lou van den Dries

Given integers a, b, define an increasing sequence

$$G_0(a,b) \subseteq G_1(a,b) \subseteq \cdots \subseteq G_n(a,b) \subseteq \ldots$$

of finite subsets of $\mathbb{Z}$ as follows: $G_0(a,b) = \{0, 1, a, b\}$, and

$$G_{n+1}(a,b) = G_n(a,b) \cup \big\{\text{sums, differences, integer quotients, remainders, and products of two numbers in } G_n(a,b)\big\}.$$

Let $g(a,b)$ be the least n such that $\gcd(a,b) \in G_n(a,b)$. There is a very easy double logarithmic *upper bound* (logarithms to base 2):

$$g(a,b) \leq 4 \log\log a \qquad (a > b > 1).$$

This talk will focus on a more difficult *lower bound*:

Result *There are infinitely many* (a, b) *with* $a > b > 1$ *such that*

$$g(a, b) \geq \frac{1}{4}\sqrt{\log \log a}.$$

The proof uses arithmetic properties of integer solutions to the Pell equation $x^2 - 2y^2 = 1$. There are also connections to irrationality and transcendence. Motivation for finding such bounds comes from *arithmetic complexity*. I will mention some open problems in this area.

Trivial stable structures with non-trivial reducts

David Evans

We offer a new viewpoint on some of the generic structures constructed using Hrushovski's predimensions and show that they are natural reducts of quite straightforward trivial, one-based stable structures. In order to describe a special case we begin with a description of a reasonably natural combinatorial object.

By a *digraph* we mean a set of vertices together with an anti-symmetric, irreflexive binary relation on the vertices (– the directed edges). Consider the class $\mathcal{D}$ of digraphs in which every vertex has at most two directed edges coming out of it (i.e. has at most two out-neighbours). In such a digraph, call a subset of vertices *closed* if any out-neighbour of a vertex in the subset is already in the subset. Using a Fraïssé-style amalgamation argument, it is easy to show that there is a unique (up to isomorphism) countable digraph $D \in \mathcal{D}$ with the properties that: D is a union of a chain of finite closed subdigraphs; any finite digraph in $\mathcal{D}$ embeds as a closed subdigraph of D; any isomorphism between finite, closed subdigraphs of D extends to an automorphism of D. It can be shown that $Th(D)$, the theory of D, is stable, one-based and trivial.

Now consider the (undirected) graph H obtained by forgetting the orientation on the edges of D. The result is:

Theorem 1. *The graph H is isomorphic to the 'ab initio' Hrushovski structure constructed using the predimension 'twice number of vertices minus number of edges' on finite graphs.*

So H is ω-stable of Morley rank $\omega.2$ and it is neither trivial nor one-based.

We can do this more generally and obtain any of the basic *ab initio* Hrushovski structures produced by an integer-valued predimension as a reduct of a 'natural' stable, trivial one-based structure.

Two further results may be of interest:

(A) Consider the free k-algebra on two non-commuting variables (over some field k). The theory of modules for this has a model completion (Eklof-Sabbagh) and it is easy to see that the digraph D (and therefore the graph H) is interpretable in this.

(B) There is a sort of converse to the theorem. Any pseudoplane arising from a non-1-based reduct of a stable 1-based theory with nfcp has to satisfy a 'positivity of predimension' inequality. Thus we have a case where such a condition arises from some reasonably natural model-theoretic assumptions.

It is not at all clear how (or whether) this way of looking at the Hrushovski constructions fits with the 'analytic' viewpoint of Zil'ber. One might ask whether our 'oriented' structures (such as the directed graph D) are interpretable in any meaningful additional analytic structure supported by the (quasi-) analytic models which realise the Hrushovski structures.

Algebraic groups with extra structure

Piotr Kowalski
(joint work with A. Pillay)

We consider a field with an additional operator $\mathcal{D}$ of one of the following kinds: derivation, automorphism, Hasse derivation (several, commuting, iterative). In the case of $\mathcal{D} = \delta$ being a derivation Buium has a notion of δ-variety which is an algebraic variety with extension of δ to the structure sheaf. The same definition works in the Hasse case, so we obtain Hasse D-varieties. When $\mathcal{D} = \sigma$, the definition needs to be modified – a σ-variety is just a pair (X, f) where X is an algebraic variety and $f : X \to X^\sigma$ is an algebraic morphism.

In all the cases we obtain the category of $\mathcal{D}$-varieties, group objects there (called $\mathcal{D}$-groups) and trivial $\mathcal{D}$-varieties naturally coming from the field of constants. We are interested when a $\mathcal{D}$-variety is $\mathcal{D}$-isotrivial, i.e. $\mathcal{D}$-isomorphic to a trivial one.

Theorem 1. *Suppose G is a $\mathcal{D}$-group (*separable *in the case of $\mathcal{D} = \sigma$), X an irreducible $\mathcal{D}$-subvariety containing identity and generating G. Then there exists a connected normal $\mathcal{D}$-subgroup $N \triangleleft G, \mathrm{Stab}(X)$ such that G/N is $\mathcal{D}$-isotrivial.*

As a corollary to Theorem 1 we get that a finite Morley rank group definable in DCF_0 is interpretable as a reduct of an algebraic group. We also get fast proofs of Manin-Mumford conjecture (Pillay) and a part of its positive characteristic analogue (Kowalski).

Theorem 2. *If $\mathcal{D}$ is a Hasse derivation, then there is at most one $\mathcal{D}$-structure (up to $\mathcal{D}$-isomorphism) on a projective variety.*

Buium has proved the full version of Theorem 2 for derivations (i.e. all projective δ-varieties are δ-isotrival).

Expansions of o-minimal structures by trajectories of definable vector fields

Chris Miller

An expansion (in the sense of first-order logic) of the real field $\mathbb{R}$ is called *o-minimal* if every definable set has finitely many connected components. Such structures provide natural settings for studying so-called tame objects of real-analytic geometry—see [VDD], [VDD-Miller] for surveys—such as nonoscillatory trajectories of real-analytic planar vector fields. It turns out that even some infinitely spiralling trajectories of such vector fields have a reasonably well-behaved model theory; this motivates the notion of *d-minimality*, a generalization of o-minimality that allows for definable sets to have countably many connected components (but rules out sets that are somewhere dense-codense); see Section 3.4 in [Miller2] for the precise definition. More generally, we are interested in expanding a given o-minimal structure on $\mathbb{R}$ by collections of *trajectories* (i.e., solution curves) of vector fields definable in the structure. As in ODE theory, we find it useful to investigate first the most basic cases: linear vector fields on arbitrary $\mathbb{R}^n$, and analytic planar vector fields under a certain nondegeneracy condition.

