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Disordered Systems: Random Schrödinger Operators and Random Matrices

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Abstract. The spectral analysis of random operators plays a central role for the understanding of quantum systems with disorder. Two distinct types of ensembles are of particular significance: Random Schrödinger operators and random matrices. The workshop brought together experts from both areas to discuss recent results and future directions of research.

Mathematics Subject Classification (2000): 15xx, 34xx, 35xx, 47xx, 60xx, 81xx, 82xx.

Introduction by the Organisers

It is natural to model the spectral behaviour of disordered quantum systems by the typical spectral properties of random operators. Depending on the underlying physics different kinds of ensembles have been introduced. Two types of ensembles, random Schrödinger operators and random matrices, have proved to be of particular interest because of their wide ranges of applicability and because of their rich mathematical structures. On first sight these two types of random operators may appear to be close relatives. However, as it turns out, their typical spectral properties differ significantly. Moreover, the methods that have been developed for their respective analysis have little in common. One may say that over the years two different cultures have evolved leading to two almost disjoint mathematical communities. It was the goal of this workshop to stimulate exchange between these two communities by highlighting important recent developments in both areas.

The workshop brought together 44 researchers from 9 different countries. This report contains the extended abstracts of the 30 lectures that were delivered during the meeting.

The transition from ordered to disordered system was the main topic of four lectures.

There were 14 talks about random Schrödinger operators. One of the major topics was the theory of Anderson localization/delocalization. The density of states, especially the investigation of Lifshitz tails was the subject of several contributions. The third topic in this field concerns the theory of level statistics which is a very active field at the moment. This area is much inspired by the theory of random matrices.

Twelve of the lectures were related to the theory of random matrices. In these talks a variety of ensembles, results, open problems, and methods of proof were discussed that yield a somewhat representative picture of the current state of the art in this field. The ensembles considered include the two classical cases of Wigner ensembles (independent entries) and of invariant ensembles (invariant under appropriate changes of basis). Some of the lectures dealt with more general classes of ensembles that are motivated by applications in physics and statistics. Most of the results that were presented are concerned with local correlations of eigenvalues, with the distribution of the largest eigenvalue and with (central) limit theorems for spectral quantities. One talk was devoted to the Asymmetric Simple Exclusion Process, generalizing a much celebrated result that provided a connection to random matrix theory.

Finally, it is our happy task to thank all the participants for the lively discussions, the staff of Oberwolfach for providing such perfect and pleasant working conditions and Bernd Metzger for collecting and editing the extended abstracts.

Workshop: Disordered Systems: Random Schrödinger Operators and Random Matrices

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Abstracts

Central Limit Theorem for Linear Eigenvalue Statistics of the Wigner and the Sample Covariance Random Matrices

LEONID A. PASTUR
(joint work with A. Lytova)

We consider $n \times n$ real symmetric random matrices $M = n^{-1/2}W$ with independent (modulo symmetry condition) entries and the (null) sample covariance matrices $n^{-1}A^T A$ with independent entries of $m \times n$ matrix A. Assuming first that the 4th cumulant κ_4 (excess) of entries of W and A is zero and that the 4th moments of entries satisfy a Lindeberg type condition we prove that linear statistics of eigenvalues of the above matrices satisfy the Central Limit Theorem (CLT) as $n \to \infty$, $m \to \infty$, $m/n \to c \in [0, \infty)$ with the same variance as for matrices with the Gaussian entries if the test functions of statistics are smooth enough (essentially of the class $\mathbb{C}^5$).

We prove then the CLT for linear eigenvalues statistics of the above matrices with non-zero excess of entries and less regular test functions (essentially $\mathbb{C}^4$). However, in this case the variance of the limiting normal law differs from that for $\kappa_4 = 0$ and contains an additional term proportional to κ_4. The proofs of all limit theorems follow essentially the same scheme, based on a systematic use of certain differential formula (a version of the integration by parts) and related "interpolation trick" allowing us to deduce the results for general entries from those for the Gaussian entries.

Generalized eigenvalue-counting estimates for the Anderson model

ABEL KLEIN
(joint work with Jean-Michel Combes, François Germinet)

Consider the random self-adjoint operator

$$H_\omega = H_0 + \omega \Pi_\varphi \quad \text{on} \quad \mathcal{H}, \tag{1}$$

where H_0 is a self-adjoint operator on the Hilbert space $\mathcal{H}$, $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$, and ω is a random variable with a non-degenerate probability distribution μ with compact support. By Π_φ we denote the orthogonal projection onto $\mathbb{C}\varphi$, the one-dimensional subspace spanned by φ. Let $P_\omega(J) = \chi_J(H_\omega)$ for a Borel set $J \subset \mathbb{R}$. There is a fundamental spectral averaging estimate: for all bounded intervals $I \subset \mathbb{R}$ we have

$$\mathbb{E}_\omega \left\{ \langle \varphi, P_\omega(I)\varphi \rangle \right\} := \int \mathrm{d}\mu(\omega) \, \langle \varphi, P_\omega(I)\varphi \rangle \le Q_\mu\left(|I|\right), \tag{2}$$

where $Q_\mu(s) = 8S_\mu(s)$ for $s \ge 0$, with $S_\mu(s) := \sup_{a \in \mathbb{R}} \mu([a, a+s])$ being the concentration function of μ [9, Eq. (3.16)]. The estimate (2) is useful when the measure μ has no atoms, i.e., $\lim_{s \downarrow 0} Q_\mu(s) = 0$, which we assume from now on.

If μ has a bounded density ρ, (2) was known to hold with $Q_\mu(s) = \|\rho\|_\infty s$ (e.g, [28, 13, 5, 8, 20]. If μ is Hölder continuous, i.e., $S_\mu(s) \le Cs^\alpha$ with $\alpha \in]0,1[$, (2) was known with $Q_\mu(s) = C(1-\alpha)^{-1}s^\alpha$ [5, Theorem 6.2]. All we will require of Q_μ is the validity of (2).

We consider the generalized Anderson model given by the random Hamiltonian

$$H_{\boldsymbol{\omega}} = H_0 + V_{\boldsymbol{\omega}} \quad \text{on} \quad \ell^2(\mathbb{Z}^d), \tag{3}$$

where H_0 is a bounded self-adjoint operator and $V_{\boldsymbol{\omega}}$ is the random potential given by $V_{\boldsymbol{\omega}}(j) = \omega_j$, i.e., $V_{\boldsymbol{\omega}} = \sum_{j\in\mathbb{Z}^d} \omega_j \Pi_j$ with $\Pi_j = \Pi_{\delta_j}$. Here $\boldsymbol{\omega} = \{\omega_j\}_{j\in\mathbb{Z}^d}$ is a family of independent random variables, such that for each $j \in \mathbb{Z}^d$ the random variable ω_j has a probability distribution μ_j with no atoms and compact support. We set $Q_j = Q_{\mu_j}$.

Restrictions of $H_{\boldsymbol{\omega}}$ to finite volumes $\Lambda \subset \mathbb{Z}^d$ are denoted by $H_{\boldsymbol{\omega},\Lambda}$, a self-adjoint operator of the form

$$H_{\boldsymbol{\omega},\Lambda} = H_{0,\Lambda} + \sum_{j\in\Lambda} \omega_j \Pi_j \quad \text{on} \quad \ell^2(\Lambda), \tag{4}$$

with $H_{0,\Lambda}$ a self-adjoint restriction of H_0 to the finite-dimensional Hilbert space $\ell^2(\Lambda)$. Given a Borel set $J \subset \mathbb{R}$, we write $P_{\boldsymbol{\omega}}^{(\Lambda)}(J) = P_{H_{\boldsymbol{\omega}}}^{(\Lambda)}(J) = \chi_J(H_{\boldsymbol{\omega},\Lambda})$ for the associated spectral projection. We set $Q_\Lambda(s) := \max_{j\in\Lambda} Q_j(s)$.

The Wegner estimate [28] measures the probability that $H_{\boldsymbol{\omega},\Lambda}$ has an eigenvalue in an interval I:

$$\mathbb{P}\left\{\operatorname{tr} P_{H_\omega}^{(\Lambda)}(I) \ge 1\right\} \le \mathbb{E}\left\{\operatorname{tr} P_{H_\omega}^{(\Lambda)}(I)\right\} \le Q_\Lambda\left(|I|\right)|\Lambda|. \tag{5}$$

The Wegner estimate holds for the generalized Anderson model. It is an immediate consequence of (2).

Minami [23] estimated the probability that $H_{\boldsymbol{\omega},\Lambda}$ has at least two eigenvalues in an interval I. Assuming that all μ_j have bounded densities ρ_j, Minami proved that

$$2\,\mathbb{P}\left\{\operatorname{tr} P_{H_\omega}^{(\Lambda)}(I) \ge 2\right\} \le \mathbb{E}\left\{\left(\operatorname{tr} P_{H_\omega}^{(\Lambda)}(I)\right)^2 - \operatorname{tr} P_{H_\omega}^{(\Lambda)}(I)\right\} \le \left(\pi\rho_\infty^{(\Lambda)}\,|I|\,|\Lambda|\right)^2, \tag{6}$$

where $\rho_\infty^{(\Lambda)} := \max_{j\in\Lambda}\|\rho_j\|_\infty$. Minami's proof required H_0 to have real matrix elements, i.e., $\langle\delta_j, H_0\delta_k\rangle \in \mathbb{R}$ for all j,k. This restriction was recently removed by Bellissard, Hislop and Stolz [4] and by Graf and Vaghi [17]. They also estimated the probability that $H_{\boldsymbol{\omega},\Lambda}$ has at least n eigenvalues in I for all $n \in \mathbb{N}$, assuming, as Minami, that all μ_j have bounded densities ρ_j,

In [6] we introduced a new approach to eigenvalue-counting inequalities, obtaining a simple and transparent proof of Minami's estimate, based on (2) and a consequence of the min-max principle applied to rank one perturbations. Our proof also generalizes Minami's estimate and its extensions to n eigenvalues in two ways: we allow for singular measures and for n arbitrary intervals.

We start with our extension of Minami's estimate, i.e., (6).

Theorem 1. *Fix a finite volume $\Lambda \subset \mathbb{Z}^d$. For any two bounded intervals I_1, I_2 we have*

$$\mathbb{E}\left\{\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_1)\right)\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_2)\right) - \min\left\{\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_1), \operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_2)\right\}\right\} \le 2\, Q_\Lambda\left(|I_1|\right) Q_\Lambda\left(|I_2|\right) |\Lambda|^2 . \tag{7}$$

If $I_1 \subset I_2$, we have

$$\mathbb{E}\left\{\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_1)\right)\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_2) - 1\right)\right\} \le Q_\Lambda\left(|I_1|\right) Q_\Lambda\left(|I_2|\right) |\Lambda|^2 . \tag{8}$$

In particular, for all bounded intervals I we have

$$\mathbb{E}\left\{\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)\right)\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I) - 1\right)\right\} \le \left(Q_\Lambda\left(|I|\right) |\Lambda|\right)^2 . \tag{9}$$

We now turn to the general case of n arbitrary intervals, extending the results of [4, 17]. Given $n \in \mathbb{N}$, we let $\mathcal{S}_n$ denote the group of all permutations of $\{1, 2, \ldots, n\}$, and recall that $|\mathcal{S}_n| = n!$. Given a finite volume $\Lambda \subset \mathbb{Z}^d$ and bounded intervals $I_1, \ldots, I_n$ (not necessarily distinct), we pick $\sigma_{\boldsymbol{\omega}} = \sigma_{\boldsymbol{\omega}}^{(\Lambda)}(I_1, \ldots, I_n) \in \mathcal{S}_n$ such that

$$\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(1)}) \le \operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(2)}) \le \ldots \le \operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(n)}), \tag{10}$$

in which case we have

$$\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(1)})\right)\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(2)}) - 1\right) \cdots \left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(n)}) - (n-1)\right) \ge 0. \tag{11}$$

We let $\mathcal{S}_n(I_1, \cdots I_n)$ be the collection permutations $\sigma \in \mathcal{S}_n$ such that $\sigma = \sigma_{\boldsymbol{\omega}}$ for some $\boldsymbol{\omega}$, and let $M(I_1, \cdots I_n)$ denote the cardinality of $\mathcal{S}_n(I_1, \cdots I_n)$. Note that $1 \le M(I_1, \cdots I_n) \le n!$. We have $M(I_1, \cdots I_n) = n!$ if the n intervals are incompatible, i.e., $I_j \subset I_k$ implies $j = k$, and $M(I_1, \cdots I_n) = 1$ if $I_1 \subset I_2 \subset \cdots \subset I_n$.

