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Arbeitsgemeinschaft mit aktuellem Thema: Higher Torsion Invariants in Differential Topology and Algebraic K-Theory

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Abstract. The purpose of this Arbeitsgemeinschaft was to study and—as far as possible—compare three generalisations of the classical Franz-Reidemeister torsion to families $E \to B$ of manifolds. Dwyer, Weiss and Williams construct higher torsion as a byproduct of a family index theorem for the fibrewise A-theory Euler-class. Igusa and Klein use generalised fibrewise Morse functions to construct a classifying map from B to a Whitehead space, and to pull-back universal classes from there. Bismut and Lott prove an analytic family index theorem for flat vector bundles and obtain the higher analytic torsion as a transgression.

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Introduction by the Organisers

The classical Franz-Reidemeister torsion and its cousins, the Whitehead torsion and Ray-Singer analytic torsion, are topological invariants of manifolds with local coefficient systems (or flat vector bundles) that can distinguish homotopy equivalent spaces that are not homeomorphic. The purpose of this Arbeitsgemeinschaft was to learn about several natural generalisations of these classical invariants to families of manifolds.

Regard a family $p\colon E \to B$ of compact manifolds M, equipped with a flat vector bundle $F \to M$. Then the fibrewise cohomology groups $H^\bullet(E/B;F)$ form flat vector bundles over the base B. The starting point for our investigations are analogues of the Atiyah-Singer family index theorem that relate F to $H^\bullet(E/B;F)$. To a flat vector bundle $F \to M$, one associates Kamber-Tondeur characteristic classes $c_\bullet(F)$ in $H^{\mathrm{odd}}(M;\mathbb{R})$, which vanish if F carries a parallel metric. By

Bismut-Lott [1], one has

$$\sum_i (-1)^i c_\bullet\big(H^i(E/B;F)\big) = \int_{E/B} e(TM)\, c_\bullet(F) \quad \in H^*(B;\mathbb{R})\ ,$$

where $e(TM)$ is the Euler class of the vertical tangent bundle, and the right hand side is the Becker-Gottlieb transfer in de Rham cohomology. If one specifies some additional geometric data, then all classes above are naturally represented by specific differential forms. On the level of differential forms, the equation above only holds up a correction term $d\mathcal{T}$. Here $\mathcal{T}$ is the higher analytic torsion, which depends naturally on the fibration and the geometric data. If both $H^\bullet(E/B;F)$ and F admit parallel metrics, then $\mathcal{T}$ gives rise to a secondary characteristic class $\mathcal{T}(E/B;F) \in H^{\mathrm{even},\geq 2}(B;\mathbb{R})$.

Dwyer-Weiss-Williams [2] construct Reidemeister torsion for a smooth fiber bundle $p\colon E \to B$ as a byproduct of a family index theory. If p is any fiber bundle with fibers compact topological manifolds and base a CW complex, then the family index theory states that $\chi(p)$, the A-theory Euler characteristic of p is determined by the A-theory Euler class of $\tau_{fib}(p)$, the tangent bundle along the fiber. Here A-theory is algebraic K-theory of spaces in the sense of Waldhausen. More precisely, by applying fiberwise Poincare duality, and then an assembly map to the A-theory Euler class, one gets the A-theory Euler characteristic. If p is a smooth bundle, then one gets a stronger smooth index theorem where the A-theory Euler class is replaced by the Becker-Euler class, which lives in the (twisted) stable cohomotopy of E. When B is a point this result is equivalent to the classical Poincare-Hopf theorem.

The third approach is due to Igusa-Klein [3], and is somewhat different in nature. Here, one regards a generalised fibrewise Morse function on $M \to B$. Together with a flat vector bundle $F \to M$, this gives rise to a classifying map from B to a Whitehead space, and the higher Franz-Reidemeister torsion is the pullback of a universal class on the Whitehead space.

There are conjectural relations between all three definitions of higher torsion. In a special case, Igusa has characterized higher Franz-Reidemeister torsion axiomatically; checking these axioms for either of the other higher torsions would prove equality. For some bundles, equality of higher Franz-Reidemeister torsion and higher analytic torsion can be shown analytically using the Witten deformation. Finally, one expects that higher Franz-Reidemeister torsion can be recovered from Dwyer-Weiss-Williams torsion.

It turns out that higher torsion invariants are somewhat finer than classical FR torsion, since they detect higher homotopy classes of the diffeomorphism group of high-dimensional manifolds that vanish under the forgetful map to the homeomorphism group. In particular, these invariants distinguish differentiable structures on a given topological fibre bundle $M \to B$, where one may even fix differentiable structures on M, B and the typical fibre. There are also applications of higher torsions to problems in graph theory and moduli spaces of compact surfaces. Some of these were sketched throughout this Arbeitsgemeinschaft.

The talks were grouped as follows.

(1) The first talk gave a short introduction to classical torsion invariants.
(2) In talks 2–7, we discussed the Dwyer-Weiss-Williams homotopy theoretical approach.
(3) Parametrized Morse theory, Kamber-Tondeur classes and Igusa-Klein torsion were discussed in talks 8–16, and some applications were given.
(4) Finally, based on talks 10 and 11, we introduced analytic torsion in the talks 17–19.

The meeting took place from April 2nd till April 8th 2006 and was organized by Sebastian Goette (Regensburg), Kiyoshi Igusa (Brandeis) and Bruce Williams (Notre Dame). It was attended by 43 participants, coming mainly from Europa and the USA.

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Arbeitsgemeinschaft mit aktuellem Thema: Higher Torsion Invariants in Differential Topology and Algebraic K-Theory

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Abstracts

Classical torsion invariants

John Francis

Torsion invariants distinguish the homotopy category of spaces and the simple homotopy category of spaces. The higher torsion invariants of Igusa-Klein, Bismut-Lott, and Dwyer-Weiss-Williams measure the difference between the higher homotopy groups of these categories. Classical torsion describes the difference on π_0, that is, at the level of components or isomorphism types. The purpose of this talk was to introduce, define, and briefly describe the basic properties of the classical Whitehead, Franz-Reidemeister, and analytic torsion invariants.

Whitehead torsion arose as a variant of the algebraic K-theory of rings. Whitehead defined it in order to describe a generic homotopy equivalence between polyhedra. More precisely, a simple homotopy equivalence may be expressed as a composition of elementary contractions and expansions. Not every homotopy equivalence $f : X \to Y$ is homotopic to a simple homotopy equivalence; the Whitehead torsion is precisely the obstruction to homotoping an arbitrary homotopy equivalence into a simple homotopy equivalence. Since f is a homology equivalence, the cone on f is an acyclic chain complex. Whitehead torsion is a secondary invariant of an acyclic chain complex encoding the way in which it is acyclic: the matrix A giving an isomorphism between the even and odd degrees of the chain complex represents an element in the algebraic K-group $K_1(\mathbb{Z}[\pi_1 X])$ well defined up to a quotient. The resulting obstruction group is called the Whitehead group. See [1].

If a $\mathbb{Z}[\pi_1 X]$-chain complex is not acyclic, one may be able to make it acyclic by base change, that is, tensoring with an orthogonal representation of the fundamental group of X. Geometrically, this can be thought of as taking cohomology with coefficients in a flat vector bundle or locally constant sheaf. The resulting group of obstructions then becomes isomorphic to $\mathbb{R}$, with the isomorphism given by taking the determinant of the matrix A. Franz-Reidemeister torsion is obtained this way. Franz and Reidemeister famously applied this invariant to distinguish lens spaces which are homotopic but not combinatorially, or simply homotopy, equivalent. (Note: the homeomorphism invariance of simply homotopy type is far from obvious. It was only fully resolved by Chapman in the 1970s.)

Ray and Singer defined a similar invariant to the Franz-Reidemeister torsion, substituting the de Rham complex with coefficients in a flat bundle for simplicial chain complex. They defined the analytic torsion in terms of the corresponding secondary data, the eigenvalues of the Laplacian, as encoded by the zeta function. The Cheeger-Muller theorem states that analytic torsion is the same as Franz-Reidemeister torsion.

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Waldhausen $K-$Theory $A(X)$

Dale Husemöller

Since Quillen, $K-$theory is a homotopy type K and $K-$groups $K_i = \pi_i(K)$. Quillen introduced two approaches for making such a homotopy type:

(1) the plus contruction $X \xrightarrow{f} X_N^+$, where f is acyclic and $N = ker(\pi_1(f))$ and the $K-$theory of a ring is given by $BGL(R)^+ \times K_0(R) = K(R)$.

Waldhausen $K-$theory $A(X)$ of a space X can be described as

$$B(\lim_{q,n\to\infty} \mathrm{Aut}(\bigvee^n S^q))^+ = A(*).$$

This gives the split fibre sequence

$$\Omega^\infty S^\infty \cong BG \cong B\Sigma_\infty{}^+ \longrightarrow A(*) \longrightarrow K(\mathbb{Z}),$$

where the first map is the inclusion and the second H_*-homology of the homotopy equivalence.

(2) $cw-$category construction and nerve:
A $cw-$category is a triple $\mathcal{C}, c\mathcal{C}, w\mathcal{C}$, where $c\mathcal{C}$ is the subcategory of cofibrations in $\mathcal{C}$ and $w\mathcal{C}$ of weak equivalences. Waldhausen introduced a simplicial category $S_\bullet\mathcal{C}$, which is again a $cw-$category of sequences of cofibrations together with choices of the cofibres.

Definition 1. $\Omega\,|wS_\bullet\mathcal{C}| = K(\mathcal{C})$
Here $|\quad|$ denotes the geometric realization of topological simplicial space of nerves of $wS_n\mathcal{C}$, the simplicial subcategory of weak equivalences in $S_\bullet\mathcal{C}$.
$A(X) = K(R(X))$,
where $R(X)$ is a suitable $cw-$category of retractive spaces $r : Y \rightleftharpoons X : s$ over X.

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Homotopy Limits as spaces of sections I

Vladislav Chernysh, Ingo Schröder

1. Preliminaries

Let $\mathcal{C}$ be a small category, $\mathcal{S}$ the category of compactly generated weak Hausdorff spaces, and $\mathcal{S}^{\mathcal{C}}$ the functor category. To the small category $\mathcal{C}$ we can associate an underlying simplicial set whose n-simplices are the sequences

$$u = (C_0 \overset{a_1}{\leftarrow} \dots \overset{a_n}{\leftarrow} C_n) \in \mathcal{C}.$$

This simplicial set's geometric realisation shall be denoted by $|\mathcal{C}| \in \mathcal{S}$.

For any object $C \in \mathcal{C}$ form the over category $\mathcal{C}/C$ whose objects are the morphisms $(C \leftarrow C_0) \in \mathcal{C}$ and whose morphisms are the appropriate commutative triangles. $\mathcal{C}/C$ can be considered as a simplicial set whose n-simplices are the sequences

$$(C \overset{a}{\leftarrow} C_0 \overset{a_1}{\leftarrow} \dots \overset{a_n}{\leftarrow} C_n).$$

A morphism $f : C \to D$ in $\mathcal{C}$ induces a functor $\mathcal{C}/C \to \mathcal{C}/D$ and hence a map $\mathcal{C}/f : |\mathcal{C}/C| \to |\mathcal{C}/D|$. Combining this for all C, we obtain a functor $|\mathcal{C}/?| : \mathcal{C} \to \mathcal{S}$.

Now let W, $X : \mathcal{C} \to \mathcal{S}$ be two functors. Then the function space $\hom(W, X) \in \mathcal{S}$ is defined as the geometric realisation of the simplicial set whose n-simplices are the morphisms

$$\Delta[n] \times W \to X \in \mathcal{S}^{\mathcal{C}}.$$

2. Definition of the homotopy limit and colimit

Let $F : \mathcal{C} \to \mathcal{S}$ be a functor. Then, following [2], the homotopy limit $\operatorname{holim} F$ of F is defined as the equalizer of the maps

$$\prod_{C \in \mathcal{C}} \hom(|\mathcal{C}/C|, FC) \overset{a,\, b}{\rightrightarrows} \prod_{C \xrightarrow{f} D} \hom(|\mathcal{C}/C|, FD),$$

where a and b are induced by

$$\hom(|\mathcal{C}/C|, FC) \overset{Ff}{\to} \hom(|\mathcal{C}/C|, FD),$$

$$\hom(|\mathcal{C}/D|, FD) \overset{\mathcal{C}/f}{\to} \hom(|\mathcal{C}/C|, FD).$$

In other words, the homotopy colimit is the space of natural transformations from the functor $C \mapsto |\mathcal{C}/C|$ to F with the induced topology. Such a natural transformation is called a characteristic for F.

$\operatorname{holim} F$ is natural in F and $\mathcal{C}$: A natural transformation $f : F \to F'$ induces a map $\operatorname{holim} f : \operatorname{holim} F \to \operatorname{holim} F'$, and a functor $g : \mathcal{D} \to \mathcal{C}$ between small categories induces a functor $g^* : \mathcal{S}^{\mathcal{C}} \to \mathcal{S}^{\mathcal{D}}$ between the functor categories and thus a natural map $\operatorname{holim} g : \operatorname{holim} F \to \operatorname{holim} g^* F$.

Similarly, the homotopy colimit $\operatorname{hocolim} F$ is the geometric realisation of the simplicial space

$$[n] \mapsto \coprod_{u:[n]\to\mathcal{C}} Fu(0).$$

The homotopy colimit of F comes with a canonical map to $|\mathcal{C}|$, because $|\mathcal{C}|$ is the homotopy colimit of the terminal functor from $\mathcal{C}$ to $\mathcal{S}$. An element of $\operatorname{holim} F$ induces a map $\operatorname{hocolim}|\mathcal{C}/?| \to \operatorname{hocolim} F$. The composition $\operatorname{hocolim}|\mathcal{C}/?| \to \operatorname{hocolim} F \to |\mathcal{C}|$ is a homotopy equivalence since the fibres are contractible. Thus, we have a canonical map from $\operatorname{holim} F$ to $\operatorname{map}_{|\mathcal{C}|}(\operatorname{hocolim}|\mathcal{C}/?|, \operatorname{hocolim} F)$. But the homotopy equivalence $\operatorname{hocolim}|\mathcal{C}/?| \to |\mathcal{C}|$ induces a homotopy equivalence

$$\operatorname{map}_{|\mathcal{C}|}(|\mathcal{C}|, \operatorname{hocolim} F) \to \operatorname{map}_{|\mathcal{C}|}(\operatorname{hocolim}|\mathcal{C}/?|, \operatorname{hocolim} F).$$

Thus, we obtain a canonical map from $\operatorname{holim} F$ to $\operatorname{map}_{|\mathcal{C}|}(|\mathcal{C}|, \operatorname{hocolim} F)$. By a theorem of Dwyer [3], this map becomes a weak homotopy equivalence if F takes all maps to weak homotopy eqivalences.