For my talk, I will discuss the proof (essentially an exercise in definability combined with some basic linear algebra and ODE theory) of the following:

Theorem 1. *Let $\mathcal{M}$ be a finite set of real linear vector fields. Then there is a finite $W \subseteq (\mathbb{R}\setminus\mathbb{Q}) \cup i(\mathbb{R}\setminus\{0\}) \subseteq \mathbb{C}$ such that, for each family $\mathcal{G}$ of trajectories of members of $\mathcal{M}$, there exists $W' \subseteq W$ (depending on $\mathcal{G}$) such that $\mathcal{G}$ is interdefinable over $\mathbb{R}$ with at least one of e^x (the real exponential function), $\mathbb{Z}$ (the set of all integers) or $(x^w)_{w\in W'}$, where x^{a+ib} denotes the map*

$$\begin{array}{ccc} (0,\infty) & \to & \mathbb{R}^2 \\ t & \mapsto & t^a(\cos b\log t, \sin b\log t) \end{array}$$

(The set W is obtained explicitly from the set of all eigenvalues of the elements of $\mathcal{M}$.)

Is the above a trichotomy? The answer is not yet known, but:

- All instances of the theorem do occur (as an easy consequence of the proof).
- For $s,t,b\in\mathbb{R}$ with $b\neq 0$, we have $e^{s+it}=e^s(e^{t/b})^{ib}$, so $(\mathbb{R},e^x,x^{ib})$ defines *complex* exponentiation, hence also $\mathbb{Z}$.
- $(\mathbb{R},e^x)$ is o-minimal, see [Wil] (or [Spe]), so defines neither $\mathbb{Z}$ nor x^{ib} for $b\neq 0$.
- $\big(\mathbb{R},(x^r)_{r\in\mathbb{R}}\big)$ is polynomially bounded [Miller1], so a proper reduct (in the sense of definability) of $(\mathbb{R},e^x)$.
- If $b\neq 0$, then $(\mathbb{R},x^{ib})$ is d-minimal see Section 3.4 in [Miller2], so does not define $\mathbb{Z}$ (since $\mathbb{Q}$ is dense-codense in $\mathbb{R}$).

It is *not* yet known if there exist $W\subseteq\mathbb{C}$ such that $\big(\mathbb{R},(x^w)_{w\in W}\big)$ defines $\mathbb{Z}$.

Hence, by growth dichotomy [Miller3], Pfaffian closure [Spe], and that every proper noncyclic subgroup of $(\mathbb{R}^{>0}, \cdot\,)$ is dense-codense in $\mathbb{R}^{>0}$, we have the following:

Corollary 1. *Let $\mathcal{M}$, W and $\mathcal{G}$ be as in the theorem. Let $\mathcal{R}$ be an o-minimal expansion of $\mathbb{R}$. Then at least one of the following holds:*

(1) *$(\mathcal{R}, \mathcal{G})$ is o-minimal.*
(2) *$(\mathcal{R}, \mathcal{G})$ is interdefinable with $(\mathbb{R}, \mathbb{Z})$.*
(3) *There exists $W' \subseteq W$ such that $W' \not\subseteq \mathbb{R}$ and $(\mathcal{R}, \mathcal{G})$ is interdefinable with $\big(\mathcal{R}, (x^w)_{w\in W'}\big)$.*

If both (1) and (2) fail, then $\mathcal{R}$ is polynomially bounded. If (1) fails and $(\mathcal{R}, \mathcal{G})$ defines no dense-codense subsets of the line, then it defines no irrational power functions and W' may be taken to be a singleton.

Hence, by Section 3.4 in [Miller2]:

Corollary 2. *Let $\mathcal{M}$, W and $\mathcal{G}$ be as in the theorem. Let $\mathcal{R}$ be an o-minimal expansion of $(\mathbb{R}, e^x\restriction[0,1], \sin\restriction[0,1])$. Then at least one of the following holds:*

(1) *$(\mathcal{R}, \mathcal{G})$ is o-minimal.*
(2) *$(\mathcal{R}, \mathcal{G})$ is interdefinable with $(\mathbb{R}, \mathbb{Z})$.*
(3) *There exist $A \subseteq W \cap \mathbb{R}$ and $\emptyset \neq C \subseteq \{\, e^{2\pi i/w} : w \in W \setminus \mathbb{R}\,\}$ such that $(\mathcal{R}, \mathcal{G})$ is interdefinable with $\big(\mathcal{R}, (x^a)_{a\in A}, (c^{\mathbb{Z}})_{c\in C}\big)$.*

If both (1) and (2) fail, then $\mathcal{R}$ is polynomially bounded. If (1) fails and $(\ R, \mathcal{G})$ defines no dense-codense subsets of the line, then it defines no irrational power functions and there exists $c \in \{\, e^{2\pi i/w} : w \in W \setminus \mathbb{R}\,\}$ such that $(\mathcal{R}, \mathcal{G})$ is interdefinable with $(\mathcal{R}, c^{\mathbb{Z}})$ (and thus is d-minimal).

(For $c > 0$, $c^{\mathbb{Z}}$ denotes the subgroup of $(\mathbb{R}^{>0}, \cdot\,)$ generated by c.)

It turns out that something similar holds for trajectories of certain kinds of planar analytic vector fields. Let $\mathbb{R}_{an}$ denote the expansion of $\mathbb{R}$ by all globally subanalytic subsets of real euclidean spaces (see e.g. [VDD-Miller] for definitions).

Theorem 2 (joint with P. Speissegger and D. Novikov)**.** *Let U be an open neighborhood of $0 \in \mathbb{R}^2$ and $F\colon U \to \mathbb{R}^2$ be analytic such that such that $F^{-1}\{0\} = \{0\}$ and the Jacobian of F at 0 has a nonzero eigenvalue. Let $\gamma\colon [a,\infty) \to U$ be a nontrivial solution to $y' = F(y)$ such that $\lim_{t\to+\infty} \gamma(t) = 0$.*

- *If the eigenvalues are real, then $(\mathbb{R}_{an}, \gamma([a,\infty)))$ is o-minimal.*
- *If the eigenvalues are imaginary, then $(\mathbb{R}, \gamma([a,\infty))) = (\mathbb{R}, \mathbb{Z})$.*
- *Otherwise, $(\mathbb{R}_{an}, \gamma([a,\infty)))$ is d-minimal but not o-minimal; indeed, there exists $c > 1$ (depending only on the eigenvalues) such that $(\mathbb{R}_{an}, \gamma([a,\infty)))$ is interdefinable with $(\mathbb{R}_{an}, c^{\mathbb{Z}})$.*

(The nondegeneracy condition at 0 ties the behavior of trajectories of F to that of its linear part.)

Parts of the proof extend easily to more general planar situations; this suggests a more general result, but significant obstacles arise. Much greater difficulties arise in trying to deal with nonlinear, nonplanar cases. Work is ongoing.

