Mathematisches Forschungsinstitut Oberwolfach

Report No. 13/2007

Inverse Problems in Wave Scattering

Organised by
Martin Hanke-Bourgeois (Mainz)
Andreas Kirsch (Karlsruhe)
William Rundell (College Station)

March 4th – March 10th, 2007

Abstract. The workshop treated inverse problems for partial differential equations, especially inverse scattering problems, and their applications in technology. While special attention was paid to sampling methods, decomposition methods, Newton methods and questions of unique determination were also investigated.

Mathematics Subject Classification (2000): 14Q05, 35Rxx, 65Jxx, 65N21, 74Jxx, 78Axx.

Introduction by the Organisers

Since this was only a "half workshop", the organisers intended to focus on a relatively small number of topics. We selected scattering from an obstacle and with primarily the Maxwell equations as the underlying structure. We also wanted to concentrate on analytic methods where questions of uniqueness and (if appropriate) existence were at the forefront. That is not to say that reconstructive methods were downplayed and in fact almost all of the talks did show some illustrative reconstructions, but greater attention was to the analysis of the methods rather than implementation issues. Even within this structure there was a strong focus on ideas that were in the spirit of *sampling* or *factorization methods.* These techniques work under a wide variety of physical situations and are perhaps optimal when very little is known a priori about the location, shape or material properties of the scatterer and where the most accurate solutions are not required.

The participants were the usual geographical blend but what stood out in looking over the audience was the youthful median age. For many in the group this was their first visit to Oberwolfach and, in some cases, their first time to meet each other. With the smaller number of participants we were able to have a relaxed schedule and yet allow most to speak (about three quarters did so). Thus talks were scheduled for 9:30am, 11am, 4pm and 5pm except for the Wednesday and Friday when we only had morning talks. The duration was scheduled for a maximum of 40 minutes to allow for comments and discussions (and frequently the additional post-talk questions and comments took us up to the next speaker). We also had one evening session where a relatively new topic (the question of existence and properties of so called "transmission eigenvalues") was discussed by the whole workshop. Almost all participants attended all the lectures; the exceptions were due to interest in talks from the other workshop. We thus conclude that the format was successful from the perspective of the participants. Another reason for the high attendance was the excellent overall quality of the talks; it is clear that most of the speakers had taken great care over the preparation and there was considerable evidence that talks were modified as the conference progressed in order to present a fresh set of ideas to the audience.

It should be noted that at the last workshop run by the present organisers special sessions were held in the evening. One of these considered some new ideas on one of the oldest conjectures in the area - whether or not a single incident plane wave is sufficient to recover an arbitrary obstacle. There has been a history of partial results, but no substantial progress had been made on the problem for twenty years. Three of the participants at the previous meeting managed to make substantial progress in the year after the workshop. (The conjecture is still unproven in the most general setting but a first result proved the conjecture for convex, polygonal obstacles, a subsequent one removed the need for convexity. These used very different techniques and further progress is expected.) We think that devoting part of the time to look at a specific focused topic has enormous merit at Oberwolfach and we only hope for a similar outcome from the present workshop.

As usual, the service provided by the staff was exemplary. This plays an enormous part in the "Oberwolfach experience" and allows the participants to concentrate on the research aspects.

Finally, we should note that at the previous workshop the organisers complained about the state of current atmospheric modelling/computation leading to an erroneous weather forecast that added a surprise rainstorm to the Wednesday hike. We are sorry to report that the computational geophysicists still have work to do as they were again wrong, but we much prefer a forecast of rain yet in fact receive fine weather.

Workshop: Inverse Problems in Wave Scattering

Table of Contents

Abstracts

Decomposition methods in inverse obstacle scattering revisited

Rainer Kress

The propagation of time-harmonic acoustic waves with frequency ω in a homogeneous isotropic medium with constant speed of sound c is governed by the Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1}$$

with the positive wave number $k = \omega/c$ for the space dependent part u of the the velocity potential. For the scattering of an incident plane wave $u^i(x) = e^{ik\,x\cdot d}$ propagating in the direction of the unit vector d by a sound-soft obstacle represented through a bounded domain $D \subset \mathbb{R}^3$ with a connected complement, the total wave u is given by the superposition $u = u^i + u^s$ of the incident wave u^i and the scattered wave u^s. The total wave satisfies the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial D \tag{2}$$

and the scattered wave the Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial r} - iku^s = o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \tag{3}$$

uniformly for all directions. For the sake of simplicity we only consider the Dirichlet boundary condition for a sound-soft scatterer. However, our analysis extends to the case of the Neumann boundary condition for a sound-hard scatterer and the impedance boundary condition. For simplicity, we assume that the boundary ∂D of the scatterer D is C^2 smooth.

The Sommerfeld radiation condition characterizes radiating solutions to the Helmholtz equation and, in particular, implies an asymptotic behavior for the scattered wave of the form

$$u^s(x) = \frac{e^{ik|x|}}{|x|}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \quad \hat{x} := \frac{x}{|x|}, \tag{4}$$

uniformly with respect to all directions. The function u_∞ is known as the *far field pattern* of the scattered wave and is an analytic function of $\hat{x}$ on the unit sphere $\Omega := \{x \in \mathbb{R}^3 : |x| = 1\}$. The *inverse scattering problem* that we are concerned with is to determine the shape and location of the scatterer D from a knowledge of the far field pattern u_∞ for one incident plane wave. For detailed presentations of the current state of research in inverse obstacle scattering we refer to the monograph [1] and the survey [2].

The main idea of so-called decomposition methods is to break up the inverse obstacle scattering problem into two parts: the first part deals with the ill-posedness by constructing the scattered wave u^s from its far field pattern u_∞ and the second part deals with the nonlinearity by determining the unknown boundary ∂D of the scatterer as the location where the boundary condition (2) is satisfied in a least-squares sense. In the potential method of Kirsch and Kress, presented at the

Oberwolfach Conference on Inverse Problems in 1986 [4], enough a priori information on the unknown scatterer D is assumed so one can place a closed surface Γ inside D. Then the scattered wave u^s is sought as a single-layer potential

$$u^s(x) = \int_\Gamma \frac{e^{ik\,|x-y|}}{|x-y|}\,\varphi(y)\,ds(y), \quad x \in \mathbb{R}^3 \setminus \bar{D}, \tag{5}$$

with an unknown density $\varphi \in L^2(\Gamma)$. In this case, given the far field pattern u_∞, the density φ is now found by solving the integral equation of the first kind

$$S_\infty \varphi = u_\infty \tag{6}$$

with the compact integral operator

$$(S_\infty \varphi)(\hat{x}) := \int_\Gamma e^{-ik\,\hat{x}\cdot y}\varphi(y)\,ds(y), \quad \hat{x} \in \Omega. \tag{7}$$

Due to the analytic kernel of S_∞, the integral equation (6) is severely ill-posed. For a stable numerical solution of (6), for example, Tikhonov regularization can be applied. Given an approximation of the scattered wave u^s by inserting a regularized solution φ of (6) into the single-layer potential (5), the unknown boundary ∂D is then determined by requiring the sound-soft boundary condition (2) to be satisfied in a least-squares sense, i.e., by minimizing the L^2 norm of the defect $\|u^i + u^s\|_{L^2(\Lambda)}$ over a suitable set of admissible surfaces Λ.

Clearly, we can expect (6) to have a solution $\varphi \in L^2(\Omega)$ if and only if u_∞ is the far field pattern of a radiating solution to the Helmholtz equation in the exterior of Γ with sufficiently smooth boundary values on Γ. Hence, the solvability of (6) is related to regularity properties of the scattered wave which, in general, cannot be known in advance for the unknown scatterer D. Nevertheless, it is possible to provide a solid theoretical foundation to the above procedure (see [1, 4, 5, 6]). The method also has been implemented numerically in two and three dimensions with satisfactory reconstructions (see [1, 3, 7, 11]). Relations of the potential method of Kirsch and Kress with sampling and probe methods have been pointed out in [13].

The main advantage of the potential method of Kirsch and Kress and other decomposition methods consists of the fact that their numerical implementation does not require a forward solver. As a disadvantage, a good a priori information on the unknown scatterer is needed both for placing the auxiliary surface Γ and for the iterative solution of the minimization problem in the second part. Furthermore, the accuracy of the reconstructions is inferior to that of regularized Newton iterations for the boundary to far field mapping $F : \partial D \to u_\infty$.

More recently a hybrid method combining ideas of decomposition methods and regularized Newton iterations has been suggested. In principle, this approach may be considered as a modification of the potential method of Kirsch and Kress in the sense that the auxiliary surface Γ is viewed as an approximation for the unknown boundary ∂D and, keeping φ fixed as a regularized solution of (6), update Γ via linearizing the boundary condition (2) around Γ. Given a far field pattern u_∞ and a current approximation Γ for the boundary surface, we solve the ill-posed integral equation (6) by Tikhonov regularization and define u^s by (5). Then we evaluate the boundary values of $u = u^i + u^s$ and its derivatives on Γ via the jump relations and find an update $\Gamma_h := \{x + h(x) : x \in \Gamma\}$ with a sufficiently small C^2 vector field h on Γ by linearizing the boundary condition $u|_{\Gamma_h} = 0$, that is, by solving

$$u|_\Gamma + \operatorname{grad} u|_\Gamma \cdot h = 0$$

for h. In an obvious way, these two steps are iterated. Clearly, this approach does not require a forward solver and connects ideas of Newton iterations and decomposition methods. From numerical examples (see [8, 9, 10, 14, 15]) it can be concluded that the quality of the reconstructions is similar to that of Newton iterations for the boundary to far field operator. For the theoretical foundation similar to the method of Kirsch and Kress it can be related to a reformulation of the inverse scattering problem as an optimization problem (see [9, 10, 14]). Furthermore, convergence results can be obtained analogous to those of Potthast [12] on Newton iterations for the boundary to far field mapping $F : \partial D \to u_\infty$.

References

[1] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory. 2nd ed., Springer, Berlin 1998.

[2] D. Colton and R. Kress, *Using fundamental solutions in inverse scattering*, Inverse Problems **22** (2006), R49–R66.

[3] M. Haas and G. Lehner, *Inverse 2D obstacle scattering by adaptive iteration*, IEEE Transactions on Magnetics **33** (1997), 1958–1961.

[4] A. Kirsch and R. Kress, *On an integral equation of the first kind in inverse acoustic scattering,* in: *Inverse Problems* (Cannon and Hornung, eds). ISNM **77** (1986), 93–102.

[5] A. Kirsch and R. Kress, *A numerical method for an inverse scattering problem*, in: *Inverse Problems* (Engl and Groetsch, eds). Academic Press, Orlando 1987, 279–290.

[6] A. Kirsch and R. Kress, *An optimization method in inverse acoustic scattering*, in: *Boundary elements IX, Vol 3. Fluid Flow and Potential Applications,* (Brebbia et al, eds.), Springer-Verlag, Berlin 1987, 3–18.

[7] A. Kirsch, R. Kress, P. Monk and A. Zinn, *Two methods for solving the inverse acoustic scattering problem*, Inverse Problems **4** (1988), 749–770.

[8] R. Kress, *Newton's Method for inverse obstacle scattering meets the method of least squares*, Inverse Problems **19** (2003), 91–104.

[9] R. Kress and P. Serranho, *A hybrid method for two-dimensional crack reconstruction*, Inverse Problems **21** (2005), 773–784.

[10] R. Kress and P. Serranho, *A hybrid method for sound-hard obstacle reconstruction*, J. Comput. Appl. Math. (to appear).

[11] R. Kress and A. Zinn, *On the numerical solution of the three dimensional inverse obstacle scattering problem*, J. Comp. Appl. Math. **42** (1992), 49–61.

[12] R. Potthast, *On the convergence of a new Newton-type method in inverse scattering*, Inverse Problems **17** (2001), 1419–1434.

[13] R. Potthast and J. Schulz, *From the Kirsch-Kress potential method via the range test to the singular sources method*, J. Phys. Conf. Ser. **12** (2005), 116–127.

[14] P. Serranho, *A hybrid method for inverse scattering for shape and impedance*, Inverse Problems **22** (2006), 663–680.

[15] P. Serranho, *A hybrid method for inverse scattering for sound-soft obstacles in 3D,* Inverse Problems and Imaging **1**, (2007).

Detecting corrosion by thermal measurements

Thorsten Hohage
(joint work with M.-L. Rapún, F.-J. Sayas)

Photothermal techniques are suitable means of inspecting composite materials with non-destructive tests [6]. We are interested in a technique that consists in heating the accessible side of the material by a defocused laser beam. The goal is to reconstruct internal properties of the material (to detect structural defects, reconstruct the size, depth, orientation of the inclusions and/or physical properties of them) from measurements of the temperature at the side that has been thermically excited. In this talk we consider the detection of the level of corrosion at the interface of two materials.

The forward problem is modeled by a heat diffusion problem in the half space $\mathbb{R}^d_- := \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_d < 0\}$ with d =2 or 3. We consider an inclusion $\Omega_- \subset \mathbb{R}^d_-$, which is a bounded open domain whose boundary $\Gamma := \partial\Omega_-$ is a C^2-curve/surface. The materials occupying Ω_- and $\Omega_+ := \mathbb{R}^d_- \setminus \overline{\Omega_-}$ have different thermal properties, i.e., their corresponding diffusivities κ_-, $\kappa_+ > 0$ are different. Then the temperature distribution

$$U(\mathbf{x}, t) := \begin{cases} U_-(\mathbf{x}, t), & \text{in } \Omega_- \times (0, \infty), \\ U_+(\mathbf{x}, t), & \text{in } \Omega_+ \times (0, \infty), \end{cases}$$

satisfies the heat equation

$$\partial_t U_\mp = \kappa_\mp \Delta U_\mp, \qquad \text{in } \Omega_\mp \times (0, \infty). \tag{1a}$$

In the exterior domain Ω_+ the total temperature $U_+ + U_{\mathrm{hom}}$ is a superposition of U_+ and a given source field U_{hom} which satisfies

$$\partial_t U_{\mathrm{hom}} = \kappa_+ \Delta U_{\mathrm{hom}}, \qquad \text{in } \mathbb{R}^d_- \times (0, \infty).$$

U_{hom} plays the role of an incident field, and U_+ the role of a scattered field. On the common interface Γ, the temperature satisfies the transmission conditions

$$U_- + f\,\partial_\nu U_- = U_+ + U_{\mathrm{hom}}, \qquad \text{on } \Gamma \times (0, \infty), \tag{1b}$$

$$\alpha\,\partial_\nu U_- = \partial_\nu U_+ + \partial_\nu U_{\mathrm{hom}}, \qquad \text{on } \Gamma \times (0, \infty), \tag{1c}$$

where $\alpha > 0$ is the ratio of the interior and the exterior conductivities. Condition (1b) is sometimes called an Engquist-Nédélec condition. The function $f(\,\cdot\,) \geq f_0 > 0$ models a corrosion factor, proportional to the width of the coating at each point of the interface Γ, and it will be the unknown of the inverse problem. The model of the time-dependent forward problem is completed by the adiabatic boundary condition

$$\partial_\nu U_+ = 0, \qquad \text{on } \Pi \times (0, \infty), \tag{1d}$$

on the upper boundary $\Pi := \partial\mathbb{R}^d_-$ and the initial conditions

$$U_-(\,\cdot\,, 0) = 0, \qquad U_+(\,\cdot\,, 0) = 0. \tag{1e}$$

We also consider the case of time-harmonic excitations, i.e., incident fields of the form $U_{\mathrm{hom}}(\mathbf{x},t) = \mathrm{Re}(u_{\mathrm{hom}}(\mathbf{x})\,\exp(-\imath\,\omega\,t))$ with a given frequency $\omega > 0$. In this case we obtain asymptotically time-harmonic solutions

$$U_{\mp}(\mathbf{x},t) = \mathrm{Re}(u_{\mp}(\mathbf{x})\,\exp(-\imath\,\omega\,t))$$

to problem (1), and the space-dependent component $(u_-, u_+) \in H^2(\Omega_-)\times H^2(\Omega_+)$ is the solution of an elliptic transmission problem

$$\text{(2a)}\qquad \Delta u_- + \lambda_-^2 u_- = 0 \qquad \text{in } \Omega_-$$

$$\text{(2b)}\qquad \Delta u_+ + \lambda_-^2 u_+ = 0 \qquad \text{in } \Omega_+$$

$$\text{(2c)}\qquad u_- + f\,\partial_\nu u_- - u_+ = u_{\mathrm{hom}}, \qquad \text{on } \Gamma,$$

$$\text{(2d)}\qquad \alpha\,\partial_\nu u_- - \partial_\nu u_+ = \partial_\nu u_{\mathrm{hom}}, \qquad \text{on } \Gamma,$$

$$\text{(2e)}\qquad \partial_\nu u_+ = 0 \quad \text{on } \Pi$$

where $\lambda_\mp := (1+\imath)\sqrt{\omega/(2\kappa_\mp)}$. The incident field is typically given by a point source at some point $\mathbf{x}_0 \in \Pi$:

$$\text{(2f)}\qquad u_{\mathrm{hom}}(\mathbf{x},\mathbf{x}_0) := \begin{cases} (\imath/4)\,H_0^{(1)}(\lambda_+|\mathbf{x}-\mathbf{x}_0|) & \text{for } d=2, \\ \exp(\imath\lambda_+|\mathbf{x}-\mathbf{x}_0|)/(4\pi|\mathbf{x}-\mathbf{x}_0|) & \text{for } d=3. \end{cases}$$

The inverse problem is to find f given the incident field and measurements of $u_+(\mathbf{x})$ for $\mathbf{x} \in \Pi_{\mathrm{obs}} \subset \Pi$ or measurements of $U_+(\mathbf{x},t)$ for $\mathbf{x} \in \Pi_{\mathrm{obs}}$ and t in some time interval. We refer to [1, 2] for related inverse problems.