Thus, in these circumstances, we consider elements of $\operatorname{holim} F$ as sections of the fibration associated to the projection $\operatorname{hocolim} F \to |\mathcal{C}|$. The homotopy colimit of F is homotopy equivalent to the total space of that fibration.

3. Example: Euler Characteristic

Our main example will be the Euler characteristic of a bundle. Let B be a space, say the geometric realization of a simplicial set $\mathcal{B}$. Let $p : E \to B$ be a fibration and $\mathcal{C}$ a full subcategory of the category of spaces and homotopy equivalences such that for any simplex $x \in \mathcal{B}$, the pullback E_x of X under the characteristic map $\Delta[|x|] \to B$ belongs to C. This gives us an obvious functor from the category $Simp(\mathcal{B})$ of simplices of $\mathcal{B}$ to $\mathcal{C}$.

Let $F : \mathcal{C} \to \mathcal{S}$ be a functor taking all maps to homotopy equivalences. Then we define $F_B(E) \to B$ as the fibration associated with the composite projection $\operatorname{hocolim}_x F(E_x) \to |Simp(\mathcal{B})| \to B$, where x runs through the simplices of $\mathcal{B}$. The homotopy equivalence $|Simp(\mathcal{B})| \to B$ is induced by Kan's last vertex map [1].

If $F : \mathcal{C} \to \mathcal{S}$ comes with a characteristic χ, that gives us an element $\chi(p) \in \operatorname{holim}_x F(E_x)$ which in turn gives, up to homotopy, a section of $F_B(E) \to B$. $\chi(p)$ is called the Euler characteristic of p.

4. The case factoring over Cat

Let F be a functor from a small category $\mathcal{C}$ to $\mathcal{S}$ which factors over the category Cat of small categories. Let $\bar{F} : \mathcal{C} \to Cat$ be the corresponding functor: $F(C) = |\bar{F}(C)|$.

In this case there is an easy way to construct characteristics. We can determine a point in the homotopy limit of F by a rule selecting for each object $C \in \mathcal{C}$ an object $\chi(C) \in \bar{F}(C)$ and for any morphism $\alpha : C_1 \to C_2$ a morphism $\chi(\alpha) : \bar{F}(\alpha)(\chi(C_1)) \to \chi(C_2)$, such that $\chi(\alpha)$ is an identity morphism whenever α is, and that a 1-cocycle condition holds, i.e. $\chi(\alpha\beta) = \chi(\alpha)\bar{F}(\alpha)(\chi(\beta))$ when α, β are composable.

Then for a fixed $D \in \mathcal{C}$ the rule taking $f : C \to D$ to $\bar{F}(f)(\chi(C))$ is a functor from $\mathcal{C}/D$ to $\bar{F}(D)$, inducing a natural transformation $|\mathcal{C}/?| \to F$.

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Homotopy limits as spaces of sections and construction of $\chi(p)$, II.

Julia Weber

The aim of this talk is the construction of the A-theoretic Euler characteristic. We also describe its relation to the usual Euler characteristic and to Wall's finiteness obstruction. Finally, we study its image under linearization twisted by a representation.

1. The A-theoretic characteristic

Given any Waldhausen category $\mathcal{D}$ (category with cofibrations and weak equivalences, cw-category), one can construct an infinite loop space $K(\mathcal{D}) := \Omega|wS.\mathcal{D}|$ called the K-theory space of $\mathcal{D}$ [2]. There is also a map $|w\mathcal{D}| \to K(\mathcal{D})$ which is reminiscent of group completion.

Using the Waldhausen category of finitely dominated retractive spaces, we define A-theory.

Definition 1. Given a topological space X, we define

$$A(X) := K(\mathcal{R}^{fd}(X))$$

to be the *A-theory* of X. Here $\mathcal{R}^{fd}(X)$ is the *category of homotopy finitely dominated retractive spaces* over X.

Retractive spaces over X consist of a space Y, a map $X \xrightarrow{s} Y$ and a map $Y \xrightarrow{r} X$ such that $rs = \mathrm{id}_X$ and s is a closed embedding having the homotopy extension property. Morphisms $Y \xrightarrow{g} Y'$ are maps over and under X, i.e., such that $gs = s'$ and $r'g = r$. Taking cofibrations to be closed embeddings having the homotopy extension property and weak equivalences to be homotopy equivalences, the category of retractive spaces over X becomes a Waldhausen category.

A retractive space Y over X is called *homotopy finitely dominated* if there exist retractive spaces Y', W and Z over X which fit into a diagram

$$\begin{tikzcd} & Z \arrow[d, "\sim"] & \\ Y' \arrow[r] \arrow[rr, bend right, "\sim"'] & W \arrow[r] & Y \end{tikzcd}$$

and where Z is a CW-space relative to X with finitely many cells.

The construction of A-theory is functorial: A continuous map $f\colon X_1 \to X_2$ induces a functor $f_*\colon \mathcal{R}^{fd}(X_1) \to \mathcal{R}^{fd}(X_2)$.

On objects, this functor is given by the pushout construction. Given a retractive space Y over X_1, with inclusion map $s\colon X_1 \to Y$ and retraction map $r\colon Y \to X_1$, define the space f_*Y to be the pushout of f along the cofibration s. This comes with a cofibration $s'\colon X_2 \to f_*Y$. By the pushout property, the maps $fr\colon Y \to X_2$ and $\mathrm{id}\colon X_2 \to X_2$ induce a retraction map $r'\colon f_*Y \to X_2$.

This functor in turn induces a map of infinite loop spaces $A(X_1) \to A(X_2)$.

We now restrict ourselves to the category $\mathcal{C}$ of finitely dominated spaces X, i.e., spaces X for which there exists a compact CW complex Z, an inclusion $i\colon X \to Z$ and a retraction $r\colon Z \to X$ such that rs is homotopic to id_X.

We are ready to define the characteristic χ for the A-theory functor on $\mathcal{C}$. It suffices to construct a characteristic χ for the functor $X \mapsto |w\mathcal{R}^{fd}(X)|$, which we can then compose with the map $|w\mathcal{R}^{fd}(X)| \to A(X)$ to obtain the desired A-theoretic characteristic.

We formulate the following rule: To a space $Y \in \mathcal{C}$, we assign $Y^! \in \mathcal{R}^{fd}(Y)$, the retractive space over Y given by $Y^! := Y \amalg Y = S^0 \times Y$ with restriction map $r := \mathrm{pr}_Y$ and inclusion map $s\colon Y \to S^0 \times Y, y \mapsto (1, y)$.

For any morphism $e\colon X \to Y$ in $\mathcal{C}$, we have an induced object $e_*(X^!) \in \mathcal{R}^{fd}(Y)$ given by $\{-1\} \times X \amalg \{1\} \times Y$. We define a morphism $e^!\colon e_*(X^!) \to Y^!$ by $e \amalg \mathrm{id}_Y$. The cocycle condition $(ef)^! = e^! e_*(f^!)$ can be verified. As seen in the last talk [1, Example I.1.3], this rule yields a characteristic for the functor $X \mapsto |w\mathcal{R}^{fd}(Y)|$.

We now define $\mathcal{C}_h$ to be the category of finitely dominated spaces with homotopy equivalences as morphisms, and we consider the functor $A|_{\mathcal{C}_h}$.

Definition 2. Let $E \xrightarrow{p} B$ be a fibration with finitely dominated fibers, where B is a CW-complex. We define the *parametrized A-theory Euler characteristic of* p to be the element $\chi(p) \in \mathrm{holim}_{x \in simp(B)} A(E_x)$.

There is a projection $\mathrm{hocolim}_{x \in simp(B)} A(E_x) \to B$ which is a quasi-fibration, and we define $A_B(E) \to B$ to be the associated fibration. By [1, Remark 1.5] we can consider $\chi(p)$ as a section

$$\begin{tikzcd} A_B(E) \arrow[d, shift left] \\ B \arrow[u, bend left, "\chi(p)"] \end{tikzcd}$$

of the fibration $A_B(E) \to B$. This fibration is obtained essentially by applying the functor A to the fibers of E.

2. Special cases

2.1. Retrieving the classical Euler characteristic and Wall's finiteness obstruction. We consider the case where the base space B is a point, i.e., we have a fibration $E \to E \to *$. Then $\chi(p)$ is the map from a point to $A(E)$ given by the retractive space $E \amalg E$ over E in $\mathcal{R}^{fd}(E)$. The linearization map $L\colon A(E) \to K(\mathbb{Z}\pi_1 E)$ is 2-connected, so it induces an isomorphism on π_0. The linearization map is induced by a map of Waldhausen categories which maps the retractive space $E \amalg E$ to $C_*(\tilde{E})$, the singular chain complex of the universal covering space of E. Algebraically, the fact that E is homotopy finitely dominated corresponds to being a direct summand of a finitely generated free chain complex, so projective. Hence the chain complex $C_*(\tilde{E})$ is chain homotopy equivalent to a chain complex of projective modules.

We have $\widetilde{K_0}(\mathbb{Z}\pi_1 E) := \operatorname{coker}\big(K_0(\mathbb{Z}) \to K_0(\mathbb{Z}\pi_1 E)\big)$. The inclusion splits, and we have

$$\begin{array}{rcl} K_0(\mathbb{Z}\pi_1 E) & \cong & \widetilde{K_0}(\mathbb{Z}\pi_1 E) \oplus K_0(\mathbb{Z}) \\ {[C_*(\tilde{E})]} & \mapsto & \big(\tilde{o}(E), \chi(E)\big). \end{array}$$

Here $\chi(E)$ denotes the classical Euler characteristic of E, and $\tilde{o}(E)$ is Wall's finiteness obstruction. The space E is homotopic to a finite CW-complex if and only if $\tilde{o}(E) = 0$. Namely, both statements are equivalent to $C_*(\tilde{E})$ being a finitely generated free $\mathbb{Z}\pi_1 E$-chain complex.

2.2. Retrieving the Whitehead torsion. Let X be a connected finite CW-complex. We consider the universal fibration with fiber homotopy equivalent to X, the fibration $X \to EG(X) \xrightarrow{p} BG(X)$, where $G(X)$ stands for the topological monoid of self-homotopy equivalences of X.

The Whitehead torsion τ can be seen as a crossed homomorphism from $\pi_0 G(X)$ to the Whitehead group $\mathrm{Wh}(\pi_1 X)$, a map

$$\begin{array}{rcl} \tau\colon \pi_0 G(X) & \to & \mathrm{Wh}(\pi_1 X) := K_1(\mathbb{Z}\pi_1 X)/\{\pm g | g \in \pi_1 X\} = \pi_1(\mathrm{Wh}(X)) \\ {[f]} & \mapsto & \tau(f) \end{array}$$

satisfying the property that $\tau(gf) = \tau(g) + g_*\tau(f)$. (Here $\mathrm{Wh}(X)$ is the Whitehead space of X, it has the property that $\pi_1(\mathrm{Wh}(X)) = \mathrm{Wh}(\pi_1 X)$.)

Note that for any group G acting on an abelian group D, the crossed homomorphisms from G to D are in one-to-one correspondence with sections of the fibration $D \rtimes G \xrightarrow{\mathrm{pr}_G} G$.

In our case, the characteristic $\chi(p)$ can be thought of as a section of the fibration $EG(X) \times_{G(X)} A(X) \to BG(X)$. We have a $G(X)$-equivariant map $A(X) \to \mathrm{Wh}(X)$ which induces a map $EG(X) \times_{G(X)} A(X) \to EG(X) \times_{G(X)} \mathrm{Wh}(X)$. Composing the above section with this map and taking π_1 of this composition we obtain a map

$$\pi_0 G(X) = \pi_1\big(BG(X)\big) \to \pi_1\big(EG(X) \times_{G(X)} \mathrm{Wh}(X)\big) = \pi_0 G(X) \ltimes \pi_1(\mathrm{Wh}(X))$$

which corresponds to a crossed homomorphism from $\pi_0 G(X)$ to $\mathrm{Wh}(\pi_1 X)$. This is the Whitehead torsion.

3. Linearized characteristics

Let R be a regular ring, and let P be a finitely generated projective module over R. We set $\pi := \pi_1(E)$. Let $\rho\colon \pi \to \mathrm{Aut}_R(P)$ be a representation. This representation corresponds to a bundle V over E of finitely generated projective R-modules isomorphic to P.

We define the map $\chi_\rho(p)\colon B \to K(R)$ to be the following composition:

$$\begin{tikzcd}
B \arrow[rrr, "\chi_\rho(p)"] \arrow[dr, "\chi(p)"] & & & K(R) \\
& A_B(E) \arrow[r] & A(E) \arrow[ur, "\lambda_V"] \arrow[r, "L"] & K(\mathbb{Z}\pi) \arrow[u, "\text{ind. by } -\otimes_{\mathbb{Z}\pi} P"']
\end{tikzcd}$$

Proposition 3. [1, Proposition 6.7] *If R is a regular ring and the base space B is connected, then*

$$\chi_\rho(p) = \sum (-1)^i [\iota \circ k(i)] \in [B, K(R)].$$

Here $k(i)$ classifies the bundle on B with fiber $H_i(p^{-1}(b); V)$, i.e.,

$$B \xrightarrow{k(i)} |\operatorname{iso}(\mathcal{NP}_R)| \xrightarrow{\iota} K(\mathcal{NP}_R) \simeq K(R),$$

where $\mathcal{NP}_R$ stands for the category of nearly projective modules over R, those which have a finite projective resolution.

So we have a description of $\chi_\rho(p)$ in terms of $\pi_1(B)$ acting on the homology of the fibers of p with coefficients in P.

Sketch of proof. We have the following commutative diagram:

$$\begin{tikzcd}
|w\,\mathrm{ch}(\mathcal{NP}_R)| \arrow[r, "H_*"] \arrow[d, "\iota"] & |w\,\mathrm{tch}(\mathcal{NP}_R)| \arrow[r, "\sim"] \arrow[d, "\iota"] & \Pi'_{i\in\mathbb{Z}} |w\mathcal{NP}_R| \arrow[d, "\iota"] \\
K(\mathrm{ch}(\mathcal{NP}_R)) = K(R) \arrow[r, equal] & K(R) & \Pi'_{i\in\mathbb{Z}} K(R) \arrow[l, "\sum(-1)^i"']
\end{tikzcd}$$

Here tch stands for trivial chain complexes, these that have only trivial differential. The restricted product Π' of pointed spaces consists of those points in the honest product where only finitely many factors are nontrivial.