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The model theory of compact complex spaces: essential saturation and Kähler-type spaces

Rahim Moosa

Given a compact complex analytic space X, we consider the first-order structure $\mathcal{A}(X)$ whose underlying universe is X and where there is a predicate for each analytic subset of each cartesian power of X. Zil'ber [Zil] has observed that $\mathcal{A}(X)$ admits quantifier elimination, is ω_1-compact, and is of finite Morley rank. A survey of the model theory of compact complex spaces can be found in [Mo1].

Note that $\mathcal{A}(X)$ is not saturated since every element of X is named by a predicate. However, for certain compact complex spaces this seems to be an artifact of the choice of language. For example, suppose $X = V(\mathbb{C})$ where V is a projective algebraic variety over $\mathbb{Q}$, and let X_{alg} denote the structure whose underlying universe is X and where there is a predicate for the $\mathbb{C}$-points of every algebraic subvariety of every cartesian power of V defined over $\mathbb{Q}$. Then X_{alg} is saturated and (by Chow's theorem) a set is definable in $\mathcal{A}(X)$ if and only if it is definable (with parameters) in X_{alg}. This motivates the following definition from [Mo2]: A compact complex space X is *essentially saturated* if there exists a countable collection of predicates $\mathcal{L}_\circ$ from $\mathcal{A}(X)$ such that a set is definable in $\mathcal{A}(X)$ if and only if it is definable in $(X, \mathcal{L}_\circ)$. The structure $(X, \mathcal{L}_\circ)$ is saturated.

We obtain a geometric characterisation: *X is essentially saturated if and only if the components of the Douady spaces of the cartesian powers of X are all compact.* The Douady space is the universal parameter space for flat families of analytic subsets. For example, the Douady space of a projective algebraic variety is its Hilbert scheme. In particular, every compact Kähler-type space is essentially saturated (these are the holomorphic images of compact Kähler manifolds introduced and studied by Fujiki [Fu1]). No example of an essentially saturated non-Kähler-type space is known (to this author). There are compact complex spaces that are not essentially saturated (Hopf surfaces, for example).

Essential saturation allows model-theoretic techniques (such as canonical bases, orthogonality, and internality) to be applied directly, without having to pass to nonstandard models. As an example, we obtain a model-theoretic proof of the existence of relative algebraic reductions for maps between Kähler-type spaces, a result originally due to Campana [Cam] and Fujiki [Fu2]. This is the beginning of an *analysis* (in the sense of geometric stability theory) of a Kähler-type space in projective space, and is very much in the spirit of Fujiki's work on the bimeromorphic classification problem for Kähler-type spaces (see [Fu2]).

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Coverings of groups and weak generic types

Ludomir Newelski

Assume G is an $\aleph_0$-saturated group. We call a definable set $X \subseteq G$ weak generic if for some definable (left) generic set $Y \subseteq G$, the set $Y \setminus X$ is not generic. A type $p(x)$ is weak generic if every definable set containing the set of realizations of $p(x)$ is weak generic.

Let W be the set of complete weak generic types over $\emptyset$ (or over G, if you like). W is closed and non-empty.

Theorem 1. *Assume $\varphi(x)$ is weak generic and $X = \varphi(G) \cap W(G)$. Then finitely many (left) translates of the set $X \cdot X^{-1}$ cover G.*

Assume G is covered by countably many 0-type-definable sets $X_n, n < \omega$.

Corollary 1. *For some n, finitely many left translates of the set $X_n \cdot X_n^{-1}$ cover G.*

It turns out that in some special cases in the situation described above some finitely many of the sets X_n generate the group G in just 2 steps.

Theorem 2 (M.Petrykowski). *If G is amenable, then for some n, $G = X_{<n} \cdot X_{<n}^{-1}$, where $X_{<n} = \bigcup_{i<n} X_i$.*

However in general 2 steps are not enough, and a suitable example was found by Petrykowski. Not surprisingly, this example involves a free group (as a natural example of a non-amenable group). Consider the situation, where for some n, the set $X_{<n} \cdot X_{<n}^{-1}$ is co-countable in G (that is, the set $G \setminus X_{<n} \cdot X_{<n}^{-1}$ is contained in $acl(\emptyset)$ so that so little of the group G remains to be generated by the remaining sets X_n). In this situation one could wonder if for some larger n, $G = X_{<n} \cdot X_{<n}^{-1}$. However I have found an example of a group G covered by some sets X_n such that already $X_0 \cdot X_0^{-1}$ is co-countable, but still 2 steps are not enough to generate the group G by finitely many of the sets X_n.

Earlier I had a weaker version of Theorem 3, dealing just with the case of abelian G. The current result is much stronger, it also has a much easier proof. However, I generalized my proof from the abelian case to the general case and got the following result.

Theorem 3. *For some n, G is covered by some finitely many conjugates of the set $X_{<n} \cdot X_{<n}^{-1}$.*

While Theorems 1, 3 and Corollary 2 may be a bit surprising, Theorem 4 is really mysterious to me. Namely, Corollary 2 and Theorem 3 refer to a notion of "largeness", which is clearly understood there (by means of weak generic types or a measure). No such explanation is known to me in Theorem 4. Maybe there is some weak combinatorial counterpart of the Banach mean on each group?

Bergman proved that each symmetric goup G has the property, that if G is generated by a set X, then it is generated by this set in some finitely many steps. He asked, whether we can find a countable group with this property. Our results refer to the "Bergman property" restricted to type-definable set X. Type-definable subsets of G may be regarded as "closed" (from the point of view of the space of types). One could ask if in the situation described above, still some finitely many sets X_n generate the group G in some finite number of steps if the sets X_n are just Borel, or even just invariant (under the authomorphisms of G). This question seems to tackle the problem of how much similar an arbitrary group is to a topological compact group. Indeed, our results show some similarity.

Actually, the proof of Theorem 3 yields also, that 2 steps are enough in case, where G is stable (since then we have a finitely additive left-invariant measure on the field of definable subsets of G, and that is is all we need for the proof). However it is not clear if 2 steps are enough in case of groups in o-minimal structures (or simple ones). Most probably: yes. Marcin Petrykowski is working on this problem, and the first step seems to be describing weak generic types in groups definable in o-minimal expansions of the reals. Here, surpsirisngly, sometimes these types are stationary, sometimes not, and there seems to be a lot of work to be done. Petrykowski has looked at the groups $\mathbb{R}^+, \mathbb{R}^+ \times \mathbb{R}^+$ and $\mathbb{R}^*_{>0} \times \mathbb{R}^*_{>0}$. For example, in the last case the fact, that the weak generic types are stationary (over a model) is equivalent to the o-minimal expansion of the reals being polynomially bounded.

Also, Theorem 4 yields that when G is a pure group (that is, it has no other logic structure than the group operation), then 2 steps are enough (since conjugation is an authomorhpism then).