For the time-harmonic inverse problem we prove the following uniqueness result:

Theorem 1. *Assume that* $\mathrm{int}(\Pi_{\mathrm{obs}}) \neq \emptyset$*, let* Π_{src} *be a non-empty subset of* Π*, let* $f^{(1)}, f^{(2)} > 0$ *be two corrosion factors, and denote the corresponding solutions to (2) by* $u_\pm^{(1)}(\,\cdot\,,\mathbf{x}_0)$ *and* $u_\pm^{(2)}(\,\cdot\,,\mathbf{x}_0)$ *for all* $\mathbf{x}_0 \in \Pi_{\mathrm{src}}$*. Suppose that* $u_+^{(1)}|_{\Pi_{\mathrm{obs}}\times\Pi_{\mathrm{src}}} \equiv u_+^{(2)}|_{\Pi_{\mathrm{obs}}\times\Pi_{\mathrm{src}}}$*. Then*

$$\text{(3)}\qquad f^{(1)} = f^{(2)} \qquad \text{on } \tilde{\Gamma} := \{\mathbf{x}\in\Gamma : \exists \mathbf{x}_0 \in \Pi_{\mathrm{src}}\;\; \partial_\nu u_-^{(1)}(\mathbf{x},\mathbf{x}_0) \neq 0\}.$$

If in addition $\mathrm{int}(\Pi_{\mathrm{src}}) \neq \emptyset$*, then* $\tilde{\Gamma} = \Gamma$*, i.e.,* $f^{(1)} = f^{(2)}$ *everywhere on* Γ*.*

We discuss two methods for the numerical solution of the time-harmonic inverse problem. The first one is based on the proof of the uniqueness theorem above and avoids the solution of the forward problem. For this method a convergence result can be shown as the noise level tends to 0.

As an alternative, we study the iteratively regularized Gauß-Newton method. For this end, we introduce the operator F mapping the unknown corrosion function f to the measured data and characterize its Fréchet derivative. The forward problem is solved by a boundary integral equation method [7]. Although the regularized Newton method is more time consuming as it involves the solution of the forward problem in each step, it turns out that it yields considerably more accurate results.

The second part of the talk is concerned with the time-dependent problem. We use the approach proposed and analyzed in [3], which is based on the computation of the Laplace transform of the solution with respect to time

$$u(\mathbf{x}, s) = \int_0^\infty e^{-st}\, U(\mathbf{x}, t)\, dt, \qquad \mathbf{x} \in \mathbb{R}^d_-, \quad \operatorname{Re} s > 0. \tag{4}$$

Note that $u(\,\cdot\,, s)$ satisfies (2) with $\lambda^2_\mp = -s/\kappa_\mp$. This system of equations is uniquely solvable for any $s \in \mathbb{C} \setminus \{t \in \mathbb{R} : t \leq 0\}$. Then, one can recover the time-dependent solution by the inversion formula

$$U(\mathbf{x}, t) \;=\; \frac{1}{2\pi\imath} \int_{\mathcal{C}} e^{st} u(\mathbf{x}, s)\, ds, \qquad t > 0, \tag{5}$$

$\mathcal{C}$ being any path connecting $-\imath\,\infty$ and $\imath\,\infty$. More precisely, following [5] we use the hyperbolic pathes parametrized by $\gamma(\theta) = \mu(1 - \sin(\beta + \imath\theta))$, $\theta \in \mathbb{R}$, where $\mu > 0$ and $0 < \beta < \pi/2$ are tuning parameters. Then, $U(\mathbf{x}, t)$ is approximated by a truncated trapezoidal rule

$$U(\mathbf{x}, t) \approx \sum_{j=-m}^{m} \omega_j\, e^{t\, s_j}\, u(\mathbf{x}, s_j) \tag{6}$$

with nodes $s_j := \gamma\left(\frac{\log(m)}{m}\, j\right)$ and weights $\omega_j := \frac{\log(m)}{2\pi\imath m}\, \gamma'\left(\frac{\log(m)}{m}\, j\right)$. This methods leads to $2m + 1$ stationary problems of the form (2) to compute $u(\mathbf{x}, s_j)$. Due to the symmetry with respect to the real axis only $m + 1$ problems have to be solved. The convergence as $m \to \infty$ is of order $O(\exp(-cm/\ln(m)))$ (see [3, 5]).

For the inverse time-dependent problem one can show a result analogous to Theorem 1 if data are available on a non-degenerate time interval. However, in this case the problem is formally overdetermined. Numerical experiments suggest that at least several measurement points are necessary for a fixed point source to reconstruct the corrosion function. With a proper choice of the time interval and the measurement points, time dependent experiments can yield significantly more accurate reconstructions of the corrosion function than time-harmonic experiments for the same number of measurements.

References

[1] I. Akduman and R. Kress, *Direct and inverse scattering problems for inhomogeneous impedance cylinders of arbitrary shape. Radio Sci.* **38** (2003), 1055–1064.

[2] F. Cakoni, D. Colton and P. Monk, *The determination of the surface conductivity of a partially coated dielectric*, SIAM J. Appl. Math. **65** (2005), 767–789.

[3] T. Hohage and F.-J. Sayas, *Numerical solution of a heat diffusion problem by boundary integral methods using the Laplace transform*, Numer. Math. **102** (2005), 67–92.

[4] T. Hohage, M.-L. Rapun and F.-J. Sayas, *Detecting corrosion using thermal measurements*, Inverse Problems **23** (2007), 53–72.

[5] M. López-Fernández and C. Palencia, *On the numerical inversion of the Laplace transform of certain holomorphic mappings*, Appl. Numer. Math. **51** (2004), 289–303.

[6] A. Mandelis, *Diffusion waves and their uses*, Physics today **53** (2000), 29–34.

[7] M.-L. Rapún and F.-J. Sayas, *Boundary integral approximation of a heat diffusion problem in time–harmonic regime*, Numer. Algorithms **41** (2006), 127–160.

Tangential cone condition for electrical impedance tomography

Andreas Rieder
(joint work with Armin Lechleiter)

In the convergence analysis of Newton-like regularization schemes (see, e.g., [1] [4], [5]) for nonlinear ill-posed problems the linearization error is often controlled by the *tangential cone condition*: an operator $F\colon \mathsf{D}(F) \subset X \to Y$ satisfies the tangential cone condition in x if there is a ball $B_\rho(x) \subset \mathsf{D}(F)$ of radius ρ centered about w and a constant $\omega < 1$ such that

$$\|F(v) - F(w) - F'(w)(v - w)\|_Y \le \omega \|F(w) - F(v)\|_Y \quad \text{for all } w, v \in B_\rho(x)$$

where F' denotes the Fréchet derivative of F. Although the tangential cone condition is meaningfully defined on normed spaces most applications require X and Y to be Hilbert spaces.

The tangential cone condition is as useful as severe. For instance, it implies that $\mathsf{N}(F'(\cdot)) = \mathsf{N}(F'(x))$ in $B_\rho(x)$, that is, the null space of $F'(\cdot)$ is invariant in $B_\rho(x)$. Furthermore, $F(v) = F(w)$ whenever $v - w \in \mathsf{N}(F'(x))$ and $v, w \in B_\rho(x)$. Therefore only few meaningful examples of nonlinear operators are known satisfying a tangential cone condition. Here we add the forward operator of electrical impedance tomography (EIT) to the collection of meaningful examples (see [6] for the missing details).

EIT entails the reconstruction of the conductivity distribution $\gamma\colon B \to]c_0, \infty[$, $c_0 > 0$, in a simply connected Lipschitz domain $B \subset \mathbb{R}^2$ from measurements of the electric potential u on the boundary ∂B which is induced by applying the electric current distribution f on ∂B (see, e.g., [3]). Due to the principle of conservation of charge we prescribe $\int_{\partial B} f \mathrm{d}S = 0$ and ground the potential u by $\int_{\partial B} u \mathrm{d}S = 0$.

The governing equation in weak formulation is

$$(1) \qquad \int_B \gamma \nabla u \nabla v \mathrm{d}x = \int_{\partial B} f v \mathrm{d}S \quad \text{for all } v \in H^1_\diamond(B)$$

and it has a unique solution $u = u(f) \in H^1_\diamond(B) := \{v \in H^1(B) : \int_{\partial B} v \mathrm{d}S = 0\}$.

The inverse EIT problem can now be phrased as: given the *Neumann-to-Dirichlet operator*

$$\Lambda\colon f \mapsto u|_{\partial B}$$

find the conductivity γ. By classical results from the theory of partial differential equations $\Lambda\colon H^{-1/2}_\diamond(\partial B) \to H^{1/2}_\diamond(\partial B)$ is bounded. Thus, we have to solve the nonlinear equation $F(\gamma) = \Lambda$ where

$$F\colon \mathsf{D}(F) \subset L^\infty(B) \to \mathcal{L}(H^{-1/2}_\diamond(\partial B), H^{1/2}_\diamond(\partial B)), \quad \gamma \mapsto \Lambda,$$

with $\mathsf{D}(F) := L^\infty_+(B) = \{\gamma \in L^\infty(B) : \gamma \ge c_0\}$. Note that $F(\gamma)f = u|_{\partial B}$.

Since we are interested in a Hilbert space setting we restrict both, the pre-image and the image space of F. Henceforth we consider

$$F\colon H^q_+(B) \subset H^q(B) \to \mathcal{H}\big(L^2_\diamond(\partial B)\big), \quad q > 1,$$

where $\mathcal{H}\big(L^2_\diamond(\partial B)\big)$ is the space of Hilbert-Schmidt endomorphisms on $L^2_\diamond(\partial B)$ (for a proof that $F(\gamma)$ is indeed Hilbert-Schmidt see, e.g., [2]). Technical reasons require certain regularity of the conductivities, which explains the choice of $H^q(B)$, $q > 1$.

Let $\gamma \in \operatorname{int}(H^q_+(B))$. Then, $F'(\gamma) \in \mathcal{L}\big(H^q(B), \mathcal{H}\big(L^2_\diamond(\partial B)\big)$ is given by

$$F'(\gamma)[h]f := w|_{\partial B} \in L^2_\diamond(\partial B)$$

with $w = w(h,f) \in H^1_\diamond(B)$ such that

$$\int_B \gamma \nabla w \nabla \varphi \,\mathrm{d}x = -\int_B h \nabla u \nabla \varphi \,\mathrm{d}x \quad \text{for all } \varphi \in H^1_\diamond(B). \tag{2}$$

Moreover,

$$\big\| \big(F(\gamma+h) - F(\gamma) - F'(\gamma)[h]\big) f \big\|_{L^2(\partial B)} \le C \, \|\gamma\|_{H^q(B)} \|h\|_{H^q(B)} \|\nabla w\|_{L^2(B)} \text{ [1]}$$

for $h \in H^q(B)$ small enough: $\gamma + h \in H^q_+(B)$. Inspecting the latter estimate we conclude: if

$$\|\nabla w(h,f)\|_{L^2(B)} \le C \, \|w(h,f)\|_{L^2(\partial B)} \tag{3}$$

uniformly in h and f then, by an inverse triangle inequality,

$$\begin{aligned} &\|F(\gamma+h) - F(\gamma) - F'(\gamma)[h]\|_{\mathcal{H}(L^2(\partial B))} \\ &\qquad\qquad \le C \, \|\gamma\|_{H^q(B)} \|h\|_{H^q(B)} \|F(\gamma+h) - F(\gamma)\|_{\mathcal{H}(L^2(\partial B))}. \end{aligned} \tag{4}$$

For h so small that $C \, \|\gamma\|_{H^q(B)} \|h\|_{H^q(B)} < 1$, inequality (4) is a tangential cone condition locally about γ.

A first step to prove (3) is the following lemma.

Lemma 1. *Let $h \in H^q(B)$, $q > 1$, and let $f \in L^2_\diamond(\partial B)$. If $w|_{\partial B} = 0$ then $w|_B = 0$.*

Proof. Plugging $\varphi = u$ into (2) and taking into account the defining equation (1) for u and that $w|_{\partial B} = 0$ we find

$$-\int_B h |\nabla u|^2 \,\mathrm{d}x = \int_B \gamma \nabla w \nabla u \,\mathrm{d}x = \int_{\partial B} f w \,\mathrm{d}S = 0.$$

If h *does not change sign* then $u|_{\operatorname{supp} h} = 0$ and

$$0 = -\int_B h \nabla u \nabla \varphi \,\mathrm{d}x = \int_B \gamma \nabla w \nabla \varphi \,\mathrm{d}x \quad \text{for all } \varphi \in H^1_\diamond(B)$$

yielding $w = 0$ in B. In case h changes sign the argument is much more involved (see [6] for details). $\square$

Finally, we outline how the lemma implies (3). As w depends bilinearly on h and f we may assume that $\|h\|_{H^q(B)} = \|f\|_{L^2(\partial B)} = 1$. Further, assume (3) not to hold. Then, there is a sequence $\{(h_j, f_j)\}$ with $\|h_j\|_{H^q(B)} = \|f_j\|_{L^2(\partial B)} = 1$ such that

$$\frac{\|w(h_j,f_j)\|_{L^2(\partial B)}}{\|\nabla w(h_j,f_j)\|_{L^2(B)}} \to 0 \text{ as } j \to \infty. \tag{5}$$

[1] C always denotes a generic constant possibly attaining different values at each instance it appears.

By boundedness we may assume the whole sequence $\{(h_j, f_j)\}$ to converge weakly to $(\widetilde{h}, \widetilde{f})$ in $H^q(B) \times L^2_\diamond(\partial B)$. Further, by compact embedding we may assume the whole sequence $\{(h_j, f_j)\}$ to converge strongly to $(\widetilde{h}, \widetilde{f})$ in $H^{q-\epsilon}(B) \times H^{-1/2}(\partial B)$ ($\epsilon > 0$ such that $q - \epsilon > 1$).