Theorem 2. *Fix a finite volume $\Lambda \subset \mathbb{Z}^d$, let $n \in \mathbb{N}$, and consider n bounded intervals $I_1, \ldots, I_n$ (not necessarily distinct). Then, setting $\sigma_{\boldsymbol{\omega}} = \sigma_{\boldsymbol{\omega}}^{(\Lambda)}(I_1, \ldots, I_n)$, we have*

$$\mathbb{E}\left\{\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(1)})\right)\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(2)}) - 1\right) \cdots \left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(n)}) - (n-1)\right)\right\} \le M(I_1, \cdots I_n) \left(\prod_{k=1}^{n} Q^{(\Lambda)}\left(|I_k|\right)\right) |\Lambda|^n . \tag{12}$$

In the special case when $I_1 \subset I_2 \subset \cdots \subset I_n$, we have

$$\mathbb{E}\left\{\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_1)\right)\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_2) - 1\right) \cdots \left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_n) - (n-1)\right)\right\} \le \left(\prod_{k=1}^{n} Q^{(\Lambda)}\left(|I_k|\right)\right) |\Lambda|^n . \tag{13}$$

In particular, for any bounded interval I we have

(14)
$$\mathbb{E}\left\{\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)\right)\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)-1\right)\cdots\left(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)-(n-1)\right)\right\} \leq \left(Q^{(\Lambda)}\left(|I|\right)|\Lambda|\right)^{n}.$$

As a corollary, we get probabilistic estimates on the number of eigenvalues of $H_{\boldsymbol{\omega},\Lambda}$ in intervals.

Corollary 3. *Fix a finite volume $\Lambda \subset \mathbb{Z}^d$. For all $n \in \mathbb{N}$ and I a bounded interval, we have*

$$\mathbb{P}\left\{\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I) \geq n\right\} \leq \frac{1}{n!}\left(Q^{(\Lambda)}\left(|I|\right)|\Lambda|\right)^{n}. \tag{15}$$

Furthermore, for all bounded intervals $I_1, \cdots I_n$ we get

$$\begin{aligned}\mathbb{P}&\left\{\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(1)}) \geq 1, \operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(2)}) \geq 2, \cdots, \operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_{\sigma_{\boldsymbol{\omega}}(n)}) \geq n\right\} \\ &\leq M(I_1, \cdots I_n)\left(\prod_{k=1}^{n} Q^{(\Lambda)}\left(|I_k|\right)\right)|\Lambda|^{n},\end{aligned} \tag{16}$$

and, in the special case when $I_1 \subset I_2 \subset \cdots \subset I_n$, we have

$$\begin{aligned}\mathbb{P}&\left\{\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_1) \geq 1, \operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_2) \geq 2, \ldots, \operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I_n) \geq n\right\} \\ &\leq \left(\prod_{k=1}^{n} Q^{(\Lambda)}\left(|I_k|\right)\right)|\Lambda|^{n}.\end{aligned} \tag{17}$$

The (standard) Anderson model is given by $H_{\boldsymbol{\omega}}$ as in (3), with $H_0 = -\Delta$, the centered discrete Laplacian, and $\boldsymbol{\omega} = \{\omega_j\}_{j\in\mathbb{Z}^d}$ a family of independent identically distributed random variables with joint probability distribution μ, which we assume to have no atoms and compact support. Localization for the Anderson model has been well studied, mostly for μ with a bounded density ρ, cf. [13, 12, 10, 27, 11, 2, 1] and many others, as well as for probability distributions μ that are Hölder continuous [5, 11, 18, 3, 14], i.e., $Q_\mu(s) \leq Us^\alpha$ for s small, for some constants U and $\alpha \in]0,1[$. If the probability distribution μ has a bounded density, Minami's estimate (6) was a crucial ingredient in Klein and Molchanov's proof of simplicity of eigenvalues [22] and in Klein, Lenoble and Müller derivation of a rigorous form of Mott's formula for the ac-conductivity [21]. Using (9) these proofs extend to the case when μ is only Hölder continuous, that is,

$$Q_\mu(s) \leq Us^\alpha \quad \text{for all} \quad s \in [0, s_0]. \tag{18}$$

The details appear in [6].

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On Bernoulli decomposition of random variables and recent various applications

François Germinet

We first recall a recent Bernoulli decomposition of any given non trivial real random variable. While our main motivation is a proof of universal occurence of Anderson localization in continuum random Schrödinger operators, we review other applications like Sperner theory of antichains, anticoncentration bounds of some functions of random variables, as well as singularity of random matrices. These are joint results with M. Aizenman, L. Bruneau, A. Klein, S. Warzel.

1. Bernoulli decomposition

Let X be a real random variable that is non degenerate (i.e. non constant). Throughout this review, we shall make use of the following property (that clearly implies that X is non degenerate)

(H) *There exists* $\rho \in]0, \frac{1}{2}[$ *such that* $\mathbb{P}(X < x^-) > \rho$ *and* $\mathbb{P}(X > x^+) > \rho$ *for some real numbers* $x^- < x^+$.

Definition 1. Let X be a real random variable. Let f, δ be measurable functions, such that $f :]0,1[\to \mathbb{R}$ is monotone increasing and $\delta :]0,1[\to [0, +\infty[$. Let $p \in]0,1[$. We say that (f, δ, p) is a Bernoulli decomposition of X if (in law)

$$X = f(t) + \delta(t)\varepsilon, \tag{1}$$

where t and ε are two independent random variables, such that t has the uniform distribution in $]0,1[$; and ε is a Bernoulli with parameter p.

Theorem 2 ([1]). *Let* X *be a real non degenerate random variable.*
1. For any $p \in]0,1[$ *there exists a Bernoulli decomposition* (f, δ, p) *of* X.
2. There exists $p \in]0,1[$ *so that* X *admits a Bernoulli decomposition* (f, δ, p) *with* $\inf_{t \in]0,1[} \delta(t) > 0$.
3. Assume Property (H). *There exists* $p \in]\rho, 1-\rho[$ *so that* X *admits a Bernoulli decomposition* (f, δ, p) *with* $\inf_{t \in]0,1[} \delta(t) > 0$.
4. Assume Property (H). *Then the Bernoulli decomposition* $(f, \delta, p = \frac{1}{2})$ *satisfies* $\mathbb{P}_t(\delta(t) > x_+ - x_-) \geq 2\rho$.

2. Antichains and Sperner Theory

A possible motivation for looking into Sperner theory is the following quite natural question arising in arithmetics, see e.g. [2] and references therein. Consider distinct prime numbers $p_1, \cdots, p_N$ and the integer $M = p_1^{k_1} \cdots p_N^{k_N}$, with $k_i \geq 1$, $i = 1, \cdots, N$. Let $\mathcal{D} \simeq \bigotimes_{i=1}^N \{0, 1, \cdots, k_i\}$ be the set of divisors of M. We endow this set with a (discrete) probabilistic structure by considering $\mathbb{P} = \bigotimes_{i=1}^{N} \mu_i$ where for any i, μ_i is a discrete probability measure on $\{0, 1, \cdots, k_i\}$.

Let $\mathcal{A} \subset \mathcal{D}$ be so that for any r, r' in $\mathcal{A}$, neither $r|r'$ nor $r'|r$. The question is: what is the maximal size of such a set $\mathcal{A}$? Recasted in probabilistic terms, we would like to provide a bound on $\mathbb{P}(\mathcal{A})$.

We start with the simplest case, that is $k_i = 1$ for all i. The configuration space is $\{0,1\}^N$, and we consider a collection of Bernoulli random variables $\boldsymbol{\eta} = \{\eta_1, ..., \eta_N\}$. The set of configurations is partially ordered by the relation:

$$\boldsymbol{\eta} \prec \boldsymbol{\eta}' \quad \Longleftrightarrow \quad \text{for all } \ i \in \{1, ..., N\}: \quad \eta_i \leq \eta_i' \,. \tag{2}$$

A set $\mathcal{A} \subset \{0,1\}^N$ is said to be an *antichain* if it does not contain any pair of configurations which are comparable in the sense of "$\prec$". The original Sperner's Lemma [13] states that for any such set: $|\mathcal{A}| \leq \binom{N}{[\frac{N}{2}]}$. By an immediate computation the latter is bounded by $C2^N/\sqrt{N}$. The LYM inequality enables one to extend that bound to non even Bernoulli variables (but still identical). The same bound extends to antichains on larger alphabet: $\{0, 1, \cdots, k\}^N$ with $1 \leq k < \infty$ for equidistributed weights [2] as well as for general weights [6]. The following result, see [1, Remark 3.1], extends those bounds to non identical measure with (possibly) infinite support.

Theorem 3. *Set $\mathcal{D} = \mathbb{Z}^N$ and let μ_i, $i = 1, \cdots, N$, be discrete probability measures on $\mathbb{Z}$. Set $\mathbb{P} = \bigotimes_{i=1}^{N} \mu_i$. Assume there is $\rho \in]0, \frac{1}{2}[$ such that for any $i = 1, \cdots, N$, there exists $m_i \in \mathbb{Z}$ s.t.*

$$\mu_i(] - \infty, m_i]) > \rho \text{ and } \mu_i([m_i + 1, \infty[) > \rho. \tag{3}$$

Then there exists $C < \infty$ (independent of N), such that for any antichain $\mathcal{A} \subset \mathcal{D}$,

$$\mathbb{P}(\mathcal{A}) \leq \frac{C}{\sqrt{\rho N}} \,. \tag{4}$$

Theorem 3 has the following natural extension to (anti)concentration bounds of random variables. Let X be a real random variable and $Q_X(s) = \sup_{x \in \mathbb{R}} \mathbb{P}(X \in [x, x+s])$ its (Levy) concentration function. We have the

Theorem 4. [1] *Let $\boldsymbol{X} = (X_1, \ldots, X_N)$ be a collection of independent random variables whose distributions satisfy, for all $j \in \{1, ..., N\}$:*

$$\mathbb{P}\left(\{X_j < x_-\}\right) > \rho \quad \text{and} \quad \mathbb{P}\left(\{X_j > x_+\}\right) > \rho \tag{5}$$

for some $\rho > 0$ and $x_- < x_+$, and $\Phi : \mathbb{R}^N \mapsto \mathbb{R}$ be a function such that for some $\varepsilon > 0$

$$\Phi(\boldsymbol{u} + v\boldsymbol{e}_j) - \Phi(\boldsymbol{u}) \ > \ \varepsilon \tag{6}$$

for all $v > x_+ - x_-$, all $\boldsymbol{u} \in \mathbb{R}^N$, and $j = 1, \ldots, N$, with $\boldsymbol{e}_j$ the unit vector in the j-direction. Then, there exists $C < \infty$ (independent of N) s.t. the random variable $Z = \Phi(\boldsymbol{X})$ obeys the concentration bound

$$Q_Z(\varepsilon) \ \leq \ \frac{C}{\sqrt{\rho N}}. \tag{7}$$

3. Singularity of random matrices

Let $M_n = (a_{ij})_{ij}$ be a random $n \times n$ matrix, where the a_{ij} are independent (non necessarily identically distributed) real random variables. We assume that the random variables a_{ij} satisfy the non-degeneracy property

(H') *There exists* $\rho \in]0, \frac{1}{2}[$ *such that for any* $i, j = 1, \cdots, n$, $\mathbb{P}(a_{ij} > x_{ij}^{+}) > \rho$ *and* $\mathbb{P}(a_{ij} < x_{ij}^{-}) > \rho$ *for some real numbers* $x_{ij}^{-} < x_{ij}^{+}$.

