Arbeitsgemeinschaft mit aktuellem Thema:

Twisted K-theory

Mathematisches Forschungsinstitut Oberwolfach

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Organizers:

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Introduction:

Ordinary (co)homology theories always come along with twisted versions of themselves; the most basic example is cohomology twisted by the orientation bundle, which shows up when one discusses Poincaré duality (and more general push-forward maps) for non-orientable manifolds.

Twisted K-theory is important from this point of view. Even if a manifold is orientable in the ordinary sense, it's K-theory does not satisfy Poincaré duality. However, this is the case if one considers twisted K-theory (one has to twist with the bundle of complex Clifford algebras of the cotangent bundle). In fact, some constructions of twisted cohomology theories are quite classical and can be done in the context of parametrized stable homotopy theory (which on the other hand is itself constantly developing further).

The modern interest in twisted K-theory stems, however, from mathematical physics, in particular from string theory. In this theory D-branes are objects whose charges are measured, in the presence of a B-field, by twisted K-theory. The topological backgrounds of B-fields β on a space X are classified by three dimensional integral cohomology classes. Representatives of the B-field (we will call them twists during the Arbeitsgemeinschaft) are precisely the data needed to define twisted K-theory $K^\beta(X)$. In one version of the theory one associates to a twist a non-commutative C^*-algebra whose K-theory is by definition the twisted K-theory. It makes sense to think of the space time with B-field as a non-commutative space (or space-time) in the sense of Connes.

Twisted K-theory in mathematics has evolved to an interdisciplinary area which combines elements of topology, non-commutative geometry, functional analysis, representation theory, mathematical physics and other. In the Arbeitsgemeinschaft we want to present various aspects of the foundations of twisted K-theory and the key calculations. We will discuss the construction of twisted equivariant K-theory in different contexts (e.h. homotopy theory, non-commutative geometry, groupoids or stacks) and the verification of the basic functorial properties. We will see how the different models are related.

In order to get used to the definitions we will see some calculations of twisted K-theory using methods from algebraic topology (Mayer-Vietoris sequences and some spectral sequences). The Umkehr- (or integration or Gysin map) for twisted K-theory is of particular importance and will be illustrated through an interpretation of the classical Borel-Weil-Bott theorem. The culmination of the Arbeitsgemeinschaft is the calculation of the equivariant twisted K-theory of compact Lie groups due to Freed-Hopkins-Teleman and the interpretation of this result in the context of representation theory of loop groups.

The program of the Arbeitsgemeinschaft is designed to contain various points of re-entry presented by talks on different aspects of twisted K-theory, e.g. T-duality, non-commutative geomtry and topological stacks.

In the string theory context, some aspects of mirror symmetry are also reflected in twisted K-theory; under certain situations, a non-commutative space-time (i.e. a space with B-field) will have a dual with isomorphic K-theory, possibly in shifted degrees. Mathematically, this is worked out as T-duality and will be studied also as a computational tool in during this AG.

In connection with T-duality, but also in the equivariant situation not only ordinary spaces but more singular objects naturally show up. This results in the need to work out equivariant twisted K-theory, twisted K-theory for orbifolds, and for even more singular spaces. A convenient framework to develop this in the necessary generality is the language of stacks. We will introduce them and use them in part of our talks.

The non-commutative geometry aspect of twisted K-theory will be in particular developed in the talks about the construction of the Chern character which relates twisted K-theory to twisted (de Rham) cohomology. We will see that sheaf theory is useful here and understand that developments in twisted K-theory are in fact topological versions of similar results and constructions in algebraic geometry.

An interesting feature of the definition of twisted K-theory in terms of cycles and relations is that those cycles appear naturally in geometric and analytic constructions. As mentioned above, starting from representations of compact Lie groups, an explicit construction of cycles given by families of Dirac type operators will be discussed in connection with the Borel-Weil-Bott theorem. In a more elaborate way, this also works for (projective) representations of loop groups, as was first discovered in the physics related literature. Using this idea the calculation of the equivariant twisted K-theory of a compact Lie group G (acting on itself by conjugation) by Freed-Hopkins-Teleman (FHT) can be explicitly interpreted in terms of cycles. In this way the twisted K-theory is identified with the K-group of projective positive energy representations of the loop group $\mathcal{L}G$. The twist corresponds to the "level" of the representation. It is one of the major goals of the AG to prove this FHT theorem, and to develop the necessary interesting mathematics which is used there.

It turns out that the twisted K-theory on the one side, and the K-group of projective positive energy representations on the other side, both have a subtle products (in K-theory the Pontryagin product induced by the multiplcation map, and the Fusion product (Verlinde algebra) on the other side), and the FHT-isomorphism respects these multiplications (may be, under some additional assumptions). We will address the construction of the K-theoretic product, but will not be able to prove multiplicativity of the FHT-isomorphism. In the Arbeitsgemeinschaft we will actually discuss a related product for the twisted K-theory of orbifolds with is closely related to the quantum product in orbifold cohomology.

Talks:

The Arbeitsgemeinschaft will start with the introduction of equivariant, twisted, and equivariant twisted K-theory of spaces; and we will work out the basic properties and make some explicit calculations.

We will then introduce the notion of stacks and then define twisted K-theory for certain stacks, in particular for orbifolds. We continue with calculations via the Chern character.

From a somewhat different angle, we will spend one talk to look at twisted K-theory via parametrized stable homotopy theory.

Another, somewhat independent talk, explains T-duality and its relevance for twisted K-theory.

The last 5 talks (13-17) are then devoted to the FHT-theorem. The first two, independent of each other, give the constructions of twists and twisted K-theory classes in the context of loop group representations.

The last three talks are then devoted to the discussion of the formulation and proof of the main theorem of Freed-Hopkins-Teleman, calculation the twisted equivariant K-theory of G in terms of projective representation of its loop group.

We hope to include one additional talk on the physic background, if expertise in this direction will participate in the AG an is willing to prepare such a talk.

Remark: *A general remark for the speakers: as usual, there are so many interesting constructions and results one wants to cover, that the description of some of the talks might actually contain more than can be presented in a good way. Should this turn out during the preparation, one should contact the organizers and discuss which parts can be cut, rather than try to squeeze in too much (and as a result make the talk unpalatable). We plan to have 80 minute talks. The focus clearly is that a non-expert also has a chance to understand what is going on.*

1. **Equivariant K-theory**

   The goal this first talk of the school is to explain equivariant K-theory as an example of an equivariant cohomology theory. The speaker should concentrate on explaining the definitions and constructions - proofs are not required. We will assume basic knowledge of K-theory as an example of a generalized cohomology theory. The talk should introduce the axioms for equivariant cohomology theories [MM04, Section 2] for compact groups. As an instructive example it should be explained that every generalized cohomology theory admits an equivariant extension by the Borel construction. For the purpose of the AG the induction-restriction structure of an equivariant cohomology theory plays a less important role and could be omitted in order to save time.