Since both mappings $H^{-1/2}(\partial B) \ni f \mapsto u \in H^1(B)$ and $H^{q-\epsilon}(B) \times H^{-1/2}(\partial B) \ni (h, f) \mapsto w \in H^1(B)$ are continuous we obtain that

$$\lim_{j\to\infty} \|u(f_j) - u(\widetilde{f})\|_{H^1(B)} = 0 \quad \text{and} \quad \lim_{j\to\infty} \|w(h_j, f_j) - w(\widetilde{h}, \widetilde{f})\|_{H^1(B)} = 0.$$

Hence, the limits $u(\widetilde{f})$ and $w(\widetilde{h}, \widetilde{f})$ are related via (2):

$$\int_B \gamma \nabla w(\widetilde{h}, \widetilde{f}) \nabla \varphi \, \mathrm{d}x = -\int_B \widetilde{h} \nabla u(\widetilde{f}) \nabla \varphi \, \mathrm{d}x \quad \text{for all } \varphi \in H^1_\diamond(B). \tag{6}$$

Furthermore,

$$\|w(\widetilde{h}, \widetilde{f})\|_{L^2(\partial B)} = 0$$

which is a consequence from (5) ($\|\nabla w(h_j, f_j)\|_{L^2(B)}$ is uniformly bounded). As $(\widetilde{h}, \widetilde{f}) \in H^q(B) \times L^2_\diamond(\partial B)$ we may apply our lemma resulting in $w(\widetilde{h}, \widetilde{f}) = 0$ in B. In view of (6),

$$0 = -\int_B \widetilde{h} \nabla u(\widetilde{f}) \nabla \varphi \, \mathrm{d}x \quad \text{for all } \varphi \in H^1_\diamond(B),$$

so that $\widetilde{h} \nabla u(\widetilde{f}) = 0$ in B. However, $\nabla u(\widetilde{f})$ can only vanish on a set of measure zero (unique continuation property). Thus, $\widetilde{h} = 0$ contradicting $\|h_j\|_{H^q(B)} = 1$, that is, the crucial bound (3) holds true.

References

[1] B. Blaschke(Kaltenbacher), A. Neubauer, and O. Scherzer, *On convergence rates for the iteratively regularized Gauss-Newton method*, IMA J. Numer. Anal. **17** (1997), 421–436.

[2] M. Brühl, Gebietserkennung in der elektrischen Impedanztomographie, PhD thesis (1999), Department of Mathematics, University of Karlsruhe, download: `www.ubka.uni-karlsruhe.de/cgi-bin/psgunzip/1999/mathematik/1/1.pdf`.

[3] M. Cheney, D. Isaacson, and J. C. Newell, *Electrical impedance tomography*, SIAM Review **41** (1999), 85–101.

[4] M. Hanke, A. Neubauer and O. Scherzer, *A convergence analysis of the Landweber iteration for nonlinear ill-posed problems*, Numer. Math. **72** (1995), 21–37.

[5] A. Lechleiter and A. Rieder, *A convergence analysis of the Newton-type regularization CG-REGINN with application to impedance tomography*, IWRMM Preprint 07/01 (2007), Department of Mathematics, University of Karlsruhe, download: `www.mathematik.uni-karlsruhe.de/iwrmm/seite/preprints/media/preprint%20nr.%2007-01.pdf`.

[6] A. Lechleiter and A. Rieder, *On the ill-posedness of impedance tomography: the tangential cone condition*, work in progress, Department of Mathematics, University of Karlsruhe.

Regularization of the factorization method and an application in impedance tomography

Armin Lechleiter
(joint work with Nuutti Hyvönen, Harri Hakula)

The factorization method is a well established tool to recover the support of inhomogeneities in inverse problems for partial differential equations. Through many sophisticated ideas and successive technical refinement the method has evolved during the last decade, starting from the first formulation of the linear sampling method by Colton and Kirsch in [4] and the fundamental paper of Kirsch [9]. Applications of the method include inverse acoustic and electromagnetic scattering [11], inverse problems for periodic structures [2], inverse elliptic problems such as impedance or optical tomography [6, 8] and inverse problems in elasticity [3], this list being certainly incomplete and still rapidly growing. We refer to [5] for a concise overview of the method's state of the art and the many open questions related to it.

From both the analytical and computational point of view, the formulation of the method is probably most attractive when Picard's criterion can be used. Suppose to begin with that $F = GTG^*$ is a factorization of the compact data operator F of some inverse problem at hand and that G characterizes the searched-for inclusion $D \subset \mathbb{R}^d$, $d = 2$ or 3, in the following way: A point $y \in \mathbb{R}^d$ belongs to D if and only if $\phi_y \in \mathrm{Range}(G)$, where ϕ_y is a special testfunction (usually related to the Green's function of the problem), given for any point $y \in \mathbb{R}^d$. Moreover, we assume that the ranges of $F_\sharp^{1/2}$ and G equal each other. The modified version $F_\sharp$ of F is defined as $|\mathrm{Re}\, F| + \mathrm{Im}\, F$ and possesses a complete orthonormal eigensystem $(\lambda_j, \psi_j)_{j\in\mathbb{N}}$, see [10] for details. Using Picard's criterion we conclude that

$$y \text{ belongs to } D \Leftrightarrow \sum_{j=1}^{\infty} \frac{|\langle \phi_y,\, \psi_j\rangle|^2}{\lambda_j} < \infty. \tag{1}$$

The question we are concerned with is how the Picard criterion in (1) behaves when only an approximate version F^δ of the data operator F is accessible where $\|F - F^\delta\| = \delta$. Our aim is to show that the characterization in (1) holds asymptotically as $\delta \to 0$, that is, there is a cut-off index $N(\delta)$ (which can be given explicitely) such that $N(\delta) \to \infty$ as $\delta \to 0$ and

$$y \text{ belongs to } D \Leftrightarrow \sum_{j=1}^{N(\delta)} \frac{|\langle \phi_y,\, \psi_j^\delta\rangle|^2}{\lambda_j^\delta} \text{ is bounded as } \delta \to 0. \tag{2}$$

From (2) we conclude that evaluating the finite sum on the right-hand side for $\delta > 0$ small enough and $y \notin D$ results in considerably larger numerical values than evaluation for $y \in D$.

For the asymptotic analysis as $\delta \to 0$ we note that the estimate

$$\big\| |F| - |F^\delta| \big\| \leq C(1 + \ln \|F - F^\delta\|)\|F - F^\delta\|$$

from [14] yields that closeness of F and F^δ implies closeness of $F_\sharp$ and $F_\sharp^\delta$. Therefore we can also assume that $\|F_\sharp - F_\sharp^\delta\| = \delta$ and denote an eigensystem of $F_\sharp^\delta$ by $(\lambda_j^\delta, \psi_j^\delta)$.

Obviously, the characterization in (2) relies on perturbation theory. First, we note that $\mathrm{dist}(\sigma(F_\sharp), \sigma(F_\sharp^\delta)) \le \|F_\sharp - F_\sharp^\delta\|$, where $\sigma(F_\sharp)$ denotes the spectrum of $F_\sharp$. Second, the Picard criterion (1) also necessitates to carefully investigate the behavior of the eigenspaces of $F_\sharp^\delta$. Therefore we define the spectral projection on the eigenspaces of the jth eigenvector of $F_\sharp$ and $F_\sharp^\delta$,

$$P_j = \sum_{\lambda_n = \lambda_j} (\cdot, \psi_n)\psi_n \qquad \text{and} \qquad P_j^\delta = \sum_{\lambda_n^\delta = \lambda_j^\delta} (\cdot, \psi_n^\delta)\psi_n^\delta,$$

respectively. Cauchy's theorem states that

$$P_j = \frac{1}{2\pi i} \int_\gamma (\xi - F_\sharp)^{-1}\,\mathrm{d}\xi$$

for any contour γ around λ_j such that the rest of $\sigma(F_\sharp)$ is outside of γ. This representation formula allows the following estimate: If $\mathrm{dist}(\lambda_j, \sigma(F_\sharp) \setminus \{\lambda_j\}) = 2d$ and $\|F_\sharp - F_\sharp^\delta\| \le \delta < d$, then

$$\|P_j - P_j^\delta\| \le \frac{\delta}{d - \delta}.$$

The latter tools and some tedious estimates principally yield the asymptotic characterization (2). For the details as well as for a definition of the truncation index $N(\delta)$ we refer to [12]. Two comments are in order: The truncation index $N(\delta)$ implies regularization of the series criterion and the analysis shows that this regularization is crucial. Moreover, as it was remarked by Rainer Kress in the discussion after the talk, norm convergence of $F - F^\delta$ can be replaced by supposing pointwise convergence together with collectively compactness of $\{F^\delta\}_{0 \le \delta \le \delta_0}$. This relies on perturbation theory studied in the monograph [1].

The above theory can be used to construct a factorization method for the complete electrode model of impedance tomography. This problem has been tackled first in [7] using an inf-criterion. By stronger convergence results between different electrode models than in [7], together with the above analysis, we are able to show that in fact the series criterion of the factorization method, which is easier to deal with, is also applicable in this situation [13].

More precisely, consider first the continuum model in a bounded domain $\Omega \subset \mathbb{R}^d$ for a conductivity γ and a boundary current $f \in H_\diamondsuit^{-1/2}(\partial B)$. The direct problem is then to find a potential $u \in H_\diamondsuit^1(\Omega)$ solving $-\nabla(\gamma\nabla u) = 0$ in Ω subject to the co-normal boundary condition $\partial_\gamma u = f$ on ∂B. If γ has the form

$$\gamma = \begin{cases} \gamma_0 + \gamma_1 & \text{in } D \Subset \Omega \\ \gamma_0 & \text{in } \Omega \setminus \overline{D} \end{cases}$$

for uniformly positive smooth functions γ and γ_1, the inverse problem of determining D from the Neumann-to-Dirichlet operator $\Lambda :\ f \mapsto u|_{\partial\Omega}$ is well known to be solvable via factorization, see e.g., [6]. The complete model takes into account p discrete perfect conducting electrodes where current is injected and voltage is measured and moreover a contact impedance effect with resistivity $z = (z_1, \ldots, z_p)$ underneath the electrodes: Given a discrete current vector $I \in \mathbb{C}^p_\diamondsuit$ we want to find a pair $(u, U) \in H^1(\Omega) \times C^p_\diamondsuit$ such that

$$-\nabla(\gamma\nabla u) = 0 \qquad \text{in } B \qquad\qquad u + z_j \partial_\gamma u = U_j \qquad \text{on } E_j$$

$$\int_{E_j} \partial_\gamma u \, \mathrm{d}S = I_j \qquad\qquad \partial_\gamma u = 0 \text{ on } \partial B \setminus \cup_j E_j.$$

Let us denote by $\Sigma_p : I \mapsto U$ the discrete Neumann-to-Dirichlet operator for the complete model and by P_p an appropriately chosen projector from $L^2_\diamondsuit$ into $\mathbb{C}^p_\diamondsuit$ when p electrodes are used in the measurements. Moreover, denote by Λ_0 and Σ_{0p} the Neumann-to-Dirichlet operators for the continuous and discrete case with conductivity γ_0 instead of γ, respectively. Under some further geometric conditions on the sequence of electrode configurations one can prove that $\|(\Lambda - \Lambda_0) - (\Sigma_p - \Sigma_{0p})\|_{L^2 \to L^2} \to 0$ as $p \to \infty$. We refer to [13] for details as well as for numerical examples for the thereby constructed factorization methods which underline the validity of our results.

References

[1] P. M. Anselone and J. Davis, Collectively compact operator approximation theory and applications to integral equations, Prentice-Hall, 1971.

[2] T. Arens and N.I. Grinberg, *A complete factorization method for scattering by periodic structures*, Computing **75** (2005), 111–132.

[3] A. Charalambopoulus, A. Kirsch, K.A. Anagnostopoulus, D. Gintides and K. Kiriaki, *The factorization method in inverse elastic scattering from penetrable bodies*, Inverse Problems **23** (2007), 27–51.

[4] D. Colton and A. Kirsch, *A simple method for solving inverse scattering problems in the resonance region*, Inverse Problems **12** (1996), 383–393.

[5] N. Grinberg and A. Kirsch, The factorization method for inverse problems, in progress.

[6] M. Hanke and M. Brühl, *Recent progress in electrical impedance tomography*, Inverse Problems **19** (2003), S65–S90.

[7] N. Hyvönen, *Complete electrode model of electrical impedance tomography: approximation properties and characterization of inclusions*, SIAM J. Appl. Math. **64** (2004), 902–931.

[8] N. Hyvönen, *Locating transparent regions in optical absorption and scattering tomography*, accepted to SIAM J. Appl. Math..

[9] A. Kirsch, *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Problems **14** (1998), 1489–1512.

[10] A. Kirsch, *The factorization method for a class of inverse elliptic problems*, Math. Nachr. **278** (2005), 258–277.

[11] A. Kirsch, *An integral equation for Maxwell's equations in a layered medium with an application to the factorization method*, J. Integral Equations Appl., to appear.

[12] A. Lechleiter, *A regularization technique for the factorization method*, Inverse Problems **22** (2006), 1605–1625.

[13] A. Lechleiter, N. Hyvönen and H. Hakula, *The factorization method applied to the complete electrode method of impedance tomography*, download: `www.mathematik.uni-karlsruhe.de/iag1/~lechleiter/media/approximationcem4.pdf`.

[14] G.M. Vainikko, *The discrepancy principle for a class of regularization methods. (English. Russian original)*, U.S.S.R. Comput. Math. Math. Phys. **22**, No.3 (1982), 1–19.

On uniqueness in inverse scattering with finitely many incident waves

Johannes Elschner
(joint work with Masahiro Yamamoto)

The inverse scattering problem of determining a bounded obstacle by its far field pattern is fundamental for exploring bodies by acoustic or electromagnetic waves, and its uniqueness presents important and challenging open questions since many years. The first part of the talk gives an overview on recent uniqueness results, due to Alessandrini and Rondi [1], Liu and Zou [6], Yamamoto and the speaker [2], for the problem of determining sound-soft and sound-hard polygonal/polyhedral obstacles by a finite number of incident waves. These results can be extended to scatterers with impedance and mixed type (Dirichlet/Neumann) boundary conditions [7], [8].

Then the problem of recovering a two-dimensional perfectly reflecting diffraction grating from measurements on a horizontal line above the structure is considered. This occurs in several applications in diffractive optics and leads to the inverse Dirichlet and Neumann problems for the periodic Helmholtz equation in 2D. We present uniqueness results within the class of polygonal grating profiles by a minimal number of incident plane waves [3], which improve those of [5] where the Rayleigh frequencies were excluded and the profiles were assumed to be graphs of piecewise linear functions.

Finally, these results are extended to inverse transmission problems for the Helmholtz equation in 2D (work in progress with M. Yamamoto). In particular, it can be shown that uniqueness in the inverse TE and TM transmission problems for diffraction gratings with one incident wave holds within the class of piecewise linear profile functions. In the case of more general grating profiles (given by the graph of a periodic Lipschitz function), only local uniqueness results are known [4].

References

[1] G. Alessandrini and L. Rondi, *Determining a sound-soft polyhedral scatterer by a single far-field measurement*, Proc. Amer. Math. Soc. **133** (2005), 1685–1691.

[2] J. Elschner and M. Yamamoto, *Uniqueness in determining polygonal sound-hard obstacles with a single incoming wave*, Inverse Problems **22** (2006), 355–364.