Our definition of $\chi_\rho(p)$ can be seen as the composition $B \to |w\mathcal{NP}_R| \xrightarrow{\iota} K(R)$, where the first map classifies the bundle $C_*(p^{-1}(b); V)$. Here C_* stands for the singular chain complex.

The above commutative diagram shows that

$$\iota\big(C_*(p^{-1}(b);V)\big) = \sum(-1)^i\iota\big(H_i\big(C_*(p^{-1}(b);V)\big)\big) = \sum(-1)^i\iota\big(H_i(p^{-1}(b);V)\big).$$

Hence we have $\chi_\rho(p) = \sum(-1)^i\iota \circ k(i) \in [B, K(R)]$. □

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Controlled topology and disassembly of $\chi(p)$.

Tibor Macko, Michael Joachim

The higher R-torsion for bundles $M \to E \xrightarrow{p} B$ of compact manifolds is obtained by refining the Euler characteristic $\chi(p)$ defined for any fibration with finitely dominated fibers in the previous talk. The $\chi(p)$ is a section of the fibration $A_B(E) \to B$ and the first step in refining it is pulling it back across the *assembly map*. The existence of the pullback, denoted by $\chi^{\%}(p)$, can be seen as a parametrized version of the homeomorphism invariance of the Whitehead torsion for finite CW-complexes first proved by Chapman and the fact that finite ENR's are finite CW-complexes first proved by West.

A functor $F\colon \mathcal{C} \to CW-$Spectra from the category $\mathcal{C}$ of spaces homotopy equivalent to finite CW-complexes to CW-Spectra is called *excisive* if it preserves homotopy pushouts. If F is a homotopy and excisive functor, then $X \mapsto \pi_*(F(X))$ defines a generalized homology theory with the coefficient spectrum $F(*)$. If $F\colon \mathcal{C} \to CW-$Spectra is any homotopy functor, then there always exists a homotopy and excisive functor $F^{\%}\colon \mathcal{C} \to CW-$Spectra and a natural transformation $\alpha\colon F^{\%} \to F$, inducing $F^{\%}(*) \xrightarrow{\simeq} F(*)$ with a certain universal property; the α is called the *assembly*, see [2].

We use the controlled topology model of the assembly map in the case when $F(X) = A(X)$ from Talk 2. The model is obtained by considering various categories of retractive spaces over the *control space* $\mathbb{J}X = (X \times [0,\infty], X \times [0,\infty))$. These are the category $\mathcal{R}^{ld}(\mathbb{J}X)$ of locally finitely dominated retractive spaces over $\mathbb{J}X$ and the category $\mathcal{RG}^{ld}(\mathbb{J}X)$ with the same objects and the morphisms retractive map germs; with the structure of a Waldhausen category (=*cw*-category in the sense of Talk 2.), where the weak equivalences are "controlled" in some sense.

Theorem.[1, Thm 7.5.] *The functor $X \mapsto K(\mathcal{RG}^{ld}(\mathbb{J}X))$ is a homotopy and excisive functor with the coefficient spectrum $S^1 \wedge A(*)$.*

Theorem.[1, Prop 7.6.] *The following sequence induced by a sequence of exact functors of Waldhausen (=cw-) categories is a homotopy fibration sequence:*

$$K(\mathcal{R}^{fd}(X)) \to K(\mathcal{R}^{ld}(\mathbb{J}X)) \to K(\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}X)).$$

The assembly map $\alpha\colon A^{\%}(X) \to A(X)$ is the connecting map in the above sequence, that means:

$$\alpha\colon \Omega K(\mathcal{R}\mathcal{G}^{ld}(\mathbb{J}X)) \to K(\mathcal{R}^{fd}(X)).$$

For Xa compact ENR we then use the controlled topology description of the assembly map α to find an explicit preimage of the A-theory Euler characteristic $\chi(X)$ which was constructed in the third talk. The construction of the preimage is natural which then allows to find the desired pullback $\chi^{\%}(p)$ of the parametrized Euler characteristic $\chi(p)$ in case $p : E \to B$ is a fiber bundle with fiber a compact topological manifold M and structure group the group of self-homeomorphisms of M.

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Euler Sections and Poincaré Duality

Sadok Kallel

This talk discusses some aspects of the Dwyer-Weiss-Williams paper [3] relative to Poincaré duality in space form, its inverse and its effect on euler sections. This duality takes Euler sections of a bundle associated to the one-point compactification of the tangent bundle of a smooth (or even topological) manifold M to an appropriate Becker-Gottlieb transfer element.

Before stating the theorem in full generality it might be enlightening to review a baby version [4], [6]. Suppose M is a smooth closed manifold with tangent bundle τ. We denote by τ^+ the one point-compactification of this bundle with fiber S^n, $n = \dim M$. Let $\mathrm{SP}^\infty(M)$ denote the infinite symmetric product of M which we identify with the free abelian group generated by points of M with the basepoint $* \in M$ the identity element. We can then construct a map (via the so-called *scanning* construction)

$$\wp^{-1} : \mathrm{SP}^\infty(M) \longrightarrow s\Gamma(\tau^+) \tag{1}$$

where the space on the right stands for the space of stable sections of τ^+ or in other words the space of sections of the bundle $s\tau^+$ obtained from τ^+ after applying the functor SP^∞ fibrewise (so that the new fiber now $\mathrm{SP}^\infty(S^n)$

is an Eilenberg-MacLane space $K(\mathbb{Z},n)$). The theorem is that $\wp^{-1}$ is a homotopy equivalence. When M is orientable, the bundle $s\tau^+$ trivializes so that $s\Gamma(\tau^+) = map(M, \mathrm{SP}^\infty S^n)$ is the space of all maps into the fiber [6]. Since $\pi_*(\mathrm{SP}^\infty(M)) \cong \tilde{H}_*(M;\mathbb{Z})$ and $\pi_*(map(M, \mathrm{SP}^\infty S^n)) \cong H^{n-*}(M,\mathbb{Z})$, it is now very clear why $\wp$ (or its inverse) is labeled the Poincaré duality map. The equivalence in (1) can be stated similarly for a manifold with boundary.

A generalization of (1) to other homology theories is the content of §3 of [3] which we now describe. Let $\mathbf{J} = \{J_n\}_{n\geq 0}$ be an Ω-spectrum, M a topological manifold (which we assume closed for simplicity), τ^+ the bundle over M with fiber M_z over z where M_z is the unreduced mapping cone of the inclusion $M\backslash z \longrightarrow M$. This bundle has a preferred section (the cone points) and so we can smash fiberwise with the *coefficient* spectrum $\mathbf{J}$. If we denote by $q : \tau^+ \longrightarrow M$ the projection map, we obtain a new fibration with fiber

$$\Omega^\infty(q^{-1}(y) \wedge \mathbf{J}) := \lim_n \Omega^n(q^{-1}(y) \wedge J_n) \tag{2}$$

When $\mathbf{J} = \mathbf{S}$ the sphere spectrum we obtain a fibration with fiber QS^n ($n = \dim M$), and when $\mathbf{J} = \mathbf{H}\mathbb{Z}$ the Eilenberg-MacLane spectrum $\{K(\mathbb{Z},n)\}_{n\geq 0}$ we obtain a fibration with fiber $\mathrm{SP}^\infty(S^n)$ (recovering the previous example). The sections of this fibration are denoted by $s\Gamma(\tau^+, \mathbf{J})$. The main statement now is that there is an explicit map

$$\wp^{-1} : \Omega^\infty(M_+ \wedge \mathbf{J}) \longrightarrow s\Gamma(\tau^+, \mathbf{J}) \tag{3}$$

with an explicit homotopy inverse $\wp$. Here M_+ is M with a disjoint basepoint. An analogous statement holds when M has boundary but one has to slightly modify the definition of $s\Gamma$ and also set $M_+ = M/\partial M$. Now note that by design, the *homotopy* groups of $\Omega^\infty(M_+ \wedge \mathbf{J})$ are precisely $h_*(M)$ (the generalized homology theory associated to $\mathbf{J}$), while it can be verified that the homotopy groups of $s\Gamma(\tau^+, \mathbf{J})$ are the cohomology of M in the *parameterized* spectrum $\mathbf{J}$ in the sense of [1] (see [7]). The equivalence (3) describes Poincaré duality with twisted coefficients.

In our Oberwolfach talk we first sketch the construction of the maps $\wp$ and $\wp^{-1}$ and then make the useful link between Euler sections and transfer maps (§5 of [3]). This last part we next explain (following an older version of [3])).

Suppose M is a space equipped with a Riemannian vector bundle ζ or rank n. Denote by ϵ_k the trivial k-dimensional bundle over M. Taking automorphisms fibrewise produces an inclusion of bundles over M

$$\begin{tikzcd} O(\epsilon_{k+1}/\epsilon_k) \arrow[rr] \arrow[dr] & & O(\zeta \oplus \epsilon_{k+1})/O(\zeta \oplus \epsilon_k) \arrow[dl] \\ & M & \end{tikzcd}$$

The fibers of the bundle on the left are homeomorphic to S^k while those on the right are copies of S^{n+k}. Taking adjoints fibrewise produces an inclusion of bundles over M

$$\begin{tikzcd} M \times S^0 \arrow[rr] \arrow[dr] & & \Omega^k_M(O(\zeta \oplus \epsilon_{k+1})/O(\zeta \oplus \epsilon_k)) \arrow[dl] \\ & M & \end{tikzcd}$$

Observe that the top horizontal map is like giving two sections to the bundle $\Omega^k_M(O(\zeta \oplus \epsilon_{k+1})/O(\zeta \oplus \epsilon_k))$ over M whose fiber over $y \in M$ is $\Omega^k(p^{-1}(y)^+ \wedge S^k)$ where p is the bundle projection. One section is the zero section while the other is the *Becker-Euler* section. Stabilizing $k \to \infty$ we obtain again the bundle $s\zeta^+$ described earlier and the homotopy class of the Becker-Euler section becomes an element b_n in the n-th cohomology of M with twisted coefficients in the sphere spectrum $\mathbf{S}$. The Poincaré dual of this section as in (3) is an element tr of $\Omega^\infty(M_+ \wedge \mathbf{S}) = QM_+$. The first main observation is that this element agrees with the Becker-Gottlieb transfer of the trivial projection $M \longrightarrow *$ ("agreement" here means up to some contractible space of choices made in the construction). Note that $\pi_0 QM_+ = \mathbb{Z}$ and that tr lies in the component given by the Euler characteristic of M. Indeed the above theorem can be thought of as a sophisticated space level formulation of the classical Poincaré-Hopf theorem stating that for a compact manifold M with fundamental class $[M, \partial M]$, $e(\tau^M) \cap [M, \partial M] = \chi(M)$, where $e(\tau^M)$ is the Euler class of the tangent bundle.

There is an extended parameterized version of the above stated result: if $p : E \longrightarrow B$ is a bundle with compact n-manifold fibers, then the fibrewise Poincaré dual $\wp(b_n)$ for the vertical tangent of p also agrees with the Becker-Gottlieb transfer $B \longrightarrow Q^+_B E$ of p, where $Q^+_B E$ is the bundle over B with fiber $Q(p^{-1}(y)^+)$. Special cases of this parameterized Poincaré-Hopf type theorem are in [2] and [5].

Similarly (and this is of special relevance to the theme of this Arbeitsgemeinschaft), it is possible to identify the Poincaré dual of the $\mathbf{A}(*)$-Euler class which is defined as follows. First of all $\mathbf{A}(*)$ is the Waldhausen spectrum of which i-th term is the space $Top(i+1)/Top(i)$. If we assume that ζ as a bundle over M is a Euclidean n-bundle (i.e. a bundle with fibers homeomorphic to $\mathbb{R}^n$), then proceeding as above we get a similar inclusion of bundles $O(\epsilon_{k+1}/\epsilon_k) \longrightarrow Top(\zeta \oplus \epsilon_{k+1})/Top(\zeta \oplus \epsilon_k)$ over M leading after taking fibrewise adjoints to

$$\begin{tikzcd} M \times S^0 \arrow[rr] \arrow[dr] & & \Omega^k_M(Top(\zeta \oplus \epsilon_{k+1})/Top(\zeta \oplus \epsilon_k) \arrow[dl] \\ & M & \end{tikzcd}$$

Here again we obtain two section one of which is the zero section and the other yields in the limit over k a section of a bundle as in (2) with fibers $\Omega^\infty(p^{-1}(y)^+ \wedge \mathbf{A}(*))$ and of which homotopy class is in the n-th cohomology of M with twisted coefficients in $\mathbf{A}(*)$. The Poincaré dual of this class is an element in $\Omega^\infty(M_+ \wedge \mathbf{A}(*))$ and the index theorem of [3] states that this element is carried out via the assembly map $\Omega^\infty(M_+ \wedge \mathbf{A}(*)) \longrightarrow A(M)$ to some carefully constructed $\mathbf{A}(*)$-Euler characteristic (see previous talks). Here $A(M)$ is the K-theory of the category of homotopy finite retractive spaces over M.

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Index theorems and a definition of torsions

Markus Szymik

The classical Hopf index theorem states that the Poincaré dual of the Euler characteristic of a manifold is the Euler class of its tangent bundle. In the first part of the talk, I have stated and proven the index theorem of Dwyer, Weiss and Williams for topological and smooth families of manifolds [1].

The index theorem for a topological family $p\colon E \to B$ states that the parametrized $A^{\%}$-theory Euler characteristic $\chi^{\%}(p)$ in $\Gamma(A^{\%}_B(E) \to B)$ is Poincaré dual to the corresponding Euler class of the relative tangent bundle. As soon as all the ingredients are points in or maps between suitable homotopy limits, the proof is a formal manipulation of those.

The index theorem for smooth families states that the parametrized $A^{\%}$-Euler characteristic $\chi^{\%}(p)$ is the image of the transfer under the unit map from stable cohomotopy to $A^{\%}$-theory. The proof builds on the result for topological families and uses (1) that the Euler class of a vector bundle is the image of the Becker class under the unit map, and (2) that the Becker class is Poincaré dual to the transfer.

In the second part of the talk, different notions of higher Reidemeister torsion have been defined, again following Dwyer, Weiss, and Williams [1]. If $p\colon E \to B$ is a fibration with homotopy finitely dominated fibres, and if $\lambda\colon A(E) \to K(R)$ is a map into the K-theory of a ring R, the parametrized A-theory Euler characteristic $\chi(p)$ in $\Gamma(A_B(E) \to B)$ determines a map $B \to K(R)$, and a null homotopy of that map determines a section of the homotopy fibre, the higher *homotopy Reidemeister torsion* of p with respect to λ. The same line of thought, with A replaced by $A^{\%}$ and stable cohomotopy, yields the higher *topological* and *smooth Reidemeister torsion* for topological and smooth families, respectively.