   In the talk the definition of equivariant K-theory via G-vector bundles [Seg68, Beginning of Section 2] and C^*-algebras should be given [Bla98, Section 11]. Mention the example of $X = \{*\}$ [Seg68, Example (i) p.132] or more generally [Seg68, Proposition 2.2]. Describe the Thom homomorphism and the Thom isomorphism theorem in this context [Seg68, Proposition 3.2], and use this to define equivariant K-theory as graded equivariant cohomology theory [Seg68, Proposition 3.5]. A quite readable alternative reference is the book [Fri78].

In the talk it should be explained that equivariant K-theory is different from the theory obtained by applying the Borel construction to non-equivariant K-theory. Using the completion theorem [AS69] this can be seen by calculating the example $K_G(*)$ and $K_G^{Borel}(*)$ (the equivariant K-theory and the Borel construction K-theory of the point with the trivial G-action) and observing that the completed representation ring differs from the uncompleted one.

The relation between the vector bundle and the C^*-algebra approach to K-theory is given by the equivariant Swan theorem [Bla98, Theorem 11.4.2]. The passage from equivariant to non-equivariant K-theory via the crossed product construction [Bla98, Theorem 11.7.1] should be mentioned as a construction in the sprit of non-commutative geometry.

A third definition of equivariant K-theory uses G-homotopy classes of maps to a suitable G-classifying space ([AS04, p. 28], compare also [Seg70, Section 5]) . The model $Fred_{G-cts}$ of [AS04, Appendix 3] of this classifying space (for K^0) should be described. In fact, the construction of twisted equivariant K-theory classes in [FHTa] uses a twisted version of this approach.

2. Twisted K-theory - basic definitions

Twisted cohomology theories are based on a suitable notion of twists. Isomorphism classes of twists are usually classified in some homotopy theoretic way. But their automorphisms play an equally important role since they act non-trivially on the twisted cohomology groups. An axiomatic framework for this, which is particularly well suited for twisted K-theory, is presented in [BS05, Section 3.1]. The Mayer-Vietoris sequence for twisted K-theory of S^3 (just using the axioms and the claim how the differential looks like) of [BS05, Section 4.1] should be presented.

Our first model of twisted K-theory will be based on bundles of compact operators or of projective Hilbert spaces, respectively. The speaker should introduce these objects and discuss their classification in terms of third Cech cohomology (via the Dixmier-Douady class), following e.g. [AS04, Section 2]. Note that the classification of automorphisms [AS04, Proposition 2.2] is as important as the classification of the bundles themselves. The extension (taking orientation twists into account) of [AS04, Proposition 2.3] should be explained. Do *not* include a discussion of more general twists at this point.

The K-theory twisted by such projective Hilbert space bundles should be defined first in terms of a classifying space [AS04, Definition 3.3], then in terms of the K-theory of the continuous trace algebra of sections of the associated bundles of compact operators [AS04, Definition 3.4]. An alternative proof of the equivalence of the homotopy theoretic with the C^*-algebra point of view can be found in [Ros89, Proposition 2.1].

The talk should adress methods of comptation. This includes in particular the Mayer-Vietoris sequence and Bott periodicity [AS04, Section 4]. As an important tool the Atiyah-Hirzebruch spectral sequence in twisted K-theory should be constructed in the usual way from the filtration of the space by skeleta [AS, Section 4]. Of particluar importance is the derivation of the d_3-differential, [AS, Proposition 4.6]. The relation beween Massey products and the higher differentials of this spectral sequence [AS, Section 5] should be discussed.

As an illustration some explicit calculations of twisted K-theory should be presented. Choose from [BS05, Section 4] (in particular [BS05, Section 4.3]), and the examples with interesting Massey products from [AS, Section 11].

3. **Twisted K-Theory - the equivariant case**

Let G be a compact group acting on a space X. The goals of this talk are the definition G-equivariant twisted K-theory and the classification of equivariant twists. This will be achieved by a combination of the passage to equivariant K-theory with the introduction of G-actions on twists. The main reference for this talk is [AS04, Section 6].

An equivariant twist will be a stable equivariant projective bundle $P \to X$. The definition of equivariant twisted K-theory $K_P(X)$ [AS04, Definition 6.1] uses homotopy classes of sections of the bundle of Fredholm operators (note that there are two versions for K^0 and K^1, respectively) associated to P. Alternatively, the algebra of sections of the bundle of compact operators associated to P is a G-C^*-algebra. Its G-equivariant K-theory gives another definition of $K_P(X)$. The speaker should work out the equivalence of these two definitions following the non-equivariant example in [AS04].

The main emphasis of the talk should be put into the classification of equivariant twists [AS04, Proposition 6.3]. Isomorphism classes of equivariant twists are classified by the (Borel-) equivariant cohomology $H^3_G(X;\mathbb{Z})$. The speaker should sketch the proof of this result given in [AS04]

A natural source of projective bundles are Fock space constructions, since it often occurs that symmetries can be implemented projectively, only. The Fock space construction of twists on G itself should be briefly discussed following [AS04, p. 31–32]. For this talk put the emphasis on the determination of the classifying cohomology classes in $H(G;\mathbb{Z})$ (i.e. forgetting the equivariance). Include the relation to the cohomology of the loop group.

4. **Cubic Dirac operators, Thom isomorphism, and the Orbit correspondence**

   The goal of this section is to construct classes in twisted equivariant K-theory via a suitable family of Dirac operators (the cubic Dirac operators) associated to a representation of a compact Lie group; essentially, this is the content of [FHTa, Section 1].

   The talk could start with the introduction of the groups Pin^c [FHTa, (1.1)]. One should then introduce the canonical 3-form Ω of a Lie group (it gives a canonical class to twist K-theory and cohomology). Then, the cubic Dirac operator and corresponding Dirac family [FHTa, (1.10), (1.12)] can be defined. Here, one should explain the difference to the ordinary spin-Dirac operator.

   One should proceed with the representation theoretic decomposition of the (cubic) Dirac operator and use this to compute the kernel of D_μ [FHTa, Proposition 1.19].

   The speaker should give the precise K-theoretic interpretation of the cubic Dirac operator construction by explaining the diagram [FHTa, (1.26)] (put emphasis on the Thom homomorphism interpretation). Furthermore, the connection to the classical Borel-Weil-Bott theorem should be worked out (discuss the relations between weights, coadjoint orbits, and irreducible representations). This material is scattered in [FHTa, Section 1.3, Section 1.4].

The talk could finish with examples of the calculations, in particular [FHTa, Example 1.34, 1.37, 1.35].

