Arbeitsgemeinschaft mit aktuellem Thema: PERCOLATION

Mathematisches Forschungsinstitut Oberwolfach

8-12 October, 2007

Organizers:

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Introduction:

Percolation as a mathematical theory was introduced by Broadbent and Hammersley [8, 9] about fifty years ago to model the spread of a gas or a fluid through a porous medium. To model the randomness of the medium, they took the edges between nearest-neighbors on $\mathbb{Z}^d$ and made all edges independently *open* (to the passage of the gas or fluid) with probability p or *closed* with probability $1 - p$. Since then, many variants of this simple model have been studied, attracting the interest of both mathematicians and physicists.

Mathematicians are interested in percolation because of its deceiving simplicity which hides difficult and elegant results. For the physicists, percolation is maybe the simplest statistical mechanical model undergoing, as the value of the parameter p is varied, a phase transition with all the standard features typical of critical phenomena (scaling laws, a conformally invariant scaling limit, universality). On the applied side, percolation has been used to model the spread of a disease, a fire or a rumor, the displacement of oil by water, the behavior of random electrical circuits.

The work of mathematicians has concentrated on the behavior of the model both at the critical point p_c and away from it. Thanks to those efforts, we now have a good understanding of the *subcritical* ($p < p_c$) and *supercritical* ($p > p_c$) phases, at least on regular lattices (for some notable examples, see [30, 1, 10, 19] and the books [22, 18, 6]). A complete and rigorous understanding of the behavior at the critical point has proved more difficult and until recently seemed to be out of reach, despite various important achievements (see, e.g., [31, 24] and again [22, 18, 6] as general references).

Meanwhile, the problems encountered by mathematicians did not prevent the physicists from studying the critical point and its vicinity using theoretical physics methods. This enterprise was particularly successful in two dimensions where the tools of Conformal Field Theory (CFT) produced many predictions describing the behavior of the model at p_c or as $p \to p_c$, including various critical exponents.

Recently, the introduction by Oded Schramm [34] of the Schramm-Loewner Evolution (SLE) has provided a new powerful and mathematically rigorous tool to study scaling limits of critical lattice models. While CFT focuses on correlation functions, SLE describes the behavior of macroscopic random curves present in those models, such as percolation cluster boundaries. There is a one-parameter family of SLEs, indexed by a positive real number κ, and they appear to be essentially the only possible candidates for the scaling limits of interfaces of two-dimensional critical systems that are believed to be conformally invariant.

In particular, thanks to the work of Lawler, Schramm, Werner on SLE and of Smirnov [36], in recent years tremendous progress has been made in the study of two-dimensional critical percolation. The main power of SLE stems from the fact that it allows to compute different quantities; for example, percolation crossing probabilities and various percolation critical exponents [26, 37], confirming the predictions made by physicists using CFT methods.

Although the description of the geometry of the continuum scaling limit and the rigorous determination of several critical exponents in two dimensions represent maybe the single most exciting recent development, since its introduction percolation has continued to produce a wealth of beautiful results and has been an important paradigm for the behavior of other random systems and an important tool for the study of various other models, and it is still a very active area of research, strategically placed at the interface between probability and statistical physics. The present Arbeitsgemeinschaft will focus on some recent developments, including but not limited to two dimensions, trying at the same time to provide a concise introduction to some of the main techniques and results on which modern percolation theory is built, including SLE. The program will include time for informal discussions.

Talks:

1. **Introductory talk 1**
   In this talk Rob van den Berg will give an introduction to percolation and some of the main tools that have traditionally been used in its analysis, such as the Harris-FKG and BK inequalities and Russo's formula. Various fundamental results of percolation theory will be presented. An excellent reference is [18].

2. **Introductory talk 2**
   In this second introductory talk the speaker will present the Russo-Seymour-Welsh theorem (see [31]) and the Burton-Keane proof [10] of uniqueness of the infinite cluster in the supercritical phase.