Theorem 5. [5] *Let M_n be an $n \times n$ matrix whose coefficients are independent random variables satisfying* (H'). *Then* $\mathbb{P}(\det M_n = 0) \leq C_\rho/\sqrt{n}$, *for some* $C_\rho < \infty$.

The study of the singularity of random matrices goes back, at least, to Komlós [11][3][12]. For even Bernoulli's, it is conjectured that that $\mathbb{P}(\det M_n = 0) \leq C_\alpha \alpha^n$ for all $\alpha > \frac{1}{2}$. Such an exponential behaviour have been obtained and successively improved in [10, 14, 15] up to $c = \frac{3}{4}$. If one turns to general entries, Komlós proved in [12] that $\mathbb{P}(M_n \text{ is singular}) = o(1)$ for independent *and identically distributed* non degenerate random variables. Furthermore, as pointed out by Tao and Vu in [14, Section 8], it follows from their analysis that $\mathbb{P}(M_n \text{ is singular}) = o(1)$ for independent non degenerate entries, provided Property (H') holds. The note [5] provides an elementary proof of Theorem 5.

4. Application to random Schrödinger operators

In this application, we consider random Schrödinger operators on $\mathrm{L}^2(\mathbb{R}^d)$ of the type

$$H_{\boldsymbol{\omega}} := -\Delta + \sum_{\zeta \in \mathbb{Z}^d} \omega_\zeta \, u(x - \zeta), \tag{8}$$

where Δ is the d-dimensional Laplacian operator, and

(I) the single site potential u is a nonnegative bounded measurable function on $\mathbb{R}^d$ with compact support, uniformly bounded away from zero in a neighborhood of the origin, more precisely,

$$u_- \chi_{\Lambda_{\delta_-}(0)} \leq u \leq u_+ \chi_{\Lambda_{\delta_+}(0)} \quad \text{for some constants } u_\pm, \delta_\pm \in]0, \infty[; \tag{9}$$

(II) $\boldsymbol{\omega} = \{\omega_\zeta\}_{\zeta \in \mathbb{Z}^d}$ is a family of independent identically distributed random variables, whose common probability distribution μ is non-degenerate with bounded support, and satisfies $\{0, 1\} \in \operatorname{supp} \mu \subset [0, 1]$.

Localization is proved at the bottom of the spectrum for the Anderson Hamiltonian without any extra hypotheses. Spectral localization is proved in [1] based on an extension of [4] given in [9]. If one wants more detailed informations about the region of localization, the following result holds, based on the concentration bound given in Theorem 4.

Theorem 6. [9] *Let $H_{\boldsymbol{\omega}}$ be an Anderson Hamiltonian on* $\mathrm{L}^2(\mathbb{R}^d)$ *as above with hypotheses* (I), (II). *Then there exists $E_0 = E_0(d, u_\pm, \delta_\pm, \mu) > 0$ such that $H_{\boldsymbol{\omega}}$ exhibits Anderson localization as well as dynamical localization in the energy interval $[0, E_0]$. More precisely:*

- *(Anderson localization) There exists $m = m(d, V_{\mathrm{per}}, u_\pm, \delta_\pm) > 0$ such that the following holds with probability one:*
  - *$H_{\boldsymbol{\omega}}$ has pure point spectrum in $[0, E_0]$.*
  - *If ϕ is an eigenfunction of $H_{\boldsymbol{\omega}}$ with eigenvalue $E \in [0, E_0]$, then ϕ is exponentially localized with rate of decay m, more precisely,*

$$\|\chi_x \phi\| \leq C_{\boldsymbol{\omega},\phi}\, e^{-m|x|} \qquad \text{for all} \quad x \in \mathbb{R}^d. \tag{10}$$

  - *The eigenvalues of $H_{\boldsymbol{\omega}}$ in $[0, E_0]$ have finite multiplicity.*
- *(Dynamical localization) For all $s < \frac{3}{8}d$ we have*

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}} \left\| \langle x \rangle^{\frac{m}{2}} e^{-itH_\omega} \chi_{[0,E_0]}(H_\omega)\chi_0 \right\|_2^{\frac{2s}{m}} \right\} < \infty \quad \text{for all} \;\; m \geq 1. \tag{11}$$

The full proof of Theorem 6 is presented in [9]. In particular it combines the multiscale analysis of Bourgain and Kenig [4] together with the concentration bound of [1] (Theorem 4 above). This yields Anderson localization (using [8] for finite multiplicity). To get dynamical localization, one builds on ideas that are by now standard, see e.g. [7, 8].

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The Skew-Shift Model and Spectral Pseudo-Randomness

David Damanik

We discuss recent results concerning Schrödinger operators in $\ell^2(\mathbb{Z})$ with potentials generated by the skew-shift. That is, let $\Omega = \mathbb{T}^2$, $T(\omega_1, \omega_2) = (\omega_1+\omega_2, \omega_2+\alpha)$, where $\alpha \notin \mathbb{Q}$, and $f \in C(\mathbb{T}^2)$. Define potentials,

$$V_\omega(n) = f(T^n\omega), \quad \omega \in \Omega,\ n \in \mathbb{Z}$$

and Schrödinger operators

$$[H_\omega\psi](n) = \psi(n+1) + \psi(n-1) + V_\omega(n)\psi(n).$$

It is expected that this operator family is pseudo-random in the sense that it has spectral properties very much akin to those known to holds for random potentials. In particular, one expects the spectrum to be a finite union of intervals and the spectral measures to have a tendency to be pure point.

The following pair of results, however, shows that the expected properties fail generically.

Theorem 1 (Avila-Bochi-D.). *Consider the skew-shift case. Then, there is a dense G_δ subset $\mathcal{F}_s$ of $C(\mathbb{T}^2)$ such that for $f \in \mathcal{F}_s$, the spectrum of H_ω contains no intervals.*

Note that by minimality of T and continuity of f, the spectrum of H_ω is independent of ω. This result was shown in [1]. In fact, one can work in much greater generality and replace the skew-shift T by a strictly ergodic homeomorphism on a compact metric space which fibers over an almost periodic dynamical system.

Theorem 2 (Boshernitzan-D.). *Consider the skew-shift case with α having unbounded partial quotients. Then, there is a dense G_δ subset $\mathcal{F}_c$ of $C(\mathbb{T}^2)$ such that for every $f \in \mathcal{F}_c$ and almost every $\omega \in \mathbb{T}^2$, the spectrum of H_ω is purely continuous.*

This result is proved in [2]. If one introduces a coupling constant $\lambda \in \mathbb{R}$ and considers instead potentials of the form $V_\omega(n) = \lambda f(T^n\omega)$, then a "for every λ" can be added at the end of the formulation of Theorem 2.

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Random colourings of aperiodic graphs: Ergodic and spectral properties

Peter Müller
(joint work with Christoph Richard)

The talk focuses around the preprint [14]. It is motivated by the results of [10, 15], which concern spectral properties of Laplacians on bond-percolation graphs in the integer lattice $\mathbb{Z}^d$. A Lifshits-tail behaviour of the integrated density of states was found in the non-percolating phase [10], while the percolating cluster gives rise to a van Hove asymptotics in the case of Neumann boundary conditions at cluster borders [15]. Very recently, these results have been extended to percolation on amenable Cayley graphs [1, 3]. There, it is invariance under the appropriate group action that replaces translational invariance with respect to $\mathbb{Z}^d$ in the setting of [10, 15]. However, the arguments in [10] suggest that the Lifshits tail for the Neumann Laplacian in the non-percolating phase should hold even in the absence of a symmetry group. Here, we extend it to such a case, namely percolation on aperiodic graphs, for which there is a description in terms of nice dynamical systems.

We consider a simple graph $G = (\mathcal{V}_G, \mathcal{E}_G)$, whose vertex set is a uniformly discrete subset of $\mathbb{R}^d$. Uniform discreteness means that there exists a minimal separating distance $r > 0$ between any two different elements of $\mathcal{V}_G$. Simplicity excludes multiple edges between the same pair of vertices and self-loops. We also require G to be of *finite local complexity*, that is, for every given finite radius $R > 0$, one gets only finitely many different subgraphs (up to translations in $\mathbb{R}^d$) when restricting G to any ball $B_R(v)$, $v \in \mathcal{V}_G$, of radius R centred around one of the vertices of G; see e.g. [14] for a precise definition.

For G as above, the completion with respect to a suitable metric of the set of all its $\mathbb{R}^d$-translates yields a compact metric space $X_G := \overline{\{x + G : x \in \mathbb{R}^d\}}$, the *hull* of G; see e.g. [16, 17]. Thanks to compactness the hull supports at least one probability measure μ which is ergodic with respect to $\mathbb{R}^d$-translations [11].

Let G be a graph with bounded degree sequence, $d_{\max} := \sup\{d_G(v) : v \in \mathcal{V}_G\} < \infty$. Here, $d_G(v)$ denotes the vertex degree of v, that is, the number of edges in G attached to v. The *Neumann Laplacian* (or combinatorial or graph Laplacian) associated with G is the bounded, self-adjoint operator on the Hilbert space $\ell^2(\mathcal{V}_G)$ defined by

$$(\Delta_G \varphi)(v) := \sum_{u \in \mathcal{V}_G \colon \{v,u\} \in \mathcal{E}_G} \big[\varphi(v) - \varphi(u)\big] \tag{1}$$

for all $v \in \mathcal{V}_G$ and all $\varphi \in \ell^2(\mathcal{V}_G)$.

For a given graph G and a given bond probability $p \in]0,1[$, we denote by $(\Omega_G, \mathbf{P}_G^p)$ the probability space associated with all Bernoulli bond-percolation subgraphs $G^{(\omega)} := (\mathcal{V}_G, \mathcal{E}_G^{(\omega)})$ of G. By definition, we have $\mathcal{E}_G^{(\omega)} \subseteq \mathcal{E}_G$ for all $\omega \in \Omega_G$ and the event $e \in \mathcal{E}_G^{(\omega)}$ holds with probability p for all $e \in \mathcal{E}_G$. Moreover, different edges of $\mathcal{E}_G$ are kept (or rejected) independently of each other.

Next we fix an energy $E \in \mathbb{R}$ and a graph G_0 of finite local complexity and with a uniformly discrete vertex set. The *integrated density of states* of the Neumann Laplacian on bond-percolation subgraphs of G_0 is defined by

$$(2) \qquad N(E) := \lim_{n\to\infty} \left\{ \frac{1}{\mathrm{vol}\big(B_n(0)\big)} \sum_{v \in \mathcal{V}_{G^{(\omega)}} \cap B_n(0)} \big\langle \delta_v, \chi_{]-\infty,E]}(\Delta_{G^{(\omega)}})\delta_v \big\rangle \right\}.$$

Here, $\mathrm{vol}(\cdot)$ refers to Lebesgue measure on $\mathbb{R}^d$, χ_I is the indicator function of some set I, $\langle\cdot,\cdot\rangle$ is the canonical Hilbert-space scalar product on $\ell^2(\mathcal{V}_G)$ and δ_v denotes the canonical basis vector in $\ell^2(\mathcal{V}_G)$ with $\delta_v(u) = 1$ if $v = u$, and 0 otherwise. It is shown in [14] that the limit (2) exists and is non-random for μ-almost every graph $G \in X_{G_0}$ and $\mathbf{P}_G^p$-almost every $\omega \in \Omega_G$; see also (ii) below. Here is the central result.

Theorem 1. *Let G_0 be a connected, infinite graph of finite local complexity, with a uniformly discrete vertex set and maximal edge length $\ell_{\max} := \sup\big\{|u-v| : \{u,v\} \in \mathcal{E}\big\} < \infty$. Fix a bond probability $p \in]0,1[$ for which there exist constants $\gamma_p, \lambda_p \in]0,\infty[$ (depending only on G_0 otherwise) such that the cluster-size distribution obeys*

$$(3) \qquad \mathbb{P}_G^p\big(\omega \in \Omega_G : |C_v^{(\omega)}| \geq n\big) \leq \gamma_p\, \mathrm{e}^{-n\lambda_p}$$

for all $n \in \mathbb{N}$, all $G \in X_{G_0}$ and all $v \in \mathcal{V}_G$. Here, $C_v^{(\omega)}$ denotes the cluster of $G^{(\omega)}$ containing v. Then, the integrated density of states N of the Neumann Laplacian exhibits a Lifshits tail

$$(4) \qquad \lim_{E\downarrow 0} \frac{\ln\big|\ln[N(E) - N(0)]\big|}{\ln E} = -1/2,$$

with Lifshits exponent $1/2$ at the lower edge of the spectrum.