5. **Non-equivariant twisted K-homology of Lie groups**

   The goal of this talk is to explicitly carry out calculations of twisted K-theory (actually, we switch to the dual twisted K-homology here), using tools from algebraic topology. One should closely follow [Dou]. The main result to be presented therefore is [Dou, Theorem 1.1, 1.2], including the main ideas of the proof. It is suggested to concentrate on a special case (e.g. $SU(n)$) and treat this in detail. The talk should start with a description of the Rothenberg spectral sequence [Dou, Section 2.2]. To carry out the explicit calculations, one first has to work out the E_2-term, for this, one should follow [Dou, Section 3.1]. The talk should then proceed to the main theorem [Dou, Theorem 3.2] and its proof. It remains to calculate the "cyclic order" of the K-theory groups calculated so far. This can be done with different methods, depending on the Lie group in question. The basic idea is to use "generating varieties" to represent the K-homology classes we want to understand; this general principle should be discussed (following [Dou, Section 4.1]), and then worked out explicitly for $SU(n)$, as in [Dou, Section 4.2].

   If time permits, one should cover other Lie groups, as well, in particular illustrate the different methods which can be used (as explained in [Dou, Section 4.1.1]).

6. **Introduction to topological stacks**

   The language of topological stacks is a very effective language in order to express the following things:

   (a) Let $H \subset G$ be a normal subgroup which acts freely on a G-space X. Then the G-space X and the G/H-space X/H should be considered as equivalent. E.g. their equivariant cohomologies are isomorphic by the induction structure. Indeed, if we consider the

G-space X as a stack $[X/G]$, then we have an equivalence of stacks $[X/G] \cong [(X/H)/(G/H)]$.

(b) Similar in sprit is the equivalence of stacks $[X/G]$ and $[X \times_G K/K]$ if $G \subset K$.

(c) The language of stacks is very natural when one considers orbifolds (or orbispaces). Roughly speaking an orbispace is a stack which looks locally like a stack of the form $[X/G]$, where X is a G-manifold (or space) for a finite group G. Considering orbifolds (orbispace) as stacks makes in particular clear what the right notion of a morphism is (see the discussion of the notion of good maps in [LU04]). Namely, morphisms are representable maps.

(d) The language of stacks is natural if one considers gerbes. The model of a gerbe with band $U(1)$ is the stack $BU(1) := [*/U(1)]$. In general a map $G \to X$ from a stack to a space is a gerbe with band $U(1)$ if it is locally on X of the form $V \times BU(1) \to V$. Considering gerbes in this way makes clear how to define morphims between $U(1)$-banded gerbes. They are isomorphisms of stacks over X which are compatible with the local trivializations.

The goal of this talk is to give an introduction to stacks in the topological and smooth context. The main reference should be [Hei05b], but one could also consult [Noo]. For more examples showing the relevance of the stack language to the main topic of the school, the twisted equivariant K-theory of compact Lie groups, one can also look at [BS].

Start with a definition of a stack as a groupoid valued presheaf (on some site of topological spaces or manifolds) satisfying certain descend conditions. Explain that stacks form a two-category and explain the notion of fibre products in this framework (use [Hei05b, Sec.1]).

The introduction to stacks should be mainly based on the following key examples. Show how spaces are viewed as stacks via the Yoneda embedding. Then explain the quotient stack $[X/G]$ and show the facts 1. and 2. above (use [Hei05b, Sec .2]) Finally explain the lifting gerbe of a PU- or $SO(n)$-principal bundle (see [BSS, Sec. 1.3]).

Explain how properties of maps between stacks are defined (in particular properties like representable, proper, smooth, closed). Introduce the notion of an atlas and a topological (smooth) stack (use [Hei05b, Sec. 2]). In the literature, stacks are often represented by groupoids. Explain that groupoids are related with topological (smooth) stacks in the same way as manifolds with distinguished atlas are related with just manifolds (see [Hei05b, Sec. 3]).

Define orbifolds (orbispaces) as a particular kind of topological stacks (namely those which admit an atlas leading to a proper étale (topological) smooth groupoid).

Then follow [Hei05b, Sec. 5] and define the notion of a gerbe over a topological (smooth) stack X. Explain the classification of $U(1)$-banded gerbes by $H^2(X, U(1)_{cont})$ and the Dixmier-Douady class (explain [Hei05b, Prop. 5.8]).

If time permit and in order to introduce other relevant literature explain the relation of this definition with the notion of a bundle gerbe [Mur96], [MS00], the picture [Hit01] and gerbes as central extensions of groupoids [TXLG, Sec 1.].

7. **Twisted K-theory for orbifolds and local quotient stacks**

   In previous talks we have seen various constructions of twisted and equivariant twisted K-theory. In this talk we adopt the stacky point of view and extend the definition of twisted K-theory to nice stacks. If a compact group G acts on a space X, then in the stacky picture the G-equivariant twisted K-theory of X is the twisted K-theory of $[X/G]$.

   The main goal of the talk is to describe the translation of the construction of [TXLG] to the stack framework. Let $f : G \to X$ be a gerbe with band $U(1)$ over a topological or differentiable stack. We assume that X admits an atlas $a : A \to X$ which gives rise to a locally compact groupoid $\mathcal{A} : A \times_X A \Rightarrow A$ admitting a Haar system λ. Passing to a refinement, if necessary, we can assume that the restriction of the gerbe to A is trivial, i.e. that we have a lift $b : A \to G$ and a two-isomorphism $f \circ b \rightsquigarrow a$. Then $b : A \to G$ is an atlas with associated groupoid $\mathcal{B} : A \times_G A \Rightarrow A$, and the canonical map $\mathcal{B} \to \mathcal{A}$ is a central $U(1)$-extension of groupoids. The groupoid $\mathcal{B}$ has an induced Haar system $\tilde{\lambda}$.

   We are now in the set-up of [TXLG]. Using the Haar system λ one defines a convolution product on the compactly supported continuous functions $C_c(\mathcal{B}^1)$ on $\mathcal{B}^1$. The inversion of $\mathcal{B}$ induces a $*$-operation.

The convolution algebra $C_c(B^1)$ has natural $*$-representations π_b on the Hilbert spaces $L^2(\mathcal{B}^b, \tilde{\lambda}^b)$, $b \in \mathcal{B}^0 = \mathcal{A}^0$. On defines the reduced C^*-norm $\| \dots \| := \sup_{b \in \mathcal{B}^0} \|\pi_b(\dots)\|$. The closure of $C_c(\mathcal{B}^1)$ in this norm is the reduced groupoid C^*-algebra of $\mathcal{B}$ with Haar system $\tilde{\lambda}$ and will be denoted by $C^*(\mathcal{B}, \tilde{\lambda})$ (compare [TXLG, Sec 3.1]).

The central $U(1)$ acts on $C^*(\mathcal{B}, \tilde{\lambda})$ by automorphism. We let ${}^G C^*(\mathcal{A}, \lambda) \subset C^*(\mathcal{B}, \tilde{\lambda})$ be the C^*-subalgebra on which $U(1)$ acts by the identity character. The definition of G-twisted K-theory of X is (compare [TXLG, Def. 3.4])

$$ {}^G K_c(X) := K({}^G C^*(\mathcal{A}, \lambda)) \ . $$

A priori this definition depends on the choice of the atlas $a : A \to X$, the lift $b : A \to G$, $f \circ b \rightsquigarrow a$, and the choice of an Haar system.