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Generalized Morse functions

Bernhard Hanke

Recall that if M is a closed smooth manifold, then there exists a *Morse function* $M \to \mathbb{R}$. This means that f is smooth and all critical points of f are nondegenerate. If f is generic with this property and we fix a Riemannian metric on M as well as framings of the negative eigenspaces TM_x^- of $\mathrm{hess}(f)$ at every critical point $x \in M$, then f induces a CW-decomposition of M.

In this note we aim at a generalization of these statements to families of manifolds: Given a smooth fibre bundle

$$M \hookrightarrow E \xrightarrow{p} B$$

over a closed smooth manifold B, does there exist a smooth function

$$F : E \to \mathbb{R}$$

whose restriction to each fibre $E_b = p^{-1}(b)$, $b \in B$, is a Morse function? Simple examples show that this may not always be possible: If B is simply connected, then the critical points of $F|_{E_b}$ can be used to construct a section of p.

We therefore must weaken the Morse condition. The correct definition can be motivated as follows. By Thom transversality a generic choice of F induces a section

$$E \xrightarrow{\phi_F} \mathrm{Vert}^*\,,\ x \mapsto DF(x)|_{\mathrm{Vert}}\,,$$

which is transversal to the zero section $E \subset \mathrm{Vert}^*$. Here, Vert^* is the dual space of the vertical tangent bundle of E. Let

$$V := \phi_F^{-1}(E) \subset E$$

be the space of critical points of the restrictions of F to the fibres of p. This is a smooth manifold of dimension $\dim B$ and $x \in V$ is a Morse singularity of $F|_{E_{p(x)}}$, if and only if x is a regular point for $p|_V$. A mild weakening of the last requirement is described in the following definition. Let $k := \dim B = \dim V$.

Definition 1. A point $x \in V$ is called a *fold type singularity* of $p|_V$, if $p|_V$ is given by

$$(x_1, \ldots, x_k) \mapsto (x_1^2, x_2, \ldots, x_k)$$

in suitable coordinates around x and $p(x)$.

A point $x \in V$ is a fold type singularity of $p|_V$, if and only if x is a degenerate critical point of $F|_{E_{p(x)}}$ of birth-death type. We give the relevant definition.

Definition 2. Let
$$f : M^n \to \mathbb{R}$$
be a smooth function. We call $x \in M$ a *birth-death singularity* of f, if f is given by
$$(x_1, \ldots, x_n) \mapsto x_1^3 + x_2^2 + \ldots + x_l^2 - x_{l+1}^2 - \ldots - x_n^2$$
in suitable coordinates around x and $f(x)$. The function f is called a *generalized Morse function*, if all degenerate critical points of f are of birth-death type.

Let $\mathcal{H}(M)$ be the space of generalized Morse functions $M \to \mathbb{R}$ equipped with the C^∞-topology. The condition of being generalized Morse defines a partial relation (i.e. a subspace)
$$\mathcal{R}_{\mathcal{H}}(M) \subset J^3(M;\mathbb{R})$$
in the bundle of 3-jets of smooth functions on M. Recall that the total space of this bundle consists of taylor expansions up to third order of smooth functions $U \to \mathbb{R}$ at points $x \in U$, where $U \subset M$ is an open subset. The following important result says that the space of generalized Morse functions satisfies a (parametrized) h-principle.

Theorem 3 (Eliashberg-Mishachev, [1])**.** *The prolongation map*
$$J^3 : \mathcal{H}(M) \to \Gamma_{\mathcal{H}}(M) := \{\phi \in \Gamma(J^3(M;\mathbb{R})) \mid \operatorname{im}\phi \subset \mathcal{R}_{\mathcal{H}}(M)\},$$
which associates to each function its canonical 3*-jet, is a weak homotopy equivalence.*

The proof of this fact uses the investigation of so called wrinkled maps. An important corollary of the Eliashberg-Mishachev theorem is

Corollary 4. *There exists a smooth function $F : E \to \mathbb{R}$ which is fibrewise generalized Morse.*

If we want to make sure that this function is unique up to homotopy, we need to impose more structural data on the functions under consideration. These are also important to get induced CW-decompositions of the fibres of p.

Definition 5. A *framed function* on M is a generalized Morse function $f : M \to \mathbb{R}$ together with a framing $(\zeta_1, \zeta_2, \ldots, \zeta_i)$ of the nonpositive eigenspace $TM_x^{\leq 0}$ of $\operatorname{hess}(f)$ for each critical point x of f. Furthermore, we require that at each degenerate critical point, ξ_i points into the positive cubic direction of f.

Let $\mathcal{L}(M)$ be the space of framed functions on M and $\Gamma_{\mathcal{L}}(M)$ be the space of framed sections in $\Gamma_{\mathcal{H}}(M)$.

Theorem 6 (Igusa, [2])**.** *The space $\Gamma_{\mathcal{L}}(M)$ is weakly contractible. Furthermore, the prolongation map $J^3 : \mathcal{L}(M) \to \Gamma_{\mathcal{L}}(M)$ is a* $(\dim M)$*-equivalence.*

Igusa's methods are different from those used in the paper [1]. It is an important open problem to decide if the space of framed functions on M satisfies an h-principle.

Corollary 7. *If $\dim B < \dim M$ then there is up to homotopy a unique fibrewise framed function $E \to \mathbb{R}$.*

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A-infinity functors and fibrewise Morse functions

Jeffrey Giansiracusa

Start with a bundle $M \to E \to B$ of smooth manifolds. The purpose of this talk was to illustrate how one gets from a fibrewise Morse function on E to a family of chain complexes over B equipped with a system of *higher homotopies*. This will produce a map into a Whitehead space in which the universal torsion invariants live. The main references for this talk are [1], [2], and [3] chapter 2.

1. From Morse functions to chain complexes

First we triangulate the base B. Now suppose that $f : E \to \mathbb{R}$ is a fibrewise Morse function which is oriented in the sense that at each critical point $p \in E_b$ we have an orientation for the subspace of T_pE_b on which the Hessian of f is negative definite. Furthermore, let us suppose we also have a fibrewise Riemannian metric g which satisfies as much Morse-Smale transversality with respect to the triangulation as possible. Lastly, fix a coefficient ring R.

Over each vertex $v \in B$ the Morse function and metric produce a *cellular chain complex*, also known as the Morse complex, $C_*(v)$. This is a free R-module with basis corresponding to the critical points of f over v. The basis is graded by the index of the critical points and it comes with a partial ordering generated by the gradient trajectories between critical points. The differential is determined on generators by $d : p \mapsto \lambda_1 q_1 + \lambda_2 q_2 + \cdots$ where $ind(q_i) = ind(p) - 1$ and the coefficient λ_i is the signed count of gradient trajectories from p to q_i The transversality assumption ensures that there is no trajectory between critical points of the same index.

Now consider a 1-simplex $v \to w$ in the base. We expect to get a chain isomorphism $C_*(u) \to C_*(v)$. Over the 1-simplex one sees a bijection from the basis over v to the basis over w, but this is in general *not* a chain isomorphism because the differentials can change. This is because the trajectories which define the differentials can jump discontinuously even as the Morse function varies smoothly. However, one can add a small perturbation ψ_1 to the bijection to get a chain isomorphism. The components of ψ_1 are given by counting the trajectories over the simplex between critical points of the *same* index. With the partial ordering of the basis given by looking at trajectories over the entire simplex, the ψ_1 is strictly *upper triangular*.

These jumping trajectories can be understood by making an observation about the space of Morse functions. There is an open dense subset for which the trajectories only change index by 1, but on a codimension 1 subspace one encounters trajectories going between points of the same index. Within this there is a codimension 2 subspace on which some trajectories go up by one index. And in general there is a codimension n subspace in which one finds trajectories going up by degree $n-1$. Therefore a transversal n-parameter family of Morse functions will contain a finite number of trajectories of degree $n-1$.

The chain isomorphisms arising from 1-simplices are not strictly associative, but 2-simplices provide associating homotopies. Over a 2-simplex (with vertices u, v, and w) there may be some finite number of trajectories which go up from index i to index $i+1$. These determine a degree 1 chain homotopy $\psi_2 : C_*(u) \to C_*(w)$ between the chain isomorphism over $u \to w$ and the composition of the isomorphisms over $u \to v$ and $v \to w$. Again if we take the partial ordering of the basis by looking at the trajectories over the entire 2-simplex then ψ_2 is strictly upper triangular.

Similarly, simplices of higher dimensions provide higher homotopies. A n-simplex produces a strictly upper triangular degree $n-1$ map from the complex over the first vertex to the complex over the last vertex with entries determined by counting degree $n-1$ trajectories. The way that these higher homotopies fit together is codified in the definition of a *twisting cochain*, but before I get to that I'd like to comment on a slightly different way of thinking about the above.

The idea is that a Morse function determines a cellular structure. In good circumstances, each cell is only attached to cells of strictly lower dimension. But as one varies the Morse function in a 1-parameter family the cell structure can change because an i-cell A can slide over another i-cell, B and thus change the homotopy class of its attaching map. This corresponds to the occurence of a trajectory between two index i critical points at some point in the family of Morse functions. Clearly the new attaching map of A can only involve those lower dimensional cells which are in the images of either B's attaching map or A's old map. This is where the strictly upper triangular condition comes from. More generally, with an n-parameter family of Morse functions one can have the cell structure change my sliding i-cells over $(i+n-1)$-cells in nontrivial ways and one sees the upper triangular condition arising in the same way.

2. Twisting cochains

Suppose $\mathcal{B}$ is a category and K is functor from $\mathcal{B}$ to free R-modules (with graded poset basis).

Definition 2.1. A *p-cochain* ψ on $\mathcal{B}$ with coefficients in K is an assignment

$$\psi(X_0 \leftarrow \cdots \leftarrow X_p) \in Hom(KX_p, KX_0)$$

for each p-simplex in the nerve. The *coboundary* of ψ is the $p+1$-cochain

$$\begin{aligned}\delta\psi(X_0 \overset{f_1}{\leftarrow} \cdots \overset{f_{p+1}}{\leftarrow} X_{p+1}) &= (f_1)_*\psi(X_1 \leftarrow \cdots \leftarrow X_{p+1}) \\ &\quad + (-1)^p (f_{p+1})^* \psi(X_0 \leftarrow \cdots \leftarrow X_p) \\ &\quad + \sum_i (-1)^i \psi(X_0 \leftarrow \cdots \leftarrow X_{i-1} \overset{f_i \circ f_{i+1}}{\longleftarrow} X_{i+1} \leftarrow \cdots \leftarrow X_{p+1}).\end{aligned}$$

Definition 2.2. A *twisting cochain* ψ on $\mathcal{B}$ is a sum of cochains $\Sigma_p \psi_p$, where ψ_p is a strictly upper-triangular p-cochain of degree $p-1$, such that $\delta\psi = \psi' \cup \psi$, where $\psi' = \Sigma_p (-1)^p \psi_p$.

(Here the cup product of $\alpha \cup \beta$, with deg $\alpha = i$, deg $\beta = j$ is given by evaluating α on the front i simplex and β on the back j-simplex.)

Unpacking the definition, one sees that $\psi_0(X)$ is a differential on KX, $(K + \psi_1)(X \leftarrow Y)$ is a chain map $KY \to KX$, ψ_2 is a chain homotopy making these chain maps associative up to homotopy, and the higher components give higher associating homotopies. Twisting cochains can be thought of as the higher homotopies part of A_∞ functors. More precisely, it is easy to check that $K + \psi$ is an A_∞ functor if and only if ψ is a twisting cochain.

In the manner described in the first section a fibrewise Morse function produces a twisting cochain on $\mathcal{B}$.

3. Superconnections

In this section we define superconnections and indicate roughly how they are a de Rham version of twisting cochains. This was needed to set the stage for some of the later talks.

Let $V = \bigoplus_{n \geq 0} V_n$ be a graded complex vector bundle over B.

Definition 3.1. A $\mathbb{Z}$-graded superconnection D is a degree 1 operator

$$D : \Omega^*(B; V) \to \Omega^*(B; V)$$

satisfying the Leibniz rule $D(\alpha \otimes s) = d\alpha \otimes s + (-1)^{\deg \alpha} \alpha \otimes Ds$ for $\alpha \in \Omega^*(B)$ and $s \in \Omega^*(B; V)$.

Proposition 3.2. *D is of the form $\nabla - A$ where ∇ is an ordinary connection on V and $A \in \Omega^*(B; End^*(E))$ is of total degree 1.*

The *curvature* of D is defined to be D^2, and the superconnection is said to be *flat* if the curvature vanishes.

Let us suppose that ∇ is flat. Then D being flat means that $(\nabla - A)^2 = 0$, which is equivalent to $\nabla(A) = A^2$. Hopefully this strikes one as similar to the equation $\delta\psi = \psi' \cup \psi$ which characterizes twisting cochains.

Briefly, we have the following dictionary:

$$\psi \text{a twisting cochain} \Leftrightarrow A \in \Omega^*(B; End^*(V))$$

$$K \text{the coefficient functor} \Leftrightarrow \nabla \text{a flat connection on } V$$

$$(K + \psi)\text{an } A_\infty \text{ functor} \Leftrightarrow (\nabla - A)\text{a graded superconnection}$$

References

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Kamber-Tondeur classes

Georg Tamme, Volker Neumaier

Chern-Weil theory for complex vector bundles gives classes in the De Rahm cohomology of the base. Hereby one chooses a connection on the vector bundel and puts it's curvature into a $GL_n(\mathbb{C})$-invariant polynomial. One obtaines a closed form on the base and it's associated class does not depend on the choice of the connection, see [3]. So for flat vector bundels, i.e. vector budels having a connection with vanishing curvature, this gives just trivial information.
To define interesting classes for flat vectorbundels F with a fixed flat connection ∇, we choose a hermitian metric g on F which gives rise to an adjoint connection ∇^*. Now, again, we take a $GL_n(\mathbb{C})$-invariant polynomial and put in the difference $\nabla^* - \nabla$, which is given by an endomorphism-valued differential form of degree one. This turns out to be a closed form on the base, whose cohomology class is indpendent of the choice of the metric. The Kamber Tondeur forms $c(F, g)$ and classes $c(F)$ are arising in this way. Given two metrics g_1, g_2 on F, the corresponding Kamber Tondeur forms differ by the coboundary of a form $c(F, g_1, g_2)$ which is naturally defined up to exact forms. One more interesting fact is, that the higher Kamber Tondeur classes are rigid under smooth deformation of the flat connection.
In a similiar way one can define Kamber Tondeur forms for flat superconnections on superbundels, compare [1].