Remarks.

- *Requiring exponential decay of the cluster-size distribution restricts the allowed values for the bond probability p to the non-percolating regime. The lemma below gives a sufficient condition for (3) to hold. Uniformity of the constants γ_p and λ_p in G and v is not necessary, however. It can be replaced by some weaker* L^1*-condition.*
- *Each finite cluster and each isolated vertex of $G^{(\omega)}$ contributes zero as a non-degenerate eigenvalue so that $N(0) > 0$ equals their density.*
- *The Lifshits exponent $1/2$ in the theorem does not depend on the spatial dimension d of the underlying space. This comes from the fact that the asymptotics for $E \downarrow 0$ is determined by the longest* unbranched *clusters (i.e. chains) of the percolation graphs.*

Our proof of the theorem involves three preparatory steps, each of which is interesting in its own. The first step belongs to the realm of dynamical systems theory, the second to spectral theory and the third to percolation theory.

(i) *Construct the appropriate ergodic dynamical systems, explicitly characterise ergodic measures and prove an ergodic theorem.* Given an ergodic measure μ on the dynamical system X_{G_0} of the "base" graphs, we will explicitly construct an ergodic measure for corresponding randomly coloured graphs, following ideas of [9]. The main result of this step is an ergodic theorem for dynamical systems associated with randomly coloured graphs. It extends [9], where colourings of aperiodic Delone graphs with strictly ergodic dynamical system have been studied. Our setting covers the full range from periodic structures to random tilings. Moreover, we do not require relative denseness of the vertex sets, thereby including examples such as the visible lattice points [4] in our setup. Apparently, some of the technical problems we had to overcome are closely related to ones in [5], where diffraction properties of certain random point sets, including percolation subsets, have been investigated very recently.

(ii) *Derive ergodic spectral properties of covariant, finite-range operators on randomly coloured aperiodic graphs.* We prove self-averaging of the integrated density of states of such operators. Furthermore we show that the spectrum is almost surely non-random and coincides with the set of growth points of the integrated density of states. For the particular case of uniquely ergodic systems, our theorems guarantee that this holds for *all* base graphs. We provide elementary proofs of these results. In the absence of a colouring, corresponding results have been derived in [7, 8, 12, 13], mainly in the strictly ergodic or in the uniquely ergodic case.

(iii) *Establish exponential decay of the cluster-size distribution in the non-percolating phase for general graphs.* We derive an elementary exponential-decay estimate for the probability to find an open path from the centre to the complement of a large ball. Unfortunately, this estimate holds only for sufficiently small bond probabilities. For these probabilities, the decay of the cluster-size distribution then follows by verifying that the corresponding arguments in [6] apply also in our more general setting.

Lemma 2. *Let G_0 be a graph with a uniformly discrete vertex set and a finite maximal edge length $\ell_{\max} < \infty$. Then,* (3) *holds for every $p \in [0, \frac{1}{d_{\max}-1}[$.*

Exponential decay throughout the non-percolating phase for quasi-transitive graphs has been proved recently [2]. Within our more general setup, an extension to higher bond probabilities up to criticality remains a challenging open question, see also the discussion in [9].

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Spectral consequences of geometric disorder

Daniel Lenz
(joint work with Steffen Klassert, Peter Stollmann)

Here, we report on joint work with Peter Stollmann [5, 6] and with Steffen Klassert and Peter Stollmann [3] (see [2] as well). This work concerns the spectral theory of geometric disorder. This means we consider the following situation: Fix a measurable, bounded, compactly supported function

$$v : \mathbb{R}^N \longrightarrow \mathbb{R}.$$

Then, any suitable discrete $\Lambda \subset \mathbb{R}^N$ gives rise to a selfadjoint operator

$$H_\Lambda = -\Delta + \sum_{x \in \Lambda} v(\cdot - x)$$

on $L^2(\mathbb{R}^N)$. Such models arise in the quantum mechanical treatment of disordered solids. The set Λ can be thought to model the positions of the atoms of the solid in question.

The spectral properties of H_Λ depend on the geometric features of Λ. The basic idea is that the more disordered Λ is the more singular the spectrum of H_Λ becomes. In fact, crystallographic Λ have been known for a long time to lead to purely absolutely continuous band spectrum. Sets with high disorder such as random displacement models and Poisson models lead to point spectrum as investigated for $N = 1$ in [1, 10] and for arbitrary N in [7, 8].

We are concerned with singular continuous spectrum. For $r, R > 0$ let $D_{r,R}$ be the set of all subsets Λ of $\mathbb{R}^N$ satisfying

- the minimal distance between different points in Λ is $2r$,
- the maximal distance of a point of $\mathbb{R}^N$ to Λ is R.

The set $D_{r,R}$ is a compact metric space in a natural way. Somewhat loosely, the main result of [5] can be phrased as follows.

Result [5]: Let R be sufficiently larger than r. Then, for generic $\Lambda \in D_{r,R}$, the spectrum of H_Λ contains a nonempty interval with purely singular continuous spectrum.

To prove this result we provide a slight strengthening (and an alternative proof) of a result of Simon known as "Wonderland theorem" [9]. The Wonderland theorem was used in [9] to obtain generic singular continuous spectrum within the class of bounded continuous potentials decaying at infinity.

In one dimension, we also have a non-generic result on absence of absolutely continuous spectrum for certain classes of geometric disorder [3]. More precisely, we assume that we are given a closed, translation invariant, minimal family Ω in $D_{r,R}$ of finite local complexity. This latter condition means that there are only finitely many local configurations of a fixed size. Then, in particular, the following holds.

Result [3]: If Ω is not periodic, the operator H_Λ has empty absolutely continuous spectrum for every $\Lambda \in \Omega$.

The proof uses Kotani theory and a result of Last/Simon [4] on constancy of the absolutely continuous spectrum for minimal systems.

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Local eigenvalue statistics for orthogonally invariant matrix models

Mariya Shcherbina

We consider ensembles of real symmetric ($\beta = 1$) or hermitian ($\beta = 2$) matrices $M = \{M_{jk}\}_{j,k=1}^n$ with the probability law:

$$P_{n,\beta}(d_\beta M) = Z_{n,\beta}^{-1} \exp\left\{-\frac{\beta n}{2} \mathrm{Tr}\ V(M)\right\} d_\beta M,$$

where $V \in C(\mathbb{R})$ (the potential of the model) satisfy the condition

$$V(\lambda) \geq 2(1+\epsilon)\log(1+|\lambda|), \quad \epsilon > 0,$$

and $Z_{n,\beta}$ is a normalization constant. An important feature of these ensembles is that their joint eigenvalue distribution can be written directly as

$$p_{n,\beta}(\lambda_1, \ldots, \lambda_n) = \prod_{j=1}^n e^{-n\beta V(\lambda_j)/2} \prod_{1\leq j<k\leq n} |\lambda_j - \lambda_k|^\beta,$$

where $Q_{n,\beta}$ is a new normalization constant. This allows to study their correlation functions

$$p_{l,\beta}^{(n)}(\lambda_1, ..., \lambda_l) = \int_{\mathbb{R}^{n-l}} p_{n,\beta}(\lambda_1, ...\lambda_l, \lambda_{l+1}, ..., \lambda_n) d\lambda_{l+1} ... d\lambda_n$$

without using of moments or of the Stieltjes transform technics.

It is known (see [1, 2]) that the integrated density of states (IDS) of the matrix models with any β

$$N_{n,\beta}(\Delta) = \int_\Delta p_{1,\beta}^{(n)}(\lambda) d\lambda \qquad (1)$$

converges, as $n \to \infty$, to some limiting measure $N(d\lambda)$, and $N(d\lambda)$ is a unique minimizer of some functional, defined on the set of non-negative unit measures $\mathcal{M}_1(\mathbb{R})$. Moreover, if V' is a Hölder function, then $N(d\lambda)$ has a density ρ called usually Density of States (DOS).

To study the local eigenvalue statistics means to find $\lim_{n\to\infty}(n^{1-\gamma})^l p_l^{(n)}(\lambda_0 + s_1/n^\gamma, \ldots, \lambda_0 + s_l/n^\gamma)$, where γ is chosen from the condition $\int_{|\lambda-\lambda_0|\leq n^{-\gamma}} \rho(\lambda) d\lambda \sim 1$. Differently from IDS these limits are expected to be universal, e.i. independent of V and depending only on β and of the type of λ_0 which defines γ. It is easy to see that $\gamma = 1$, if $\rho(\lambda_0) \neq 0$ (bulk local eigenvalue statistics), $\gamma = 2/3$, if $\rho(\lambda) \sim C\sqrt{|\lambda - \lambda_0|}$ (edge local eigenvalue statistics), $\gamma = 1/3$, if $\rho(\lambda) \sim C(\lambda - \lambda_0)^2$ (extreme point local eigenvalue statistics).

For Hermitian matrix models $\beta = 2$ all these types of local eigenvalue statistics are well studied now (see [3], [4], [5], [6]), mainly because of the links with polynomials $\{P_k^{(n)}\}_{k=1}^n$ which are orthogonal on the real line with the weight e^{-nV}:

$$\int P_k^{(n)}(\lambda)P_m^{(n)}(\lambda)e^{-nV(\lambda)}d\lambda = \delta_{k,m} \tag{2}$$

For $\beta = 2$ all correlation functions can be expressed in terms of the reproducing kernel of the system (2)

$$K_n(\lambda,\mu) = \sum_{l=0}^{n-1} \psi_l^{(n)}(\lambda)\psi_l^{(n)}(\mu), \quad \psi_l^{(n)}(\lambda) = P_k^{(n)}(\lambda)e^{-nV(\lambda)/2}. \tag{3}$$

For real symmetric matrix models to study the local eigenvalue statistics we need to study a 2×2 matrix kernel whose entries (see [7]) are defined in terms of the scalar kernel:

$$S_n(\lambda,\mu) = -\sum_{i,j=0}^{n-1} \psi_i^{(n)}(\lambda)(\mathcal{M}^{(0,n)})^{-1}_{i,j}(n\epsilon\psi_j^{(n)})(\mu), \tag{4}$$

with

$$\epsilon\psi_j^{(n)} = \frac{1}{2}\int \mathrm{sign}(\lambda-\mu)\psi_j^{(n)}(\mu)d\mu, \quad M_{j,l} = n(\psi_j^{(n)}, \epsilon\psi_l^{(n)}),$$

and $\mathcal{M}^{(0,n)} = \{M_{j,l}\}_{j,l=0}^n$. One of the main problems here is to prove that $||(\mathcal{M}^{(0,n)})^{-1}||$ is bounded uniformly in n. If it is done, then the kernel (4) can be represented in the form

$$S_n(\lambda,\mu) = K_n(\lambda,\mu) + r_n(\lambda,\mu), \tag{5}$$

where K_n is defined by (3) and

$$r_n(\lambda,\mu) = n\sum_{j,k>0} F_{j,k}\psi_{n-j}^{(n)}(\lambda)\epsilon\psi_{n-k}^{(n)}(\mu), \quad |F_{j,k}| \le e^{-d(|j|+|k|)}.$$

It can be proved that $n^{-\gamma}r_n(\lambda_0 + s/n^\gamma, \lambda_0 + t/n^\gamma)$ tends to zero in the bulk and produces finite rank operators in the case of the edge or extreme points. Hence, since the local behavior of K_n is well studied, it is clear that (5) allows to study the local eigenvalue statistics for $\beta = 1$.