It should be explained that $K({}^G C^*(\mathcal{A}, \lambda))$ does not depend on the choice of the Haar system. Changing the choice of the atlas and the lift leads to Morita equivalent groupoids. In the talk the notion of Morita equivalence of groupoids and C^*-algebras should be introduced, and it should be explained that Morita equivalent groupoids lead to Morita equivalent C^*-algebras. Using the Morita invariance of C^*-algebra K-theory it should be discussed how this implies that the definition of twisted K-theory is independent of choices. Try to be careful to define the group ${}^G K_c(X)$ up to *canonical* isomorphism. Adress also the problem of contravariant functoriality of ${}^G K_x(X)$ in the pair $G \to X$.

The second part of the talk should relate this picture of twisted K-theory with other pictures:

(a) The family of Hilbert spaces $L^2(\mathcal{B}^b, \tilde{\lambda}^b)$, $b \in \mathcal{B}^0$ can be considered as a family of projective Hilbert spaces $\mathbb{PH}$ on $\mathcal{A}$. One of the main results of [TXLG, Sec 4.] is a description of ${}^G K_c(X)$ in terms families of Fredholm operators on $\mathbb{PH}$. This picture can be useful in order to prove the contravariant functoriality.

(b) For torsion twists there is a description of ${}^G K_c(X)$ in terms of twisted vector bundles, see [TXLG, Sec 5.], [AR03].

For the relation between gerbes and projective Hilbert space bundles one can also consult [Hei05b, Sec. 6], [KMRW98].

8. **The twisted Chern character via non-commutative geometry** Over a smooth manifold M one can use Chern-Weil theory in order to define the Chern character $\mathbf{ch} : K^*(M) \to H_{dR}(M)$. Let $E \to M$ be a vector bundle with connection ∇ and curvature R. We can define the closed form $\mathtt{ch}(\nabla) := \mathtt{Tr}\ \mathtt{exp}(\frac{1}{2\pi i} R^\nabla)$. Let $[E] \in K^0(M)$ and $[\mathtt{ch}(\nabla)] \in H_{dR}(M)$ be the corresponding K-theory and de Rham cohomology classes. Then we have

$$\mathbf{ch}([E]) = [\mathtt{ch}(\nabla)] \ .$$

The goal of this talk is to extend the construction of the Chern character to the twisted case. The role of vector bundles as cycles for K-theory could be taken over by bundle gerbe modules. A natural idea would then be to extend the classical Chern-Weil theory to these objects. A construction of a Chern character along these lines is given in [MS03], [BCM+02].

Actually, the talk should focus on the following more recent and structured construction [MS06]. Assume for simplicity that M is compact. Let $\alpha : P \to M$ be a locally trivial smooth PU-bundle. The α-twisted K-theory of M is then defined as the K-theory of the C^*-algebra A of continuous sections of the bundle of C^*-algebras $P \times_{PU} \mathcal{K} \to M$:

$$K(M;\alpha) := K(A) \ .$$

Let $\mathcal{L}^1 \subset \mathcal{K}$ denote the subalgebra of trace class operators and set $\mathcal{A} := C^\infty(M, P \times_{PU} \mathcal{L}^1)$. We have an inclusion $\mathcal{A} \subset A$ as a smooth dense subalgebra [MS06, Sec. 4.2] such that the natural map $K(\mathcal{A}) \to K(A)$ is an isomorphism. The first step is the Connes-Chern character [MS06, Sec. 4.3]

$$\mathbf{ch}^{Connes} : K(\mathcal{A}) \to HP(\mathcal{A})$$

to the periodic cyclic cohomology of $\mathcal{A}$. In the talk the definition of periodic cyclic cohomology and the construction of the Connes-Chern character (see [MS06, Sec. 3 and Sec 4.3]) should be reviewed.

The target of the Chern character is the twisted de Rham cohomology. Let $c \in \Omega^3(M)$ be a closed three form which represents the image of the characteristic class of P in de Rham cohomology. It can be obtained from connection data on P, and by abuse of notation the symbol "c"

will also subsume these additional choices[1]. The c-twisted de Rham cohomology $H^*(M;c)$ is defined in [MS06, Sec.Def. 4.5]. The first main theorem to be explained in detail is the isomorphism [MS06, Thm. 1.1]

$$\Phi_c : HP(\mathcal{A}) \xrightarrow{\sim} H(M;c) \ .$$

The talk should address the construction of the map as well as the verification that it is an isomorphism. On the way the extension of the Hochschild Kostant-Rosenberg theorem

$$HH(\mathcal{A}) \cong \Omega(X)$$

($HH(\mathcal{A})$ denotes Hochschild cohomology) should be explained and related to the classical case (see [HKR62], [Con85] and follow the discussion in [MS06, Sec. 3.2]). The Chern character is then defined as the composition

$$\mathbf{ch}_c : K(X,P) \overset{\mathbf{ch}^{Connes}}{\rightarrow} HP(\mathcal{A}) \xrightarrow{\Phi_c} H(M;c) \ .$$

In the talk it should be explained why it is a real isomorphism (follow the Mayer-Vietoris sequence argument [MS06, Sec. 6]).

9. **The twisted Chern character — the orbifold case**

The goal of this talk is to discuss the results of [TXa]. Let $X^1 \Rightarrow X^0 \to M$ be a representation of an orbifold M by an étale and proper groupoid X in smooth manifolds, and $Y \to X$ be a central S^1-extension representing a gerbe $G \to M$ with band $U(1)$. We form the line bundle $L := Y^1 \times_{S^1} \mathbb{C} \to X^1$. In [TXa, Sec. 2.4], after introducing a convolution $*$-algebra $\mathcal{A} := C_c^\infty(X^1;L)$ and its C^*-closure A, the G-twisted K-theory of M is defined by

$$K(M;G) := K(A) \ .$$

Using the results of [TXLG04] it shoud be indicated why this definition is independent of the choice of the groupoid representation $Y \to X$ of $G \to M$.

[1] The transformations Φ_c and $\mathbf{ch}_c$ below depend on these choices and not only on the three form.

The target of the Chern character is the twisted de Rham cohomology of the inertia orbifold ΛM of M with coefficients in the inner local system $\tilde{L} \to \Lambda M$.

In the talk the two pictures the de Rham cohomology of an orbifold M as the de Rham cohomology of the simplicial manifold $X^\bullet$ associated to the groupoid X and as the cohomology of the de Rham complex $\Omega(M)$ should be presented, use [TXa, Sec. 3.2]. Next, the notions of a connection, a curving and the 3-curvature Ω [TXa, Def 3.3] on a $U(1)$-central extension $Y \to X$ should be introduced. It should be explained that the de Rham cohomology class of Ω represents the Dixmier-Douady class of the gerbe $G \to M$.