References

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Finite-dimensional Torsion Classes

Thilo Kuessner

For a flat family of chain complexes $(E = \oplus E_i, \partial) \to B$, one wants to show that the secondary characteristic classes of E and of the fiberwise homology $H(E)$ agree. Even better, a higher torsion form $T \in \Omega^*(B)$ with $dT = ch^0(E) - ch^0(H)$ will be constructed. Finally, an application to fibrewise Morse functions is explained.

The approach will be to consider the formal sum $\partial + \nabla$ as a flat superconnection and to apply secondary characteristic classes for this superconnection to interpolate between the secondary characteristic classes of E and H.

A superconnection of total degree -1 is an odd operator

$$A = \sum\nolimits_{j\geq 0} A_j : C^\infty(B, E) \to \oplus_{\geq 0}\Omega^j(B, E),$$

such that A_1 is a connection and $A_i \in \Omega^i(B, \mathrm{Hom}(E_j, E_{j-1+i}))$ if $i \neq 1$. It is flat if $A^2 = 0$, i.e. $A_0^2 = 0, [A_0, A_1] = 0, [A_0, A_2] + A_1^2 = 0, \ldots$. In particular, if A is a flat connection and ∂ a parallel differential, then $\partial + \nabla$ is a flat superconnection.

Let g be a hermitian metric, A^* the adjoint of A for g, and $X = \frac{1}{2}(A - A^*) \in \Omega(B, \mathrm{End}E)$. For each (odd) holomorphic function f, one defines a characteristic class (even a differential form) by $\alpha := (2\pi i)^{\frac{1-N}{2}}\ strf(X) \in \Omega(B)$. ($N$ is to be understood as $N = k$ on $\Omega^k(B)$.) We will consider $f(z) = ze^{z^2}$, which gives ch^0.

Lemma 1. *: The cohomology class $\left[ch^0(E, A, g)\right]$ is independent of the metric.*

Proof: Let $g_t = tg_1 + (1-t)g_0$ and use it to define $X_t = \frac{1}{2}(A - A_t^*)$. One computes $\frac{d}{dt} strf(X_t) = (2\pi i)^{-\frac{N}{2}} d\ str\left(\frac{1}{2}\left(g_t^{-1}\right)\frac{dg_t}{dt}f'(X_t)\right)$, hence $f(X_1)$ and $f(X_0)$ differ by a closed form. □

For $b \in B$ we denote $H(E, \partial)_b$ the homology of $(E, \partial)_b$. By Hodge theory, H is a subspace of E with induced connection and metric.
Rescaling: for $t \in \mathbb{R}$ we define a metric g_t by $g_t = t^N g$, i.e. $g_t = t^i g$ on E_i.

Theorem 2. *(Bismut-Lott):*

$$\lim_{t\to\infty} ch^0(E, A, g_t) = ch^0\left(H, \nabla^H, g^H\right),$$

$$\lim_{t\to 0} ch^0(E, A, g_t) = ch^0(E, \nabla, g).$$

In view of Lemma 1 this proves that $\left[ch^0(E, \nabla)\right] = \left[ch^0\left(H, \nabla^H\right)\right]$. To construct a Stammform for their difference, we consider

$$\hat{f}(A, g_t) := (2\pi i)^{-\frac{N}{2}}\ str\left(\frac{N}{2}f'(X_t)\right)$$

and note that the proof of Lemma 1, together with $\left(g_t^{-1}\right)\frac{dg_t}{dt}=\frac{N}{t}$, implies that $\frac{1}{t}d\hat{f}\left(A,g_t\right)=\frac{d}{dt}str\ f\left(A,g_t\right)$. This implies that

$$T:=\int_0^\infty \hat{f}\left(A,g_t\right)-\lim_{t\to\infty}\hat{f}\left(A,g_t\right)-\lim_{t\to 0}\hat{f}\left(A,g_t\right)\frac{dt}{t}$$

$$=\int_0^\infty\left(\hat{f}\left(A,g_t\right)-\frac{1}{2}\chi'\left(H\right)f'\left(0\right)-\left(\chi'\left(E\right)-\frac{1}{2}\chi'\left(H\right)\right)f'\left(\frac{i\sqrt{t}}{2}\right)\right)\frac{dt}{t}$$

satisfies $dT=\left[ch^0\left(H,\nabla^H\right)\right]-\left[ch^0\left(E,\nabla\right)\right]$. (The constant terms are added to assure convergence of the integral.)
If E and H carry parallel metrics, it follows that T_f is closed and thus defines a cohomology class in $H^*\left(B\right)$.

Fibrewise Morse theory. Let $p:M\to B$ be a fibration. Let $h:M\to\mathbb{R}$ be a fibrewise Morse function, and let $\left(C_i\right)\to B$ be the finite covering by fibrewise critical points of index i. Let F be a flat vector bundle over M and consider the fibrewise Thom-Smale complex (with coefficients in F) $E=\left(p\mid_C\right)_*\left(F\otimes o\left(W^u x\right)\right)$.

If h satisfies the fibrewise Smale condition, then the fibrewise differential ∂ on E is defined and one may consider the flat superconnection $\partial+\nabla$. If h does not satisfy the fibrewise Thom-Smale condition, then Goette still constructs a flat superconnection $A=\nabla+a, a\in\Omega^*\left(B,End^{*-1}E\right)$ of total degree -1 satisfying the axioms of family Thom-Smale complexes.
For such family Thom-Smale complexes, the higher torsion form can be defined as above, but by using the metric $g_t=t^h g$ for $t\leq 1$, $g_t=t^N g$ for $t\geq 1$. (In the case that h satisfies the fibrewise Thom-Smale condition, one can assume that h is fibrewise self-indexing, hence $h=N$ on C.)

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Polylogarithms

Marco Hien

For $k\geq 1$, the k-th polylogarithm is defined to be the convergent power series

$$\mathrm{Li_k}(z):=\sum_{n=1}^{\infty}\frac{z^n}{n^k}\text{ for }z\in\mathbb{C}\text{ such that }|z|<1\ ,$$

for example $\mathrm{Li}_1(z)=-\log(1-z)$. They satisfy $\frac{d}{dz}\mathrm{Li}_k(z)=z^{-1}\mathrm{Li}_{k-1}(z)$ and hence allow extensions to multivalued analytic functions on $\mathbb{C}\setminus\{0,1\}$. The polylogarithm functions appear in various areas of mathematics and had been discussed in detail in a previous Arbeitsgemeinschaft, October 2004 (Oberwolfach Report No. 48/2004).

The k-th polylogarithm satisfies the distribution relation

$$\mathrm{Li}_k(z^n) = n^{k-1} \cdot \sum_{\zeta^n=1} \mathrm{Li}_k(\zeta \cdot z)$$

for $n \in \mathbb{Z}$. These relations moreover characterize the polylogarithms. In particular, the following proposition holds:

Proposition ([2]). *Let $f : S^1 \setminus \{1\} \to \mathbb{R}$ be a continuous function such that*

$$f(z^n) = n^{k-1} \cdot \sum_{\zeta^n=1} f(z \cdot \zeta)$$

holds for any $n \in \mathbb{Z}$ and all $1 \neq z \in S^1$. Then there is a real number $a \in \mathbb{R}$, such that

$$f(z) = a \cdot \mathrm{Re}\big(\frac{1}{i^{k-1}}\mathrm{Li}_k(z)\big) \text{ for all } z \in S^1 \setminus \{1\} \ .$$

Now, consider a $U(1)$-principal bundle $E \to B$ and let $\mu_n \subset U(1)$ denote the subgroup of all n-th roots of unity. To any element $u \in \mu_n$ one associates a hermitian flat line bundle $\mathcal{F}_u$ on the S^1-fibre bundle $E/\mu_n \to B$ whose vertical monodromy $\pi_1(S^1) \cong \mathbb{Z} \to U(1)$ maps the generator 1 to u.

A function f on E/μ_n is called a fibrewise positive generalized Morse function, if the restriction f_b to any fibre over $b \in B$ has critical points of order three at most with the additional condition that the third derivative of f_b is positive at any degenerate critical point. Any such function (which is uniquely determined up to contractible choice) induces a map

$$w_{n,u} : B \to |\mathcal{W}_\bullet \mathbb{C}|$$

to the Whitehead-space represented by its simplicial model $\mathcal{W}_\bullet\mathbb{C}$ explicitly constructed in terms of acyclic complexes modelled on the chain complexes arising in the Morse theory of f.

There is natural way to construct universal classes $\tau_k^{FR} \in H^{2k}(|\mathcal{W}_\bullet\mathbb{C}|, \mathbb{R})$ by the methods presented in the previous talk. The *higher Franz-Reidemeister torsion* associated to $E/\mu_n \to B$ together with the flat hermitian line bundle $\mathcal{F}_u$ is defined as the pull-back of the universal classes along $w_{n,u}$, i.e.

$$\tau_k^{FR}(E/\mu_n, u) := (w_{n,u})^* \tau_k^{FR} \in H^{2k}(B, \mathbb{R}) \ .$$

These constructions can be generalized to give higher torsion classes $\tau_k^{FR}(E/\mu_n, z)$ for any $1 \neq z \in S^1$ varying smoothly in z.

The elementary observation that, given a $\xi \in S^1 \setminus \{1\}$ with $\xi^m \in \mu_n$ and a fibrewise positive generalized Morse function f on E/μ_{nm}, the composition of f with the m-fold covering

$$E/\mu_n \longrightarrow E/\mu_{nm} \xrightarrow{f} \mathbb{R}$$

is again a fibrewise positive generalized Morse function, leads to the transfer formula

$$\tau_k(E/\mu_n, \xi^m) = \sum_{\zeta^m=1} \tau_k(E/\mu_{nm}, \zeta \cdot \xi)$$

for the higher torsion classes of circle bundles which by continuity generalizes to arbitrary parameters z. Applied to the universal case, the proposition above leads to the main theorem of this talk

Theorem ([1]). *Let $E \to B$ be a smooth $U(1)$-principal bundle, $n \in \mathbb{N}$ and $k \geq 0$. Then*

$$\tau_k^{FR}(E/\mu_n, z) = a_k \cdot n^k \cdot \mathrm{Re}\big(\frac{1}{i^k}\mathrm{Li}_{k+1}(z)\big) \cdot c_1(E)^k$$

for some constant $a_k \in \mathbb{R}$ which depends on k only.

Computations of explicit cases (using methods beyond the goals of this talk) give $a_k = -\frac{1}{k!}$.

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Definition of higher Franz-Reidemeister torsion, I

Nathalie Wahl

For a manifold bundle $M \to E \to B$ and a unitary representation $\rho : \pi_1 E \to U(r)$, the higher Franz-Reidemeister torsion of (E, ρ) is a collection of cohomology classes $\tau_k(E, \rho) \in H^{2k}(B; \mathbb{R})$. The 'natural' definition of these classes uses the filtration induced by a framed function on the total singular chain complex of a manifold. This is the approach described in this talk. The references for this talk are [1, Chapter 4] and [2, Chapter 2].

Let R be a ring. Given a manifold bundle $M \to E \xrightarrow{p} B$ and a representation $\rho : \pi_1 E \to G \subseteq R^\times$, we want to construct a map from the base B to a space of filtered chain complexes whose cohomology contains universal torsion classes. The map can then be used to pull back the classes to B. The construction starts as follows:

1) Stabilize the bundle, replacing $E \to B$ by $E \times D^N \to B$ for some N large enough so that the dimension of the fiber is larger than the dimension of the base. (Stabilization does not affect the torsion.)

2) Choose a fiberwise framed function $f : E \to \mathbb{R}$. After stabilization, such a function exists and is unique up to homotopy by the framed function theorem (talk 8).

3) Triangulate B. We denote by $\mathrm{Simp}\, B$ the category of simplices of B.

4) Consider the functor from $\mathrm{Simp}\, B$ to the category of R-complexes which takes a simplex x to the total singular chain complex $C_*(p^{-1}(x); R)$ with twisted R-coefficients.

We want to add to this functor the extra data given by the filtration induced by f on the complexes. We first describe the filtration.

The Morse components of the singular set of $f\big|_{p^{-1}(x)}$, i.e. those consisting solely of Morse critical points, form a poset $P(x)$ with the ordering induced by the existence of trajectories between points of the different components. The poset is moreover graded by the index of the critical points.

Each complex $C_*(p^{-1}(x);R)$ is filtered by the lattice $\Lambda P(x)$ of closed subposets of $P(x)$. (To a closed subposet of $P(x)$ corresponds a submanifold of $p^{-1}(x)$ and we take the singular chain complex of this submanifold.) A *ΛP-filtered chain complex* is defined to be a pair (E,λ) where E is a chain complex filtered by ΛP and λ is a collection of cohomology classes

$$\lambda_A(z) \in H^d(E^B, E^A; R)$$

for each (A,z) such that $A \in \Lambda P$, $z \in P$ of degree d, and $B = A \cup \{z\} \in \Lambda P$, where E^A denotes the Ath filtration of E. The graded group $H^*(E^B,E^A;R)$ is moreover assumed to be non-trivial only in degree d where it is generated by $\lambda_A(z)$. In the case of a filtration obtained from a Morse function, these cohomology classes are a choice of generator for the 'Morse layer' defined by the critical point z.

The space $FC(R,G)$ of filtered R-complexes is the classifying space of a category with objects the ΛP-filtered chain complexes and morphisms allowing a change of poset, possibly forgetting some 'collapsing pairs' to account for the birth-death singularities, together with an action of G to allow a change of identification of the local coefficients. (A combinatorial version of this category is defined more precisely in the following talk.) It is defined in such a way that the functor described in (4) above can be refined to a functor

$$\xi(E,\rho) : \operatorname{Simp} B \to FC(R,G).$$

We denote by $FC^h(R,G)$ the subspace of acyclic chain complexes. Note that the image of $\xi(E,\rho)$ lies in $FC^h(R,G)$ precisely when $H_*(M;R)$ is trivial.

There are two main theorems about filtered chain complexes.

Theorem 1. *There is a fibration sequence*

$$FC^h(R,G) \to \Omega^\infty S^\infty(BG_+) \to \mathbb{Z} \times BGL(R)^+$$

When $(R,G) = (M_r(\mathbb{C}), U(r))$, the Kamber-Tondeur forms (also known as Borel regulators) in $H^{2k+1}(BGL^\delta(\mathbb{C}))$ (where $\mathbb{C}$ is taken with the discrete topology) transgress to classes $\tau_k \in H^{2k}(FC^h(M_n(\mathbb{C}), U(r)))$ which are the universal Franz-Reidemeister torsion classes. In particular, taking $\tau_k(E,\rho) = \xi(E,\rho)^*(\tau_k)$ defines higher Franz-Reidemeister torsion for bundles $M \to E \to B$ and representation $\rho : \pi_1 E \to U(r)$ such that $H_*(M; M_r(\mathbb{C}))$ is trivial. However this definition is not well-suited to calculations. A more 'practical' —and more general— definition will be given in the following talk.