There are some (equivalent) model-theory-free versions of these results. Now assume G is an arbitrary group and X an arbitrary compact topological space covered by countably many sets $X_n, n < \omega$.

Theorem 4. *Assume $f : G \to X$ and the sets X_n are closed. Then there is a finite set $A \subset G$ such that for some $n < \omega$,*

$$G = A \cdot f^{-1}[U] \cdot f^{-1}[U]^{-1}$$

for every open set U containing X_n.

Theorem 5 (M.Petrykowski)**.** *Assume $f : G \to X$ and G is amenable. Then for some $n < \omega$ we have that $G = f^{-1}[U] \cdot f^{-1}[U]^{-1}$ for every open $U \supset X_{<n}$.*

Theorem 6. *Assume $f : G \to X$ and the sets X_n are closed. Then for some finite n and some finite set $A \subset G$ we have that for every open $U \supset X_{<n}$, G is covered by the set of A-conjugates of $f^{-1}[U] \cdot f^{-1}[U]^{-1}$.*

Also, there are some corresponding generalizations of these results for the case of coverings of types rather than groups. The idea is to think of a type as an "affine version" of a (non-existing) group. This leads to the notions of c-free and weakly c-free extensions, generalizing non-forking (in a new way) to the unstable cases. Since in this situation, working with a type, we "dream" of a group, naturally we define the notion of an "abelian" type, and (with some more doubts) "amenable" type. For these types we have the corresponding counterparts of the results above. Likewise, we have also "model-theory-free" versions on edge colourings of graphs. The results on coverings of types have already found some applications in some results of Ziv Shami on binding groups.

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Definably compact abelian groups

Margarita Otero

(joint work with M. Edmundo)

We consider $\mathcal{R}$ an o–minimal expansion of a real closed field and sets definable (with parameters) in $\mathcal{R}$. A typical example of a definable set in a semialgebraic set. A group G is definable if both the set and the (graph of the) group operation are definable. By results of Pillay in [Pi] such groups can be equipped with a manifold structure. Pillay also proved that for a definable group to be definably connected is equivalent to not having definable subgroups of finite index. We may suppose that a definble group G is a submanifold of M^k for some $k \geq 0$. Also, by results of Peterzil and Steinhorn in [PS] in this context to be definably compact is equivalent to be closed and bounded. The o–minimal fundamental group of a definable set is defined in the usual way except that we consider definable paths and definable homotopies. In [BO1] Berarducci and Otero, extending some results of Delfs and Knebusch in [DK], proved that the o–minimal fundamental group of a definable set is a finite presented group. The o–minimal singular homology of a definable set is also defined in the usual way except that we consider definable singular simplices; Woerheide proved in [Wo] that it satisfies the corresponding of the Eilenberg-Steenrod axioms to the o-minimal context . With these data we can state the following structure result:

Theorem 1. *Let G be a definable group of dimension n. Suppose G is abelian, definably connected and definably compact. The following holds:*
(a) *the o–minimal fundamental group of G is isomorphic to $\mathbb{Z}^n$;*
(b) *the k–torsion subgroup of G is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$, and*
(c) *the o–minimal cohomology algebra over $\mathbb{Q}$ of G is isomorphic to the exterior algebra over $\mathbb{Q}$ with n generators of degree 1.*

Note that if G would have being a compact connected abelian Lie group of dimension n, we would have had (a), (b) and (c) (without "o–minimal") of the theorem because of the classification of Lie groups (G would have being an n-dimensional torus). But we do not have such a classification in the o–minimal (or semialgebraic) case.

The main ingredients of the proof of the above theorem are as follows: Firstly, we introduce the concept of a definable covering map and prove the corresponding path lifting and homotopy lifting properties in the o-minimal context. Then, considering the definable covering map $p_k \colon G \to G \colon x \mapsto kx$, and its group of deck transformations we prove that there is $s \geq 0$ satisfying (a) and (b) of the theorem (with s instead of n). Next step is to introduce the notion of o–minimal cohomology and prove that $H^*(G;\mathbb{Q})$ is the exterior algebra over $\mathbb{Q}$ with a finite number r of odd degree generators. To finish the proof remains to prove that the $s = r = n\,(= \dim G)$, and the degree of the generators (of the exterior algebra) is one. We then make use of the existence of a fundamental class for G (proved in [BO2]) and introduce the notion of degree of a definable map (under suitable hypothesis). We can compute the degree of the map p_k using the cohomology algebra and get $deg\, p_k = k^r$. On the other hand p_k being a definable covering map we get $deg\, p_k \leq k^s$ and hence $r \leq s$. Finally proving an analogue of Hurewitz theorem for the o–minimal setting we get the result.

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Complex analytic geometry, an o-minimal viewpoint

Ya'acov Peterzil and Sergei Starchenko

Let $\mathbf{R}$ be a real closed field and $\mathbf{K} = \mathbf{R}(\sqrt{-1})$ its algebraic closure. As in the classical case (by classical we always mean $\mathcal{R} = \mathbb{R}$, $\mathbf{K} = \mathbb{C}$) the field $\mathbf{K}$ can be identified with $\mathbf{R}^2$, and every subset of $\mathbf{K}^n$ can be identified with a subset of $\mathbf{R}^{2n}$.

Since $\mathbf{R}$ is a real closed field it has a natural ordering and topology. It induces the product topology on K, making it into a topological field.

As in the classical case, for a function $f : \mathbf{K} \to \mathbf{K}$ and $a \in \mathbf{K}$ we say that f is *K-differentiable at a* if $\lim_{z\to a} \frac{f(z)-f(a)}{z-a}$ exists in $\mathbf{K}$.

In general, the real closed field $\mathbf{R}$ need not be Dedekind complete, as an ordered set, nor even archimedean, and the topology which it induces on $\mathbf{K}$ is far from being locally compact or connected. Thus $\mathbf{K}$-differentiable functions can be very wild. To avoid it we will restrict ourselves to the category of sets and functions definable in some o-minimal expansion $\mathcal{R}$ of $\mathbf{R}$.

From now on we will fix an o-minimal expansion $\mathcal{R}$ of $\mathbf{R}$. By definable we always mean definable in $\mathcal{R}$ with parameters.

Definition 1. Let $U \subseteq \mathbf{K}$ be an open set. A function $f : U \to \mathbf{K}$ is *K-holomorphic on U* if f is definable and $\mathbf{K}$-differentiable at all points of U.

Example 1. If we take $\mathcal{R}$ to be the structure $\mathbb{R}_{an}$ then definable functions are exactly (globally) subanalytic functions. In this case, $\mathbb{C}$-holomorphic means complex analytic and, as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, subanalytic.