The subset $SX \subset X^1$ of morphisms with equal source and range is the set the objects of the inertia groupoid ΛX (see [TXa, Sec. 2.3]). It represents the inertia orbifold ΛM [LU04], [CR04]. The restriction of $\tilde{L} := L_{|SX}$ is called the inner local system associated to the gerbe $G \to M$. It carries a canonical flat connection $\nabla^{\tilde{L}}$ [TXa, Prop. 3.9]. With these notions the twisted delocalized de Rham cohomology $H_{deloc}(M; G)$ can now be defined as the cohomology of the complex $(\Omega_c(\Lambda M, \tilde{L})((u)), \nabla^{\tilde{L}} - 2\pi i u \Omega)$, where u is a formal variable of degree -2. We use the adjective delocalized in order to indicate that it is defined on ΛM. It should be indicated how this definition depends on the choices of the groupoid representations and the connection and curving.

The construction of the Chern character now proceeds in the following steps:

(a) The inclusion $\mathcal{A} \to A$ induces an isomorphism in K-theory $K(\mathcal{A}) \xrightarrow{\sim} K(A)$.

(b) There is Connes' Chern character (see 8) $\mathbf{ch}^{Connes} : K(\mathcal{A}) \to HP(\mathcal{A})$.

(c) In [TXa, Sec 4.3] an explicit chain map $\tau : CC(\mathcal{A})((u)) \to \Omega_c(\Lambda M, \tilde{L})((u))$ is constructed. It induces a map $\tau : HP(\mathcal{A}) \to H_{deloc}(M; G)$.

(d) The Chern character is the composition

$$K(M, G) \cong K(\mathcal{A}) \xrightarrow{\mathbf{ch}^{Connes}} HP(\mathcal{A}) \xrightarrow{\tau} H_{deloc}(M; G) \ .$$

In the talk it should be explained how this Chern character depends on the choices of the groupoid representations and the connection and curving. Furthermore, it should be explained why it becomes an isomorphism after tensoring by $\mathbb{C}$.

10. **Twisted K-theory via parametrized homotopy theory and twisted integration**

    The goal of the first part of this talk is to embed twisted K-theory into the framework of parametrized homotoy theory. The second part is devoted to the Gysin (Integration-, Umkehr-) map in twisted K-theory,

    Twisted K-theory for a space B can be considered as a construction which is applied to and uses categories of spaces (and later spectra) *over the space* B. This is a very classical notion, denoted ex-spaces (or ex-spectra, respectively).

    The talk should start with recalling the necessary notions of and about ex-spaces and ex-spectra (or spectra over a space), as given e.g. in [CP84, p. 220]. In particular, one should define the corresponding stable homotopy category, and state the main results of [CP84] about it.

    One should then describe the notion of a twisted cohomology theory in this context, as in [Dou, Section 1.1.1, Section 2.1]; with its main properties. Here, one should also discuss the more general possible notion of twist for K-theory in this context —but then concentrate on the “elementary” twists classified by maps to $K(\mathbb{Z}, 3)$, as in [Dou, Section 2.1], and describe the particular choices one can make to describe twisted K-theory. TMF should not be mentioned.

    Very careful descriptions of good categories of ex-spaces and ex-spectra, also equivariant, are given in [MS]. It is, however, not necessary to cover this in the talk.

    For the definition of twisted K-theory, with an appropriate K-theory spectrum, the speaker should follow [Joa97, Section 3.2, 3.3], but translate from real to complex K-theory. Here, one uses the definition of twisted K-theory of [Joa97, Section 4.1]. The talk should specify the necessary level of rigidity in the construction of the K-theory spectrum to make the corresponding definitions; in particular in view of the equivariant generalizations. This would also be a good place to show that twisted $Spin^c$ bordism fits well into this picture.

The construction of K-theory spectra usually relies on spaces of operators on Hilbert spaces. The correct choice of these spaces, in particular of the topology to be used, is quite delicate. Explicit description can be found in [Joa03, BJS03] (where we are only interested in the case of the C^*-algebra $\mathbb{C}$). These difficulties and their possible solutions should only be mentioned. A more readable alternative is in particular [Joa01, Theorem 3.4].

Using the explicit form of the spectrum which can be associated functorially to a bundle of projective Hilbert spaces, the speaker should work out a comparison between the ex-spectrum definition of twisted K-theory for precisely these twists (use the constructions in [Joa97], and compare with the sections of the associated bundle of Fredholm operators).

A basic construction which is often used and applied in calculations of equivariant twisted K-theory groups is a wrong-way or Gysin or push-forward map in (equivariant twisted) K-theory (three names for the same concept) for suitable "K-oriented" maps, together with the properties of this map.

The talk should include an axiomatc description of the properties which are required, following [BS, 2.2.10]. One has to add the normalization requirement that for the trivial bundle, the Thom isomorphism is just the suspension isomorhpism.

One alternative for a presentation of the push-forward could be in terms of a suitable model of K-theory via parameterized homotopy theory and sections of bundles of K-theory spectra [Wal06].

Alternatively, twisted integration could be defined and described in explicit cocycle models of twisted (equivariant) K-theory.

To put our construction into context, one has to mention the (equivariant twisted) Thom isomorphism in general. This includes the non-twisted case, namely

(a) Introduction of (equivariant) $Spin^c$ structures, extension of the Thom isomorphism to bundles with a $Spin^c$ structure: definition of $Spin^c$-structure ([Kar78, 4.25]), Thom homomorphism [Kar78, 5.12,5.13], isomorphism Theorem [Kar78, 5.14] with obvious equivariant generalizations.

(b) Discussion of (equivariant) K-oriented maps (at least fibrations and embeddings), and the Umkehr map in equivariant K-theory, following e.g. [Kar78, 5.22] with the (obvious) equivariant generalizations.

The Thom isomorphism in the twisted case is worked out in [CW, Section 3.2 and 3.3]. The push-forward in (non-equivariant) twisted K-theory for general proper maps is defined in [CW, Section 4]. The idea is to factor the map into an embedding and a submersion. In the first case the construction of the push-forward uses the Thom isomorphism for the normal bundle and the open embedding of the normal bundle as a tubular neighborhood of the image. The push-forward in the submersion case is defined using Dirac operators. The latest version [CW, Section 4] contains all necessary verifications of well-definedness.

A possible reference for the twisted equivariant case could be [ARZ, Section 3] which uses groupoids. One should (after recalling the setting and the definition of twisted K-theory of the paper) concentrate on [ARZ, Proposition 3.6] and its proof, and then show how "standard" constructions (turning a map into an embedding) can be generalized to groupoids [ARZ, Lemma 3.9] and used to define a Gysin map for "complex oriented" maps [ARZ, Theorem 3.10]. One should discuss that in particular the following points have to be clarified: how does the construction behave under passage to Morita equivalent groupoids? Functoriality? Projection formula? The speaker should also try to generalize to maps which preserve a given $Spin^c$-structure.

A third possibility would be to follow [TXb, Section 4.6]. However, this construction uses heavily the machinery of Kasparow's KK-theory. Since this is not covered and used in the rest of the talks we suggest to skip this, or only mention the corresponding construction and idea [TXb, Theorem 4.19, Corollary 4.20]. It should be pointed out that here, functoriality is discussed [TXb, Corollary 4.20].