[3] J. Elschner and M. Yamamoto, *Uniqueness in determining polygonal periodic structures*, Z. Anal. Anwendungen **26** (2007), 165–177.

[4] J. Elschner and M. Yamamoto, *Uniqueness results for an inverse periodic transmission problem*, Inverse Problems **20** (2004), 1841–1852.

[5] J. Elschner, G. Schmidt, and M. Yamamoto, *An inverse problem in periodic diffractive optics: global uniqueness with a single wavenumber*, Inverse Problems **11** (2003), 235–244.

[6] H. Liu and J. Zou, *Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers*, Inverse Problems **22** (2006), 515–524.

[7] H. Liu and J. Zou, *On unique determination of partially coated polyhedral scatterers with far field measurements*, Inverse Problems, **23** (2007), 297–308.

[8] H. Liu and J. Zou, *Uniqueness in determining multiple polygonal or polyhedral scatterers of mixed type*, Preprint No. 337, The Chinese University of Hong Kong, 2006.

The linear sampling method for inverse rough surface scattering

Simon N. Chandler-Wilde
(joint work with Peter Monk)

In this talk we first recall results over the last ten years on the direct rough surface scattering problem, the problem of calculating the field scattered by an unbounded surface that is the graph of a function. We then move on to consider the following specific two-dimensional inverse problem: a point source moves along a finite horizontal line above the rough surface and, for every position along this line, measurements of the scattered waves are made, also on a (possibly identical) finite horizontal line. We discuss uniqueness and present some theoretical and computational results for a version of the linear sampling method.

Locating transparent cavities in optical absorption and scattering tomography

Nuutti Hyvönen

In optical absorption and scattering tomography (OAST), a physical body is illuminated by a flux of near-infrared photons and the out coming flux is measured on the surface of the body. The idea is to reconstruct the optical properties, such as absorption and scatterer, inside the body by using the measured pairs of input and output fluxes. OAST has a few possible clinical applications, the most important of which are, arguably, screening for breast cancer and the development of a cerebral imaging modality for mapping structure and function in newborn infants. For more medical and instrumental details we refer to [1, 2, 6].

Since brain consists of strongly scattering tissue with embedded cavities filled with nearly transparent cerebrospinal fluid, a forward model of OAST for the human head can be constructed by sewing up the diffusion approximation of the radiative transfer equation with geometrical optics [8]. As noted in [5], a current disadvantage with this radiosity-diffusion model is that the boundaries of the transparent cavities must be known in advance when reconstructing the physiologically interesting quantities, i.e., the absorption and the scatterer in the strongly scattering tissue. Although the ultimate goal is to develop algorithms for simultaneous reconstruction of the transparent cavities and the optical properties in the brain tissue, in this work we tackle a preliminary inverse problem: We assume that the absorption and the scatterer in the diffusive background of the examined body are known and try to locate the transparent regions through boundary measurements.

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be the body under investigation and assume that the symmetric and uniformly strictly positive definite diffusion tensor $\kappa : \Omega \to \mathbb{R}^{n\times n}$ and the strictly positive absorption coefficient $\mu : \Omega \to \mathbb{R}$ are known in advance. If the boundary measurements are static in time, the radiosity-diffusion forward problem of OAST can be formulated as follows [7, 8]: Find the photon density φ

that satisfies the elliptic boundary value problem

$$
\begin{aligned}
\nabla \cdot \kappa \nabla \varphi - \mu \varphi &= 0 && \text{in } \Omega \setminus \overline{D}, \\
\nu \cdot \kappa \nabla \varphi &= f && \text{on } \partial\Omega, \\
G\varphi + \nu \cdot \kappa \nabla \varphi &= 0 && \text{on } \partial D,
\end{aligned} \tag{1}
$$

where ν is the unit normal pointing out of $\Omega \setminus \overline{D}$ and D consists of the transparent cavities. Furthermore, the operator $G : L^2(\partial D) \to L^2(\partial D)$, appearing in the nonlocal 'inner' boundary condition of (1), is defined through $G = 2\gamma(I - \mathcal{G})^{-1}(I + \mathcal{G})$, where γ is a dimension-dependent positive constant and

$$
\begin{aligned}
(\mathcal{G}\Phi)(x) \quad = \quad & \frac{n-1}{|S^{n-2}|} \int_{\partial D} v(x,y) \frac{(\nu(x)\cdot(x-y))(\nu(y)\cdot(x-y))}{|x-y|^{n+1}} \times \\
& e^{-\tilde{\mu}|x-y|} \Phi(y) dS(y).
\end{aligned}
$$

Here the constant $\tilde{\mu}$ is the absorption coefficient in D and $v : \partial D \times \partial D \to \{0, 1\}$ is a visibility function,

$$
v(x,y) = \begin{cases} 1, & \text{if } tx + (1-t)y \in D \text{ for } 0 < t < 1, \\ 0, & \text{otherwise.} \end{cases}
$$

Under suitable smoothness conditions, the forward problem (1) has a unique solution $\varphi \in H^1(\Omega \setminus \overline{D})$ that depends continuously on the input $f \in L^2(\partial\Omega)$ [8].

When considering reconstruction of the transparent cavities D, we assume that the operator mapping the conormal derivative of the photon density to its Dirichlet boundary value can be measured on $\partial\Omega$, i.e., we assume to know the map

$$
\Lambda : f \mapsto \varphi|_{\partial\Omega}, \quad L^2(\partial\Omega) \to L^2(\partial\Omega),
$$

where φ is the unique solution of (1). Since κ and μ are known in the whole of Ω, we can, in addition, compute the Neumann-to-Dirichlet map corresponding to Ω with no embedded cavities:

$$
\Lambda_0 : f \mapsto \varphi_0|_{\partial\Omega}, \quad L^2(\partial\Omega) \to L^2(\partial\Omega),
$$

where φ_0 is the solution of the problem obtained by deleting the 'inner' boundary condition in (1) and letting the diffusion equation be satisfied everywhere in Ω.

By applying the factorization method of Andreas Kirsch [12, 13] to our problem setting, we get an explicit (but conditional) characterization of the transparent cavities D through the boundary measurements of OAST [11]. Notice that the factorization technique has been used within OAST in [3, 4, 9, 10], as well. To be able to describe our result in more detail, we need to introduce the photon density h_y corresponding to a point source at $y \in \Omega$ and the homogeneous Neumann boundary condition on $\partial\Omega$:

$$
\begin{aligned}
\nabla \cdot \kappa \nabla h_y(x) - \mu h_y(x) &= \delta(x-y) && \text{in } \Omega, \\
\nu \cdot \kappa \nabla h_y &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

Under somewhat restrictive conditions on the transparent cavities and on the background optical parameters κ and μ, one can show that [11]

$$y \in D \Longleftrightarrow h_y|_{\partial\Omega} \in \mathcal{R}\left(|\Lambda - \Lambda_0|^{1/2}\right), \tag{2}$$

where $\mathcal{R}\left(|\Lambda - \Lambda_0|^{1/2}\right)$ denotes the range of the self-adjoint and positive definite square root $|\Lambda - \Lambda_0|^{1/2} : L^2(\partial\Omega) \to L^2(\partial\Omega)$. Since $h_y|_{\partial\Omega}$ and Λ_0 can be computed and Λ can, in principle, be measured, the validity of the relation on the right hand side of the above equivalence can be tested. As a consequence, (2) provides an explicit characterization of D.

With the help of Tikhonov regularization, one can build a reconstruction algorithm based on the characterization (2). This matter is considered in detail in [11].

References

[1] S. R. Arridge, *Optical tomography in medical imaging*, Inverse Problems **15** (1999), R41–R93.

[2] S. R. Arridge and J. C. Hebden, *Optical imaging in medicine: II. Modelling and reconstruction*, Phys. Med. Biol. **42** (1997), 841–853.

[3] G. Bal, *Reconstructions in impedance and optical tomography with singular interfaces*, Inverse Problems **21** (2005), 113–131.

[4] B. Gebauer, *The factorization method for real elliptic problems*, Journal for Analysis and its Applications **25** (2006), 81–102.

[5] A. P. Gibson, J. C. Hebden and S. R. Arridge, *Recent advances in diffuse optical tomography*, Phys. Med. Biol. **50** (2005), R1–R43.

[6] J. C. Hebden, S. R. Arridge and D. T. Delpy, *Optical imaging in medicine: I. Experimental techniques*, Phys. Med. Biol. **42** (1997), 825–840.

[7] J. Heino and E. Somersalo, *Estimation of optical absorption in anisotropic background*, Inverse Problems **18** (2002), 559–573.

[8] N. Hyvönen, *Analysis of optical tomography with non-scattering regions*, Proc. Edinb. Math. Soc. **45** (2002), 257–276.

[9] N. Hyvönen, *Characterizing inclusions in optical tomography*, Inverse Problems **20** (2004), 737–751.

[10] N. Hyvönen, *Application of a weaker formulation of the factorization method to the characterization of absorbing inclusions in optical tomography*, Inverse Problems **21** (2005), 1331–1343.

[11] N. Hyvönen, *Locating transparent regions in optical absorption and scattering tomography*, accepted to SIAM J. Appl. Math..

[12] A. Kirsch, *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Problems **14** (1998), 1489–1512.

[13] A. Kirsch, *The factorization method for a class of inverse elliptic problems*, Math. Nachr. **278** (2005), 258–277.

Optical tomography on simple Riemannian manifolds

Stephen R. McDowall

Optical tomography refers to the use of near-infrared light to determine the optical absorption and scattering properties of a medium. One prescribes a distribution of particles (photons in this case) entering the body at its boundary and measures the resulting flux of particles leaving the body. One then seeks to determine the absorption and scattering properties interior to the medium from knowledge of the "albedo" operator, the map from the incoming to the outgoing distributions of particles. In the stationary Euclidean case the dynamics are modeled by the radiative transport equation which assumes that, in the absence of interaction, particles follow straight lines. We are concerned here with the situation of particles moving in an ambient field represented by a Riemannian metric. The consequence is that in the absence of interaction a particle will now follow the geodesics of the metric.

We first describe the problem in some generality. Let M be a bounded open domain in $\mathbb{R}^n$ with smooth boundary and let g be a Riemannian metric on M. If $f(x,v)$ represents the density of particles at position x with velocity vector v in the unit tangent sphere at x, $\Omega_x M$, then the stationary linear transport equation is

$$-\mathcal{D}f(x,v) - \sigma_a(x,v)f(x,v) + \int_{\Omega_x M} k(x,v',v)f(x,v')\,dv'_x = 0. \tag{1}$$

The operator $\mathcal{D}$ is the derivative along the geodesic flow which in the case of g being Euclidean is simply $\mathcal{D}f(x,v) = v \cdot \nabla_x f(x,v)$. The coefficient σ_a is the absorption coefficient and k is the scattering kernel; σ_a describes the probability of a particle with position x and velocity v being absorbed and k describes that of a particle with position x and velocity v' being "scattered" to velocity v. We restrict ourselves to the case where all particles travel at unit speed and hence use the unit sphere bundle ΩM rather than the full tangent bundle TM. The measure dv'_x in (1) is the Euclidean volume form on the tangent sphere $\Omega_x M$ determined by the metric g at x. Define the incoming and outgoing bundles

$$\Gamma_\pm = \{(x,v) \in \Omega M \;:\; x \in \partial M,\ \pm\langle v,\nu\rangle > 0\}$$

on the boundary ∂M of M, where ν is the outward unit normal vector to ∂M and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{g_x}$ is the inner product with respect to the metric g at x. If f_- is a distribution of particles defined on Γ_- let f be the solution to (1), should it exist, with boundary condition $f|_{\Gamma_-} = f_-$. Then the albedo operator is the map $\mathcal{A}$ which maps f_- to the outgoing flux of particles,

$$\mathcal{A}f_- = f|_{\Gamma_+}.$$

The inverse problem, the problem of optical tomography, is to determine uniquely σ_a and k from knowledge of $\mathcal{A}$.

The case when the ambient metric is Euclidean is treated in [2] where it is shown that the albedo operator determines the x-ray transform of the absorption coefficient in all dimensions and determines the collision kernel k in dimensions 3 and greater (see also [1]. In [7], a more precise analysis results in unique determination of k in dimension two, under the assumption that k is small relative to σ_a, with an explicit constant given. In [3] the author of this note proved the analogous result to that of [2], namely that in the presence of a known Riemannian metric $\mathcal{A}$ uniquely determines σ_a and k in dimensions three and greater, and only σ_a in dimension two. In [4] the author used the approach of [7] to show that under similar smallness assumptions, k is uniquely determined, as is also the metric, on (simple) Riemannian surfaces. The determination of the metric follows from the results of [5], [6].

A precise statement of the result in [4] follows. We make the following assumptions on the geometry of (M, g):

M1. The sectional curvature of (M, g) is bounded above by κ_0.
M2. If $\kappa_0 > 0$ we assume that (M, g) has no *focal points*, that is, for every geodesic $\gamma : [a, b] \to M$ and every non-zero Jacobi field $J(t)$ along γ satisfying $J(a) = 0$, we have $\|J(t)\|$ is a strictly increasing function on $[a, b]$. Note that if $\kappa_0 \leq 0$ then (M, g) necessarily has no focal points.
M3. In the case that $\kappa_0 > 0$ we assume that the diameter A of (M, g) satisfies $A < \pi/(2\sqrt{\kappa_0})$. There is no restriction on the diameter of (M, g) when $\kappa_0 \leq 0$, other than it is finite.

It follows that (M, g) is "simple." In particular, for any $x \in \bar{M}$ the exponential map $\mathrm{Exp}_x : \mathrm{Exp}_x^{-1}(\bar{M}) \to \bar{M}$ is a diffeomorphism. Consequently M is diffeomorphic to a disk. We make the following assumptions on (σ_a, k):

A1. Even in the Euclidean case and when $k = 0$, $\mathcal{A}$ does not uniquely determine σ_a (see [2], [7]) and so for the inverse problem we assume that σ_a depends only on x.
A2. $\sigma_a \in L^\infty(M)$, $k \in L^\infty(\{(y, v', v) \in M \times \Omega_y M \times \Omega_y M\})$, and $\|k\|_{L^\infty} \leq (2\pi\, \mathrm{diam}(M))^{-1}$.

We define the class

$$\mathcal{U}_{\Sigma,\varepsilon} = \Big\{ \big(\sigma_a(x), k(x, w', w)\big) \ : \ \|\sigma_a\|_{L^\infty} \leq \Sigma, \ \|k\|_{L^\infty} \leq \varepsilon, \\ \text{and } (\sigma_a, k) \text{ satisfy A1, A2} \Big\}. \tag{2}$$

Theorem 1. *Let (M, g) be a two dimensional Riemannian manifold with smooth, strictly convex boundary, satisfying assumptions M1–M3: (M, g) has curvature bounded above by κ_0, has no focal points, and in the case that $\kappa_0 > 0$ has diameter bounded by $\pi/(2\sqrt{\kappa_0})$.*

(1) *If (σ_a, k) satisfy A1, A2 then the metric g is uniquely determined by the associated albedo operator $\mathcal{A}$.*
(2) *Given $\Sigma > 0$ there exists $\varepsilon > 0$ such that any pair $(\sigma_a, k) \in \mathcal{U}_{\Sigma,\varepsilon}$ is uniquely determined, within $\mathcal{U}_{\Sigma,\varepsilon}$, by the associated albedo operator $\mathcal{A}$. Furthermore, ε can be chosen to be $\varepsilon = Ce^{-2A\Sigma}$ where C depends only on (M, g) and $A = \mathrm{diam}(M, g)$.*

References

[1] M. Choulli and P. Stefanov, *Inverse scattering and inverse boundary value problems for the linear Boltzmann equation*, Comm. Partial Differential Equations **21** (1996), 763–785.
[2] M. Choulli and P. Stefanov, *An inverse boundary value problem for the stationary transport equation, Osaka J. Math.* **36** (1999), 87–104.
[3] S. McDowall, *An inverse problem for the transport equation in the presence of a Riemannian metric*, Pacific J. Math. **216** (2004), 303–326.
[4] S. McDowall, *Optical tomography on simple Riemannian surfaces*, Comm. Partial Differential Equations **30** (2005), 1379–1400.
[5] L. Pestov and G. Uhlmann, *Two dimensional compact simple Riemannian manifolds are boundary distance rigid*, Ann. of Math. **2** (2005), 1093–1110.
[6] L. Pestov and G. Uhlmann, *The boundary distance function and the Dirichlet-to-Neumann map*, Math. Res. Lett. (2004), 285–297.
[7] P. Stefanov and G. Uhlmann, *Optical tomography in two dimensions*, Methods Appl. Anal. **10** (2003), 1–9.