Finally, the following theorem relates the filtered chain complex construction to the construction described in the next talk:

Theorem 2. *Let (E, λ) be a ΛP-filtered chain complex. Then E is a direct sum of two ΛP-filtered chain complexes*

$$E = D \oplus L$$

where L is minimal in the sense that $L^{A\cup\{z\}}/L^A$ is free of rank 1 in $\deg(z)$ and 0 otherwise for all pairs (A, z) as above. Moreover, L can be taken to be any ΛP-filtered subcomplex of E minimal in this sense.

Note that D is acyclic. In the next talk, we will work with the Morse cellular complex, which is a minimal complex, and D will be replaced by a collection of higher homotopies (in the form of a twisting cochain).

Theorem 2 can be used to prove the equivalence between the cellular complex approach and the filtered chain complex approach, namely to show that

$$Wh(R, G) \xrightarrow{\simeq} FC(R, G) \quad \text{and} \quad Wh^h(R, G) \xrightarrow{\simeq} FC^h(R, G)$$

where $Wh^{(h)}(R, G)$ will be defined in the next talk.

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Definition of higher Franz-Reidemeister torsion, II

Nathalie Wahl

In this talk, we give a definition of higher Franz-Reidemeister torsion more suited to computations. The reference for the talk is [2, Chapter 2].

As in the previous talk, fix a ring R and a group $G \subseteq R^\times$. Let $\mathcal{P}$ be the category of graded posets with degree 0 order preserving bijections as morphisms. Define $\mathcal{P}(G)$ to be the category of graded posets with morphisms $\alpha = (\alpha, \tilde{\alpha}) : P \to Q$ of the form

$$\begin{tikzcd} P \times G \arrow[r, "\tilde{\alpha}"] \arrow[d] & Q \times G \arrow[d] \\ P \arrow[r, "\alpha"] & Q \end{tikzcd}$$

where $\tilde{\alpha}$ is G-equivariant and α is a morphism in $\mathcal{P}$. Such a morphism induces a map $\alpha_* : PR \to QR$ between the free R-modules generated by P and Q, defined by $\alpha_*(\sum x_i r_i) = \sum \pi(\tilde{\alpha}(x_i, 1), r_i)$, where $\pi((y, g), r) = y(gr)$. We denote by $\alpha_\# :$ $\mathrm{End}(PR) \to \mathrm{End}(QR)$ the morphism induced by conjugation with α_*.

A Δ^k*-family of R-complexes* is a pair (P, ϕ), where P is a poset and $\phi = \sum_p \phi_p$ is a twisting cochain subordinate to P (see talk 9). So over the jth vertex of the simplex Δ^k we have the free R-module RP with a differential given by $\phi_0(i)$, over each edge $\phi_1(i,j)$ gives a chain isomorphism and the ϕ_k's for $k \geq 2$ give higher homotopies.

In fact, in the case of a fiberwise framed function on a manifold bundle $E \xrightarrow{p} B$, the higher homotopies can be obtained through the following result:

Theorem 1. *Let x be a simplex of B. There exists a Δ^k-family of chain complexes $(P(x), \phi(x))$ which is uniquely determined up to simplicial homotopy by the fact that its total complex (the twisted tensor product in the sense of* [1]*)*

$$C_*(\Delta^k) \otimes_{\phi(x)} P(x)R$$

together with the filtration given by $\Lambda P(x)$ and by the faces of x, is filtered quasi-isomorphic to the total singular complex $C_*(p^{-1}(x); R)$ with twisted R-coefficients and so that the basis element of each layer z in the twisted tensor product is dual to the class $\lambda_A(z)$ of the filtered chain complex.

The *Whitehead category* $Wh_\bullet(R,G)$ is a simplicial category with objects in $Wh_k(R,G)$ the Δ^k-families of R-complexes and morphisms $(P,\phi) \to (Q,\psi)$ given by a decomposition $P = A \sqcup B$ and a morphism $\alpha : A \to Q$ in $\mathcal{P}(G)$ such that
(1) $\alpha_\#(\phi\big|_A) = \psi$
(2) $B = S_1 \sqcup \cdots \sqcup S_n$ with $S_i = \{x_i^+, x_i^-\}$ so that $\phi_0(j)(x_i^+) = x_i^-$ for all $0 \leq j \leq k$.
We denote by $\operatorname{Simp} Wh_\bullet(R,G)$ the simplicial category of simplices of $Wh_\bullet(R,G)$.

An (R,G)-*expansion functor* on a simplicial set $X_\bullet$ is a functor

$$\xi : \operatorname{Simp} X_\bullet \to \operatorname{Simp} Wh_\bullet(R,G)$$

which has the form $\xi(x) = (P(x), \phi(x)) \in \operatorname{Obj} Wh_k(R,G)$ for $x \in X_k$, so that each $a : y \to x$ in $\operatorname{Simp} X_\bullet$ with $y \in X_j$ induces a map

$$g_{yx} : \xi(y) = \big(P(y), \psi(y)\big) \to \big(P(x), \phi(x)\big|_y\big) = \xi(x)\big|_y$$

in $Wh_j(R,G)$ satisfying $g_{zx} = g_{yx}\big|_z \circ g_{zy}$ and $g_{xx} = id$.

Let $M \to E \to B$ be a manifold bundle and $f : E \to \mathbb{R}$ a fiberwise framed function. Let $\rho : \pi_1(E) \to G \subseteq R^\times$ be a representation. Triangulate B and choose for each simplex x a path from the basepoint of E to each Morse component of the singular set of $f\big|_x$. The construction given in talk 9 using the cellular Morse complex together with Theorem 1 above, gives a Δ^k-family of chain complexes over each simplex of B, and those fit into an (R,G)-expansion functor

$$\xi(E,\rho) : \operatorname{Simp} B \to \operatorname{Simp} Wh_\bullet(R,G)$$

(disregarding the fact that birth-death singularities do not give independent sub-posets as in condition (2) in the definition of the Whitehead category —this problem is solved with 'complicated algebra').

Let $Wh_\bullet^h(R,G)$ and $Wh_\bullet^{h[0,1]}(R,G)$ denote respectively the subcategory of acyclic complexes and the subcategory of acyclic complexes non-trivial only in degree 0 and 1.

Theorem 2 (2-index theorem). *For any ring R and $G \subseteq R^\times$,*

$$Wh^h_\bullet(R,G) \simeq Wh^{h[0,1]}_\bullet(R,G)$$

An acyclic complex non-trivial in only two degrees is given by an invertible matrix which depends on a choice of a basis for the complex. In particular, $Wh^{h[0,1]}_\bullet(R,G)$ is equivalent to a space of invertible matrices with coefficients in R, well-defined up to left and right multiplication by G-monomial matrices (i.e. permutation matrices with coefficients in G). When $(R,G) = (M_r(\mathbb{C}), U(r))$, one can define a smooth version of this space of matrices and produce explicit universal Franz-Reidemeister torsion classes in its cohomology. This defines the torsion as in the previous talk when $H_*(M;R)$ is trivial:

$$\tau_k(E,\rho) = \xi(E,\rho)^*(\tau_k) \in H^{2k}(B,\mathbb{R})$$

where $\tau_k \in H^{2k}(Wh^h(M_r(\mathbb{C}), U(r)), \mathbb{R})$ are these universal classes. (The construction of these classes is analogous to that of the classes defined in terms of super connections in talks 10–11.)

If K is a field and π a group, we say that a finite dimensional $K[\pi]$-module M is *upper triangular* if it has a filtration by $K[\pi]$-modules

$$M = M_n \supset M_{n-1} \supset \cdots \supseteq 0$$

such that π acts trivially on each M_i/M_{i-1}.

Higher Franz-Reidemeister torsion can be defined more generally using the following result:

Theorem 3. *Let $\xi : \operatorname{Simp} B \to \operatorname{Simp} Wh_\bullet(R,G)$ be a functor such that either of the following holds, where $v \in B$ is any basepoint:*
(1) The homology groups $H_n(\xi(v);R)$ are projective R-modules with trivial $\pi_1 B$-action.
(2) R is a field and each $H_n(\xi(v);R)$ is an upper triangular $\pi_1 B$-module.
Then there exists a functor

$$C(\xi) : \operatorname{Simp} B \to \operatorname{Simp} Wh^h_\bullet(R,G)$$

containing ξ as a subfunctor which is defined canonically up to homotopy (except on π_0).

The functor $C(\xi)$ is obtained from a cone construction which kills the homology of $\xi(v)$. The trivial $\pi_1(B)$-action allows to do it in one go (constructing a map over B from the A_∞-functor defined by the homology to the functor ξ). In the upper triangular case, one kills the homology layer by layer.

References

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Axioms of higher torsion

Bernard Badzioch

A smooth bundle of manifolds $F \to E \to B$ is unipotent if the rational homology groups of the fiber $H_*(F, \mathbb{Q})$ admit a filtration such that the action of $\pi_1 B$ on the quotients of the filtration is trivial. Unipotence of a bundle $E \to B$ is a sufficient condition for existence of higher Franz-Reidemeister torsion invariants [1] of that bundle $\tau^{2k}_{FR}(E) \in H^{4k}(B, \mathbb{R})$. The basic properties of these invariants are as follows.

1) (naturality) If $f\colon B' \to B$ is any map then $\tau^{2k}_{FR}(f^*E) = f^*\tau^{2k}_{FR}(E)$.

2) (additivity) If $E = E_1 \cup E_2$ where E_1, E_2 are unipotent bundles over B with the same vertical boundary then $\tau^{2k}_{FR}(E) = \tau^{2k}_{FR}(DE_1) + \tau^{2k}_{FR}(DE_2)$ where $DE_i = E_i \cup E_i$ is the fiberwise double of E_i.

3) (transfer) if $F \to E \to B$ is a unipotent bundle and $S^n \to D \to E$ is a bundle of spheres associated to an $SO(n+1)$ bundle over E then

$$\tau^{2k}_{FR,B}(D) = \chi(S^n)\tau^{2k}_{FR}(E) + \mathrm{tr}^E_B(\tau^{2k}_{FR,E}(D)).$$

Here $\tau^{2k}_{FR,B}(D)$ and $\tau^{2k}_{FR,E}(D)$ stand for the torsion invariants of the bundles $D \to B$ and $D \to E$ respectively, $\chi(S^n)$ is the Euler characteristic of S^n, and tr^E_B is the Becker-Gottlieb transfer associated to the bundle $E \to B$.

As it turns out [2] the above properties determine the the invariants τ^{2k}_{FR} uniquely up to a pair of scalars. More precisely, if τ^{2k} is any assignment which associates to every smooth, unipotent bundle $F \to E \to B$ a cohomology class $\tau^{2k}(E) \in H^{4k}(B, \mathbb{R})$ in such way that τ^{2k} satisfies the formulas 1)-3) then τ^{2k} determines scalars $a, b \in \mathbb{R}$ such that $\tau^{2k}(E) = a\tau^{2k}_{FR}(E)$ if the dimension of the fiber F is odd, and $\tau^{2k}(E) = b\tau^{2k}_{FR}(E)$ if the dimension of F is even.

An example of an invariants of unipotent bundles satisfying the properties 1)-3) is given by Miller-Morita-Mumford classed M_{2k} which – as a consequence – turn out to be just scalar multiples of the higher Franz-Reidemeister torsion invariants. Since for all bundles $E \to B$ with odd dimensional fibers we have $M_{2k}(E) = 0$, therefore in this case $a = 0$. For a bundle with even dimensional fibers we have

$$M_{2k}(E) = \frac{2(2k)!}{(-1)^k\zeta(2k+1)}\tau^{2k}_{FR}(E)$$

where ζ is the Riemann zeta function.

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Computation of higher Franz-Reidemeister torsion and the Framing Principle

Ulrich Bunke

Higher Reidemeister-Franz torsion is a characteristic class type invariant for smooth fibre bundles which can distinguish diffeomorphism types in fibre-homotopy equivalence classes of bundles. Its definition is quite complicated and was developed in detail in the book [1]. The rough idea of its construction is as follows. Let $E \to B$ be a smooth fibre bundle possibly with fibrewise boundary. Then we choose a generic family of fibrewise generalized Morse functions $f : E \to [0, 1]$. A function is called a generalized Morse function if it has has at most Morse- and birth-dead singularities. It induces on each fibre E_b, $b \in B$, a filtration of the total singular complex of E_b. In this way we get a family of filtered chain complexes which is classified by a map $\xi_f : B \to Wh$, where Wh is the classifying of filtered chain complexes usually called a Whitehead space. The Whitehead space can be analyzed by methods of algebraic K-theory. In particular, the Borel regulator classes in $H^{4k+1}(BGL(\mathbb{R})^+; \mathbb{R})$ induce universal torsion classes $\tau_{2k} \in H^{4k}(Wh; \mathbb{R})$. The higher Franz-Reidemeister torsion classes of $E \to B$ are then defined by

$$\tau_{2k}(E, \partial_0 E) = \xi_f^*(\tau_{2l})$$

(here $\partial_i E = f^{-1}(i)$, $i = 0, 1$). This rough idea is incorrect in several respects. First of all the Whitehead space Wh which carries the universal torsion classes is actually the classifying space for families of filtered acyclic complexes. But the fibres of $(E, \partial_0 E)$ are almost never acyclic. But under certain additional assumptions (e.g. if $\pi_1(B)$ acts trivially or upper-triangularly on $H^*(E, \partial_0 E, \mathbb{R})$) one can associate to f a family of acyclic chain complexes using a canonical mapping cone construction. We will denote the classifying map of the canonical mapping cone by $\tilde{\xi}_f : B \to Wh$. The other problem related to the topic of the talk is that the space of generalized fibrewise Morse functions on $E \to B$ is not connected, and the choice of f in different components may give different classes $\tilde{\xi}_f^*(\tau_{2k})$. In order to solve this problem on requires that the family of fibrewise generalized Morse functions used in the definition of higher Franz-Reidemeister torsion admits a framing of their unstable bundles at the critical sets. The basic observation is that there is a unique non-empty component of fibrewise framed generalized Morse functions [2]. A more precise definition of the higher Franz-Reidemeister torsion of the bundle $(E, \partial E_0) \to B$ is now

$$\tau_{2k}(E, \partial_0 E) := \tilde{\xi}_f^*(\tau_{2l}) ,$$

where f is a generic family of framed fibrewise generalized Morse functions.