The uniform bounds of $||(\mathcal{M}^{(0,n)})^{-1}||$ were found only in a few cases: for $V(\lambda) = \lambda^2/2$ in [7], for $V(\lambda) = \frac{1}{4}\lambda^4 - \frac{a}{2}\lambda^2$ in [8], for $V(\lambda) = \lambda^{2m} + o(1)$ for the standard matrix models in [9, 10], and for the Laguerre type ensemble with $V(\lambda) = \lambda^{2m} + o(1)$ in [11]. In all these cases the representation (5) allows to study bulk and edges type of local eigenvalue statistics and to prove that they are universal.

In the talk we present the result of [12], valid for any even real analytic V with one interval support of the limiting spectrum σ ($\sigma = [-2,2]$) with generic behavior

Theorem 1. *If V is an even real analytic function in $[-2-d, 2+d]$, the support of the limiting IDS (see 1) $\sigma = [-2,2]$, and DOS ρ has the form*

$$\rho(\lambda) = \frac{1}{2\pi}P(\lambda)\sqrt{4-\lambda^2}, \quad P(\lambda) > 0, \tag{6}$$

then for even n $||(\mathcal{M}^{(0,n)})^{-1}||$ is bounded uniformly in n.

A function $P(\lambda)$ plays a key role in the proof of Theorem 1. In particular, the formula for $(\mathcal{M}^{(0,n)})^{-1}_{j,k}$ with $j,k \geq n-\log^2 n$ can be represented in terms of P:

$$(\mathcal{M}^{(0,n)})^{-1}_{j,k} = \frac{1}{2}((\mathcal{R}^{(-\infty,n)})^{-1}\mathcal{D}^{(-\infty,n)})_{j,k} - \frac{1}{2}a_j b_k + O(n^{-\alpha}),$$

where the matrices $\mathcal{R}^{(-\infty,n)} = \{R_{j-k}\}^n_{j,k=-\infty}$ and $\mathcal{D}^{(-\infty,n)}\{D_{j,k}\}^n_{j,k=-\infty}$ have the entries:

$$R_{j-k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P^{-1}(2\cos x)dx, \quad D_{j,k} = \delta_{j,k+1} - \delta_{j,k-1}.$$

and

$$a_j = ((\mathcal{R}^{(-\infty,n)})^{-1}e_{n-1})_j, \quad b_k = ((\mathcal{R}^{(-\infty,n)})^{-1}r^*)_k, \quad r^*_{n-i} = R_i.$$

For others j,k the entries of $\mathcal{M}^{(0,n)})^{-1}_{j,k}$ coincides with that of the matrix of $\frac{d}{d\lambda}$.

This representation yields (5) and allows to prove the theorem:

Theorem 2. *Under conditions of Theorem 1 for any $\lambda_0 \in [2-\epsilon, 2+\epsilon]$*

$$(\rho(\lambda_0))^{-l} p_{l,n}(\lambda_0 + x_1/\rho(\lambda_0)n, \dots \lambda_0 + x_l/\rho(\lambda_0)n$$

converges weakly to the same limit as for GOE given in terms of the matrix kernel

$$\widehat{Q}_s(x,y) = \begin{pmatrix} S(x-y) & S'(x-y) \\ \int_{x-y}^{\infty} S(t)dt - \epsilon(x-y) & S(y-x) \end{pmatrix}, \quad S(x) = \frac{\sin \pi x}{\pi x}.$$

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Integral Formulas for the Asymmetric Simple Exclusion Process

Craig A. Tracy
(joint work with Harold Widom)

The *asymmetric simple exclusion process* (ASEP), introduced by Frank Spitzer [7][1] nearly forty years ago, has become the *default stochastic model for transport phenomena* [12]. Many have called it the *Ising model of nonequilibrium statistical physics.*

An *exclusion process* is a stochastic model for *interacting particles* on a lattice S, say $S = \mathbb{Z}^d$.

(I) A *state* η is a map $\eta : S \to \{0,1\}$ such that

$$\eta(x) = \begin{cases} 1 & \text{if site } x \in S \text{ is occupied by a particle,} \\ 0 & \text{if site } x \in S \text{ is vacant.} \end{cases}$$

Thus the ensemble of states is $\Omega = \{0,1\}^S$.

(II) Let $\{p(x,y)\}$ denote a set of transition probabilities on $S \times S$. Introduce *dynamics*: $t \to \eta_t \in \Omega$:
  (a) Each particle $x \in S$ waits exponential time with parameter 1, independently of all other particles;
  (b) at the end of that time, it chooses a $y \in S$ with probability $p(x,y)$; and
  (c) if y is vacant, it goes to y; while if y is occupied, it stays at x and the clock is reset to zero.

Remarks:

(I) Without the "exclusion" of part 2(c), we would have particles moving on S according to *independent*, continuous time Markov chains on S that have unit exponential holding times. Because time is continuous, we need not worry about two or more clocks ringing at the same time.

(II) The classic references are the books by Liggett [4, 5] where a rigorous construction of the exclusion process is given for a system with an infinite number of particles assuming only mild conditions on $\{p(x,y)\}$.

The one-dimensional ASEP is the choice $S = \mathbb{Z}$ and

$$p(x,y) = \begin{cases} p & \text{if } y = x+1, \\ q = 1-p & \text{if } y = x-1, \\ 0 & \text{otherwise} \end{cases}$$

with $p \neq q$.

The case $p = 1$ is called the T(totally)ASEP. Johansson [2] was the first to establish a connection between TASEP and random matrices. Start with an initial *step configuration* of particles located at $\mathbb{Z}^-$, then the probability that at time t

[1]For some photographs of Frank Spitzer see the MFO collection at
http://owpdb.mfo.de/person_detail?id=3964

the particle initially at $-m$ has moved at least $n\,(\geq m)$ times equals

$$C_{m,n}\int_{[0,t]^m}\prod_{0\leq i<j<m}(\tau_i-\tau_j)^2\prod_i(\tau_i^{n-m}\,e^{-\tau_i})d^m\tau \tag{1}$$

which in turn equals distribution of the *largest eigenvalue* in the unitary Laguerre ensemble. From (1), Johansson [2] derived the following limit theorem: If $Y(k,t)$ is the number of particles to the right of k at time t, then

$$\lim_{t\to\infty}\mathbb{P}\left(\frac{Y([ut],t)-c_1t}{c_2t^{1/3}}\leq\xi\right)=1-F_2(-\xi),\ \ 0\leq u<1, \tag{2}$$

where[2]

$$c_1=\frac{1}{4}(1-u)^2,\ \ c_2=(1-u)^{2/3}(1+u)^{-1/3},$$

and F_2 is the distribution function of the largest eigenvalue in the Gaussian Unitary Ensemble (GUE) in the edge scaling limit [9] . A recent review of TASEP and its connection with random matrices can be found in Spohn [8].

What about the case $p<1$? The process is not obviously a *determinantal process* so none of the techniques from random matrix theory seem to apply. However, using some ideas from integrable systems going under the name of *Bethe Ansatz* [1, 3, 11, 6], we derive exact formulas for the analogous distribution functions. But we hasten to add, there is no Ansatz in our work!

We state our result in the special case of a step initial condition:

$$\eta_0(x)=\begin{cases}1 & \text{if}\quad x=1,2,\ldots\\ 0 & \text{if}\quad x=0,-1,-2,\ldots.\end{cases} \tag{3}$$

The result below is unpublished and it follows from the Corollary on page 838 of [10].

Let $\tau:=p/q<1$ (so there is drift to the left),

$$\varepsilon(\xi):=\frac{p}{\xi}+q\,\xi-1,$$

and $(\lambda;\tau)_m:=(1-\lambda)(1-\lambda\tau)\cdots(1-\lambda\tau^{m-1})$, $m\in\mathbb{Z}^+$. Assume the initial state is (3). If $x_m(t)$ denotes the position of the mth particle from the left at time t (so $x_m(0)=m\in\mathbb{Z}^+$), then

$$\boxed{\mathbb{P}\left(x_m(t)\leq x\right)=\frac{1}{2\pi i}\int_{\mathcal{C}_R}\frac{\det(I-\lambda qK)}{(\lambda;\tau)_m}\,\frac{d\lambda}{\lambda}.} \tag{4}$$

Here K is a trace-class operator on $L^2(\mathcal{C}_R)$ defined by

$$(Kf)(\xi)=\frac{1}{2\pi i}\int_{\mathcal{C}_R}\frac{(\xi')^x e^{\varepsilon(\xi')t}}{p+q\xi\xi'-\xi}\,f(\xi')\,d\xi'$$

where $\mathcal{C}_R$ is a circle centered at the origin of radius $R\gg 1$ (all poles coming from the denominator of the kernel are inside the circle).

[2]The constant c_1 was computed earlier by Rost, see [4, 5].

Remarks:

(I) The contour integral (4) can be simply evaluated by the method of residues. For example, $\mathbb{P}(x_1(t) \le x) = 1 - \det(I - qK)$.

(II) The asymptotic regime of conjectured *universal current fluctuations* [8], e.g. the analogue of (2), remains presently out of reach, but there is reason for optimism now that we have expressed $\mathbb{P}(x_m(t) \le x)$ in terms of Fredholm determinants. For fixed m we have (non-rigorously at this writing) limit theorems for

$$\mathbb{P}(x_m(t) \le (p-q)t + yt^{1/2}) \text{ as } t \to \infty.$$

(III) In our proof [10] a number of miraculous identities and cancellations occur. It would be enlightening to understand these in a more structural way.

(IV) Our general result [10] applies to initial conditions with particles are located at

$$y_1 < y_2 < \cdots < y_n < \cdots \to \infty$$

This restriction does not permit us to study the case of *stationary* ASEP, e.g. an initial condition that is a product of Bernoulli measures.

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Random string: The transition from homogenization to localization

Stanislav A. Molchanov

The talk presented the initial stage of the universality theory for random systems with local interactions (random Schrödinger operators, random membrames or strings etc.). In the bulk of the spectrum of any such operator one can expect Poisson statistics of the eigenvalues under mild conditions. At the edge of the spectrum the situation depends on the type of the operator. For the Schrödinger case the density of states typically has the form of a "Lifshitz tail" due to large deviations and the bottom of the spectrum is defined by the particular features of the potential. In the case of the random string, the phenomenon of localization leads to the following picture: the eigenvalues on the bottom of the spectrum are equal to the eigenvalues of the homogenized operator plus small Gaussian correction. The higher levels are given by the zeros of the appropriate (and universal) random stationary entire function.

On the Spectral Properties of Large Wigner Random Matrices with Non-symmetrically Distributed Entries

Alexander B. Soshnikov
(joint work with Sandrine Péché)

We consider the n-dimensional random real symmetric matrix A with matrix entries being independent up from the diagonal. In addition, we assume that the matrix entries are centralized, have common variance $\sigma^2 = 1$ (except on the diagonal where the variance is bounded by some constant), and that the matrix entries are bounded by some constant M that does not depend on n.
It was proven by Soshnikov in 1999 that when the matrix entries are symmetrically distributed sub-Gaussian random variables (i.e. $\mathbb{E}[a_{ij}^{2k}] \leq (const * k)^k$, uniformly in $1 \leq i \leq j \leq n$ and $k \geq 1$, and the odd moments vanish), the distribution of the largest eigenvalues exhibits universal behavior in the limit $n \to \infty$. In other words, after proper rescaling, the distribution of the largest eigenvalues satisfies the Tracy-Widom law in the limit. The non-symmetric case (for example, when the third moment does not vanish) appears to be more difficult to analyze. Until recently, the best bound on the spectral norm of A was due to Van Vu, who extended the earlier result of Füredi and Komlós and proved in 2005 that with probability going to one $||A|| \leq 2\sqrt{n} + Kn^{-1/4}\log(n)$, where K is some constant. Our main result strengthens Van Vu's bound. Namely, we prove that for any arbitrary small $\varepsilon > 0$ one has $||A|| - 2\sqrt{n} = O(n^{-6/11+\varepsilon})$.