Towards convergence of the linear sampling method

Tilo Arens
(joint work with Armin Lechleiter)

In an inverse scattering problem, the aim is to obtain a reconstruction of a scatterer or value of a refractive index from the knowledge of the far field pattern of the scattered wave for one or several known incident fields. For the purpose of this talk, we consider the simplest type of a scattering problem, where a scalar (i.e. acoustic) field u^i is incident on a bounded obstacle $D \subset \mathbb{R}^3$. The total field is assumed to vanish on the boundary of D.

Mathematically, the problem is formulated as an exterior boundary value problem for the Helmholtz equation, i.e. the total field u is a solution of

$$\Delta u + k^2 u = 0 \qquad \text{in} \quad \mathbb{R}^3 \setminus \overline{D}, \qquad\qquad u = 0 \qquad \text{on} \quad \partial D,$$

and the scattered field $u^s = u - u^i$ must satisfy the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial u^s}{\partial |x|} - ik\, u^s \right) = 0$$

uniformly for all directions of observation $x/|x|$.

The scattered field u^s can, by Rellich's Lemma, be uniquely associated with its far field pattern u^∞ which is the leading order behavior of the expansion of u^s in inverse powers of $|x|$ for $|x| \to \infty$. In the case of a plane wave

$$u^i(x) = \exp(ikd \cdot x), \qquad x \in \mathbb{R}^3, \quad d \in \mathbb{S}^2,$$

as incident field, we shall denote the far field pattern of the corresponding scattered field by $u^\infty(\hat{x}, d)$, where $\hat{x} \in \mathbb{S}^2$ denotes the direction of observation and $d \in \mathbb{S}^2$ the direction of incidence of the plane wave. Then, the *far field operator* $F : L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ may be defined by

$$Fg = \int_{\mathbb{S}^2} u^\infty(\cdot, d)\, g(d)\, ds(d), \qquad g \in L^2(\mathbb{S}^2).$$

The Linear Sampling Method attempts to reconstruct the scatterer D directly from the knowledge of F. For all $z \in \mathbb{R}^3$, the ill-posed *far field equation*

$$Fg = \Phi^\infty(\cdot, z) \tag{1}$$

is approximately solved using a regularization strategy and the norm of the computed approximate solution serves as an indicator for whether z is inside D or not. Here,

$$\Phi^\infty(\hat{x}, z) = \frac{1}{4\pi} \exp(-ik\hat{x} \cdot z), \qquad \hat{x} \in \mathbb{S}^2,$$

denotes the far field pattern generated by a point source at $z \in \mathbb{R}^3$ in free field conditions. The method was first published in the papers [4, 5] and has since been successfully applied to a large number of related problems. A recent review of the results available can be found in [3].

However, these results only represent reports that this method numerically reliably produces good reconstructions. An existence theorem for an approximate solution of (1) with certain additional properties is usually given, but no guarantee is provided that this is the solution computed. Only for the Dirichlet scattering problem above has a convergence analysis been given in [1] which is based on the Factorization method first introduced in [6].

In [1], Tikhonov regularization of F is combined with applying a further operator $H : L^2(\mathbb{S}^2) \to H^{1/2}(\partial D)$ to obtain a regularization strategy for the operator $G = H^{-1}F$, i.e. the operator family $R_\alpha = H\,(\alpha + F^*\,F)^{-1}F^*$, $\alpha > 0$, is studied. A first issue is the influence of noise on the reconstruction. The standard theory for regularizations defined using spectral decompositions as used in [1] cannot be employed directly in this case, as the operator itself is perturbed.

Consider a perturbed operator F^δ such that $\|F - F^\delta\| \le \delta$. Taking into account the perturbation of the eigenvalues and eigenvectors of the operator and using estimates from [8] yields that an admissible choice of the regularization parameter $\alpha(\delta)$ is obtained under the condition that

$$\alpha(\delta) \to 0 \qquad \text{and} \qquad \frac{\delta}{\alpha(\delta)^{3/2}} \to 0 \qquad (\delta \to 0).$$

An open question remains in that it is unclear whether this condition may be met by an a priori choice of the regularization parameter.

Furthermore, in [1] convergence was only proved for $z \in D$. The case $z \notin \overline{D}$ may be approached by considering a perturbed domain D_δ which is the union of D and a ball of radius δ centered at z. In this case, the far field operator for D_δ is also only a small perturbation of F. Making use of recent results on the behavior of the Factorization method in the case of noisy data [7], it is possible to show that the indicator function used in [1] applied for D_δ is close to the the one applied for D. Hence, the behavior observed for the indicator function outside of the domain D is explained. A simpler version of the proof of this result as the one given in the talk is contained in the paper [2].

References

[1] T. Arens, *Why Linear Sampling works*, Inverse Problems **20** (2004), 163–173.
[2] T. Arens and A. Lechleiter *The Linear Sampling Method revisited*, in preparation.
[3] F. Cakoni and D. Colton *Qualitative Methods in Inverse Scattering Theory. An Introduction.* Springer, Berlin, 2006.
[4] D. Colton and A. Kirsch, *A simple method for solving inverse scattering problems in the resonance region*, Inverse Problems **12** (1996), 383–393.
[5] D. Colton, M. Piana and R. Potthast, *A simple method using Morozov's discrepancy principle for solving inverse scattering problems*, Inverse Problems **13** (1997), 1477–1493.
[6] A. Kirsch, *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Problems **14** (1998), 1489–1512.
[7] A. Lechleiter, *A regularization technique for the Factorization Method*, Inverse Problems **22** (2006), 1605–1625.
[8] G.M. Vainikko, *The discrepancy principle for a class of regularization methods. (English. Russian original)*, U.S.S.R. Comput. Math. Math. Phys. **22**, No.3 (1982), 1–19.

An asymptotic factorization method for inverse electromagnetic scattering

Roland Griesmaier

We consider a simple but fully three dimensional model problem for the electromagnetic exploration of perfectly conducting objects within an unbounded two-layered background medium. This model problem is motivated by the project [13] of the German Federal Ministry of Education and Research; cf. also [9, 16] for further studies in this direction. It leads to an inverse scattering problem for Maxwell's equations which is well-known to be nonlinear and ill-posed. Recently new solution methods for inverse scattering problems such as linear sampling methods, introduced first by Colton and Kirsch in [7], and factorization methods, proposed first by Kirsch in [15], have been developed which avoid the issue of nonlinearity. In order to handle the ill-posedness it is generally advisable to incorporate all available a priori knowledge about the unknown inclusions and to try to determine very specific features. Commercial off-the-shelf metal detectors used for humanitarian demining work at relatively low frequencies, e.g. the "Foerster Minex 2FD 4.500" works at 2.4 kHz and 19.2 kHz; cf. [11, pp. 82]. In vacuum this corresponds to wave lengths of more than 15 km. Therefore we may assume that the size of the scatterers is small with respect to the wavelength of the incident field. So we try to reconstruct the centers of finitely many perfectly conducting small scatterers that are well separated from each other, from the interface, and from the measurement device. Here, general purpose reconstruction methods are likely to fail, since due to the smallness of the inhomogeneities the associated scattered fields are very small. Hence, unless one knows exactly the patterns that should be looked for, noise will largely dominate the information contained in the measured data.

We decompose the space $\mathbb{R}^3 = \mathbb{R}^3_- \cup \Sigma_0 \cup \mathbb{R}^3_+$ in a hyperplane Σ_0 corresponding to the surface of ground, and the two halfspaces $\mathbb{R}^3_\pm$ above and below Σ_0

representing air and ground, respectively. We assume that both half spaces are filled with homogeneous materials with dielectricity ε and permeability μ given by

$$\varepsilon(\boldsymbol{x}) := \begin{cases} \varepsilon_+, & \boldsymbol{x} \in \mathbb{R}^3_+, \\ \varepsilon_-, & \boldsymbol{x} \in \mathbb{R}^3_-, \end{cases} \qquad \mu(\boldsymbol{x}) := \begin{cases} \mu_+, & \boldsymbol{x} \in \mathbb{R}^3_+, \\ \mu_-, & \boldsymbol{x} \in \mathbb{R}^3_-. \end{cases}$$

The associated (discontinuous) wave number is denoted by k. Measurements and excitations are restricted to a bounded sheet $\mathcal{M} \subset \Sigma_d$, where $\Sigma_d \subset \mathbb{R}^3_+$ is the hyperplane parallel to the surface of ground at height $d > 0$. We suppose that the scatterers are of the form $D_{\delta,l} = \boldsymbol{z}_l + \delta B_l$, $1 \leq l \leq m$, where B_l is a bounded domain containing the origin, the points $\boldsymbol{z}_l \in \mathbb{R}^3_-$ indicate the positions of the "centers" of the scatterers, and the "average" inhomogeneity size is specified by the parameter $\delta > 0$. In order to study the inverse problem we first examine the corresponding direct scattering problem in detail and derive an asymptotic expansion of the scattered field in terms of the incident field, the centers of the scatterers and their geometry, as the size of the inhomogeneities tends to zero. We define the (near-field) measurement operator G_δ which maps magnetic dipole densities on the measurement device $\mathcal{M}$ onto the corresponding scattered fields $\boldsymbol{H}^s|_{\mathcal{M}}$ on $\mathcal{M}$,

$$G_\delta : \boldsymbol{L}^2(\mathcal{M}; \mathbb{C}^3) \to \boldsymbol{L}^2(\mathcal{M}; \mathbb{C}^3), \quad G_\delta \boldsymbol{\varphi} := \boldsymbol{H}^s|_{\mathcal{M}}.$$

We derive a factorization of G_δ similar to the one developed in [9]. Next we use layer potential techniques to describe the three operators occurring in this factorization, expand them separately as the size of the inhomogeneities tends to zero, and use these expansions to calculate the leading order term in the asymptotic formula for the scattered field. This generalizes the approach we used in [1] for an inverse obstacle problem in electrostatics and yields the following asymptotic formula. Let $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\mathcal{M}; \mathbb{C}^3)$ and let

$$\boldsymbol{H}^i = k_+^2 \int_{\mathcal{M}} \mathbb{G}^m(\cdot, \boldsymbol{y}) \boldsymbol{\varphi}(\boldsymbol{y}) \, \mathrm{d}\sigma(y)$$

be the corresponding incident field, where $\mathbb{G}^m$ denotes the magnetic dyadic Green's function for the Maxwell's equations in two-layered background medium. Then

$$G_\delta \boldsymbol{\varphi} = \delta^3 \sum_{l=1}^m \Big(k_-^2 \mathbb{G}^m(\cdot, \boldsymbol{z}_l) M^0_{B_l} \boldsymbol{H}^i(\boldsymbol{z}_l) + \frac{\mu_-}{\mu_+} \mathbf{curl}_x \mathbb{G}^e(\cdot, \boldsymbol{z}_l) M^\infty_{B_l} \mathbf{curl} \boldsymbol{H}^i(\boldsymbol{z}_l) \Big) + \mathcal{O}(\delta^4)$$

in $\boldsymbol{L}^2(\mathcal{M}; \mathbb{C}^3)$, as $\delta \to 0$. Here $\mathbb{G}^e$ denotes the electric dyadic Green's function for the Maxwell's equations in two-layered background medium, and $M^0_{B_l}$ and $M^\infty_{B_l}$ are so-called polarization tensors. Similar formulas have been derived formally in [2, 3, 4] for homogeneous background medium and in [14] for two-layered background medium; see also the prior work of Ammari, Vogelius, and Volkov [5] for a rigorous derivation in the case of small inhomogeneities in a bounded homogeneous reference domain.

Using this asymptotic expansion we can study the inverse problem of recovering the centers of the small scatterers from magnetic near-field scattering data corresponding to an incident field due to a magnetic dipole distribution on the measurement device, i.e. from the knowledge of G_δ. Inspired by [6], see also [12], we design a direct reconstruction algorithm that is closely related to factorization methods, and also MUSIC-type methods developed by Devaney [8]. We denote the leading order term in the asymptotic expansion of G_δ by $T : \boldsymbol{L}^2(\mathcal{M}; \mathbb{C}^3) \to \boldsymbol{L}^2(\mathcal{M}; \mathbb{C}^3)$,

$$T\boldsymbol{\varphi} := \sum_{l=1}^{m} \big(k_-^2 \mathbb{G}^m(\cdot, \boldsymbol{z}_l) M_{B_l}^0 \boldsymbol{H}^i(\boldsymbol{z}_l) + \frac{\mu_-}{\mu_+} \mathbf{curl}_x \mathbb{G}^e(\cdot, \boldsymbol{z}_l) M_{B_l}^\infty \mathbf{curl} \boldsymbol{H}^i(\boldsymbol{z}_l) \big).$$

Then we can prove that the range of T is finite dimensional and its dimension is $6m$, where m is the number of unknown scatterers. Furthermore, for a polarization $\boldsymbol{d} = (\boldsymbol{d}_1, \boldsymbol{d}_2) \in \mathbb{C}^3 \times \mathbb{C}^3 \setminus \{(0,0)\}$ and a sampling point $\boldsymbol{z} \in \mathbb{R}^3_-$ we define the test function

$$\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} := (\mathbb{G}^m(\cdot, \boldsymbol{z})\boldsymbol{d}_1 + \mathbf{curl}_x \mathbb{G}^e(\cdot, \boldsymbol{z})\boldsymbol{d}_2)|_{\mathcal{M}}.$$

Then we can show that $\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}} \in \mathcal{R}(T)$ if and only if $\boldsymbol{z} \in \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_m\}$. So, if we denote the angle between the test function $\boldsymbol{g}^{\boldsymbol{z},\boldsymbol{d}}$ and the (finite dimensional) range space $\mathcal{R}(T)$ by $\beta(\boldsymbol{z})$, we find that

$$\boldsymbol{z} \in \{\boldsymbol{z}_l \mid l = 1, \dots, m\} \iff \beta(\boldsymbol{z}) = 0 \iff \cot \beta(\boldsymbol{z}) = \infty.$$

Approximating the singular value decomposition of T by the singular value decomposition of the (compact) measurement operator G_δ we obtain an approximation $\beta^\delta(\boldsymbol{z})$ of $\beta(\boldsymbol{z})$. If we plot $\cot \beta^\delta(\boldsymbol{z})$, we expect to see large values for points $\boldsymbol{z}$ which are close to the positions $\boldsymbol{z}_l$, $l = 1, \dots, m$. This leads to a simple and efficient visualization method for the solution of our inverse scattering problem.

A complete derivation of these results for homogeneous background medium including numerical reconstructions can be found in [10]. A corresponding work for two-layered background medium is in preparation.

References

[1] H. Ammari, R. Griesmaier and M. Hanke, *Identification of small inhomogeneities: asymptotic factorization*, Math. Comput., to appear (electronically published on http://www.ams.org/journals/mcom/0000-000-00/).