In [Ps1] we showed that many result from the classical complex analysis (e.g. Maximum Principle, Infinite $\mathbf{K}$-differentiability) can be extended to the category of $\mathbf{K}$-holomorphic functions.

In the case of many variables we use the following definition.

Definition 2. Let $U \subset \mathbf{K}^n$ open. A definable $f : U \to K$ is *K-holomorphic on U* if it is continuous on U, and K-holomorphic in each variable separately.

In [Ps2] we developed a theory of **K**-holomorphic functions in several variables through reduction to the 1-variable case ("taking fibers").

In this series of talks we generalize the notion of complex analytic sets to sets definable in **K**. We also demonstrate how **K**-analyticity can can be used in the classical setting.

The following Theorem on Removal of Singularities, proved in [Ps2], plays a key role.

Theorem 1. *Let* $U \subseteq \mathbf{K}^n$ *be definable and open,* $L \subseteq U$ *definable and closed, and* $f : U \setminus L \to K$ *a* **K**-*holomorphic function.*

(1) *If* $\dim L \leq \dim U - 1$ *and* f *is continuous on all of* U *then it is* K-*holomorphic on all of* U.
(2) *If* $\dim L \leq \dim U - 2$ *and* f *is locally bounded on* L *then* f *is* K-*holomorphic on all of* U.
(3) *If* $\dim L \leq \dim U - 3$. *then* f *is necessarily* K-*holomorphic on all of* U.

1. K-manifolds and K-analytic sets

Definition 3. *A* **K**-*manifold (in* $\mathcal{R}$*) of* **K**-*dimension* n is a Hausdorff topological space M , a **finite** cover $M = \cup_{i\in I} U_i$ and $\forall i \in I$, $\phi_i : U_i \to V_i \subseteq \mathbf{K}^n$ a homeomorphism with an $\mathcal{R}$-definable open set V_i, such that

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

are **K**-holomorphic. We will denote by $\dim_{\mathbf{K}}(M)$ the **K**-dimension of M.

Definition 4. Let M be a **K**-manifold. A definable $A \subseteq M$ is *a* **K**-*analytic subset of* M if for every $p \in M$ there is definable open V containing p such that $A \cap V$ is the zero locus of finitely many **K**-holomorphic functions. A is called *finitely* **K**-*analytic* if finitely many such V's cover A.

We prove that every **K**-analytic set is finitely **K**-analytic.

Theorem 2 (Main Theorem). *Let* M *be a* **K**-*manifold,* $A \subseteq M$ *a definable closed subset. Then the following are equivalent:*

(1) A *is a* **K**-*analytic subset of* M.
(2) $\dim_{\mathbf{R}}(\mathrm{Sing}(A)) \leq \dim_{\mathbf{R}} A - 2$, *and the same is true in every open subset of* M.
(3) A *is finitely* **K**-*analytic subset of* M.

Remarks: (a) Notice that (2) and (3) are first order statements, preserved in elementary extensions.
(b) (2) $\Rightarrow$ (3) implies that the functions which witness the analyticity of A are definable in $\langle \mathbf{R}, <, +, \cdot, M, A\rangle$.

As a corollary of Main Theorem we obtain the following

Theorem 3. *Assume that M is a $\mathbf{K}$-manifold, $U \subseteq M$ open and definable and $A \subset U$ an irreducible $\mathbf{K}$-analytic subset of U. If $\dim_{\mathbf{R}}(\mathrm{Cl}_M(A) \setminus A) \leq \dim_{\mathbf{R}} A - 2$ then $\mathrm{Cl}_M(A)$ is a $\mathbf{K}$-analytic subset of M.*

Definition 5. If $F \subseteq M$ is a $\mathbf{K}$-analytic subset then $M \setminus F$ is called *a Zariski open subset of M*. If X is a $\mathbf{K}$-analytic subset of M then $X \setminus F$ is a Zariski open subset of X.

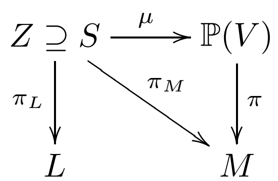
Theorem 4 (Closure Theorem)**.** *Assume that M is a definably connected $\mathbf{K}$-manifold, and A a $\mathbf{K}$-analytic subset of a Zariski open subset of M. Then $\mathrm{Cl}_M(A)$ is a $\mathbf{K}$-analytic subset of M.*

Corollary 1. *Let M, N be $\mathbf{K}$-manifolds, $F \subseteq X \subseteq M$ $\mathbf{K}$-analytic subsets of M. Assume that $X \setminus F$ is a dense susbet of $\mathrm{Reg}(X)$.*
If $f : X \setminus F \to N$ is $\mathbf{K}$-holomorphic then the closure of the graph of f is a $\mathbf{K}$-analytic subset of $M \times N$. Namely, f is a $\mathbf{K}$-meromorphic map.

2. A theorem of Campana and Fujiki

The following theorem and its corollary can be considered as an extension to $\mathbf{K}$-analytic sets of a result proved independently by Campana [Cam] and Fujiki [Fuj].

Theorem 5. *Let L, M be $\mathbf{K}$-manifolds, and $Z \subseteq L \times M$ an irreducible $\mathbf{K}$-analytic subset.*
Then there is a $\mathbf{K}$-holomorphic vector bundle $\pi : V \to M$, a $\mathbf{K}$-meromorphic map $\mu : Z \to \mathbb{P}(V)$ and Zariski open $S \subseteq Z$ such that μ is $\mathbf{K}$-holomorphic on S, the following diagram is commutative



and, for $(c, a), (c', a) \in S$, $\mu((c, a)) = \mu((c', a))$ implies $S_c = S_{c'}$ near a.

Corollary 2. *Assume in addition that Z is definably compact and for some Zariski open $C \subseteq \pi_L(Z)$ all $Z_c, c \in C$, are irreducible and pairwise distinct as subsets of M. Then μ is a $\mathbf{K}$-bimeromorphism between Z and a $\mathbf{K}$-analytic subset of $\mathbb{P}(V)$.*

Remark. *Because of the flexibility to work in any definable manifold, almost all results generalize to "$\mathbf{K}$-analytic spaces" and their subsets.*

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Generic sets in definably compact groups

Anand Pillay

(joint work with Y. Peterzil)

We discuss the notion of genericity in definably compact groups definable in a staurated o-minimal expansion $\mathcal{R}$ of a real closed field. Let G be a group as stated. A set $X \subset G$ is *left generic* if X is definable and finitely many left translates of X cover G.

Using resuls of A. Dolich on forking in o-minimal structures (or rather extracting from his work a suitable statement), we prove:

Theorem 1. *If G is definably compact, $X \subset G$ is definable and not left generic, then $G \setminus X$ is right generic.*

We also discuss a conjecture of myself on definably compact groups, and the relation with generics as well as the case of generics in G, where G *comes from* a compact Lie group.

