Arbeitsgemeinschaft mit aktuellem Thema:
ALGEBRAIC COBORDISM
Mathematisches Forschungsinstitut Oberwolfach
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Organizers:

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Introduction:

Over the years, many different types and flavors of cohomology theories for algebraic varieties have been constructed. Theories like étale cohomology or de Rham cohomology provide algebraic versions of the topological theory of singular cohomology. The Chow ring and algebraic K_0 are other (partial) examples, more directly tied to algebraic geometry.

The partial theory K_0^{alg} was extended to a full theory with the advent of Quillen's higher algebraic K-theory. It took considerably longer for the Chow ring to be extended to motivic cohomology. In the process of doing so, Voevodsky developed his category of motives, and this construction was put in a more general setting with the development by Morel-Voevodsky of $\mathbb{A}^1$ homotopy theory. This enabled a systematic construction of cohomology theories on algebraic varieties, with algebraic K-theory and motivic cohomology being only two fundamental examples.

These two cohomology theories have in common the existence of a good theory of push-forward maps for projective morphisms. Not all cohomology theories have this structure, those that do are called *oriented*. In the Morel-Voevodsky stable homotopy category, the universal oriented theory is represented by the $\mathbb{P}^1$-spectrum MGL, an algebraic version of the classical Thom spectrum MU. The corresponding cohomology theory $MGL^{*,*}$ is called *higher algebraic cobordism*.

In an attempt to better understand the theory $MGL^{*,*}$ and strongly influenced by Quillen ideas on complex cobordism, Levine and Morel constructed a theory of *algebraic cobordism* Ω^*. This is (conjecturally) related to $MGL^{*,*}$ as the classical Chow ring CH^* is to motivic cohomology and also as the geometric cobordism of manifolds is related to the cohomology theory associated to the Thom spectrum. Like CH^*, Ω^* has a purely algebro-geometric description. In addition to giving some insight into $MGL^{*,*}$, Ω^* gives a simultaneous presentation of both CH^* and K_0, exhibiting K_0 as a deformation of CH^*. Ω^* has also been used to give conceptually simple proofs of various very general "degree formulas" first formulated by Rost: these are directly obtained by proving for Ω^* the analogue of Quillen's theorem on complex cobordism. Some of these degree formulas have been used in the study of Pfister quadrics and norm varieties, properties of which are used in the proofs of the Milnor conjecture and the Bloch-Kato conjecture.

In this workshop, we will describe aspects of the topological theory of complex cobordism which are important for algebraic cobordism (Lectures 1-3) and give the construction of Ω^* and proofs of its fundamental properties (Lectures 4-7). In lectures 8-11, we show how K_0 and CH^* are described by Ω^*, how Ω^* recovers the universal formal group law, give the proof the generalized degree formula for Ω^* and use this to proof the degree formula for the Segre class. Additional applications to Steenrod operations, further degree formulas and the use of these in the study of quadrics and other varietes is given in lectures 12 and 13. Lectures 14 and 15 concern the construction of funtorial pull-backs in algebraic cobordism. The two concluding lectures (16 and 17) give a quick sketch of the Morel-Voevodsky $\mathbb{A}^1$ stable homotopy category and describe what we know about MGL and its relation to motivic cohomology and Ω^*.

Talks:

(1) **Introduction to classical cobordism** [11][9]
This talk will give an introduction to the fundamental works on cobordism by Thom and then also by Milnor, Novikov on complex cobordism.
   (a) Definition of unoriented Ω^O, oriented cobordism Ω^{SO} and complex cobordism Ω^U rings.
   (b) Thom's results: the computation of Ω^O and of $\Omega^{SO} \otimes \mathbb{Q}$).
   (c) Thom's method using Thom spaces and Serre's theory of homotopy groups modulo a class of abelian groups.
   (d) Milnor and Novikov computation of Ω^U.
   (e) Cobordism and characteristic numbers, divisibility [9].

(2) **Quillen's work on cobordism** [10]
   (a) Cobordism as a cohomology theory
   (b) Structure as an oriented theory

(c) Chern classes and the formal group law
(d) Quillen's main theorem and the fundamental isomorphism $\Omega^U(pt) = \mathbb{L}$.
(e) Quillen's theorem in the unoriented case.