Eigenvalues of spiked random matrices

Jinho Baik

Let H be a random $N \times N$ Hermitian matrix with independent Gaussian entries (Gaussian unitary ensemble) and let A be a deterministic matrix with rank r. Consider the matrix $H+A$. The interest is to study the statistics of the eigenvalues of this spiked random matrix $H+A$ when $N \to \infty$ while r remains fixed. A closed related random matrix is $X\Sigma X^*$ where X is a rectangular matrix with iid Gaussian entries (with no symmetry condition) and Σ is a fixed matrix which is a rank r perturbation of the identity matrix. Again the size of X grows to infinity while r remains fixed.

The matrix $H + A$ was first considered by Furëdi and Komlós [7] in 1981 in their study of random symmetric matrices whose entries are non-centered random variables. Hence it is the case when A is of rank 1. The matrix $X\Sigma X^*$ has been of interest in statistics [8], and the recent results on the largest eigenvalue distribution has found applications in finance, economics, genetics and wireless communications.

One can expect that when the eigenvalues of A or $\Sigma - I$ are 'small', they would not affect the eigenvalues of H much and the limiting eigenvalues distribution does not change much from the non-perturbed case. However when some of the eigenvalues of A or $\Sigma - I$ are 'large', then it may be possible that a few large eigenvalues of $H + A$ or $X\Sigma X^*$ would get excited and be separated from the rest of the eigenvalues. Such phase transition phenomena was first studied in [1] for $X\Sigma X^*$ with complex entries, in which the limiting distributions of the largest eigenvalues were obtained for all choices of the the perturbation.

Since for non-perturbed case there are various universality results for the eigenvalues, it is natural to ask the similar question to the spiked random matrices. There has been a few results [2, 9, 5, 6, 10]. In this talk, we will discuss a step toward the universality result for the following class of matrices. When H is from Gaussian unitary ensemble, $M = H + A$ has the density function

$$p(M) = const \cdot e^{-\frac{1}{2}Tr(M^2-MA)}.$$

By replacing M^2 by general function $V(M)$, we can consider the density function

$$p(M) = const \cdot e^{-Tr(V(M)-MA)}$$

on the set of Hermitian matrices. Such matrix ensemble is called random matrices with external source. Based on the work of Bleher and Kuijlaars [3], and Deams and Kuijlaars [4], when A of a finite rank, we find a closed form formula for the reproducing kernel in terms of the orthogonal polynomials for the weight $e^{-V(x)}$. Since in this formula, the size of the matrix M appears in an explicit way as was studied in the earlier universality results on unitary invariant ensembles, we expect that the formula would be the key step toward the asymptotic study of the spiked random Hermitian matrices with the above density function.

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On the Structure of Hofstadter's Butterfly

Yoram Last

(joint work with Mira Shamis)

The Almost Mathieu Operator is the discrete Schrödinger operator on $\ell^2(\mathbb{Z})$, given by

$$(H_{\alpha,\lambda,\theta}\psi)(n) = \psi(n+1) + \psi(n-1) + \lambda\cos(2\pi\alpha n + \theta)\psi(n)\,,$$

where $\alpha, \lambda, \theta \in \mathbb{R}$. We denote

$$S(\alpha,\lambda) = \bigcup_{\theta} \mathrm{Spec}\,(H_{\alpha,\lambda,\theta}).$$

Since $\mathrm{Spec}\,(H_{\alpha,\lambda,\theta})$ is independent of θ for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, it coincides with $S(\alpha,\lambda)$ for such α.

Some central characteristics for the spectrum of $H_{\alpha,\lambda,\theta}$ are given by the following theorem, which has been established due to work by many authors over the last 25 years:

Theorem 1 (Many People, 1982–2008+)**.** *For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\lambda \neq 0$, $S(\alpha,\lambda)$ is a Cantor set of Lebesgue measure $|4 - 2|\lambda||$.*

Our interest here is in the critical point $|\lambda| = 2$, where the Lebesgue measure of $S(\alpha,\lambda)$ vanishes for irrational α, and so the interest in it's fractal dimensions naturally arises. In the the late 1980's, several papers by physicists suggested that one should expect $\dim_{\mathrm{B}}(S(\alpha,2)) = \dim_{\mathrm{H}}(S(\alpha,2)) = \frac{1}{2}$ for a.e. α. In 1994, the paper of Wilkinson-Austin [3] provided numerical evidence that $\dim_{\mathrm{B}}(S(\alpha,2)) = 0.498\ldots$ for α the golden mean, and thus made the following conjecture

Conjecture 1. $\dim_{\mathrm{H}}(S(\alpha,2)) < \frac{1}{2}$ *for every* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

The paper [3] further gave numerical and analytical evidence that $\dim_{\mathrm{B}}(S(\alpha_n,2)) \to 0$ as $n \to \infty$ for irrationals α_n of the form

$$\alpha_n = [n,n,n,\dots] = \cfrac{1}{n+\cfrac{1}{n+\cfrac{1}{n+\cdots}}}$$

It thus became clear that different irrationals should lead to different fractal properties of $S(\alpha,2)$ and that characterizing the fractal dimensions of $S(\alpha,2)$ should be a reach subject. Around that time, J. Bellissard made the conjecture that there should nevertheless be some $\beta \in (0,\frac{1}{2}]$ such that $\dim_{\mathrm{H}}(S(\alpha,2)) = \beta$ for a.e. α.

Rigorous results on this subject have been scarce and until very recently, the only one was

Theorem 2 ([2])**.** *There exists a dense G_δ set of α's (explicitly, those with $q_n^4|\alpha - \frac{p_n}{q_n}| < C$ for infinitely many rationals) for which* $\dim_{\mathrm{H}}(S(\alpha,2)) \leq \frac{1}{2}$.

The main purpose of the talk is to present the following new result

Theorem 3 (Last-Shamis, 2008+)**.** *There exists a dense G_δ set of α's for which* $\dim_{\mathrm{H}}(S(\alpha,2)) = 0$.

We note that Theorem 2 is obtained as a cosequence of the following two theorems

Theorem 4 ([2])**.** *For p,q relatively prime,*

$$\frac{2(\sqrt{5}+1)}{q} < |S(p/q,2)| < \frac{8e}{q}\,.$$

Theorem 5 ([1])**.** *For a fixed λ and $|\alpha-\alpha'| < C(\lambda)$, each $E \in S(\alpha,\lambda)$ has $E' \in S(\alpha',\lambda)$ with $|E-E'| < 6|\lambda(\alpha-\alpha')|^{1/2}$.*

along with the following elementary lemma

Lemma 6. *If a set S can be covered by q intervals of total length $\frac{1}{q}$, for infinitely many q's, then* $\dim_{\mathrm{H}}(S) \leq \frac{1}{2}$.

The crucial addition leading to Theorem 3 is

Theorem 7 (Last-Shamis, 2008+)**.** *Fix p/q and let*

$$S_-(p/q,2) = \bigcap_\theta \mathrm{Spec}\,(H_{p/q,\lambda,\theta}) = \{E_1,\dots,E_q\}$$

$$J_\delta = \{E \,:\, \mathrm{dist}\,(E, S_-(p/q,2)) > \delta\}$$

Then for a fixed $\delta > 0$ and $\left|\frac{p}{q} - \frac{\tilde p}{\tilde q}\right|$ sufficiently small,

$$|S(\tilde p/\tilde q,2) \cap J_\delta| < \frac{C_1}{\delta}\, e^{-C_2\delta\tilde q}$$

where C_1 and C_2 depend only on p/q.

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Wegner estimates for non-monotoneously correlated alloy type models

Ivan Veselić

We study spectral properties of Schrödinger operators which are given as the sum $H = -\Delta + V$ of the negative Laplacian Δ and a multiplication operator V. We do not distinguish in the notation between the multiplication operator and the underlying function V. The operators can be considered in d-dimensional Euclidean space $\mathbb{R}^d$ or on the lattice $\mathbb{Z}^d$. To be able to treat both cases simultaneously let us use the symbol X^d for either $\mathbb{R}^d$ or $\mathbb{Z}^d$. On the continuum the Laplace operator is the sum of second derivatives $\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ and V is a bounded function $\mathbb{R}^d \to \mathbb{R}$. Thus H is selfadjoint on the usual Sobolev space $W^{2,2}(\mathbb{R}^d)$. In the discrete case the Laplacian is given by the rule $\Delta\phi(k) = \sum_{i=1}^d \phi(k+e_i) + \phi(k-e_i)$, where ϕ is a sequence in $\ell^2(\mathbb{Z}^d)$ and $(e_1, \dots, e_d)$ is an orthonormal basis which defines the lattice $\mathbb{Z}^d$ as a subset of $\mathbb{R}^d$. The potential is given by a bounded function $V\colon \mathbb{Z}^d \to \mathbb{R}$, and thus H is a bounded self-adjoint operator.

The operators we are considering are random. More precisely, the potential is a stochastic field $V_\omega(x) := \sum_{k\in\mathbb{Z}^d} \omega_k u(x-k), x \in X^d$, of alloy or Anderson type. Here $u\colon X^d \to \mathbb{R}$ is a bounded, compactly supported function, which we call single site potential. The coupling constants $\omega_k, k \in \mathbb{Z}^d$ form an independent, identically distributed sequence of random variables. We assume that the random variables are non-trivial and bounded and denote the associated distribution measure on $\mathbb{R}$ by μ. In the discrete case the random operator $H_\omega = -\Delta + V_\omega$ is called Anderson model, and in the continuum case H_ω is called alloy type model.

Note that the spectrum $\sigma(H_\omega)$ of H_ω depends on the randomness. However, there are certain spectral features which are shared by almost all members of the family $(H_\omega)_\omega$. In particular, there are fixed subsets $\Sigma, \Sigma_{pp}, \Sigma_{sc}, \Sigma_{ac}$ of the real line such that $\sigma(H_\omega) = \Sigma, \sigma_{pp}(H_\omega) = \Sigma_{pp}, \sigma_{sc}(H_\omega) = \Sigma_{sc}, \sigma_{ac}(H_\omega) = \Sigma_{ac}$ almost surely. (Let us emphasize that $\sigma_{pp}(H)$ denotes here the closure of the set of eigenvalues of H.) Moreover, there is a well defined spectral distribution function $N\colon \mathbb{R} \to \mathbb{R}$ of the family $(H_\omega)_\omega$ which is closely related to eigenvalue counting functions on finite cubes. To explain this in more detail, we need some more notation (which will be also useful in the remainder of the text). In the following let us denote by χ the characteristic function of the set $[0,1[^d \cap X^d$. Thus in the continuum case this set is a unit cube, and in the discrete case it is a single point. translate of χ. The cube $[-L-\frac{1}{2}, L+\frac{1}{2}]^d \cap X^d$ will be abbreviated by Λ_L, the restriction of H_ω to $\Lambda = \Lambda_L$ with selfadjoint boundary conditions (e.g. Dirichlet, Neumann or periodic ones) by H_ω^Λ, the spectral projection associated to H_ω (resp. H_ω^Λ) and an interval I by $P_\omega(I)$ (resp. by $P_\omega^\Lambda(I)$), $\mathbf{1}_\Lambda$, and the number of eigenvalues of $H_\omega^{\Lambda_L}$ in $]-\infty, E]$ by $N_\omega^L(E) := \mathrm{Tr}[P_\omega^{\Lambda_L}(\,]-\infty,E])]$. With this notation at our disposal we can define the integrated density of states, which is the spectral distribution of the family $(H_\omega)_\omega$, by $N(E) := \mathbf{E}\{\mathrm{Tr}[\chi P_\omega(]-\infty,E])]\}$. It has the following self-averaging property: for all E where N is continuous (that's a set with countable complement) the relation $\lim_{L\to\infty}(2L+1)^{-d} N_\omega^L(E) = N(E)$ holds almost surely. This implies that if E_1 and E_2 are two continuity points of N, we have

$$\lim_{L\to\infty} (2L+1)^{-d}\, \mathbf{E}\{N_\omega^L(E_2) - N_\omega^L(E_1)\} = N(E_2) - N(E_1). \tag{1}$$

Thus if one is able to show that there is a function $C\colon \mathbb{R} \to \mathbb{R}$ and an exponent $\beta \in]0,1]$, such that for all $E_1, E_2 \le E$ and for all $L \in \mathbb{N}$ the so-called Wegner bound (named after the paper [9])

$$\mathbf{E}\{N_\omega^L(E_2) - N_\omega^L(E_1)\} \le C(E)\,(2L+1)^d\,|E_2 - E_1|^\beta \tag{2}$$

holds, it follows that the integrated density of states is (locally uniformly) Hölder-continuous with exponent β. Note that this shows a posteriori, that there are no points of discontinuity of N and thus the convergence in (1) hold actually for all $E_1, E_2 \in \mathbb{R}$. This is only one of the reasons why one is interested in bounds on the averaged quantity $\mathbf{E}\{\mathrm{Tr}[P_\omega^{\Lambda_L}(]E_1,E_2])]\}$. It plays also a crucial role in arguments leading to the proof of localisation, i.e. the phenomenon that there is a subset I_{loc} of the real axis such that $I_{loc} \cap \Sigma_{pp} = I_{loc}$ and $I_{loc} \cap (\Sigma_{ac} \cup \Sigma_{sc}) = \emptyset$. In fact, usually localisation goes along with quite explicit bounds on the decay of eigenfunctions and on the non-spreading of electron wavepackets (see for instance the monograph [7] or the characterisation established in [2]).