It is of course impossible to discuss all these approaches in one talk. For a general understanding it suffices to see which basic concepts are involved. The speaker can make his personal choice for the presentation.

11. **Products in twisted K-theory**

    The multiplication map $m : G \times G \to G$ is a G-equivariant map (G acts on itself by conjugations). If X is an orbifold, then the inertia orbifold $\Lambda X \to X$ is a group orbifold over X. These examples become equal for finite groups. In both cases push-forward along the multiplication map can be used to define a product on the twisted K-theory. It is of particular importance to understand which properties of the twist and modifications are necessary to ensure associativity. For the equivariant K-theory of Lie groups the product is a hot topic of the work [FHTb]. The twisted equivariant K-theory of a Lie group is related to the positive energy representations of the loop group not just as a group, but also as a ring, if one considers the fusion product on the representation side.

    But in the present talk one should focus on the orbifold case worked out in [ARZ]. The talk should start with a review of the Pontryagin product for equivariant K-theory of finite groups [ARZ, Section 6]. The next goal is, to extend this to a product on equivariant twisted K-theory for finite groups, as in [ARZ, Definition 6.1]. We should observe that we need a pushdown map in twisted K-theory with good properties, if this is supposed to be well defined and to yield an associative product structure. The focus of the present talk should not be to construct this integration map, but to assume its existence (a discussion of this point has been given in the previous talk). However, the speaker should always point out which properties of the pushdown map are used in his constructions and proofs.

    Adem, Ruan, Zhang's construction of the Pontryagin product relies on the "external product", where twists are added. This product is induced (e.g.) by the tensor product of (projective) Hilbert spaces and Hilbert space bundles. This construction should be explained, one should follow e.g. [ARZ, page 10], and [AS04, Proposition 2.1 (vii),Proposition 2.3,(4.1)]. If time permits, the talk should include a brief discussion of the external product using the Kasparov product in KK-theory (following [TXb] — compare formula (1) of this paper, as well as Section 4.5). The speaker shouldn't spend too much time on this point, because during the Arbeitsgemeinschaft, we are not planning to investigate KK-techniques in any detail.

The goal for the rest of the talk is the construction of the general Pontryagin product in twisted K-theory of [ARZ, Section 7], and to prove its main properties, in particular associativity.

The speaker could start with [ARZ, Definnition 7.1], and discuss the structure of this formula. This should include a discussion of the twists (looking at their formal properties), following [ARZ, p. 23]. One could then present the proof of associativity (using suitable formal properties of the operations involved), to show why the additional factor with the obstruction bundle is necessary. Finally, the talk should pick up the two loose ends and explain the inverse transgression of cocycles, following [ARZ, Section 4], without going too much into the details. One should stress, however, that one gets explicit cocycles and cocycle identities. The remaining time should be spend with a (concise) description of the constructin of the obstruction bundle and of its main properties, as in [ARZ, Section 7].

12. **Twisted K-theory and T-duality**

The goal of this talk is to work out topological aspects of T-duality and applications to calculations of twisted K-theory. T-duality is a notion in (super) conformal field theory. More specifically it is the observation that type IIA and IIB-string theories become equivalent on T-dual targets. If the targets are torus bundles equipped with B-fields, then the corresponding fields are metrics, connections and the B-field. The Buscher rules describe how these fields transform under transition to the T-dual target. Starting from the Buscher rules in [BEM04] and follow-ups a geometric construction of T-dual pairs of torus bundes has been given.

The underlying topology was formalized in [Schb]. Here T-duality is a relation between torus bundles with a twist (the twist plays the role of the topological background of the B-field in the same sense as a $U(1)$-principal bundle is the topological background for the electro-magnetic field) on the total space, on a given base space (generalizations to more general spaces have also been considered). One can also look for a functor, which assigns to one such object a T-dual second one. The talk should describe the T-duality relation for circle bundles as in [Schb, Definition 2.9], and the properties [Schb, 2.10-2.14] which follow. One should compare with the Definition of [BEM04, Section 3.1], following [Schb, 2.2.6]. One should also introduce the Definition of T-duality for higher dimensional bundles [BRS, Definition 1.5]; and point out in particular that the category of twists plays an important role here.

The talk should go on to describe the T-duality transformation in twisted homology theories, following [BRS, Definition 1.5 and Section 5] and prove the corresponding T-duality isomorphism in twisted K-theory, following [Schb, Section 3.3] and [BRS, Theorem 5.3]. (More general admissibile cohomology theories should not be mentioned). One should then use this to carry out some example calculations of twisted K-theory groups, in particular following [Schb, Section 4.2, 4.3], and also [Schb, Section 4.1]. The speaker is also invited to work out additional interesting examples himself.

One should also describe the C^*-algebraic framework for T-duality (using continuous trace algebras) following [MR05, Section 3], and describe T-duality for circle bundles in this framework following [MR05, Section 4.1]. It is not necessary to mention the Chern character, but the resulting isomorphism of twisted K-theory groups should be described. Indicate (briefly) the generalizations to higher dimensional torus bundles, in particular [MR05, Theorem 4.4.2,4.4.3]. Attention: the uniqueness statement in [MR05, Theorem 4.4.2] is wrong, compare [Schb, 4.4.8].

The talk should discuss the topological approach to T-duality via classifying spaces of pairs, and via a universal T-duality transformation on this classifying space (which is a consequence of the calculation of the homotopy type of this space). The simpler case of circle bundles is discussed in [Schb, Section 2]; the talk should also briefly mention the problems in the higher dimensional case, and how it can (partially) be overcome, following [BRS, Section 2, 3, 4].

In particular, one should mention the existence and classification results for T-duals ([Schb, Theorem 2.16], [Schb, Corollary 2.11], [BRS, Theorem 6.24], [BRS, Theorem 1.18]).

13. **Equivariant twists of compact Lie groups**

One of the goals of the AG is to determine the G-equivariant twisted

K-theory of (certain) Lie groups G (acting on itself by conjugation) in terms of representations of the loop group of G.

The first step is to understand the classification of equivariant twists of G. We have seen in talk 3 that equivariant twists are classified by $H^3_G(G;\mathbb{Z})$ (we ignore gradings for the moment). The geometric realization of twists is achieved via S^1-extensions of the loop group.

The classical reference for central extensions of loop groups and their representations is [PS86]. However, more suitable for our purposes (and in parts slightly more general) is the exposition of [FHTa] (with some crucial results cited from [PS86]).

The talk should start with the classification of principal G-bundles $P \to S^1$, then introduce the gauge group $L_G P$ and study the action of it on the space of connections $\mathcal{A}_P$ of P. If P is trivial, then $LG := L_P G$ is the classical (smooth) loop group. The corresponding Lie algebra $L_P\mathfrak{g}$ should also be introduced.