For a given fibre bundle $E \to B$ it is some times easy to find a family of fibrewise generalized Morse functions, but it might be complicated to find a framed one. Let us discuss the disk bundle $D(V)$ of a real oriented euclidean vector bundle $V \to B$. The function $D(V) \ni v \mapsto f(v) := \|v\|^2$ is a family of fibrewise framed (the unstable bundle is zero-dimensional) generalized Morse functions and can be used to calculate $\tau_{2k}(D(V))$. In fact, in this case the classifying map $\tilde{\xi}_f : B \to Wh$ is constant so that

$$\tau_{2k}(D(V)) = 0 \tag{1}$$

for all $k > 0$. In order to calculate the relative torsion $\tau_{2k}(D(V), S(V))$ (where $S(V)$ denotes the unit sphere bundle of V) we would like to take the family of fibrewise Morse functions $g(v) := 1 - \|v\|^2$. The critical set $\Sigma_g \subset D(V)$ can be identified with the zero section of V, i.e. with B, and the unstable bundle $U_g \to \Sigma_g$ of g can be identified with V itself. Therefore, if V is not trivial, then g can not be framed. In this special case we can calculate the higher Reidemeister Franz torsion using additivity and the calculation for sphere bundles. The additivity gives

$$\tau_{2k}(D(V)) + \tau_{2k}(D(V), S(V)) = \tau_{2k}(D(V) \cup_{S(V)} D(V)) = \tau_{2k}(S(V \oplus \mathbb{R})) \ . \tag{2}$$

The calculation of higher Franz-Reidemeister torsion for sphere bundles gives [1, Thm 5.7.15]

$$\tau_{2k}(S(V \oplus \mathbb{R})) = (-1)^{\dim(V)+k} \frac{\zeta(2k+1)}{2} \mathbf{ch}_{4k}(V \otimes_{\mathbb{R}} \mathbb{C}) \ ,$$

where ζ is the Riemann zeta function. This formula follows from [1, Lemma 5.7.14]

$$\tau_{2k}(S(V \oplus W)) = (-1)^{\dim(W)} \tau_{2k}(S(V)) + (-1)^{\dim(V)} \tau_{2k}(S(W))$$

which can be proved using additivity, the splitting principle, and an explicit calculation of the torsion for one- and two-dimensional sphere bundles from the definitions. Using (1) and (2) togther we get

$$\tau_{2k}(D(V), S(V)) = (-1)^{\dim(V)+k} \frac{\zeta(2k+1)}{2} \mathbf{ch}_{4k}(V \otimes_{\mathbb{R}} \mathbb{C}) \ .$$

We thus have

$$\tau_{2k}(D(V), S(V)) - \tilde{\xi}_g^* \tau_{2k} = (-1)^{\dim(V)+k} \frac{\zeta(2k+1)}{2} \mathbf{ch}_{4k}(V \otimes_{\mathbb{R}} \mathbb{C}) \ .$$

This formula is a special case of the framing principle and in fact a crucial ingredient in its proof.

We now present a statement of the general framing principle. Let $(E, \partial_0) \to B$ be a smooth fibre bundle satisfying the additional assumptions mentioned above and choose a generic family of fibrewise generalized Morse functions $g : E \to [0,1]$, not necessarily framed. Since g is generic the singular set $\Sigma_g \subset E$ is a smooth submanifold. The projection $p : \Sigma \to B$ is almost everwhere (outside the birth-death locus $\Sigma_g^{bd} \subset \Sigma_g$) a local diffeomorphism. It has a canonical orientation so that the integration $p_! : H^*(\Sigma; \mathbb{R}) \to H^*(B; \mathbb{R})$ is well-defined. The submanifold $\Sigma_g^{morse} := \Sigma_g \setminus \Sigma_g^{bd}$ is graded by the index of the critical points, i.e. we have a decomposition $\Sigma_g^{morse} := \sqcup_{i \geq 0} \Sigma_g^i$. Let $U_g^i \to \Sigma_g^i$ denote the unstable bundle of g along Σ_g^i. The collection of $\dim(E) - \dim(B)$-dimensional bundles $U_g^i \oplus \mathbb{R}^{\dim(E)-\dim(B)-i}$ can be glued together along Σ_g^{bd} (this employs the special structure of a birth-dead singularity) to the stabilized unstable bundle $U_g \to \Sigma_g$ of g. The framing principle [3, Thm 4.11] states that

$$\tau_{2k}(E, \partial_0 E) - \tilde{\xi}_g^* \tau_{2k} = (-1)^k \frac{\zeta(2k+1)}{2} p_*(\mathbf{ch}_{4k}(U_g \otimes_{\mathbb{R}} \mathbb{C})) \ .$$

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The Bismut-Lott index theorem and higher analytic torsion.

Wojciech Dorabiała

The purpose of this talk was to define higher analytic torsion and state the Bismut-Lott index theorem for flat vector bundles along with its geometric refinement, following [1].

The analytic torsion arises from writing a topological theorem in terms of differential forms. The topological theorem in question is an index theorem for flat vector bundles. To state the theorem we first need to define certain characteristic classes of flat $\mathbb{Z}_2$-graded vector bundles; the construction mimics the Chern-Weil theory. Let E be a complex $\mathbb{Z}_2$-graded vector bundle over a smooth manifold M with A' a flat superconnection on E. Choose an arbitrary Hermitian metric h^E on E such that E_+ and E_- are orthogonal. Then there is another flat superconnection $A^{'*}$ on E as well, which is the adjoint of A' with respect to h^E. We can now define an element of $\Omega(B; \mathrm{End}(E))$ by

$$X = \frac{1}{2}(A^{'*} - A').$$

Definition 1. Let $f : \mathbb{C} \to \mathbb{C}$ be an odd holomorphic function. Define the characteristic differential form by

$$f(A', h^E) = (2\pi i)^{1/2} \varphi Tr_s[f(X)] \in \Omega(B)$$

where $\varphi : \Omega(B) \to \Omega(B)$ indicates the linear map such that for any homogeneous $\omega \in \Omega(B)$,

$$\varphi(\omega) = (2\pi i)^{-(\deg \omega)/2} \omega.$$

Lemma 2. *The characteristic differential form $f(A', h^E)$ is real, odd and closed. Furthermore, its de Rham cohomology class is independent of the choice of the Hermitian metric h^E, hence will be denoted by $f(A')$.*

If $E = \overset{n}{\underset{i=0}{\oplus}} E^i$ is a $\mathbb{Z}$-graded complex vector bundle on M, we can apply the above formalism to the associated $\mathbb{Z}_2$-graded vector bundle, with $\mathbb{Z}_2$ grading given by $E_+ = \underset{i=\text{even}}{\oplus} E^i$ and $E_- = \underset{i=\text{odd}}{\oplus} E^i$. Thus, it will suffice to indicate the appropriate

$\mathbb{Z}$-graded complex vector bundle on B associated to a smooth fiber bundle $Z \to M \to B$, where B is a compact, connected base with connected, closed fiber Z_b and flat complex vector bundle F on M.

Thus, we let $H(Z_b, F|_Z)$ be the $\mathbb{Z}$-graded complex vector bundle on B whose fiber over $b \in B$ is isomorphic to the graded cohomology group $H^*(Z_b, F|_{Z_b})$. It has a canonical flat connection $\bigtriangledown^{H(Z,F|_Z)}$ which preserves the $\mathbb{Z}$-grading.

As usual, TZ indicates the vertical tangent bundle of the fiber bundle $Z \to M \to B$, $o(TZ)$ the orientation bundle of TZ, and $e(TZ) \in H^{\dim(Z)}(M; o(TZ))$ denotes the Euler class of TZ.

Theorem 3 ([1])**.** *The following equation holds in $H^{\mathrm{odd}}(B;\mathbb{R})$:*

$$f(\bigtriangledown^{H(Z,F|_Z)}) = \int_Z e(TZ) \cup f(\bigtriangledown^E).$$

Suppose that we want a differential-form version of Theorem 3. We first need to equip the fiber bundle with a horizontal distribution $T^H M$.

If we let W be the infinite-dimentional $\mathbb{Z}$-graded vector bundle on B whose fiber over $b \in B$ is isomorphic to $\Omega(Z_b; F|_{Z_b})$, then there is an isomorphism of spaces of smooth sections

$$C^\infty(B; W) \approx C^\infty(M; \Lambda(T^*Z) \otimes F)$$

and an isomorphism of $\mathbb{Z}$-graded vector spaces

$$\Omega(B; W) \approx \Omega(M; F).$$

In this case, the exterior differentiation operator d^M, acting on $\Omega(M;F)$, defines a flat superconnection on W. In terms of the $\mathbb{Z}$-grading on $\Lambda(T^*B)$, d^M can be decomposed as

$$d^M = d^Z + \bigtriangledown^W + i_T$$

where

- d^Z represents vertical exterior differentiation
- $\bigtriangledown^W$ is a natural connection on W which preserves the $\mathbb{Z}$-grading
- i_T is interior multiplication by the curvature of the fiber bundle.

Let N be the number operator of W acting by multiplication by j on the component
$C^\infty(M; \Lambda^j(T^*Z) \otimes F)$. Using this, for $t \in (0, \infty)$ we define the flat superconnection on W

$$C_t^{'} = t^{N/2} d^M t^{-N/2} = \sqrt{t} d^Z + \bigtriangledown^W + \frac{1}{\sqrt{t}} i_T$$

and we will be interested below in the behavior near each end of the t interval of the associated characteristic classes.

To study differential forms on the vector bundle W, we have to define a suitable metric on W. We start by choosing a vertical Riemannian metric g^{TZ} and a Hermitian metric h^F on F. Then W acquires an L^2-inner product h^W. If we let

$C_t^{''}$ be the adjoint superconnection to C'_t with respect to h^W, then the adjoint is given by

$$C_t^{''} = \sqrt{t}(d^Z)^* + (\nabla^W)^* - \frac{1}{\sqrt{t}}(T\wedge).$$

As with X above, we then define an odd element D_t of $\Omega(B; \mathrm{End}(W))$ by

$$D_t = \frac{1}{2}(C_t^{''} - C_t^{'}).$$

As in Lemma 2, we next define a real, odd differential form on B by

$$f(C_t^{'}, h^W) = (2i\pi)^{1/2}\varphi Tr_s[f(D_t)]$$

and even differential form on B by

$$f^\wedge(C_t^{'}, h^W) = \varphi Tr_s\left[\frac{N}{2}f^{'}(D_t)\right].$$

The following technically important fact identifies the relationship between these two families of characteristic classes.

Lemma 4. *For any $t \in (0, \infty)$,*

$$\frac{\partial}{\partial t}f(C_t^{'}, h^W) = \frac{1}{t}df^\wedge(C_t^{'}, h^W).$$

In what follows, we will need the following integer valued function on B:

$$\chi'(Z; F) = \sum_{i=0}^{\mathrm{dim(Z)}}(-1)^i rk(H^i(Z; F|_Z)).$$

The behavior of the associated characteristic classes near the endpoints of the interval $t \in (0, \infty)$ is given by the following Lemma.

Lemma 5. *As $t \to 0$*

$$f(C_t^{'}, h^W) = \begin{cases} \int_Z e(TZ, \nabla^{TZ})f(\nabla^E, h^F) + \mathcal{O}(t) & \text{if } dim\ (Z) \text{ is even;} \\ \\ \mathcal{O}(\sqrt{t}) & \text{if } dim\ (Z) \text{ is odd} \end{cases}$$

$$f^\wedge(C_t^{'}, h^W) = \begin{cases} \frac{1}{4}\mathrm{dim}(Z)\mathrm{rk}(F)\chi(Z) + \mathcal{O}(t) & \text{if } dim\ (Z) \text{ is even;} \\ \\ \mathcal{O}(\sqrt{t}) & \text{if } dim\ (Z) \text{ is odd} \end{cases}$$

As $t \to \infty$

$$f(C_t^{'}, h^W) = f(\nabla^{H(Z;F|_Z)}, h^{H(Z;F|_Z)}) + \mathcal{O}(\frac{1}{\sqrt{t}})$$

and

$$f^\wedge(C_t^{'}, h^W) = \frac{\chi^{'}(Z;F)}{2} + \mathcal{O}(\frac{1}{\sqrt{t}}).$$

With all of the above definitions in hand, we are ready to define the analytic torsion.

Definition 6. The analytic torsion form

$$\mathcal{T}(T^HM, g^{TZ}, h^F) = -\int_0^\infty \Big[f^\wedge(C_t', h^W) - \frac{\chi'(Z;F)}{2} f'(0) - \Big(\frac{\dim(Z)\mathrm{rk}(F)\chi(Z)}{4} - \frac{\chi'(Z;F)}{2} \Big) f'\big(\frac{i\sqrt{t}}{2}\big) \Big] \frac{dt}{t}.$$

Remark 7. It follows from Lemma 5 that the integrand in definition 6 is integrable.

Unfortunately, the form $\mathcal{T}(T^HM, g^{TZ}, h^F)$ is not closed in general. However the following Lemma, which follows from Lemmas 4 and 5, gives the formula for the differential in any case.

Lemma 8.

$$d\mathcal{T}(T^HM, g^{TZ}, h^F) = \int_Z e\Big(TZ, \nabla^{TZ}\Big) \wedge f\Big(\nabla^F, h^F\Big) - f\Big(\nabla^{H(Z;F|z)}, h^{H(Z;F|z)}\Big).$$

Notice Lemma 8 implies that if Z is odd-dimensional and $H_p(Z; F|Z)$ vanishes for all p then
$\mathcal{T}(T^HM, g^{TZ}, h^F)$ is closed. Bismut and Lott [1] have showed that its de Rham cohomology class $T_{k-1}(M, F)$ is independent of the choices of T^HM, g^{TZ} and h^F. Thus $T_{k-1}(M, F) \in H^{k-1}(B; \mathbb{R})$ is a smooth invariant of the pair (M, F). In the case $k = 1$, $T_0(M, F)$ is represented by the single locally constant function on B which, to any point $b \in B$, assigns half of the Reidemeister torsion of the pair $(Z_b, F|_{Z_b})$.

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The Witten deformation and the analytic framing principle

Sebastian Goette

The goal of this talk is a comparison of the higher Franz-Reidemeister torsion of Igusa and Klein and the higher analytic torsion of Bismut and Lott. We consider a smooth fibre bundle $p\colon E \to B$ with compact base B and fibres M, and a flat complex vector bundle $F \to E$. If F carries a parallel metric and the bundle $H := H^\bullet(E/B; F) \to B$ is trivial, then both Igusa-Klein's $\tau(E/B; F) \in H^{\mathrm{even}}(B; \mathbb{R})$ and Bismut-Lott's $\mathcal{T}(E/B; F) \in H^{\mathrm{even}}(B; \mathbb{R})$ are well-defined. We will employ the Witten deformation as in [2] and [1] to compare both torsions.