Now we specialise to certain types of single site potentials such that the resulting Anderson/alloy type model describes a random potential where negative correlations between potential values at different points in space are allowed: Let $\kappa \in \mathbb{R}$ be positive, $v\colon X^d \to \mathbb{R}$ a function satisfying $v \ge \kappa\chi$, and $\alpha\colon \mathbb{Z}^d$ a function with compact support Λ_D such that its Fourier transform $\hat\alpha\colon [0,2\pi[^d \to \mathbb{C}$, $\hat\alpha(\theta) := \sum_{k\in\mathbb{Z}^d} \alpha_k \mathrm{e}^{-\mathrm{i}k\cdot\theta}$ does not vanish on $[0,2\pi[^d$. Then we call

$$u(x) := \sum_{k\in\mathbb{Z}^d} \alpha_k v(x-k) \tag{3}$$

a single site potential of generalised step function form. Note that the sum contains only finitely many non-vanishing terms. Due to the fact that the coeffcents $\alpha_k, k \in \mathbb{Z}^d$, may change sign, the random potential V_ω can have negative correlations between values at different sites. Now we are in the position to formulate our main result:

Theorem 1. *Let $H_\omega = -\Delta + V_\omega$ be an Anderson model on $\ell^2(\mathbb{Z}^d)$ or an alloy type model on $L^2(\mathbb{R}^d)$ with a single site potential u of generalised step function form. Assume that the distribution measure μ of the coupling constants has a density f*

of compact support and bounded variation. Then there is a continuous function $C\colon \mathbb{R} \to \mathbb{R}$ *such that for all* $E_1, E_2 \le E$ *and for all* $L \in \mathbb{N}$ *the Wegner bound*

$$\mathbf{E}\{N_\omega^L(E_2) - N_\omega^L(E_1)\} \le C(E)\,(2L+1)^d\,|E_2 - E_1| \tag{4}$$

holds.

As discussed above, estimate (4) implies that the integrated density of states $N\colon \mathbb{R} \to \mathbb{R}$ is locally uniformly Lipschitz-continuous. This in turn implies that the derivative $n(E) := \frac{dN(E)}{dE}$ exists almost everywhere on $\mathbb{R}$ and is locally uniformly bounded by $n(E) \le C(E)$. The function n is called density of states.

The Theorem recovers the main result of [8] where the same statement was proven under two additional conditions: It was assumed that there is an index $j \in \mathbb{Z}^d$ such that $|\alpha_j| > \sum_{k\in\mathbb{Z}^d, k\neq j} |\alpha_k|$ and that the density f belongs to the Sobolev space $W_c^{1,1}(\mathbb{R})$. Exactly the same statement as in the Theorem above, but only for dimensions $d = 1$ and $d = 2$ was proven in [5] in a joint paper with V. Kostrykin. There is another method to prove Wegner estimates for single site potentials that are allowed to change sign which is based on certain vector fields in the parameter space underlying the alloy type model. It was introduced in [4] by F. Klopp and improved by P. Hislop and F. Klopp in [3]. Its advantage is that it applies to arbitrary continuous, compactly supported single site potentials (which are not identically equal to zero). The regularity requitement on μ is roughly the same as in the above Theorem. This method allows one to derive Wegner estimates for any energy intervall $[E_1, E_2]$ which stays below a certain critical energy, but it does not apply to arbitrarily large E_1, E_2 (in the case of sign-changing single site potentials). The papers [4, 3, 5] contain various other results, which we do not state here, because they cannot be directly compared with our theorem above.

Let us briefly discuss the relevance of the condition that the Fourier transform $\hat{\alpha}$ does not vanish on $[0, 2\pi[^d$. It ensures that the multi-dimensional Laurent-matrix A with coefficients $\alpha_{j-k}, j, k \in \mathbb{Z}^d$, when considered as an operator from $\ell^p(\mathbb{Z}^d)$ to $\ell^p(\mathbb{Z}^d)$ has a bounded inverse B. However, in the proof of the Wegner estimate above we encounter not the infinite matrix A, but rather finite size matrices A_Λ which need to have bounded inverses B_Λ with norms uniformly bounded in $\Lambda = \Lambda_L, L \in \mathbb{N}$. The relevant norm is the column sum norm, corresponding to the operator norm on $\ell^1(\Lambda)$. If A_Λ is choosen to be a finite section multi-dimensional Toeplitz operator this leads to nontrivial open questions concerning the invertibility of truncated Toeplitz matrices, see for instance [1]. This is the reason why the results of [5] are restricted to dimension one and two, cf. also [6]. However, it turns out that one has a certain freedom in the choice of the finite volume matries A_Λ. This enables one to choose them to be finite multi-dimensional circulant matrices (rather than finite Toeplitz matrices), which have much better invertibility properties and can be used to complete the proof of the above Theorem.

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Asymptotic analysis of Riemann Hilbert Problems, D-Bar Problems, and applications to universality in random matrix theory

Kenneth McLaughlin

In this short talk I described recent work together with Peter Miller, establishing asymptotics for orthogonal polynomials under weak assumptions on the external field. Some background and motivation was provided.

Nonuniform Upper Estimates of the Density of States in Dimension One

Bernd Metzger

The problem under consideration are upper estimates of the density of states of the one-dimensional Anderson model given by the random Hamiltonian

$$[H_\omega u](n) \;=\; [(-\Delta + V_\omega)u](n) \;=\; -u(n+1) - u(n-1) + V_\omega(n)u(n) \tag{1}$$

on $l^2(\mathbb{Z})$. Here Δ is the one-dimensional discrete Laplacian without diagonal entry and $\{V_\omega(n)\}_{n\in\mathbb{Z}}$ are independent identically distributed random variables according to the uniform distribution with density $d\mu(v) = (2\lambda)^{-1}\chi_{[-\lambda,\lambda]}(v)dv$, $\lambda > 0$. In order to describe our results, we recall some fundamental properties of the integrated density of states (IDS) and of the density of states (DOS). For a more comprehensive overview we refer the interested reader to [2], [3], [4], [6] and references therein. With $\Lambda = \{-\ell, .., \ell\}$, $\ell \in \mathbb{N}$ the finite-dimensional matrix H_ω^Λ is the operator H_ω restricted to $l^2(\Lambda)$ with suitable boundary conditions [2]. The integrated density of states (IDS) is defined by

$$N(E) = \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \sharp\{\text{eigenvalues of } H_\omega^\Lambda \le E\}. \tag{2}$$

Under mild assumptions the integrated density of states is a continuous function even in dimensions larger than one. Of special interest in our context is the following result of Wegner [7]. If the single site measure is absolutely continuous with a bounded density, $N(E)$ is absolutely continuous with a bounded derivative, i.e.

$$N(E) = \int_{-\infty}^{E} \rho(E')\, dE'. \tag{3}$$

The support of the density of states measure (DOS) ρ in our setting is

$$\operatorname{supp} \rho(E) \;=\; \operatorname{supp} \mu + \sigma(-\Delta) \;=\; [-\lambda - 2, \lambda + 2], \tag{4}$$

i.e. the support of the DOS is larger than the support of the single site measure. The IDS and the DOS are important in the physical understanding of the thermodynamic behavior of solids. As a consequence not only qualitative properties but also quantitative estimates are of interest. The physical expectations with respect to one energy band are summarized in figure 1.

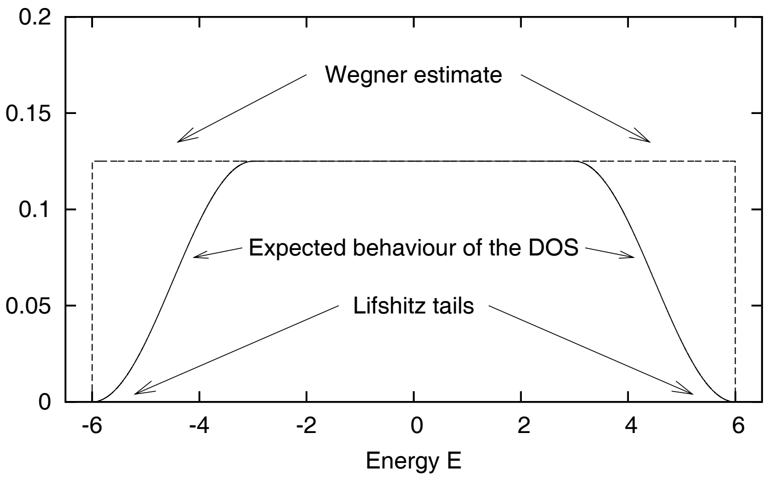

FIGURE 1. A schematic diagram summarizing the known results and the physical expectations with respect to the density of states of one energy band.

Maybe best understood is the asymptotic behavior of the IDS close to the boundary of supp ρ, i.e. in the asymptotic limit $|E| \nearrow \lambda + 2$ in our setting. Here physicists expect Lifshitz tails, e.g. at the bottom of the spectrum they expect a very strong exponential decay of the IDS like

$$N(E) \overset{E \searrow -\lambda - 2}{\sim} C_1 \;\exp(-C_2(E + \lambda + 2)^{-d/2}). \tag{5}$$

At least to our knowledge all known non-asymptotic results controlling the DOS inside supp ρ are based on [7]. This concerns the positivity of the DOS in supp ρ and uniform upper bounds

$$0 \;\leq\; \rho(E) \;\leq\; \sup_{E \in [-\lambda, \lambda]} \mu(E) \;=\; \frac{1}{2\lambda}. \tag{6}$$

For a more detailed discussion of the mathematical extensions of [7] we refer to [3], [6].
To summarize the known results the physically expected behavior of the DOS as in figure 1 is mathematically only partially understood. One step to improve the situation is maybe the following result.

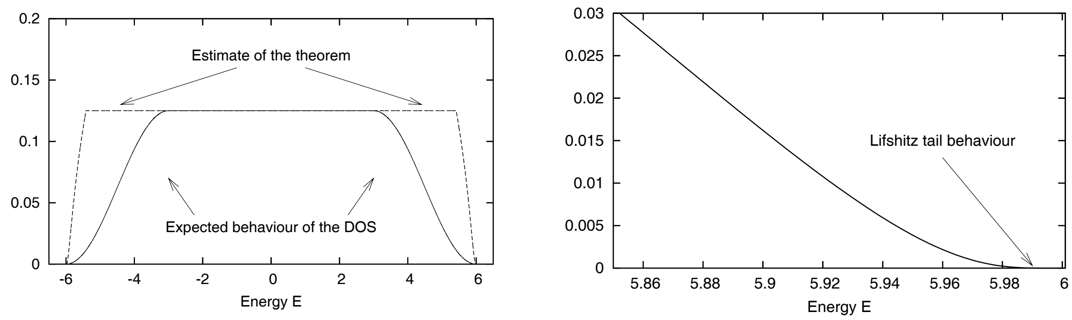

FIGURE 2. The diagrams above compare the physical expectations with respect to the DOS and the estimate of Theorem 1. Especially close to the band edge (but inside the interval $\operatorname{supp}\,\rho = [-\lambda-2, \lambda+2]$) the estimate of Theorem 1 gives an non-asymptotic expression coinciding in the asymptotic limit $|E| \nearrow \lambda + 2$ with the Lifshitz behavior.