The holonomy map $\mathcal{A}_P \to G[P]$ induces an equivalence of quotient stacks $[\mathcal{A}_P/L_P G] \to [G[P]/G]$ [BS, Lemma 2.2]. Following [Hei05a, Example 5.4.1], one should discuss how central S^1-extensions of $L_G P$ give rise to twists (i.e. S^1-gerbes) of $[G[P]/G]$, compare also [BS, 2.1.2] and [FHTa, Definition 2.8]. The speaker should prove that every twist of the quotient stack $[G[P]/G]$ is obtained this way. This does not seem to be documented in the literature, one could follow the following argument due to Jochen Heinloth (disregarding the grading):

We consider the quotient stack $[\mathcal{A}_P G/L_P G]$. Since $\mathcal{A}_P G$ is a contractible space, its cohomology is canonically isomorphic to the cohomology of $[*/L_P G]$. On the other hand, every S^1-gerbe on $[*/L_P G]$ becomes trivial on the point and therefore has an atlas consisting of the one-point space. But a groupoid with one object is a group, which means that every S^1-gerbe on $[*/L_P G]$ comes from a central S^1-extension of $L_P G$.

The notion of *admissible* central extension has to be introduced [FHTa, Definition 2.10]. The speaker should mention that this is no condition in the semisimple case, but not elaborate on the proof. However, the “group” structure on (admissible) central extensions [FHTa, p. 22] should be discussed.

If time permits, the talk could discuss in some detail the admissible central extensions in a subset of the three special cases of simply connected groups, tori, and finite groups; following [FHTa, Section 2.3].

The next goal is to introduce the notion of a positive energy representation of $L_G P$, [FHTa, Definition 2.42].

To be able to do this, one needs to introduce the Dirac type operators $d_A \in L_P\mathfrak{g}$, $d_A \in \hat{L}_P\mathfrak{g}$ corresponding to a connection $A \in \mathcal{A}_P$ [FHTa, p. 19,(2.14)]. One should explain the finite energy vectors and the decomposition [FHTa, (2.43)]. This should be followed by a description of the main properties, including the finite dimensionality of the bounded (above) energy vectors [FHTa, Lemma 2.45] (with respect to eigenspaces of E_A), and the list in the middle of page 30 of [FHTa]. To discuss this, one also has to describe the (creation and annihilation) action of $L_P\mathfrak{g}$ on the representation, and for this to introduce the finite energy loops in $L_P\mathfrak{g}$ and the decomposition of these as $\oplus_{n\in\mathbb{Z}} z^n \mathfrak{g}_{\mathbb{C}}$, following [FHTa, 2.4].

Note the the lowest energy vectors form a $\mathfrak{g}_{\mathbb{C}}$-representation. Recall the concept of weight, and lowest weight, and of lowest weight vectors. Introduce the affine Weyl group and the (energy, level, weight)-decomposition of a positive energy representation; and introduce the concept of lowest weight (and the corresponding vectors). This can be done following the survey [Scha, Section 6], or the relevant parts of [PS86].

Note perhaps that, if the principal bundle is trivial and we therefore talk about the ordinary loop group, one will usually choose the trivial connection A for this finite energy decomposition.

This allows the definition of the group of finite energy representations of level τ [FHTa, Definition 2.50].

14. **Construction of twisted K-theory classes from positive energy representations**

In the preceeding talks we have discussed various general properties of twisted K-theory. But we haven't seen a single non-trivial representative of a twisted K-theory class. The goal of this talk is to give an explicit construction of equivariant twisted K-theory classes of a compact Lie group G. This construction first appeared in physics oriented literature (compare [Mic04]) and is now one of the main ingredients of the work of [FHTa], [FHTb]. It will eventually be encoded in a homomorphism

$$\Phi : R^{\tau-\sigma}(L_P G) \to K_G^{\tau+\dim(G)}(G) \ .$$

The talk should closely follow [FHTa, Sec. 3], but also explain the connection with [FHTb, Sections 11, 12, 13]. Let $\pi : P \to S^1$ be a G-principal bundle and $L_P\mathfrak{g}$ be its associated loop algebra (the space of smooth sections of the adjoint bundle $\mathtt{Ad}_P(\mathfrak{g}) := P \times_{(G,\mathtt{Ad})} \mathfrak{g}$). An invariant scalar product on $\mathfrak{g}$ induces an L^2-metric on $L_P\mathfrak{g}$. Let H be its L^2-closure. A connection $A \in \mathcal{A}_P$ induces a skew-adjoint Dirac-type operator d_A on H. The (real !) skew-adjoint operator $J_A := iE_{d_A}(0,\infty) - iE_{d_A}(-\infty, 0)$ induces a polarization $\mathcal{J}$ of H and determines the restricted orthogonal group $O_{\mathcal{J}}(H)$. There is a distinguished central extension

$$1 \to U(1) \to Pin^c_{\mathcal{J}}(H) \to O_{\mathcal{J}}(H) \to 1$$

and a spin representation $Pin^c_J(H) \to \mathtt{End}(S)$. The action of L_PG on H preserves the polarization $\mathcal{J}$, i.e. we have a homomorphism $L_PG \to O_{\mathcal{J}}(H)$. By pull-back one obtaines a central extension

$$1 \to U(1) \to L_PG^\sigma \to L_PG \to 1 \ .$$

The first part of the talk should address the following points.

(a) Explain the construction of the spin representation the $Pin^c_{\mathcal{J}}(H)$-group.

(b) Why does L_PG preserves $\mathcal{J}$?

(c) Explain why the infinitesimal spin representation is given by [FHTa, (3.19)] or [FHTa, (8.7)], resp.

(d) Calulate the level of the spin representation. It is determined by the commutator calulation [FHTa, (3.20), (3.21)].

Consider now a positive energy representation V of $L_PG^{\tau-\sigma}$ and form $W := V \otimes S$. Introduce the operator D_0 on the finite energy vectors as in [FHTb, Sections (11.1)]. Discuss how it commutes with the action of $L_P\mathfrak{g}$ and the Clifford algebra [FHTb, Prop. 11.2]. Then define the family of operators $\mu \to D_\mu$ [FHTb, Prop. 12.2]. Show that it is equivariant with respect to the action of $L_P G^\tau$ on W and the level $k(\tau)$-action on $L_P\mathfrak{g}$. The infinitesimal invariance is shown in [FHTb, Prop. 12.2], while the global invariance is discussed in [FHTb, Prop. (13.3)].

Then turn back to [FHTa, 3.3] and interpret the family $L_P\mathfrak{g} \ni \mu \to D_\mu$ as a function $\mathcal{A}_P \ni A \to D_A$, and its bounded version $\mathcal{F}_A := D_A(1 - D_A^2)^{-1/2}$ as a family of $L_P G^\tau$-equivariant Fredholm operators, $\mathcal{F}_{gA} = g\mathcal{F}_A g^{-1}$. The main goal of the talk is reached by observing that the family of Fredholm operators $\mathcal{A}_P \ni A \mapsto \mathcal{F}_A \in B(W)$ is a cycle for the class

$$\Phi(V) \in K_G^{\tau+\dim(G)}(G[P]) \ .$$

The talk should address the verification of the necessary continuity and Fredholm properties.