3. **Minicourse SLE, part 1**
   This is the first of three lectures by Vincent Beffara intended to provide a concise introduction to the Schramm-Loewner Evolution (SLE), focusing on aspects that are relevant for the study of critical percolation in two dimensions. Excellent references are [38] and the recent book [25].

4. **Minicourse SLE, part 2**
   Second of three lectures by Vincent Beffara on SLE.

5. **Minicourse SLE, part 3**
   Last lecture by Vincent Beffara on SLE.

6. $p_c + p_c^* = 1$
   This is one of the cornerstones of percolation theory in two dimensions. The main reference for this talk is [32], but the speaker should also explain how Kesten's celebrated result [23] can be recovered from there.

7. **Cardy's formula, part 1**
   The main reference for this talk is Smirnov's celebrated paper [36] where conformal invariance of the scaling limit of critical site percolation on the triangular lattice was first established. The speaker will present the proof of the existence and conformal invariance of the scaling limit of

crossing probabilities. The proof should be split in two parts in order to present it in some detail. Other useful references are [2] and the recent book [6].

8. **Cardy's formula, part 2**
The speaker will conclude Smirnov's proof of Cardy's formula.

9. **Kesten's scaling relations**
The speaker will present the scaling relations worked out by Kesten in [24]. These results play an essential role in the rigorous determination of the percolation critical exponents in two dimensions.

10. **Convergence of 2D critical percolation to SLE(6)**
The speaker will present the main ideas of the convergence, in the scaling limit, of the critical percolation exploration path to SLE(6). This can be done following Section 6 of [14]. The speaker may also want to consult [13] which contains a complete proofs of the result. The result is crucial in the determination of the critical exponents in two dimensions and in all the applications of SLE to percolation.

11. **Convergence of 2D critical percolation to CLE(6)**
The convergence of the percolation exploration path to SLE(6) is not always sufficient to apply the full power of SLE to draw conclusions about percolation scaling limits (see, e.g., [26] and the use of Theorem 2.1 there). Besides, it is conceptually interesting and natural to consider the scaling limit of the collection of all percolation interfaces at once. Assuming the convergence of critical percolation to SLE(6), the speaker will consider the "full" scaling limit of percolation, presenting the main ideas of the convergence of critical percolation interfaces to a *Conformal Loop Ensemble* (see [39, 40, 35]). This can be done following Sections 4 and 5 of [14] ([11, 12] are also relevant references). The speaker will also present Theorem 2.1 of [26], to be used later in talk 12.

12. **One-arm exponent**
The speaker will show how to obtain the one-arm exponent for critical site percolation on the triangular lattice following [26] but without explaining Theorem 2.1, discussed in talk 11. This is an important example of how SLE can be used to compute critical exponents for two-dimensional percolation.

13. **Lace expansion and percolation in high dimensions, part 1**
The lace expansion is an important tool in the study of certain probabilistic models in high dimensions (above the upper critical dimension). The speaker will present the lace expansion in the percolation context. Accessible and self-contained introductions are contained in Sections 3 and 4 of [7] and in [20].

14. **Lace expansion and percolation in high dimensions, part 2**
The speaker will conclude the presentation started in the previous talk.

15. **Sharp-threshold theorems**
A sharp-threshold theorem was first applied to percolation by Russo [33]. There is an extensive literature about sharp-threshold theorems in more general settings and their relevance for percolative systems has been recently re-discovered (see, e.g., [3, 4, 5, 17]). The speaker will provide an introduction to the subject following, for example, [16, 21]. Alternatively, the speaker could decide to present [33] in detail.

16. **Voronoi percolation**
The speaker will use [3] (see also [6]) to present the main ideas of the recent proof of $p_c = 1/2$ for Voronoi percolation.

17. **Multiscale methods**
Multiscale methods represent an important tool in the study of percolative systems. Conceptually they rely on a renormalization scheme. An excellent reference for this talk is Section 6 of [15].

18. **Percolation on trees and nonamenable graphs**
Percolation on trees and other nonamenable graphs has attracted considerable attention. The speaker will choose and present some relevant results. Possible references for this talk are [27, 28, 29].

References

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Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

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by AUGUST 31, 2007 at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.