(3) **Oriented theories over a base field** k [4, 5, 6] This talk introduces the notion of oriented cohomology theory for smooth varieties over a field k. This definition is directly inspired by Quillen's ideas in (2)(b).
(a) Definition
(b) Examples: CH^*, $H^*_{\text{ét}}$, other ordinary theories, $K_0[\beta, \beta^{-1}]$, $\mathrm{MGL}^{2*,*}$, etc...
(c) The formal group law of an oriented theory

(4) **Algebraic cobordism: basic properties** This is a summary of the basic properties and structures of algebraic cobordism [6, Introduction].
(a) Algebraic cobordism as the universal oriented theory
(b) Extra structure: localization sequence
(c) $\Omega^*(k) = \mathbb{L}^*$
(d) Conner-Floyd and $\Omega^* \otimes \mathbb{Z} = \mathrm{CH}^*$.
(e) The analogue of Quillen's theorem: degrees and generalized degree formulas. Examples.

(5) **The construction of algebraic cobordism**
This talk gives the construction of algebraic cobordism as a Borel-Moore functor Ω_*. Cf [6] Part 1 §1 and §2

(6) **Basic properties: localization**
Cf [6] Part 2 §6. This lecture proves the fundamental localization sequence for a closed subscheme $i : Z \to X$ with open complement $j : U \to X$

$$\Omega_*(Z) \xrightarrow{i_*} \Omega_*(X) \xrightarrow{j^*} \Omega_*(U) \to 0.$$

As a preliminary step, the class of a normal crossing divisor is constructed.

(7) **Basic properties: homotopy invariance and the projective bundle formula**
Cf [6] Part 2 §7 and [6] Part 2 §8 . It should be nice to mention also here the extended homotopy property [6] Part 2 §9.

(8) $\Omega^* \otimes \mathbb{Z}[\beta, \beta^{-1}] = K_0[\beta, \beta^{-1}]$

Cf [6] Part 3 §11

(a) results on projective bundles [6, Section 11.1]
(b) universality of $K_0[\beta, \beta^{-1}]$ [6, Section 11.2]
(c) the indentification $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}[\beta, \beta^{-1}] \cong K_0[\beta, \beta^{-1}]$ [6, Corollary 11.9]
(d) Grothendieck-Riemann-Roch theorem.

(9) $\Omega^*(k)$ **and the Lazard ring**

The main theorem: $\Omega_*(k) = \mathbb{L}_*$. Cf [6] Part 3 §12 The injectivity is rather easy, either by relying on topology or by using characteristic numbers.

The surjectivity reduces to showing that the additive theory $\Omega_* \otimes_{\mathbb{L}} \mathbb{Z}$ on $X = \operatorname{Spec} k$ is just $\mathbb{Z}$. One starts by using the computations of the classes of projective space bundles from 8(a) to reduce to a birational statement, and then using the generic projection of a smooth projective variety to reduce to the case of hypersurfaces, then finally deforming to a union of hyperplanes.

(10) **Degree formulas**

Mainly a discussion of [6] Part 3 §13.3 and §13.4. The main results are the "generalized degree formula" in algebraic cobordism and its application to Rost's degree formula for the characteristic numbers constructed from the Newton class.

(11) Ω_* and CH_*

Cf [6] Part 3 §14. This lecture shows that CH_* is the universal additive theory, thus identifying CH_* with $\Omega^* \otimes \mathbb{Z}$. Additional computations and a discussion of the topological filtration on Ω_* are discussed.

(a) $\Omega^* \otimes \mathbb{Z} = \mathrm{CH}^*$ [6, Section 14.1]
(b) The topological filtration [6, Section 14.2]
(c) Computations [6, Section 14.3]

(12) **Steenrod operations and other degree formulas**

Cf. [3], see also [1], [8].

Use the basic properties of algebraic cobordism and the method of twisting an oriented theory following [3] to give a construction of Steenrod operations on the mod p Chow groups (constructed by different methods by Voevodsky [12] and by Brosnan [1]).

The same ideas give a large number of degree fomulas for certain mod p characteristic classes, redoing a construction due to Merkurjev [8].