15. **The FHT-Theorem I**

In this talk the construction [FHTb, 5.7] and the key lemma [FHTb, 5.8] should be explained. It should be applied to calculate the twisted equivariant K-theory of finite groups, tori and their extensions by finite groups.

In the following we copy [FHTb, 5.7]. Let H be a compact group, acting on a compact Hausdorff space X, τ a twisting for H-equivariant K-theory, $M \subseteq H$ a normal subgroup acting trivially on X. The following data can be extracted from this:

(a) an H-equivariant family, parametrised by X, of $\mathbb{T}$-central extensions M^τ of M;
(b) an H/M-equivariant covering space $p : Y \to X$, whose fibres label the isomorphism classes of irreducible, τ-projective representations of M;
(c) an H-equivariant, tautological projective bundle $\mathbb{P}R \to Y$, whose fibre $\mathbb{P}R_y$ at $y \in Y$ is the projectivised τ-representation of M labelled by y;
(d) a class $[R] \in {}^{\mathbb{P}R}K_H(Y)$, represented by R;
(e) a twisting τ' for the H/M-equivariant K-theory of Y, and an isomorphism of H-equivariant twistings $\tau' \cong p^*\tau - \mathbb{P}R$.

The key Lemma asserts that

$$K^{\tau'}_{H/M}(Y) \cong K^{\tau}_{H}(X) \ .$$

After explainig why this is true it should be illustrated by calculating the twisted representation ring of $H := \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. In this case X is a point and choose $M \cong \mathbb{Z}/n\mathbb{Z} \subset G$ as the diagonal. Make the objects introduced above explicit and calculate $K^{\tau}_{H}(*)$ (see also [FHTb, 10.7]).

The second example to be considered in this talk is an important step towards the main result of [FHTb]. Consider an extension

$$1 \to \mathbb{T}^n \to N \to W \to 1$$

of a n-torus by a finite group W. Introduce the notion of a regular twist τ of N and calculate $K^{\tau}_{N}(N)$ (see [FHTb, Sec. 6]). A detailed description of this calculation for $\mathbb{T}^n$ can also be found in [BS, Prop. 2.7 and its proof]. As a preparation of the next talk it is importand to introduce the affine Weyl group W_{aff} and the W_{aff}-set of τ-affine weights ${}^{\tau}P$ of $\mathbb{T}^n$.

16. **The FHT -theorem II**

The main goal of this talk is to calculate $K^{\tau}_{G}(G)$ for a compact Lie group G which acts on itself by conjugations.

Start with explaining the diagram

$$\begin{tikzcd} 1 \arrow[r] & T \arrow[r] \arrow[d] & Q_T \arrow[r] \arrow[d] & \pi_0(G) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & G_1 \arrow[r] & G \arrow[r] & \pi_0(G) \arrow[r] & 1 \end{tikzcd} \ ,$$

where T is a maximal torus of G_1 and Q_T is called a quasi torus. For $f \in Q_T$ let $N(f) \subseteq G(F)$ be the stabilizers of the components $fT \subseteq Q_T$ and $fG_1 \subseteq G$. Inclusion gives a natural map of stacks $\omega : [fT/N(f)] \to [fG_1/G(f)]$. Show that these maps induce induction (this uses a K-orientation of ω) and restriction maps in twisted K-theory

$$\omega_* : K^{\omega^*\tau}_{N(f)}(fT) \to K^{\tau}_{G(f)}(fG_1) \ , \quad \omega^* : K^{\tau}_{G(f)}(fG_1) \to K^{\omega^*\tau}_{N(f)}(fT) \ .$$

The first theorem to be proved in this talk is that $\omega_* \circ \omega^*$ is the identity on $K^{\tau}_{G(f)}(fG_1)$ [FHTb, Thm. 7.9]. A detailed proof in the case of connected groups can be found in [BS, Prop. 2.8]. The second theorem [FHTb, Thm. 7.10] identifies the image of $\omega^* \circ \omega_*$ in the group $K^{\omega^*\tau}_{N(f)}(fT)$ which has been calculated already in talk 15. State the final result in terms of the twisted K-theory of the action of the f-twisted affine Weyl group W_{aff} on the set ${}^{\tau}P^{reg}$ of regular τ-affine weights of T. Again, the case of connected groups was worked out in detail also in [BS, Prop. 2.10].

Use [FHTb, Thm. 7.10] in order to exhibit an explicit basis of $K^{\tau}_{G}(G)$. As an illustration the case of $G := SU(2)$ should be worked out.

17. **The FHT -theorem III**

The goal of this talk is to compare the classification of admissibe representations of central extensions of twisted loop groups L_fG of level τ with the calculation of $K^{\tau}_{G(f)}(fG_1)$. In order to connect the notation with talk 14 note that the quasi torus element $f \in Q_T \subset G$ determines a principal bundle $P \to S^1$ with holonomy in fG_1 and $L_PG = L_fG$. The level τ is a $U(1)$-central extension $\hat{L}_fG \to L_fG$ which is considered as a $G(f)$-equivariant twist τ of the coset fG_1 (see talk 13).

The first result to be explained is [FHTb, Thm 10.2] which states that the categories of admissible level $(\tau - \sigma)$-representations of L_fG and the category of W_{aff}-equivariant $\sigma(\tau)$-twisted vector bundles over the W_{aff}-set ${}^{\tau}P^{reg}$ are equivalent. In particular, the K-group of such representations is isomorphic to $K^{\sigma(\tau)}_{W_{aff}}({}^{\tau}P^{reg})$ (this object was explained in talk 15). Explain how the W_{aff}-equivariant vector bundle over ${}^{\tau}P^{reg}$ is extracted from the positive energy representation V by taking lowest weight vectors in $S \otimes V$.

Then reconsider the construction of the Dirac family $\mu \mapsto D_\mu$ associated to a positive energy representation V of level $\tau - \sigma$ in [FHTb, Thm 12.2]. From talk 14 it is known that it represents a class $\Phi(V) \in K^{\tau+\dim(G)}_{G}(G(P))$. The main result to be explained is [FHTb, Prop. 13.2] which states that the element in $K^{\sigma(\tau)}_{W_{aff}}({}^{\tau}P^{reg})$ representing $[V]$ according to the first part of the talk is equal to the element corresponding to $\Phi(V)$ under the map $K^{\tau+\dim(G)}_{G}(G[P]) \to K^{\sigma(\tau)}_{W_{aff}}({}^{\tau}P^{reg})$ constructed in talk 16. To this end calculate the kernel of D_μ [FHTb, Prop. 12.7]. Show that it occurs as a vector bundle on a coadjoint orbit determined by the lowest weight of V. Furthermore calculate the action of the deformation of D_μ in the normal direction to the orbit [FHTb, Prop. 12.8] and use this to relate the class $\Phi(V)$ with a Thom push forward along the embedding of the orbit [FHTb, 12.9].

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Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`bunke@uni-math.gwdg.de`

by August 10 at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.