[2] H. Ammari, E. Iakovleva, D. Lesselier and G. Perrusson, *MUSIC-type electromagnetic imaging of a collection of small three-dimensional bounded inclusions*, SIAM J. Sci. Comput., to appear.

[3] H. Ammari and A. Khelifi, *Electromagnetic scattering by small dielectric inhomogeneities*, J. Math. Pures Appl. **82** (2003), 749–842.

[4] H. Ammari, S. Moskow and M.S. Vogelius, *Boundary integral formulae for the reconstruction of electric and electromagnetic inhomogeneities of small volume*, ESAIM Control Optim. Calc. Var. **9** (2003), 49–66.

[5] H. Ammari, M.S. Vogelius and D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations*, J. Math. Pures Appl. **80** (2001), 769–814.

[6] M. Brühl, M. Hanke and M.S. Vogelius, *A direct impedance tomography algorithm for locating small inhomogeneities*, Numer. Math. **93** (2003), 635–654.

[7] D. Colton and A. Kirsch, *A simple method for solving inverse scattering problems in the resonance region*, Inverse Problems **12** (1996), 383–393.
[8] A.J. Devaney, *Super-resolution processing of multi-static data using time reversal and MUSIC*, preprint.
[9] B. Gebauer, M. Hanke, A. Kirsch, W. Muniz and C. Schneider, *A sampling method for detecting buried objects using electromagnetic scattering*, Inverse Problems **21** (2005), 2035–2050.
[10] R. Griesmaier, *An asymptotic factorization method for detecting small objects using electromagnetic scattering*, submitted.
[11] D. Guelle, A. Smith, A. Lewis and T. Bloodworth, *EUR 20837 Metal detector handbook for Humanitarian Demining*, European Commission, Luxembourg: Office for Official Publications of the European Communities, 2003.
[12] M. Hanke and M. Brühl, *Recent progress in electrical impedance tomography*, Inverse Problems **19** (2003), S65–S90.
[13] *HuMin/MD — Metal detectors for humanitarian demining — Development potentials in data analysis methodology and measurement*, Project Network, `http://www.humin-md.de/`.
[14] E. Iakovleva, S. Gdoura, D. Lesselier and G. Perrusson, *Multi-static response matrix of a 3-D inclusion in half space and MUSIC imaging*, submitted.
[15] A. Kirsch, *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Problems **14** (1998), 1489–1512.
[16] A. Kirsch, *An integral equation for Maxwell's equations in a layered medium with an application to the factorization method*, J. Integral Equations Appl., to appear.

Herglotz, Helmholtz, and far field support

John Sylvester

In practical remote sensing, faraway sources radiate fields that, within measurement precision, are nearly those radiated by point sources. Algorithms like MUSIC [1, 2] correctly identify their number, their locations, and their strengths based on monostatic observations of the near or far fields they radiate. Our motivation in this lecture is to *locate* sources which are supported on larger sets.

For any compactly supported source, F, v^+ below denotes the unique outgoing solution to

$$(\Delta + k^2)v^+ = F(x).$$

The asymptotics of v^+ are of the form:

$$v^+ \sim \frac{e^{ikr}}{\sqrt{2\pi r}}\beta(\Theta)$$

and β is called the far field radiated by F.

A source for which $\beta = 0$ is called a non-radiating source. In [4] we showed that there were many non-radiating sources, and classified those that arose from single and double layer potentials.

Theorem 1. *Suppose that ω is a distribution in $H^{-2+}(\mathbb{R}^n)$ and that* $\operatorname{supp}\omega$ *has measure 0. The source ω is non-radiating, if and only if, there is a bounded open* *set B and a $u \in H^{0+}(\mathbb{R}^n)$ satisfying the free Helmholtz equation in B, such that* $\text{supp}\omega = \partial B$ *and* $\omega = \mathcal{C}_{\partial B} u$.

The notation $\mathcal{C}_{\partial B} u$ means the Cauchy data of u restricted to ∂B. For smooth u and ∂B, it is a combination of single and double layer potentials with densities given by u and its normal derivative.

A consequence is that a far field of a solution to the inhomogeneous Helmholtz equation does not determine the source, or its support, uniquely. A second consequence is that it is also impossible to find an upper bound for the support of a source based solely on the far field it radiates (just add a non-radiating source with arbitrarily large support). While not as apparent, it is also impossible to associate a lower bound. That is, there is no smallest compact set which supports a source that radiates a given far field.

In this talk we describe how to associate with any far field a unique smallest compact convex set [3], and, more generally, a unique smallest compact *union of well-separated-convex sets* (UWSC sets) that is both big enough to support a source that can radiate that far field, and small enough that it must be contained in the UWSC-support of any source that radiates the same far field. This means that it makes theoretical sense to look for not only the number and the locations, but also the convex geometry of sources based on the far field they radiate. The only requirement is that sources be well-separated — the diameter of each convex component is strictly smaller than the distance to the other components.

We also give examples to illustrate the extent to which both the convexity and well-separated properties in UWSC are necessary, i.e. we exhibit far fields with which it is not possible to associate a unique smallest *compact set* or, in $\mathbb{R}^2$, a unique smallest *disjoint union of convex sets.*

References

[1] A. Devaney, *Super-resolution processing of multi-static data using time reversal and MUSIC*, J. Acoust. Soc. Am., accepted for publication.

[2] A. Kirsch, *The MUSIC algorithm and the factorization method in inverse scattering theory for inhomogeneous media*, Inverse Problems **18** (2002), 1025–1040.

[3] S. Kusiak and J. Sylvester, *The scattering support*, Comm. Pure Appl. Math. **56** (2003), 1525–1548.

[4] J. Sylvester, *On notions of far field support*, Inverse Problems **21** (2006), 1273–1288.

Identifying scattering obstacles by the construction of nonscattering waves

Russell Luke
(joint work with Anthony Devaney)

The inverse scattering literature abounds with methods to determine the shape of scatterers from far field data. Of principal concern here are the MUSIC algorithm [5], the Linear Sampling Method [3], the Point Source Method [13] and the connections between these methods. The connection between the MUSIC algorithm and Kirsch's Factorization Method [9] has been detailed by Cheney [2] and Kirsch [10] for scattering from point-like inhomogeneities. More recent studies [1, 8, 6, 7] approach an application of the MUSIC algorithm to scatterers of some specified size, relative to the wavelength, and are based on the finite-dimensional multi-static response matrix for point-like scatterers. Our goal here is to provide a rigorous analysis in the continuum for scatterers of arbitrary size illuminated by fields of arbitrary frequency.

Our central result, Theorem 1, is built upon the Linear Sampling Method of Colton and Kirsch [3] and shows that, on the boundary of a scatterer with Dirichlet boundary conditions, there is a unit-magnitude incident field with respect to the $H^{1/2}$-norm that has arbitrarily small pointwise magnitude. With the help of the Point Source Method of Potthast [13] we show in Corollary 2 that such an incident field does not generate a scattered field. Theorem 3 combines these results as the foundation for a MUSIC algorithm [5] for determining the shape and location of an obstacle. The technique indicates intriguing possibilities for the construction of *nonscattering fields* that might be used to shield obstacles from interrogating waves. To our knowledge this is the first rigorous analysis in the continuum and our application of the linear sampling method appears to be novel.

In the following statements, $\mathcal{F} : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ denotes the far field operator corresponding to illumination of sound-hard obstacles with incident plane waves ($\mathbb{S}$ is the unit sphere in $\mathbb{R}^m$ for $m = 2$ or 3), $\mathcal{B} : H^{1/2}(\partial\mathbb{D}) \to L^2(\mathbb{S})$ denotes the mapping of radiating solutions to the Helmholtz equation from the boundary data to the far field pattern, and $\mathcal{H} : L^2(\mathbb{S}) \to H^1_{loc}(\mathbb{R}^m)$ is the Herglotz wave operator. A key tool is the factorization of the far field pattern as $-\mathcal{B}\mathcal{H} = \mathcal{F}$.

Theorem 1 (normalized Linear Sampling)**.** *Let $\mathbb{D}$ be a domain with smooth boundary and assume that k^2 is not a Dirichlet eigenvalue for the negative Laplacian on $\mathbb{D}$. If $z \notin \mathbb{D}$, for $\epsilon > 0$ and $\delta > 0$, there exist solutions $f(\cdot; z)$ and $g(\cdot; z)$ to*

$$\|\mathcal{B}f(\cdot; z) + \Phi^\infty(\cdot; z)\|_{L^2(\mathbb{S})} < \delta \tag{1a}$$

$$\|\mathcal{F}g(\cdot; z) - \mathcal{B}f(\cdot; z)\|_{L^2(\mathbb{S})} < \epsilon \tag{1b}$$

such that

$$\lim_{\delta\to 0} \|\mathcal{F}\widehat{g}(\cdot; z)\|_{L^2(\mathbb{S})} = 0 \quad \textit{and} \quad \lim_{\delta\to 0} \|\mathcal{H}\widehat{g}(\cdot; z)\|_{H^{1/2}(\partial\mathbb{D})} = 1 \tag{1c}$$

where

$$\widehat{g}(\cdot; z) \equiv \frac{g(\cdot; z)}{\|f(\cdot; z)\|_{H^{1/2}(\partial\mathbb{D})}}. \tag{2}$$

Since the far field pattern is zero if and only if there is no scattered field, the above theorem implies that the incident Herglotz wave function $\mathcal{H}\widehat{g}$ does not scatter. That is,

Corollary 2 (nonscattering incident fields)**.** *The scattered field, $v^s_{\widehat{g}}(\cdot; z)$, corresponding to the incident Herglotz wave function $v^i_{\widehat{g}}(\cdot; z) = \mathcal{H}\widehat{g}(\cdot; z)$ in Theorem 1 has the behavior*

$$\lim_{\delta\to 0} v^s_{\widehat{g}}(x; z) = 0 \qquad \textit{and} \qquad \lim_{x\to\partial\mathbb{D}} \lim_{\delta\to 0} v^i_{\widehat{g}}(x; z) = 0 \quad \textit{for} \quad x \in \mathbb{D}^o.$$

The MUSIC algorithm is based on the observation that the set of Green's functions

$$\Phi^\infty(z, \widehat{\eta}) \equiv \lim_{r\to\infty} \Phi(z, r\widehat{\eta}) = \beta e^{\mathrm{i}k(-\widehat{\eta})\cdot z} \tag{3}$$

for z near $\partial\mathbb{D}$ and all $\widehat{\eta} \in \mathbb{S}$, are nearly orthogonal to the *noise subspace* of $\mathcal{F}$. We discuss what we mean by the noise subspace in more detail below. In precise terms we have

Theorem 3 (MUSIC)**.** *Let $\mathbb{D}$ be a domain with smooth boundary and assume that k^2 is not a Dirichlet eigenvalue for the negative Laplacian on $\mathbb{D}$. Let $(\sigma_n, \xi_n, \psi_n)$, $n \in \mathbb{N}$, be the singular system for the far field operator $\mathcal{F}$ with $\sigma_n \le \sigma_m$ for $n > m$. Given any $\gamma > 0$ there is a vector $a \in l^2$ with $\|a\|_2 = 1$ and a $\rho > 0$ such that for any $x \in \mathbb{D}^o$ satisfying* dist $(x, \mathbb{D}) < \rho$ *we have*

$$\sum_{n=1}^{\infty} \left| a_n \langle \xi_n(\cdot), \Phi^\infty(x, \cdot)\rangle_{L^2(\mathbb{S}')} \right| < \gamma. \tag{4}$$

Our main results show that there is a density $\widehat{g}$ that approaches, nontrivially, the null space of the far field operator corresponding to some fixed, smooth scatterers. A superposition of plane waves weighted by such a density is a nonscattering incident field for these scatterers. The density can be constructed from the singular functions of the far field operator and the nonscattering phenomenon understood as the orthogonality of the singular functions to the far field pattern of a point source with sources located on the boundary of the scatterer. Since the image of this density acted upon by the far field operator vanishes, we arrive at the seemingly counterintuitive conclusion that it is the *noise* subspace of the far field operator that renders the shape and location of the obstacle, not the signal subspace.

The point source method of Potthast [11, 12] rests on the approximation of the scattered field u^s by computing the correct density for the construction of a *backpropagation* operator. As already noted, constructing such a density is a nontrivial task since this requires some knowledge of the boundary of the scatterer which we assume is unknown. The linear sampling method approaches the problem of finding the shape and location of the scatterer by looking for points where the fundamental solution far field pattern is not in the range of the far field operator, but still, one must solve an ill-posed linear integral equation at each point to be so tested in some computational domain. What we have shown, however, is that it is unnecessary to create an approximate domain as with the point source method, nor is it necessary to solve many ill-posed linear integral equations as in the linear sampling method. We need only work with incident plane waves and the known singular functions of the far field operator.

These results have intriguing implications for inverse scattering and signal design. The method works very much like the Linear Sampling Method for inverse scattering in that the proposed incident field is constructed from the measured far field data and the scatterer is identified by those points in the domain where the incident field (and scattered field) are small. For signal design the method opens the door to the possibility of constructing signals that avoid certain known obstacles while irradiating others. Our application of the linear sampling method to the MUSIC algorithm is novel and clarifies the connections between many different inverse scattering approaches.

References

[1] H. Ammari, E. Iakovleva, and D. Lesselier, *Two numerical methods for recovering small inclusions from the scattering amplitude at a fixed frequency*, SIAM J. Sci. Comp. **27** (2005), 130–158.

[2] M. Cheney, *The linear sampling method and the MUSIC algorithm*, Inverse Problems **17** (2001), 591–595.

[3] D. Colton and A. Kirsch, *A simple method for solving inverse scattering problems in the resonance region*, Inverse Problems **12** (1996), 383–393.

[4] D. Colton and R. Kress, *On the denseness of Herglotz wave functions and electromagnetic Herglotz pairs in Sobolev spaces*, Math. Meth. Appl. Sci **24** (2001), 1289–1303.

[5] A.J. Devaney, *Time reversal imaging of obscured targets from multistatic data*, IEEE Transactions on Antennas and Propagation **53** (2005), 1600–1610.

[6] A.J. Devaney and E.A. Marengo, *Nonradiating sources with connections to the adjoint problem*, Phys. Rev. E **70** (2004), 037601.

[7] A.J. Devaney, E.A. Marengo and F.K. Gruber, *Time-reversal-based imaging and inverse scattering of multiply scattering point targets*, J. Acoust. Soc. Amer. **118** (2005), 3129–3138.

[8] S. Hou, K. Solna and H. Zhao, *A direct imaging algorithm for extended targets*, Inverse Problems **22** (2006), 1151–1178.

[9] A. Kirsch, *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Problems **14** (1998), 1489–1512.

[10] A. Kirsch, *The MUSIC algorithm and the factorization method in inverse scattering theory for inhomogeneous media*, Inverse Problems **18** (2001), 1025–1040.

[11] R. Potthast, *A fast new method to solve inverse scattering problems*, Inverse Problems **12** (1996), 731–742.

[12] R. Potthast, *A point-source method method for inverse acoustic and electromagnetic obstacle scattering problems*, IMA Jour. Appl. Math. **61** (1998), 119–140.

[13] R. Potthast, Point Sources and Multipoles in Inverse Scattering Theory, Chapman & Hall, London, 2001.