(13) **Some applications**
This lecture discusses a number of interesting applications of various degree formulas given in [8].
(a) Rigidity. The main consequence of the degree formula in this context gives strong restrictions on varieties related by a prime to p correspondence [8, Theorem 7.2].
(b) Consequences. the rigidity results have interesting consequences for Brauer-Severi varieties, hyupersurfaces and quadratic forms, including new proofs of results of Hoffmann, Izhboldin and Karpenko ([8, Section 7]).

(14) **Construction of pull-backs in algebraic cobordism, part 1** The main task in [7] is to define functorial pull-back maps on Ω_*. In principle, this is done following Fulton's method of deforming to the normal bundle, which reduces to the case of intersection with a Cartier divisor. For algebraic cobordism, built out of *smooth* varieties, the construction needs to go further. Rather than requiring a smooth intersection, we use the classes of normal crossing divisors constructed in the proof of localization. This allows one to intersect a modified algebraic cobordism group, designed so that all intersections with the given divisor are normal crossing. Finally one proves a moving lemma, which shows that the modified cobordism group is the same as the usual one.
(a) Refined algebraic cobordism [7, Section 2]
(b) Intersection with divisors [7, Section 3,4]

(15) **Construction of pull-backs in algebraic cobordism, part 2**
(a) The moving lemma [7, Section 5]
(b) Pull-back for l.c.i. morphisms [7, Section 6]

(16) **$\mathbb{A}^1$-homotopy approach to algebraic cobordism**
Part one: an overview of $\mathbb{A}^1$-homotopy theory. [2]. This lecture gives a quick overview of the Morel-Voevodsky $\mathbb{A}^1$-stable homotopy category of $\mathbb{P}^1$-spectra over k, discussing the construction of the categories of spaces over k, S^1-spectra over k

and $\mathbb{P}^1$-spectra over k, as well as the corresponding homotopy categories.

In each of these sections, concentrate on giving just the definitions of the relevant categories, and some key examples.

(a) Homotopy theory of spaces over k [2, section 2, 3]
(b) S^1-spectra over k [2, section 4]
(c) $\mathbb{P}^1$-spectra over k [2, section 5]

(17) **$\mathbb{A}^1$-homotopy approach to algebraic cobordism**
Part two: the Thom spectrum
(a) Towards the identification $\Omega^d(X) = MGl^{2d,d}(X)$
(b) Thom spectrum, K-theory spectrum and motivic cohomology spectrum. Atiyah-Hirzebruch spectral sequence.

References

[1] P. Brosnan, *Steenrod operations in Chow theory*, Trans. Amer. Math. Soc. **355** (2003), 1869-1903.

[2] F. Morel, An introduction to $\mathbb{A}^1$ homotopy theory, ICTP Trieste July 2002. http://www.ictp.trieste.it/~pub_off/lectures/vol15.html

[3] M. Levine, Steenrod operations, degree formulas and algebraic cobordism. http://www.math.neu.edu/~levine/publ/Publ.html

[4] M. Levine et F. Morel, *Cobordisme algébrique I*, Note aux C.R. Acad. Sci. Paris, **332** Série I, p. 723-728, 2001.

[5] M. Levine et F. Morel, *Cobordisme algébrique II*, Note aux C.R. Acad. Sci. Paris, **332** Série I, p. 815-820, 2001.

[6] M. Levine and F. Morel, Algebraic Cobordism I. revised version. http://www.math.neu.edu/~levine/publ/Publ.html

[7] M. Levine, Algebraic Cobordism II. http://www.math.neu.edu/~levine/publ/Publ.html

[8] A. Merkurjev, Alexander, Steenrod operations and degree formulas. J. Reine Angew. Math. 565 (2003), 13–26.

[9] J. Milnor and J. Stasheff, Characteristic classes, Princeton University Press.

[10] D. G. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971) 29–56.

[11] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954) 17–86.

[12] V. Voevodsky, *On 2-torsion in motivic cohomology*. http://www.math.uiuc.edu/K-theory/0502/

Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

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by February 10 at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. …
Second choice: talk no. …
Third choice: talk no. …

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.