Geometric stability theory and bimeromorphic geometry

Anand Pillay

We discuss some basic tools for understanding definable sets in theories of finite Morley rank, and how these ideas make sense in the concrete context of the many-sorted structure of compact complex spaces.

Specializing to the (*saturated*) many-sorted structure $\mathfrak{C}$ of *Kähler-type* compact complex spaces, we discuss the structure of simple manifolds, whose generic type is trivial of U-rank 1. Assuming that all such manifolds are *irreducible symplectic*, we point out that Th($\mathfrak{C}$) is nonmultidimensional.

Recent applications of model theory in module theory

Mike Prest

In the talk I presented a variety of applications, both new and old, with the aim of illustrating something of the range of methods used and the areas of application. Here I concentrate on the relatively newer results. I also omit most background and definitions, which are easily found in the literature.

Perhaps most notable are recent counterexamples of Puninski to conjectures concerning serial modules and superdecomposable pure-injective modules. These are different areas of application but the techniques have quite a bit in common, being based on a detailed understanding of the structure of the lattice of pp conditions.

The first theorem concerns serial modules. A module is serial if it is a direct sum of uniserial modules. A module is uniserial if its lattice of submodules is totally ordered. It was an open question whether a direct summand of a serial module is again serial. Puninksi showed [Pun1] that this is not always the case. Indeed, over every exceptional uniserial ring there is a counterexample. His counterexample is a pure-projective module and the proof that it has the required properties relies on his analysis in [Pun2] of pure-projective modules over such rings. In this latter paper he also proves the existence of a uniserial module, over a uniserial domain, which is not quasi-small, so answering a question of Facchini [Fac]. Another question answered in [Pun2] is whether it is true that every pure-projective module over a uniserial ring is a direct sum of finitely presented modules (as opposed to a direct summand of such a module): again, Puninski provides a counterexample and the proof that it is a counterexample relies heavily on techniques from the model theory of modules.

Analysis of pp-types in pure-projective modules has also been used recently by Puninski and Rothmaler [PR] to prove the positive result that over all hereditary noetherian rings pure-projective modules are direct sums of finitely presented modules. They also give some conditions on a serial ring for its pure-projective modules to have this form.

Moving to a very different, almost dual, situation, a pure-injective module is said to be superdecomposable if it is non-zero and has no indecomposable direct summands. It was known that, at least over countable rings, algebras of wild representation type have superdecomposable pure-injectives and that tame hereditary algebras do not have such modules. It was suspected by some (conjectured in [PreBk]) that existence of a superdecomposable module would characterise wild representation type. This was shown to be completely mistaken in another paper of Puninski [Pun3] where he showed that every (countable) non-domestic string algebra has superdecomposable pure-injectives. Continuing in this vein Puninski, Puninskaya and Toffalori showed [PPT] that if G is a finite non-trivial group then the integral group ring $\mathbb{Z}[G]$ has a superdecomposable pure-injective module.

In passing we mention the question of characterising those rings over which the pure-injective hull of a flat module is flat. Although no ring-theoretic characterisation is known, Rothmaler proved [Ro] the pleasingly symmetric result that this holds exactly when the flat cover of every pure-injective module is pure-injective. A strong thread running through the model theory of modules and parallel algebraic investigations has been the computation of Krull-Gabriel dimension, a.k.a. elementary Krull dimension, of various types of ring. For some time all computations of this dimension for finite-dimensional algebras gave the value 0 (finite representation type), 2 or ∞ (undefined) (a theorem of Herzog had already excluded the value 1 for such rings). Using model-theoretic techniques, Burke and Prest [BP] and, independently, using combinatorial/algebraic techniques Schröer [Schr], gave examples showing that all finite values ≥ 2 are achieved. It is conjectured that the Krull-Gabriel dimension of a finite-dimensional algebra is finite iff the algebra is of domestic (or finite) representation type but this is still open, as is the conjecture that every non-domestic algebra has Krull-Gabriel dimension undefined. A report on somewhat earlier results on this, and on the related isolation property, can be found in [PreHoAM] and also [PreBiel].

At the end of the talk it was mentioned that the techniques of the model theory of modules apply in any locally finitely presented abelian category: in particular one has a good model theory for additive functors, for sheaves over certain ringed spaces [PRa] and for comodules [CPR]. One may even extend this to compactly generated triangulated categories [GP1], [GP2] by following the functor category approach used by Krause [Kr].

Finally, although I did not cover these in my talk, I point to the very varied recent applications of Herzog to representations of Lie algebras, to rings and modules and to additive categories, [H1], [H2], [H3], [H5], [H4] and also to results on modules over generalised Weyl algebras as exemplified by [PP].

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Almost complex manifolds as Zariski-type structure

Thomas Scanlon

For us, an almost complex manifold is a real analytic manifold M given together with an operator $J : M \to TM \otimes T^*M$ on its tangent bundle satisfying $J^2 = -1$. A morphism $f : (M, J) \to (M', J')$ between almost complex manifolds is a map of real analytic manifold $f : M \to M'$ for which $J' \circ df = df \circ J$. An almost complex submanifold of (M, J) is a submanifold $N \subseteq M$ for which the restriction of J to TN takes TN back to itself.

Note that a complex manifold is an almost complex manifold with J taken to be multiplication by i and a morphism between complex manifolds (considered in the category of almost complex manifolds) is nothing other than an analytic map.

Given a compact almost complex manifold (M, J), we consider several proposed notions of "closed" subset of M and its Cartestian powers.

$\mathcal{H}$: A set of $S \subset M^n$ of the form $f(N)$ where $f : N \to M^n$ is a map from a complex manifold to M^n is called a *holomorphic shadow.*

$\mathcal{V}$: An *almost complex subvariety* of M^n is a closed real analytic subvariety $X \subseteq M^n$ whose smooth locus is an almost complex submanifold of M^n.

$\mathcal{S}$: A *stratified almost complex submanifold* of M^n is a closed real subanalytic set $X \subseteq M^n$ given together with a nested sequence of closed subanalytic subsets $\varnothing = X_0 \subset X_1 \subset \ldots \subset X_m = X$ for which $X_i \setminus X_{i-1}$ is an almost complex submanifold of M^n for each $i \leq m$.

Remark. *A stratified almost complex submanifold need not be a submanifold!*

$\mathcal{A}$: An *almost complex shadow* in M^n is an image in M^n of a compact almost complex manifold N under a map of alost complex manifolds $f : N \to M^n$.

To each of the above classes $\mathcal{C} = \mathcal{H}$, $\mathcal{A}$, $\mathcal{V}$, or $\mathcal{S}$ one may associate a first order language $\mathcal{L}_{\mathcal{C}}$ for which the basic n-ary relation symbols correspond to the subsets of M^n in $\mathcal{C}$ and one may regard M as an $\mathcal{L}$-structure in the obvious way.