Inequalities in Inverse Scattering for Anisotropic Media

Fioralba Cakoni
(joint work with David Colton, Houssem Haddar, Peter Monk)

We consider the scattering of electromagnetic waves by a (possibly) partially coated anisotropic dielectric with support D at fixed frequency. For a particular polarization and geometry the corresponding forward problem in $\mathbb{R}^2$-case is given by the following set of equations

$$\begin{aligned}
&\nabla \cdot A\nabla w + k^2 w = 0 && \text{in} \quad D\\
&\Delta u + k^2 u = 0 && \text{in} \quad D_e\\
&w - u = 0 && \text{on} \quad \Gamma_1\\
&w - u = -i\eta\frac{\partial u}{\partial \nu} && \text{on} \quad \Gamma_2\\
&\frac{\partial w}{\partial \nu_A} - \frac{\partial u}{\partial \nu} = 0 && \text{on} \quad \partial D = \overline{\Gamma}_1 \cup \overline{\Gamma}_2\\
&u = u^s + u^i &&\\
&\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, &&
\end{aligned}$$

where u^s is the scattered field, u^i is the given incident field and ν is the outward normal vector to the (piecewise smooth) boundary ∂D of D. In the case of plane waves the incident field is given by $u^i := e^{ikx\cdot d}$, $d \in \Omega := \{x : |x| = 1\}$. We assume that A is a real valued 2×2 matrix-valued function whose entries are piecewise continuously differentiable functions in $\overline{D}$ with (possible) jumps along piecewise smooth curves such that A is symmetric and $\bar{\xi} \cdot A\,\xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{C}^2$ and $x \in \overline{D}$ where γ is a positive constant. Furthermore we assume that $\eta \in L_\infty(\Gamma_2)$ is such that $\eta(x) \geq \eta_0 > 0$ and that $\mathbb{R}^2 \setminus \overline{D}$ is connected. It can be shown [5] that the scattered field u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O\left(r^{-3/2}\right) \tag{1}$$

as $r \to \infty$ uniformly in $\hat{x}$ where u_∞ is the *far field pattern.*

The *inverse scattering problem* we are concerned with is to determine D, η and A from a knowledge of $u_\infty(\hat{x}, d)$ for all $\hat{x}, d \in \Omega$. In [1], [4] it is proven that D is uniquely determined from the above data. Furthermore for a fixed D and A, η is also uniquely determined from the data provided that for an arbitrary choice of Γ_2, A and η there is at least one incident plane wave such that the corresponding total field u satisfies $\partial u/\partial\nu|_{\Gamma_0} \neq 0$ where Γ_0 is an arbitrary portion of the boundary. However, it is also know that the matrix A is not uniquely determined from the far field patterns for all d even if they are known for a range of frequencies k.

We now define the *far field operator* $F : L^2(\Omega) \to L^2(\Omega)$ by

$$Fg(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d)\, ds(d) \tag{2}$$

and introduce the far field equation

$$(Fg)(\hat{x}) = \gamma e^{-ik\hat{x}\cdot z} \qquad g \in L^2(\Omega), \quad z \in D \tag{3}$$

where $\gamma = \frac{e^{i\pi/4}}{\sqrt{8\pi k}}$ and $\gamma e^{-ik\hat{x}\cdot z}$ is the far field pattern of the fundamental solution $\Phi(x,z) := \frac{i}{4}H_0^{(1)}(k|x-z|)$ to the Helmholtz equation in $\mathbb{R}^2$ with $H_0^{(1)}$ being a Hankel function of the first kind of order zero. A reconstruction of D can be obtained by using the linear sampling method which characterizes the support D from a solution of the far field equation (3) (see e.g. [1]). Assuming that D is known our next goal is to provide a lower bound for η in terms of the solution of the far field equation (3). We recall that a Herglotz wave function with kernel $g \in L^2(\Omega)$ is an entire solution of the Helmholtz equation defined by

$$v_g(x) = \int_\Omega e^{ikx\cdot d} g(d)ds(d), \qquad x \in \mathbb{R}^2. \tag{4}$$

It can be shown (see [1], [4]) that for $z \in D$ and $\epsilon > 0$, there exists a Herglotz wave function $v_{g_\epsilon^z}$ with kernel $g_\epsilon^z \in L^2(\Omega)$ such that

$$\|(Fg_\epsilon^z)(\hat{x}) - \gamma e^{-ik\hat{x}\cdot z}\|_{L^2(\Omega)} \le \epsilon \tag{5}$$

and

$$\|w_z - v_{g_\epsilon^z}\|_{\mathbb{H}^1(D)} \le \epsilon \tag{6}$$

where (w_z, v_z) solves the interior transmission problem

$$\begin{aligned}
\nabla\cdot A\nabla v_z + k^2\, v_z &= 0 && \text{in} \quad D\\
\Delta w_z + k^2\, w_z &= 0 && \text{in} \quad D\\
v_z - (w_z + \Phi(\cdot,z)) &= 0 && \text{on} \quad \Gamma_1\\
v_z - (w_z + \Phi(\cdot,z)) &= -i\eta\,\frac{\partial}{\partial\nu}(w_z + \Phi(\cdot,z)) && \text{on} \quad \Gamma_2\\
\frac{\partial v_z}{\partial \nu_A} - \frac{\partial}{\partial\nu}(w_z + \Phi(\cdot,z)) &= 0 && \text{on} \quad \partial D.
\end{aligned} \tag{7}$$

Applying Green's formula to (7) we obtain that

$$\eta \ge \frac{-2k\pi|\gamma|^2 - \operatorname{Im}(w_{z_0}(z_0))}{\|\frac{\partial}{\partial\nu}(w_{z_0} + \Phi(\cdot;z_0))\|^2_{L^2(\Gamma_2)}} \ge \frac{-2k\pi|\gamma|^2 - \operatorname{Im}(w_{z_0}(z_0))}{\|\frac{\partial}{\partial\nu}(w_{z_0} + \Phi(\cdot;z_0))\|^2_{L^2(\partial D)}}. \tag{8}$$

Note that $w_z(z)$ is approximated by the Herglotz wave function $v_{g_\epsilon^z}$ with kernel $g_\epsilon^z \in L^2(\Omega)$, the regularized solution of the far field equation (3).

The linear sampling method for determining D and the above estimate for η hold true if k is not such that the homogeneous interior transmission problem, i.e (7) with $\Phi(\cdot,z) = 0$, has non trivial solutions. It is easy to see that the latter values of k are subset (possibly empty set) of the set of *transmission eigenvalues* i.e. the set of values of k for which the following problem has non trivial solutions

$$
\begin{array}{lll}
\nabla\cdot A\nabla v_z + k^2\, v_z = 0 & \text{in} & D \\
\Delta w_z + k^2\, w_z = 0 & \text{in} & D \\
v_z - (w_z + \Phi(\cdot,z)) = 0 & \text{on} & \partial D \\
\frac{\partial v_z}{\partial \nu_A} - \frac{\partial}{\partial \nu}(w_z + \Phi(\cdot,z)) = 0 & \text{on} & \partial D.
\end{array}
\tag{9}
$$

We remark that, except for the case of spherically stratified medium [5], it is not know whether transmission eigenvalues exist.

Our aim is to use transmission eigenvalues to provide inequalities that are satisfied by all matrix valued index of refractions A that give raise to the same far field data. In [2] it is shown that, provided that $\|A^{-1}(x)\|_2 \geq \delta > 1$ for all $x \in D$ and some constant δ, then

$$
\sup_D \|A^{-1}\|_2 \geq \frac{\lambda(D)}{k^2}
\tag{10}
$$

where k is a transmission eigenvalue, $\lambda(D)$ is the first eigenvalue of $-\Delta$ on D and $\|\cdot\|_2$ is the Euclidean norm of A. Hence, if k is the first transmission eigenvalue, (10) provides a lower bound for the Euclidean norm of A. It is important to notice that the transmission eigenvalues (if they exit) can be seen in the far field equation. In particular, due to the lack of injectivity and the denseness of the range of the far field operator F, when k is a transmission eigenvalue the L^2-norm of the (regularized) solution to the far field equation (3) can be expected to be large for such values of k. (This expectation are numerically verified for several examples in [2] and [3]). This provides a method for determining the smallest transmission eigenvalue from the far field data for a range of frequencies.

References

[1] F. Cakoni and D. Colton, Qualitative Methods in Inverse Scattering Theory, Springer, Berlin, 2006.

[2] F. Cakoni, D. Colton and H. Haddar, *The computation of lower bounds for the norm of the index of refraction in an anisotropic media from far field data*, J. Integral Equations and Applications, to appear.

[3] F. Cakoni, D. Colton and P. Monk, *On the use of transmission eigenvalues to estimate the index of refraction from far field data*, Inverse Problems **23** (2007), 507–522.

[4] F. Cakoni, D. Colton and P. Monk, *The determination of the surface conductivity of a partially coated dielectric*, SIAM J. Appl. Math **65** (2005), 767–789.

[5] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, Vol. 93 in Applied Mathematical Sciences, 2nd edition, Springer, New York, 1998.

A variational approach for the solution of the electromagnetic interior transmission problem for anisotropic media

Houssem Haddar
(joint work with Fioralba Cakoni)

The electromagnetic scattering problem for anisotropic media presents difficulties that are not present in the isotropic case. These difficulties are all connected to the fact that the (tensor) index of refraction is not uniquely determined from the scattering data and hence the basis inverse scattering problem to be considered is different from the corresponding isotropic case. In particular, it has been shown that only the support of the inhomogeneous media can be uniquely determined [1] and this fact has led to the problem of deriving reconstruction algorithms to recover the support from the measured scattering data [2, 3, 4, 10]. Central to the derivation of both uniqueness theorems and reconstruction algorithms has been the Interior Transmission Problem (ITP) and a better understanding of the behavior of solutions to this problem is basic to further developments in the inverse scattering problem for anisotropic media. Since all materials exhibit some degree of anisotropy and many, such as human tissue, to a large degree, such problems in inverse scattering are not only of considerable mathematical interest but also of also of central importance in numerous applications.

One possible approach to solve ITP is the use of an integral-type method. For instance, this type of method has been successfully applied to the case of an inhomogeneous medium if one assumes that the index of refraction is smooth inside the medium and has no jump across the boundary [9]. However, as presented in [7], it gives only partial answers in the case of anisotropic media (where the index of refraction N is a matrix-valued function). One needs to further assume that the imaginary part of N is definite positive but sufficiently small (without knowing how small it should be).

Our approach to treat the problem uses a variational framework where minimal regularity for N is required and where the entries of this matrix can have a jump across the boundary of the medium. A variational treatment of ITP has been proposed in [2] for the acoustic case. It is based on the study of a modified coercive ITP obtained from the original one by adding some appropriate zero order terms. This modified problem can be seen then as a compact perturbation of the original one. However, it turns out that in the Maxwell case the modified ITP is no longer a compact perturbation of the original one and therefore this approach does not apply any more. Let us however mention that in the cases where the relative permeability and permittivity are greater than one, and applying ideas from [8] that combines the use of integral equation method and a variational approach, one can overcome this difficulty. The approach in [8] could in principle be generalized to anisotropic cases.

Our alternative method follows [10] and is based on a reformulation of the problem as a fourth order boundary value problem. This procedure applies when the permeability or the permittivity is equal to one. It leads in particular to optimal existence results in the sense that in general the interior transmission problem has only L^2 solutions. The idea of transforming the problem into a fourth order partial differential equation goes back to [11] where the acoustic case is studied (see also [6]). However their technique to solve the obtained boundary value problem cannot be transposed to Maxwell's equations due to the lack of compactness of $H(\text{curl})$-like spaces into L^2.

The known results on the electromagnetic interior transmission problem for anisotropic media are contained in [10]. In this paper it was shown that if the real part of the index of refraction is positive definite and greater then one then in an appropriate function space there exists a unique solution to the interior transmission problem provided the wave number is not a transmission eigenvalue. This work is continued in [5] where the case when the real part of the index of refraction is less then one is taken into account and the countability of transmission eigenvalues is proved. These results are also generalized to the class of problems when the anisotropic media is partially coated by a thin highly conducting layer

References

[1] F. Cakoni and D. Colton, *A uniqueness theorem for an inverse electromagnetic scattering problem in inhomogeneous anisotropic media*, Proc. Edinburgh Math. Assoc. **46** (2003), 293–314.

[2] F. Cakoni, D. Colton and H. Haddar, *The linear sampling method for anisotropic media*, J. Comput. Appl. Math. **146** (2002), 285–299.

[3] F. Cakoni, D. Colton and P. Monk, *The determination of the surface conductivity of a partially coated dielectric*, SIAM J. Appl. Math. **65**, (2005) 767–789.

[4] F. Cakoni, F.MB. Fares and H. Haddar, *Analysis of two linear sampling methods applied to electromagnetic imagining of burried objects*, Inverse Problems **22** (2006), 845–867.

[5] F. Cakoni and H. Haddar, *A variational approach for the solution of the electromagnetic interior transmission problem for anisotropic media*, Inverse Problems and Imaging, at press, 2007.

[6] D. Colton, L. Päivärinta and J. Sylvester, *The interior transmission problem*, Inverse Problems and Imaging **1** (2007), 13–28.

[7] D. Colton and R. Potthast, *The inverse electromagnetic scattering problem for an anisotropic medium*, Quart. Jour. Mech. Applied Math. **52** (1999), 349–372.

[8] A. Kirsch, *An integral equation approach and the interior transmission problem for Maxwell's equations*, Inverse Problems and Imaging **1** (2007), 159–179.

[9] H. Haddar and P. Monk, *The linear sampling method for solving the electromagnetic inverse medium problem*, Inverse Problems **18** (2002), 891–906.

[10] H. Haddar, *The interior transmission problem for anisotropic Maxwell's equations and its applications to the inverse problem*, Math. Methods Applied Sciences **27** (2004), 2111–2129.

[11] B.P. Rynne and B.D. Sleeman, *The interior transmission problem and inverse scattering from inhomogeneous media*, SIAM J. Math. Anal. **22** (1991), 1755–1762.

On the convergence of the no-response test

Roland Potthast

The *no-response test* suggested by Luke-Potthast [1] is a recent scheme for the location of inhomogeneities for scattering problems from the knowledge of the far field pattern or the trace of the scattered field for *one* incident wave. The method *does not depend on the particular physical structure of the inclusion*, i.e. it does not depend on the boundary condition or whether the scatterer is penetrable, inhomogeneous or anisotropic. Here, we provide new results [3] on the convergence of the method.

We consider the scattering of some time-harmonic acoustic wave u^i by an impenetrable scatterer D in two or three dimensions. The scattered field is denoted by u^s and the total field $u = u^i + u^s$ is a solution to the *Helmholtz equation*

$$\triangle u + \kappa^2 u = 0 \tag{1}$$

in $\mathbb{R}^m \setminus \overline{D}$ with $m = 2$ or $m = 3$, where κ denotes the wave number. The scattered field is assumed to satisfy the *Sommerfeld radiation condition*

$$r^{\frac{m-1}{2}} \Big(\frac{\partial u^s}{\partial r} - i\kappa u^s \Big) \to 0, \quad r = |x| \to \infty, \tag{2}$$

uniformly in all directions $\hat{x} = x/|x|$. A radiating scattered field u^s has the asymptotic behavior

$$u^s(x) = \frac{e^{i\kappa|x|}}{|x|^{\frac{m-1}{2}}} \left\{ u^\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right\}, \ |x| \to \infty, \tag{3}$$

where u^∞ is known as *far field pattern* of u^s. By $\mathbb{S}$ we denote the boundary of the unit disk or unit ball, respectively. Via Green's representation formula the field u^s in the exterior of D can be represented as

$$u^s(x) = \int_{\partial D} \left(\Phi(x,y) \frac{\partial u^s}{\partial \nu}(y) - \frac{\partial \Phi(x,y)}{\partial \nu(y)} u^s(y) \right) ds(y), \ \ x \in \mathbb{R}^m \setminus \overline{D}, \tag{4}$$

$$u^\infty(\hat{x}) = \gamma \int_{\partial D} \left(e^{-i\kappa \hat{x}\cdot y} \frac{\partial u^s}{\partial \nu}(y) - \frac{\partial e^{-i\kappa \hat{x}\cdot y}}{\partial \nu(y)} u^s(y) \right) ds(y), \ \ \hat{x} \in \mathbb{S}, \tag{5}$$

where Φ is the free space fundamental solution to the Helmholtz equation and

$$\gamma := \begin{cases} \dfrac{e^{i\pi/4}}{\sqrt{8\pi\kappa}}, & m = 2, \\[2ex] \dfrac{1}{4\pi}, & m = 3, \end{cases} \tag{6}$$

For the *inverse problem* we assume that u^∞ is given on an open subset Λ of the unit sphere. The task is to determine the location, shape and properties of some scattering object D.