Here is some motivation for a comparison formula:

- Several computations have been done for higher Franz-Reidemeister torsion but not for higher analytic torsion, and vice versa;

- Additivity of higher analytic torsion has not yet been proved, so the axiomatic approach (see talk 15) is not (yet) applicable;
- Ma has proved a general transfer formula for higher analytic torsion in [5], which would hold for higher Franz-Reidemeister torsion as well if both were related;
- Higher analytic torsion can be defined in $\Omega^{\mathrm{even}}(B)/d\Omega^{\mathrm{odd}}(B)$ in very general cirumstances, and some methods for computating the higher Franz-Reidemeister torsion could be applied even in this setting.

Let $p\colon E \to B$ and $F \to E$ be as above, and let $TM \to E$ denote the vertical tangent bundle. Assume that $h\colon E \to \mathbb{R}$ is a fibrewise Morse function, and let $\hat{p}\colon C \to B$ denote the covering of B by the fibrewise critical points of h. Let $TM|_C = T^sM \oplus T^uM$ denote the splitting into positive and negative directions of d^2h, and let $o(T^uM)$ denote the orientation bundle of T^uM. Then we consider the vector bundle

$$V = \hat{p}_*(F|_C \otimes o(T^uM)) \to B$$

that is $\mathbb{Z}$-graded by the index of h, and inherits a flat connection ∇^V from $\nabla^{F|C}$ and an endomorphism h^V given by multiplication with $h|_C$. In analogy with talks 9 and 13, it is possible to define a flat superconnection A' on V such that $A' - \nabla^V$ is upper triangular with respect to h^V, and an $\Omega^\bullet(B)$-linear fibrewise quasiisomorphism

$$I\colon (\Omega^\bullet(E;F), d_E) \to (\Omega^\bullet(B;V), A') . \tag{1}$$

As in talk 11, one can now define a finite-dimensional analytic form $T(A', g^V, h^V) \in \Omega^{\mathrm{even}}(B)$. The objectives are

- a comparison of $\mathcal{T}(T^HE, g^{TM}, g^F)$ and $T(A', g^V, h^V)$, and
- under suitable hypotheses a comparison of $T(A', g^V, h^V)$ and $\tau(E/B; F)$.

1. The Witten deformation

We fix a horizontal complement $T^HE \subset TE$ of TM and metric g^{TM} and g^F on TM and on F as in the previous talk. Let us extend p to a bundle

$$p''\colon E'' = E \times (0,\infty) \times \mathbb{R} \to B'' = B \times (0,\infty) \times \mathbb{R} ,$$

let F'' denote the pullback of F by $p_E\colon E'' \to E$, let TM'' denote the vertical tangent bundle, and extend T^HM, g^F and g^{TM} to the new setting in the obvious way. Then we define

$$g^{TM''}_{(p,t,T)} = \frac{1}{t}\, g^{TM}_p$$

as in talk 17, and regard the Witten deformed metric

$$g^{F''}_{(p,t,T)} = e^{-2Th(p)}\, g^F_p$$

on F''. Then $g^{TM''}$ and $g^{F''}$ induce a new L^2-metric g'' on the infinite-dimensional vector bundle $\Omega^\bullet(E''/B''; F'') \to B''$. The exterior differential $d_{E''}$ still defines a superconnection on $\Omega^\bullet(E''/B'';F'')$, and as in the previous talk, we may regard the associated Kamber-Tondeur form

$$f(d_{E''},g'')\in\Omega^{\mathrm{odd}}(B'')\;.$$

Let us fix $0<\varepsilon<A<\infty$ and $T_0>0$, and consider the rectangle $R=[\varepsilon,A]\times[0,T_0]$. Because $f(d_{E''},g'')$ is a closed form, the relative version of Stokes' theorem tells us that

$$d\int_{B\times R/B} f(d_{E''},g'')=\int_{B\times\partial R/B} f(d_{E''},g'')\in\Omega^{\mathrm{even}}(B)\;. \tag{2}$$

We now compute the limits of the integral over each of the four sides of ∂R separately as $A\to\infty$, $T_0\to\infty$ and $\varepsilon\to 0$ (in that order). By [1] and [3], up to some divergent terms that cancel in the end,

(*i*) the integral over the left side ($T=0$) becomes the higher analytic torsion $\mathcal{T}(T^HE,g^{TM},g^F)$,

(*ii*) the integral over the upper side ($t=A$) becomes a correction term from the fibrewise cohomology bundle $H\to B$,

(*iii*) the integral over the right hand side ($T=T_0$) becomes the finite-dimensional analytic torsion form $T(A',g^V,h^V)$, and

(*iv*) the integral over the bottom line ($t=\varepsilon$) gives two contributions: a local correction term defined as an integral over the fibres of p of some differential form, and

(*v*) an "exotic" additive class $\hat{p}_*(J(T^sM)-J(T^uM))\,\mathrm{rk}F$.

In particular, the computation of the higher analytic torsion (*i*) can be reduced to the computation of the somewhat simpler terms (*ii*)–(*v*), where (*iii*) is the finite-dimensional analytic torsion as in talk 11.

The last term (*v*) can be interpreted as the torsion of the $\mathbb{Z}_2$-graded vector bundle $TM|_C\to C$. The similarity with Igusa's formula for non-framed functions (cf. talk 16) has led to the name "analytic framing principle".

2. A comparison of the higher torsions of Igusa-Klein and Bismut-Lott

We will now assume in addition that F carries a parallel metric g^F, and (for simplicity) that the fibrewise cohomology bundle $H\to B$ is trivial. In this situation, both $\tau(E/B;F)$ and $\mathcal{T}(E/B;F)$ can be defined. To compare the two, one has to interpret the five contributions to the right hand side of (2) described above. By our assumptions on F and H, the first contribution becomes $\mathcal{T}(E/B;F)$, the third contribution becomes a finite-dimensional torsion class $\tau(E/B;F,h)$, and the correction terms (*ii*) and (*iv*) vanish.

Flat superconnections A' on V such that $A'-\nabla^V$ is in upper triangular shape with respect to h^V as in (1) are classified up to flat, upper triangular homotopy by homotopy classes of maps $B\to\mathrm{Wh}(\mathbb{C},U)$, where $\mathrm{Wh}(\mathbb{C},U)$ is the full Whitehead space as in talks 13, 14. A cone construction then gives a map ξ to the acyclic Whitehead space $\mathrm{Wh}^h(\mathbb{C},U)$. It is proved in [4] that

- The map ξ is homotopic to the one constructed by Igusa, cf. talks 13, 14,
- The class $\tau(E/B;F,h) = \xi^*\tau(\mathbb{C},U)$, where $\tau(\mathbb{C},U)$ is Igusa's universal torsion class on $\mathrm{Wh}^h(\mathbb{C},U)$.

Thus, taking limits in (2) now gives

$$\mathcal{T}(E/B;F) = \xi^*\tau(\mathbb{C},U) + \hat{p}_* J(T^sM - T^uM)\,\mathrm{rk}F\ .$$

Together with Igusa's framing principle

$$\tau(E/B;F) = \xi^*\tau(\mathbb{C},U) - 2\hat{p}_* J(T^uM)\,\mathrm{rk}F\ ,$$

we finally obtain (up to normalisation)

$$\begin{aligned}\mathcal{T}(E/B;F) &= \tau(E/B;F) + \hat{p}_* J(TM|_C)\,\mathrm{rk}F \\ &= \tau(E/B;F) + M(E/B)\,\mathrm{rk}F\ ,\end{aligned} \tag{3}$$

where $M(E/B)$ denotes the total Miller-Morita-Mumford class. In the language of talk 15, the analytic torsion thus equals the odd part of the higher Franz-Reidemeister torsion.

The final statement of (3) no longer involves the fibrewise Morse function h. It is therefore reasonable to conjecture that (3) holds for all fibre bundles $E \to B$ and all coefficient bundles $F \to E$ satisfying the conditions above.

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Equivariant Analytic Torsion

Gregor Weingart

Equivariant analytic torsion is analytic torsion in the presence of symmetries and it is interesting at least for two different reasons. In the first place symmetry is manifest in the known examples, where it is possible to calculate the spectrum of the Laplace operator and thus the analytic torsion explicitly. Moreover many interesting fibre bundles are associated to a principal bundle over a base B with compact structure group G and a G–manifold M. In this situation it is tempting to relate the equivariant torsion of the model fibre M to the higher torsion of the fibre bundle. In my talk I wanted to illustrate these two points of view by the calculation of the analytic torsion for symmetric spaces by Köhler and the relation to the equivarint Euler characteristic found by Bunke.

Interestingly there are two seemingly unrelated notions of equivariant torsion. Recall that the analytic torsion is defined by a zeta function associated to the Laplace operator Δ on forms. Without touching Δ we can modify the zeta function

$$Z_\gamma(s) \;:=\; \frac{1}{\Gamma(s)} \int_0^\infty \big[\mathrm{Str}(\, N\,\gamma\, e^{-t\Delta}\,) \;-\; \chi'(\gamma)\,\big]\; t^s\, \frac{dt}{t}$$

where γ is an arbitrary isometry of M, N is the fermionic number operator on forms and $\chi'(\gamma) := \sum_{k\geq 0}(-1)^k\, k\,\mathrm{tr}_{H^k(M)}(\gamma)$ the derived Euler character of the cohomology. In this setup the equivariant analytic torsion is the class function $T(M,\gamma) := \exp(-\frac{1}{2}Z'_\gamma(0))$ on G.

Alternatively there is an infinitesimal approach to equivariant analytic torsion replacing the Laplace operator by its equivariant twist $\Delta_{\mathfrak{g}}$, which is a differential operator on formal power series on the Lie algebra $\mathfrak{g}$ of G with coefficients in the forms on M. The heat kernel $e^{-t\Delta_{\mathfrak{g}}}$ is defined formally using the heat kernel $e^{-t\Delta}$ of Δ as a formal parametrix. With this proviso the equivariant zeta function

$$Z_{\mathfrak{g}}(s) \;:=\; \frac{1}{\Gamma(s)} \int_0^\infty \big[\mathrm{Str}(\, N\, e^{-t\Delta_{\mathfrak{g}}}\,) \;-\; \chi'\,\big]\; t^s\, \frac{dt}{t}$$

with $\chi' := \sum_{k\geq 0}(-1)^k\, k\dim H^k(M)$ is a well–defined invariant formal power series on $\mathfrak{g}$ and $\tau_{\mathfrak{g}}(M) = -\frac{1}{2}Z'_{\mathfrak{g}}(0)$ is the infinitesimal (logarithmic) analytic torsion. The time limit for the talks did not allow me to discuss the relation between these two seemingly disparate notions of equivariant analytic torsion uncovered by the work of Bismut–Goette [1]. Instead I discussed the two theorems mentioned above:

Theorem: (K. Köhler [5]**)**
Suppose G/K is an odd–dimensional symmetric space, in particular $\mathrm{rank}\, G$ *and* $\mathrm{rank}\, K$ *have different parity. The zeta function integrand* $\mathrm{Str}(N\gamma e^{-t\Delta}) = 0$ *vanishes identically in t unless* $\mathrm{rank}\, G = \mathrm{rank}\, K + 1$ *and in this case*

$$Z_\gamma(s) \;=\; \sum_{\substack{[w]\in W_G/W_K \\ \langle w\alpha, w\alpha+2\rho\rangle > 0}} -\,\frac{\mathrm{Tr}_{\pi_{\rho+w\alpha}}(\gamma)}{\langle w\alpha, w\alpha+2\rho\rangle^s}$$

where ρ and W_G are half sum of positive weights and the Weyl group of G and $\pi_{\rho+w\alpha}$ is the irreducible representation of G of heighest weight $\rho+w\alpha$, while α is the positive generator for the group of weights of G vanishing on K.

The equivariant analytic torsion of all symmetric spaces G/K is easily calculated using this theorem, because on an oriented even–dimensional Riemannian manifold $\mathrm{Str}(N\gamma e^{-t\Delta}) = \chi'(\gamma)$ is independent of t for every isometry γ due to a spectral symmetry. In analogy of the reasoning of de Rham a direct corollary of this calculation is that two isometries of the odd–dimensional Grassmannians $\mathrm{Gr}_r(\mathbb{R}^m)$, m even and r odd, are conjugated by diffeomorphisms if and only if they are conjugated by isometries.

Turning to the infinitesimal point of view I recalled that a G–CW complex for a topological group G is a topological space M glued together from G–cells $D^n_H := G/H \times D^n$ of dimension n, where H a closed subgroup of G and D^m the closed standard ball in $\mathbb{R}^m$, along G–equivariant maps of the boundary $S_H^{m-1} := G/H \times S^{m-1}$. To every finite G–CW complex X we can associate the element

$$[X] \;:=\; \sum_{D_H^n \text{ cell of } X} [G/H] \;\in\; \bigoplus_{\substack{\text{conjugacy class}\\ \text{of closed } H\subset G}} \mathbb{Z}\,[G/H]$$

in the free $\mathbb{Z}$–module on the set of conjugacy classes of closed subgroups of G. This module becomes a ring, the so–called Euler ring $U(G)$ of G, under the multiplication $[G/H_1] \cdot [G/H_2] := [G \times G/H_1 \times H_2]$. The equivariant Euler characteristic of a compact manifold M endowed with an action of a Lie group G is the sum

$$\chi_G(M) \;:=\; \sum_{D_H^n \text{ cell of } M} (-1)^n\,[G/H] \;\in\; U(G)$$

over the G–cells of a natural cell decomposition of M as a finite G–CW complex:

Theorem: (U. Bunke [3][4]**)**
There exists an additive map $T_G : U(G) \longrightarrow [\mathrm{Sym}^+\mathfrak{g}^*]^G$ from the Euler ring of a compact Lie group G to the invariant formal power series $[\mathrm{Sym}^+\mathfrak{g}^*]^G$ on $\mathfrak{g}$ with vanishing constant term, such that T_G and T_H for a closed subgroup $H \subset G$ are intertwined by the natural restriction maps $\mathrm{res}_H^G \circ T_G = T_H \circ \mathrm{res}_H^G$ and

$$T_G(\,\chi_G(M)\,) \;\equiv\; \tau_{\mathfrak{g}}(M)$$

modulo constants. In fact T_G is completely determined by its restriction to closed subgroups $S^1 \subset G$ of G isomorphic to S^1 and the explicit values:

$$T_{S^1}(\,[S^1/S^1]\,) \;=\; 0 \qquad T_{S^1}(\,[S^1/\mathbb{Z}_r]\,) \;=\; 2\sum_{k\geq 1}\binom{4k}{2k}\zeta(2k+1)\,(\,\frac{r\,dt}{8\pi}\,)^{2k}$$

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