We report on a result of Liat Kessler:

Theorem 1. *M is a Zariski-type structure when the closed sets are taken to be the positive quantifier-free definable sets in $\mathcal{L}_{\mathcal{H}}$.*

We discuss some obstructions to extending this theorem to the other classes and the status of the pre-smoothness axiom.

What's new about Pfaffian sets?

Patrick Speissegger
(joint work with J.M. Lion)

Let $\mathcal{R}$ be an o-minimal expansion of the real field. A sequence $f = (f_1, \dots, f_k) : U \longrightarrow \mathbb{R}^k$ of differentiable functions on an open set $U \subseteq \mathbb{R}^n$ is a Pfaffian chain over $\mathcal{R}$ if there exist an open set $V \subseteq \mathbb{R}^{n+k}$ definable in $\mathcal{R}$ and continuous functions $g_{ij} : V \longrightarrow \mathbb{R}$ definable in $\mathcal{R}$, for $i = 1, \dots, k$ and $j = 1, \dots, n$, such that:

(i) the graph of f is a closed and connected subset of V;
(ii) for all $x \in U$ and all i, j,

$$\frac{\partial f_i}{\partial x_j}(x) = g_{ij}(x, f_1(x), \dots, f_k(x));$$

(iii) for all i, j, the function g_{ij} does not depend on the last $k - i$ variables.

We prove that if $\mathcal{R}$ admits analytic cell decomposition, then the expansion of $\mathcal{R}$ by all Pfaffian chains over $\mathcal{R}$ is model complete (relative to $\mathcal{R}$). It follows that this expansion of $\mathcal{R}$ is equal to the Pfaffian closure $\mathcal{P}(\mathcal{R})$ of $\mathcal{R}$. Moreover, from earlier joint work with C. Miller, we also obtain that if $\mathcal{R}$ is exponentially bounded, then so is $\mathcal{P}(\mathcal{R})$.

For the proof, we work with the seemingly more general notion of nested Rolle leaf over $\mathcal{R}$: let $U \subseteq \mathbb{R}^n$ be definable in $\mathcal{R}$ and open, and let $\Omega = (\omega_1, \dots, \omega_k)$ be a family of differential 1-forms on U definable in $\mathcal{R}$. A sequence $V = (V_1, \dots, V_k)$ of subsets of U is a nested Rolle leaf of Ω if:

(i) Ω is transverse;
(ii) $\omega_1 \wedge \dots \wedge \omega_i \wedge d\omega_i = 0$ for each $i = 1, \dots, k$;
(iii) V_1 is a Rolle leaf of ω_1 and for $i = 2, \dots, k$, the set V_i is a Rolle leaf of $\omega_i \restriction V_{i-1}$.

We actually prove that the expansion of $\mathcal{R}$ by all nested Rolle leaves over $\mathcal{R}$ is model complete (relative to $\mathcal{R}$). It follows that this expansion is equal to $\mathcal{P}(\mathcal{R})$, and all the above statements then follow.

Rigidity of semialgebraic groups

Katrin Tent
(joint work with L. Kramer)

We use a generalization of the definition of asymptotic cones due to van den Dries and Wilkie to prove the following results.

Theorem 1 (KT2). *If R is a real closed field, G is a semisimple R-isotropic algebraic group defined over R and $G(R)$ is equipped with a left-invariant norm-like metric, then the asymptotic cone of $G(R)$ is an affine Λ-buildings of the form $G(R^\alpha)/G(O)$ where R^α is a real closed field, $O \subseteq R^\alpha$ is a convex valuation ring and $\Lambda \cong R^{\alpha *}/O^*$ is an archimedean ordered abelian group.*

In particular, the asymptotic cone of a semisimple real Lie group $G(\mathbb{R})$ is of the form $G({}^\rho\mathbb{R}_\mu)/G(O)$ where ${}^\rho\mathbb{R}_\mu$ is Robinson's real closed valued field constructed from $\mathbb{R}$ using the ultrafilter μ used to define the asymptotic cone.

We apply this to prove:

Theorem 2 (KT2). *If R is a real closed field, G and H are semisimple R-isotropic algebraic groups defined over R and $G(R)$ and $H(R)$ are equipped with left-invariant norm-like metrics such that $f : G(R) \longrightarrow H(R)$ is a quasi-isometry (with respect to R), then G and H are isomorphic as algebraic groups. Furthermore, if $\bar{R}$ is the total competion of R, then there is an $\bar{R}$-rational isomorphism $g : G(\bar{R}) \longrightarrow H(\bar{R})$ which has R-bounded distance from f on $G(R)$.*

This generalizes results of Kleiner and Leeb [KL] on quasi-isometries between Riemannian symmetric spaces and the Margulis Conjecture.

As the asymptotic cones are defined with respect to an ultrafilter μ, Gromov asked whether there are finitely presented groups whose asymptotic cone depends on μ. If Γ is a uniform lattice in $G(\mathbb{R})$, then Γ is finitely presented and $Cone(\Gamma) = Cone(G(\mathbb{R}))$. It follows from our description of $Cone(G(\mathbb{R}))$ that $Cone_\mu(G(\mathbb{R})) \cong Cone_{\mu'}(G(\mathbb{R}))$ if and only if ${}^\rho\mathbb{R}_\mu \cong {}^\rho\mathbb{R}_{\mu'}$.

In joint work with S. Thomas and S. Shelah we show

Theorem 3 (KSTT). *The existence of ultrafilters μ, μ' with ${}^\rho\mathbb{R}_\mu \ncong {}^\rho\mathbb{R}_{\mu'}$ is equivalent to the negation of the Continuum Hypothesis (i.e., is equivalent to the statement $2^{\aleph_0} > \aleph_1$).*

Furthermore, if the Continuum Hypothesis holds (i.e., if $2^{\aleph_0} = \aleph_1$), then any finitely generated group has at most $2^{\aleph_0}$-many cones up to homeomorphism.

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Remarks on the complex exponential field

Alex Wilkie

I am interested here in the definability theory for the *complex exponential field*, that is, the complex field expanded by a function symbol for the usual expontial function. Zil'ber has conjectured that every subset of the complex numbers definable in this structure is either countable or co-countable and my results are intended to be a contribution towards understanding this conjecture.