The idea of the no-response test is to study superpositions of incident waves

$$(Hg)(x) := \int_\Lambda e^{i\kappa x\cdot d} g(d)\, ds(d) \tag{7}$$

with some density $g \in L^2(\Lambda)$. Consider *test domains* $G \subset \mathbb{R}^m$ with boundary of class C^2, where we assume that the homogeneous interior Dirichlet problem for G does have only the trivial solution. The technique here is also denoted as *domain sampling* in contrast to *point sampling* carried out by other sampling methods. Then, for $\epsilon > 0$ we investigate the set $\mathcal{M}(G,\epsilon)$ of densities g with the condition

$$\|Hg\|_{C^1(\overline{G})} \leq \epsilon. \tag{8}$$

For the case $D \subset G$ via Green's theorem we calculate

$$\begin{aligned}
\mu(g) &:= \int_\Lambda u^\infty(-\hat{x}) g(\hat{x}) ds(\hat{x}) \\
&= \int_\Lambda \gamma \int_{\partial D} \left(e^{i\kappa \hat{x}\cdot y} \frac{\partial u^s}{\partial \nu}(y) - \frac{\partial e^{i\kappa \hat{x}\cdot y}}{\partial \nu(y)} u^s(y) \right) \, ds(y) \, g(\hat{x}) \, ds(\hat{x}) \\
&= \gamma \int_{\partial D} \left\{ \left(\int_\Lambda e^{i\kappa \hat{x}\cdot y} g(\hat{x}) \, ds(\hat{x}) \right) \frac{\partial u^s}{\partial \nu}(y) \right. \\
&\qquad \left. - \left(\int_\Lambda \frac{\partial e^{i\kappa \hat{x}\cdot y}}{\partial \nu(y)} g(\hat{x}) \, ds(\hat{x}) \right) u^s(y) \right\} ds(y) \\
&= \gamma \int_{\partial D} \left\{ (Hg)(y) \frac{\partial u^s}{\partial \nu}(y) - \frac{\partial Hg(y)}{\partial \nu(y)} u^s(y) \right\} ds(y)
\end{aligned}$$

and derive

$$|\mu(g)| \leq c\epsilon \to 0, \;\; \epsilon \to 0, \tag{9}$$

with some constant c. The no-response test calculates some estimate for

$$\mu(G,\epsilon) := \sup_{g \in \mathcal{M}(G,\epsilon)} |\mu(g)|, \;\; \epsilon > 0. \tag{10}$$

The behaviour of general case is resolved in the following convergence theorem [3].

Theorem 1. *Let G be a domain which is homotopic to a ball B_R, $R > 0$ sufficiently large, in the sense that there exists a continuous mapping $[0,1] \ni \lambda \mapsto G_\lambda$ with $G_0 = B_R$, $G_1 = G$ and $\overline{G}_\lambda \subset G_\eta$ for $\lambda > \eta$, where ∂G_λ is of class C^2. Then*

$$|\mu(G,\epsilon)| \to 0, \;\; \epsilon \to 0, \tag{11}$$

if and only if u^s can be analytically extended into $B_R \setminus G$.

The proof is based on analyticity arguments, approximation of multipoles and Taylor series. We show that the response is small if and only if the scattered field can be analytically extended into the exterior of the test domain G. Thus, the no-response test in fact tests for *analytic extensibility.* For the multiwave situation where the far field pattern is known for all or many directions of incidence we show convergence to the support of the scatterer under consideration for basic situations in acoustics.

References

[1] D.R. Luke and R. Potthast, *The no response test - a sampling method for inverse scattering problems*, SIAM J. Appl. Math. **63** (2003), 1292–1312.

[2] R. Potthast, *A survey on sampling and probe methods for inverse problems*, Inverse Problems **22** (2006), R1–R47.

[3] R. Potthast, *On the convergence of the no response test*, SIAM J. Math. Anal., at press.

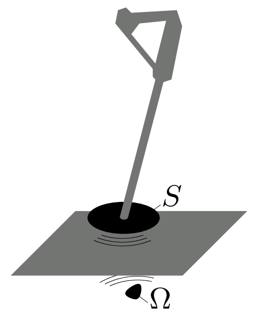
Sampling methods for low-frequency electromagnetic imaging

Bastian Gebauer

(joint work with Martin Hanke, Christoph Schneider)

For the detection of buried landmines the most frequently used devices are standard *off-the-shelf* metal detectors. These detectors exhibit (and measure) an electromagnetic field which changes in the vicinity of metallic or magnetic objects. From this change the detectors decide (on a heuristical basis) whether there is a landmine underneath them or not. To improve the reliability of these devices it is desirable to extract from the signal as much information as possible about the shape and position of magnetic, dielectric or conducting inhomogeneities. However, standard metal detectors work with very low frequencies around 20kHz, which corresponds to a wavelength of approximately 15km, while the typical objects of interest are only a few centimeters in size. The problem can therefore be expected to be severely ill-posed and to show close relations to electrical impedance tomography (EIT), which can also be considered as a problem of detecting imhomogeneities using waves of infinite wavelength. We consequently study a relatively new class of non-iterative methods that have been used with some success in EIT, namely the Linear Sampling and the Factorization Method.

As a simplified model for a metal detector we consider a two-dimensional device S, in which time-harmonic surface currents with complex amplitude J and frequency ω are being generated and on which the tangential component $\gamma_\tau E^\omega|_S$ of the resulting electric part of the scattered electromagnetic field can be measured. Idealistically we assume that we have access to *full measurements*, i. e. to the whole operator



$$M_s^\omega : \ J \mapsto \gamma_\tau E^\omega|_S.$$

M_s^ω can be factorized into $M_s^\omega = LG$, where L is the *virtual measurement operator*, that maps a tangential magnetic field, that is applied on the surface of some object Ω, to the resulting electric field on S. The range of this virtual measurement operator determines Ω, more precisely with $E_{z,d}^\omega$ being the electric field of a point current in a point z below S with arbitrary direction d one has that $\gamma_\tau E_{z,d}^\omega \in \mathcal{R}(L)$ if and only if $z \in \Omega$. An immediate consequence is that a (possibly empty) subset of Ω can be found by checking for every z below S whether $\gamma_\tau E_{z,d}^\omega$ is in $\mathcal{R}(M_s^\omega)$ or not. This is the so-called Linear Sampling Method, developed by Colton and Kirsch in [4].

Numerical examples (cf. e. g. [6]) show a much better performance than can be explained theoretically. The method seems to find the object itself and not only a subset. An explanation for this good performance is that for low frequencies

$$M_s^\omega \approx -\frac{1}{\mathrm{i}\omega}\nabla_S \Lambda_s \nabla_S^*,$$

where $-\nabla_S^*$ is the surface divergence, which maps a given current to the corresponding surface charges, Λ_s is the electrostatic measurement operator, mapping a surface charge to the generated (scattered) electrostatic potential, and ∇_S is the surface gradient, that maps the electrostatic potential to the (tangential component of the) corresponding electrostatic field. Under certain conditions on the object Ω it is known that

$$\mathcal{R}(|\Lambda_s|^{1/2}) = \mathcal{R}(L_{\mathrm{ES}})$$

(see [8] for grounded or [7] for more general objects), with L_{ES} being the electrostatic analogue to the virtual measurements L. It follows that

$$\mathcal{R}(|\nabla_S \Lambda_s \nabla_S^*|^{1/2}) = \mathcal{R}(\nabla_S L_{\mathrm{ES}}).$$

Using the electrostatic field $E_{z,d}$ of a dipole in a point z below S with direction d one can show that $z \in \Omega$ if and only if $\gamma_\tau E_{z,d} \in \mathcal{R}(\nabla_S L_{\mathrm{ES}})$. Thus Ω can be found by considering $-\mathrm{i}\omega M_s^\omega$ as an approximation to $\nabla_S \Lambda_s \nabla_S^*$ and testing for every z below S whether $\gamma_\tau E_{z,d} \in \mathcal{R}(|M_s^\omega|^{1/2})$. This is the so called Factorization Method, which was developed by Kirsch in [9] as a rigorously justified variant of the LSM and generalized to EIT by Brühl and Hanke in [2, 3]. Since for small ω also $\gamma_\tau E_{z,d}^\omega \approx \frac{1}{\omega}\gamma_\tau E_{z,d}$ and the effect of taking the square root is somewhat compensated by choosing different threshold in the numerical implementation, the good performance of the LSM can therefore be explained by the fact that the measurements are essentially electrostatic measurements for which the Factorization Method works.

However, in practice, currents are only applied and electric fields are only measured along closed coils, so that only the Galerkin projection $j^* M_s^\omega j$ to the space of divergence-free currents can be measured. In particular traces of gradient fields like $\gamma_\tau E_{z,d}^\omega \approx \frac{1}{\omega}\gamma_\tau E_{z,d}$ integrate to zero if measured along closed coils. Low-frequency asymptotics for this case show that

$$j^* M_s^\omega j \approx -\mathrm{i}\omega M_s,$$

where M_s maps an applied current to (a vector potential of) the generated magnetostatic field. Magnetostatic measurements are closely related to harmonic vector fields for which Kress derived the Factorization Method in [11]. Indeed, using the general theory in [7] one can show that

$$\mathcal{R}(|M_s|^{1/2}) = \mathcal{R}(L_{\mathrm{MS}}),$$

with the magnetostatic analogue L_{MS} to the virtual measurements L. Using a vector potential $G_{z,d}$ of the magnetostatic field of a magnetic dipole in a point z below S one then obtains that $z \in \Omega$ if and only if $\gamma_\tau G_{z,d} \in \mathcal{R}(L_{\mathrm{MS}})$. Thus for this practically relevant case of divergence free currents, one can also use the Factorization Method by considering the measurements as magnetostatic measurements and using the appropriate singular function $G_{z,d}$.

We finally note that for the case of objects with finite conductivity, the Maxwell's equations degenerate in the low-frequency limit to an equation that is (in the time-domain) parabolic in the objects Ω and elliptic outside (see [1]). For the scalar parabolic-elliptic model problem of detecting objects with a high heat capacity in a domain with low heat capacity by thermal measurements, it was shown in [5] that the Factorization Method still works when the object's heat conductivity is larger than that of the background. We therefore expect the Factorization Method also to work for the detection of diamagnetic, conducting objects.

References

[1] H. Ammari, A. Buffa and J.-C. Nédélec, *A justification of eddy currents model for the Maxwell equations*, SIAM J. Appl. Math. **60** (2000), 1805–1823.

[2] M. Brühl and M. Hanke, *Numerical implementation of two noniterative methods for locating inclusions by impedance tomography*, Inverse Problems **16** (2000), 1029–1042.

[3] M. Brühl, *Explicit characterization of inclusions in electrical impedance tomography*, SIAM J. Math. Anal. **32** (2001), 1327–1341.

[4] D. Colton and A. Kirsch, *A simple method for solving inverse scattering problems in the resonance region*, Inverse Problems **12** (1996), 383–393.

[5] F. Frühauf, B. Gebauer and O. Scherzer, *Detecting interfaces in a parabolic-elliptic problem from surface measurements*, SIAM J. Numer. Anal., to appear.

[6] B. Gebauer, M. Hanke, A. Kirsch, W. Muniz and C. Schneider, *A sampling method for detecting buried objects using electromagnetic scattering*, Inverse Problems **21** (2005), 2035–2050.

[7] B. Gebauer, *The factorization method for real elliptic problems*, Z. Anal. Anwend. **25** (2006), 81–102.

[8] P. Hähner, *An inverse Problem in electrostatics*, Inverse Problems **15** (1999) 961–975.

[9] A. Kirsch, *Characterization of the shape of the scattering obstacle using the spectral data of the far field operator*, Inverse Problems **14** (1998), 1489–1512.

[10] A. Kirsch, *The factorization method for Maxwell's equations*, Inverse Problems **20** (2004), S117–S134.

[11] R. Kress, *A factorization method for an inverse Neumann problem for harmonic vector fields*, Georgian Math. J. **10** (2003), 549–560.

Reporter: Armin Lechleiter

Participants

Dr. Tilo Arens
Institut für Algebra u. Geometrie
Universität Karlsruhe
Englerstr. 2
76131 Karlsruhe

Prof. Dr. Fioralba Cakoni
Department of Mathematical Sciences
University of Delaware
501 Ewing Hall
Newark , DE 19716-2553
USA

Prof. Dr. Simon N. Chandler-Wilde
Department of Mathematics
University of Reading
WhiteKnights
P.O.Box 220
GB-Reading , RG6 6AX

Prof. Dr. David L. Colton
Department of Mathematical Sciences
University of Delaware
501 Ewing Hall
Newark , DE 19716-2553
USA

Dr. Johannes Elschner
Weierstraß-Institut für
Angewandte Analysis und Stochastik
im Forschungsverbund Berlin e.V.
Mohrenstr. 39
10117 Berlin

Dr. Bastian Gebauer
Johann Radon Institute
Austrian Academy of Sciences
Altenberger Straße 69
A-4040 Linz

Dipl.Ing. Roland Griesmaier
Fachbereich Mathematik/Informatik
Johannes-Gutenberg-Universität
55099 Mainz

Dr. Houssem Haddar
INRIA Rocquencourt
Domaine de Voluceau
B. P. 105
F-78153 Le Chesnay Cedex

Prof. Dr. Martin Hanke-Bourgeois
Fachbereich Mathematik/Informatik
Johannes-Gutenberg-Universität
55099 Mainz

Dr. Frank Hettlich
Institut für Algebra u. Geometrie
Universität Karlsruhe
76128 Karlsruhe

Dr. Thorsten Hohage
Institut für Numerische
und Angewandte Mathematik
Universität Göttingen
Lotzestr. 16-18
37083 Göttingen

Dr. Nuutti Hyvoenen
Department of Mathematics
Helsinki University of Technology
P.O.Box 1100
FIN-02015 Hut

Prof. Dr. Andreas Kirsch
Institut für Algebra u. Geometrie
Universität Karlsruhe
Englerstr. 2
76131 Karlsruhe

Prof. Dr. Rainer Kreß
Institut für Numerische
und Angewandte Mathematik
Universität Göttingen
Lotzestr. 16-18
37083 Göttingen

Dipl.Math. Armin Lechleiter
Institut für Algebra u. Geometrie
Universität Karlsruhe
Englerstr. 2
76131 Karlsruhe

Prof. Dr. Russell Luke
Department of Mathematical Sciences
University of Delaware
409 Ewing Hall
Newark DE 19711
USA

Prof. Dr. Stephen McDowall
Dept. of Mathematics
Western Washington University
Bellingham , WA 98225-9063
USA

Dr. Roland Potthast
Department of Mathematics
University of Reading
Whiteknights, PO Box 220
GB-Berkshire , RG6 6AX

Dipl.-Math. Stefanie Reusswig
Institut für Mathematik
Universität Mainz
Staudingerweg 9
55099 Mainz

Prof. Dr. Andreas Rieder
Institut für Angewandte und Numerische
Mathematik
Universität Karlsruhe
Englerstr. 2
76131 Karlsruhe

Prof. Dr. William Rundell
Department of Mathematics
Texas A & M University
College Station , TX 77843-3368
USA

Prof. Dr. John Sylvester
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle , WA 98195-4350
USA