The corresponding theory for the real case suggests that we should first look at *restricted* exponentiation and then set up a suitable valuation theory in order to investigate the situation at infinity. However, a result of Peterzil and Starchenko (see [4]), which generalises a much earlier result of Marker (see [3]) for the algebraic case, implies that if we unnaturally restrict a function so that a bounded disc (for example) becomes definable, then the reals also become definable and so we have no hope of developing a complex dimension theory for all definable sets. Thus, we have to tread carefully and my approach to the restricted case, as I will discuss in this talk,is to avoid the notion of first-order definability as such, and simply investigate a suitable pregeometry arising from existential closure. The local theory of this pregeometry has been worked out by my former student H. Braun in his thesis (see [2]) but I am more interested here in simply demonstrating the robustness of the notion. This I do by identifying it with two other pre-geometries,one coming from differential closure and the other from the usual operation of Skolem closure in the o-minimal structure obtained by expanding the real field by the restricted exponential and sine functions. At the end of the talk I give an application of this identification by deducing Schanuel's conjecture for *generic* complex numbers from Ax's version of it (see [1]) for differential fields.

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Pseudo-analytic structures and Hrushovski's construction

Boris Zil'ber

Given a class of structures **M** with a dimension notion d, we want to consider a new function f on **M**. Now, on $(\mathbf{M}, \mathrm{f})$ we can calculate a *predimension* as follows:

$$\delta(X) = \mathrm{d}(\mathrm{X} \cup \mathrm{f}(\mathrm{X})) - \mathrm{size}(\mathrm{X})$$

Consider the subclass of structures $(\mathbf{M}, \mathrm{f})$ which satisfy:

$$\delta(X) \geq 0 \text{ for any finite } X \subset \mathbf{M}.$$

The above inequality is called *Hrushovski's inequality.*

Amalgamate all such structures to get a *universal and homogeneous* structure in this class. The resulting structure $(\tilde{\mathbf{M}}, \mathrm{f})$ will have a good dimension notion and a nice geometry.

Remark. *If* $\mathbf{M}$ *is a field and we want* $\mathrm{f} = \mathrm{ex}$ *to be a group homomorphism*

$$\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$$

then the corresponding predimension must be

$$\delta(X) = \mathrm{tr.deg}(X \cup \mathrm{ex}(X)) - \mathrm{ld(X)} \geq 0$$

Hrushovski's inequality, in the case of the complex numbers with ex = exp, is equivalent to

$$\mathrm{tr.deg}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \geq n$$

assuming that $x_1, \ldots, x_n$ are linearly independent. This is *Schanuel's conjecture.*

Consider the class of fields of characteristic 0 with a function ex as before satisfying the following:

EXP1: $\mathrm{ex}(x_1 + x_2) = \mathrm{ex}(x_1) \cdot \mathrm{ex}(x_2)$

EXP2: Ker(ex) is a cyclic additive subgroup.

Consider the subclass of such structures satisfying Schanuel-Hrushovski's condition:

$$\delta(X) = \mathrm{tr.deg}(X \cup \mathrm{ex}(X)) - \mathrm{ld(X)} \geq 0.$$

Now, the amalgamation process produces a universal and homegeneous (*generic*) structure of any given infinite cardinality $\tilde{\mathbf{K}}_{\mathrm{ex}}(\lambda)$: an *algebraically closed field with pseudo-exponentiation.* This scheme can be repeated with other (classical) analytic functions (or systems of functions) F provided we know:

I. The functional equation for F.

II. The Generalised Schanuel condition for F (GSCH).

Questions.

1. *Is there a* canonical choice *of* $\tilde{\mathbf{K}}_{\mathrm{ex}}(\lambda)$ *for each* $\lambda > \aleph_0$?
2. *What is the stability status of* $\tilde{\mathbf{K}}_{\mathrm{ex}}$?
3. *Does the following hold?*

$$\mathbb{C}_{\exp} \cong \tilde{\mathbf{K}}_{\mathrm{ex}}(2^{\aleph_0}), \ \ \textit{or } \mathbb{C}_{\exp} \equiv \tilde{\mathbf{K}}_{\mathrm{ex}}$$

Compare the two theories.

We answer all above questions in [Z1]. In particular, we give reasons to conjecture the isomorphism in 3.

Raising to powers ([Z1] and [Z2]) Analogue of $(\mathbb{C}, +, \cdot, x^r{}_{r \in \mathbb{R}})$, where $x^r = \exp(r \ln x)$.

We consider a two-sorted structure $K^{\mathcal{R}} = (V, K)$, with V an R-vector space, $K = (K, +, \cdot)$ a field of characteristic 0 and a group homomorphism

$$\mathrm{ex} : V \to K^{\times}$$

satisfying the following conditions:

GSCH: $\delta(X) = \mathrm{ld_R(X) + tr.deg(ex\,X) - ld_{\mathbb{Q}}(X) + d} \geq 0$ (e.g. $d = \mathrm{tr.deg}_R$).

EC: For any *free* and *normal* system

$$L(x_1, \ldots, x_n) = a \;\&\; P(y_1, \ldots, y_n) = 0$$

where L is R-linear and P is a polynomial over $\mathbb{Q}(\mathrm{ex}\,a)$, there exists a solution satisfying:

$$y_i = \mathrm{ex}(x_i) \quad i = 1, \ldots, n.$$

Theorem 1 ([Z2]). *The theory of a generic member $\tilde{K}^R$ of the above class is near model complete and superstable.*

Theorem 2. *Assume Schanuel's conjecture. Let $R \subseteq \mathbb{R}$. Then, the theory $\mathbb{C}^R$ (i.e $V = \mathbb{C}$, $K = \mathbb{C}$, $\mathrm{ex} = \exp$) is superstable and near model complete.*

In particular, it follows that any normal free system of exponential sums equations with real powers has a solution in $\mathbb{C}$.p

Wilkie's Theorem *Let $\mathbb{R}_{\exp,\sin}$ be the expansion of the real field by exponentiation and the restricted* sin *and $S \prec \mathbb{R}_{\exp,\sin}$ its minimal elementary submodel. Let $k_1, \ldots, k_n \in \mathbb{R} + i\mathbb{R}$ be complex numbers represented in $\mathbb{R}_{\exp,\sin}$ such that $k_1, \ldots, k_n$ are independent over S in the sense of the pregeometry of o-minimal structure $\mathbb{R}_{\exp,\sin_|}$. Let R be the subfield of $\mathbb{C}$ generated by $k_1, \ldots, k_n$. Then the structure $\mathbb{C}^R$ of raising to powers satisfies the inequality*

$$\mathrm{ld_R(X) + tr.deg(exp\,X) - ld_{\mathbb{Q}}} \geq 0.$$

Moreover, it satisfies the uniform version of Schanuel's conjecture.

Theorem 3. *Let $R \subseteq \mathbb{R}$ satisfy the assumptions of Wilkie's Theorem. Then, $Th(\mathbb{C}^R)$ is superstable and near model complete. Moreover, any normal free system of exponential sums equations with powers in R has a solution in $\mathbb{C}$.*

Problems

1. Find the functional equation, GSCH and EC for the Weierstrass function $\mathfrak{p}(\omega, x)$ as a function of two variables.
2. Prove similar statements for pseudo-Weierstrass functions and the operation of raising to powers on elliptic curves.

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