Theorem 1. *Assume $H_\omega = -\Delta + V_\omega$ as defined in (1), i.e. $V_\omega(n)$, $n \in \mathbb{Z}$ are i.i.d. random variables with respect to the uniform distribution with $\lambda > 2$. With $E \in \mathbb{R}$, the distance of E to the support of the single site distribution $d(E) = dist(E, [-\lambda, \lambda])$ and*

$$L(E) \;=\; \max\left\{ \ell \in \mathbb{N} : \ell < \frac{\pi}{2\arccos\left(\frac{d(E)}{2}\right)} - 1 \right\},$$

$$L_\lambda \;=\; \begin{cases} \frac{\ln 2}{\ln\lambda - \ln 2} & 2 < \lambda \leq 4 \\ 1 & \lambda > 4, \end{cases}$$

$$d_\lambda \;=\; 2\cos\left(\frac{\pi}{2(L_\lambda + 1)}\right) \;<\; 2,$$

we can estimate the density of states by

$$\rho(E) \leq \begin{cases} \frac{1}{2\lambda} & \text{for } \; d(E) \leq d_\lambda, \\ \lambda^{-1}\left(\frac{2}{\lambda}\right)^{L(E)} & \text{for } \; d_\lambda < d(E) < 2, \\ 0 & \text{for } \; d(E) \geq 2. \end{cases} \tag{7}$$

As shown in figure 2 the estimate of Theorem 1 interpolates between the Wegner estimates in (6) and the Lifshitz behavior in (5). In the interval $(d_\lambda, 2)$, $d_\lambda < 2$ the estimate (7) is strictly smaller than the Wegner estimate. The fundamental parameter in Theorem 1 is the distance $d(E) = \mathrm{dist}(E, [-\lambda, \lambda])$ between the energy value E and the support of the single site measure. This quantity is used to define the characteristic length $L(E)$ describing the decay of the density of states. As on can see in the proof of Theorem 1 $L(E)$ results from inverting the largest eigenvalue of the discrete Laplacian on $\{-\ell, .., \ell\}$. Observing that $\arccos(x)$ is continuous in $[-1, 1]$ and continuously differentiable in $(-1, 1)$ we have

$$L(E) \overset{d(E)\nearrow 2}{\sim} \frac{\pi}{2\arccos(d(E)/2)} \overset{d(E)\nearrow 2}{\sim} \frac{\pi}{2}(\lambda + 2 - |E|)^{-1/2}$$

i.e. in the asymptotic limit $d(E) \nearrow 2$ the estimate of Theorem 1 reproduces the Lifshitz tail behavior.

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The supersymmetry method for random matrices with local gauge symmetry

Martin R. Zirnbauer

In the general setting of quadratic Hamiltonians for disordered fermions, it is known that there exist ten symmetry classes in the sense of Dyson's 1962 classification called the threefold way. It was conjectured some time ago and proved recently [1] that these symmetry classes are in one-to-one correspondence with the ten large families of symmetric spaces. All of them have physical realizations in disordered metals or disordered superconductors or as relativistic fermions in disordered gauge field backgrounds.

For each symmetry class one can consider locally gauge invariant random matrix models (so-called N-orbital models) of a type first introduced by Wegner. The superanalytic methods discussed in this talk are most appropriate for models of that type with a large number N of orbitals. It is pointed out that by taking the large-N limit in a suitable way one can arrange for Wegner's N-orbital model to exhibit the same phenomenology as the Anderson model; in particular, one expects that there occurs an Anderson transition from localized to extended states.

We then explain in some detail the basic steps of the method of integration over commuting and anti-commuting variables, called the *supersymmetry method* for short. Unlike the method of orthogonal polynomials, the supersymmetry method does not rely on the Vandermonde form of the joint probability distribution for the random matrix eigenvalues but is formulated in terms of the characteristic function, or Fourier transform, of the random matrix ensemble.

Further development of the supersymmetry method requires geometric analysis on certain families of Riemannian supermanifolds which are called Riemannian symmetric superspaces [2]. To construct such a space one starts from a globally symmetric Riemannian manifold G/H and if $\mathfrak{g}_0 = \mathrm{Lie}(G) \otimes \mathbb{C}$ is the even part of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = (\mathfrak{h}_0 \oplus \mathfrak{p}_0) \oplus (\mathfrak{h}_1 \oplus \mathfrak{p}_1)$ with Cartan involution, one forms the associated vector bundle $G \times_H \mathfrak{p}_1 \to G/H$. Sections $f \in \Gamma(G/H, \wedge E^*) \equiv \mathcal{A}$ of the bundle of exterior powers of the dual of E are called superfunctions. By definition, a Riemannian symmetric superspace is such an algebra $\mathcal{A}$ of superfunctions equipped with a canonical action of the Lie superalgebra $\mathfrak{g}$. Invariance by the $\mathfrak{g}$-action determines a natural geometry and, in particular, a $\mathfrak{g}$-invariant Berezin integration form.

It has been argued in the physics literature that the computation of transport and spectral statistics for Wegner's N-orbital model can be reduced to computations for a supersymmetric non-linear sigma model, i.e., a functional integral of maps into a Riemannian symmetric superspace. Most of the physics intuition about extended states and the Anderson transition is gained from perturbative renormalization group analysis of these non-linear sigma models. For example, when the curvature of the Riemannian symmetric space is positive one expects all states to be localized in two dimensions. In the opposite case of negative curvature (e.g., for systems with spin-orbit scattering) one expects extended states to occur in two dimensions, provided that the disorder is weak enough.

In the last part of the talk we explain that the step of reduction from the N-orbital model to the non-linear sigma model is facilitated by a recently developed formalism [3] called 'superbosonization', which generalizes to the superworld, i.e., the case of Berezin integrals, the idea of push forward of integrals with symmetries. The new formalism is described in some detail.

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Estimates for spectral moments of random Schrödinger operators

Peter D. Hislop

(joint work with J. Bellissard, J-M Combes, F. Germinet, F. Klopp, P. Müller)

Spectral moments of random Schrödinger operators describe the transport properties of the one-particle system. These moments are defined with respect to the spectral density $\rho_\omega(E) = \lim_{\epsilon \to 0} \Im (H_\omega - E - i\epsilon)^{-1}$ for any family of covariant observables $\{A_1, \ldots, A_N\}$ by

$$K_N(E_1, \ldots, E_N) = \mathbb{E}\{Tr \chi_0 \rho_\omega(E_1) A_1 \cdots \rho_\omega(E_N) A_N \chi_0\},$$

where $\chi_0 = \delta_0$ for lattice models on $\ell^2(\mathbb{Z}^d)$ and χ_0 is the characteristic function on the unity cube for continuum models on $L^2(\mathbb{R}^d)$. The formal distributions $K_N(E_1, \ldots, E_N)$ are associated with Radon measures. It is important to determine if these measures on $\mathbb{R}^N$ have densities and, if so, what are the regularity properties of the densities. The first moment with $A_1 = 1$ is the density of states (DOS). The existence of the DOS has been extensively studied for random Schrödinger operators both on the lattice (see [15]) and the continuum, see [7, 14]. Global upper and lower bounds for the DOS for lattice models are known, see [10, 15]. Much less is known about the regularity of the DOS for models in dimensions greater than one. For lattice models at high disorder, the regularity of the DOS was proved to be dependent upon the finiteness of the moments of the characteristic function of the single-site probability distribution [5]. If the probability density admits an analytic continuation to a neighborhood of the real axis, then the DOS is real analytic [2, 8]. Virtually nothing is known about regularity of the DOS for models on $L^2(R^d)$ for $d > 1$. Transport properties are described by the higher-order moments. Certain second-order moments describe the DC conductivity. This can be seen in the derivation of the Kubo formula for the DC conductivity from linear response theory, see, for example [4]. If A_1 and A_2 are velocity operators, the resulting measure is the current-current correlation measure, see [13]. If $V_i = (i/2)[H_\omega, x_i]$ denotes the velocity operator for direction i, then the current-current correlation density $m_{ij}(E_1, E_2)$ is defined as

$$m_{ij}(E_1, E_2) = \mathbb{E}\{Tr \chi_0 \rho_\omega(E_1) V_i \rho_\omega(E_2) V_j \chi_0\},$$

It is known that the diagonal values $m_{ii}(E, E)$ give the DC conductivity in the i^{th}-direction. The existence of this density for lattice models and for energies outside a neighborhood of the diagonal in the strong localization regime was proved in [2]. In recent work with Combes and Germinet [6], we prove that for lattice and continuum models the limit as $\epsilon \to 0$ of $m_{ii}(E+\epsilon, E+\epsilon)/\epsilon^2$ exists and vanishes provided E is in the strong localization regime. This proves that the current-current correlation measure has no atoms on the diagonal, a signature of localization. Assuming that the density exists in a neighborhood of the diagonal, this proves the vanishing of the DC conductivity at energies in the strong localization regime. this result is known by other methods, see [1, 3, 9, 11, 12].

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Local dependencies, law of large numbers, and algebraicity

Ofer Zeitouni

1. Introduction

In our recent work [1] we studied convergence of the empirical distribution of eigenvalues of random band matrices, and developed a combinatorial approach, based on the moment method, to identify the limit (and also to provide central limit theorems for linear statistics). Here we describe how to develop the method further to handle a class of matrices with local dependence among entries. The details can be found in [2].

To get the flavor of our results, imagine a Wigner matrix (i.e., an N-by-N real symmetric random matrix with i.i.d. above-diagonal entries, each of mean 0 and variance $1/N$), on which a local "filtering" operation is performed: each entry not near the diagonal or an edge is replaced by half the sum of its four neighbors to northeast, southeast, southwest and northwest. We find that the limit measure is the free multiplicative convolution of the semicircle law (density $\propto \mathbf{1}_{|x|<2}\sqrt{4-x^2}$) and the arcsine law (density $\propto \mathbf{1}_{0<x<2}/\sqrt{x(2-x)}$). One can also write down the quartic equation satisfied by the Stieltjes transform of the limit measure.

2. Statement of the results

2.1. **Kernels.** Let $C = [0,1] \times S^1$, where S^1 is the unit circle in the complex plane. We fix a kernel

$$s : C \times C \to \mathbb{R}$$

which will govern the local covariance structure of our random matrix model. We impose on s the following conditions.

Assumption 2.1.1.

(I) s is a nonnegative symmetric function, i.e.

$$s(c, c') = s(c', c) \geq 0.$$

(II) s has a Fourier expansion

$$s(c, c') = \sum_{i,j \in \mathbb{Z}} s_{ij}(x, y)\xi^i \eta^j \quad (c = (x, \xi),\ c' = (y, \eta))$$

where all but finitely many of the coefficients

$$s_{ij} : [0,1] \times [0,1] \to \mathbb{C}$$

vanish identically.

(III) There is a finite partition $\mathcal{I}$ of $[0,1]$ into subintervals of positive length such that every coefficient function s_{ij} is constant on every set of the form $I \times J$ with $I, J \in \mathcal{I}$.

(IV) s is nondegenerate: $\|s\|_{L^1(C\times C)} > 0$.

2.2. **The model.** For each $N \in \mathbb{N}$, let

$$X^{(N)} = [X_{ij}^{(N)}]_{i,j=1}^N$$

be an N-by-N random hermitian matrix. We impose the following conditions, where s satisfies Assumption 2.1.1.

Assumption 2.2.1.

(I) (a) $\forall N \in \mathbb{N} \quad \forall i, j \in \{1, \ldots, N\} \quad EX_{ij}^{(N)} = 0.$

(b) $\forall k \in \mathbb{N} \quad \sup_{N=1}^{\infty} \max_{i,j=1}^{N} E|X_{ij}^{(N)}|^k < \infty.$

(II) There exists $K > 0$ such that for all $N \in \mathbb{N}$, the following hold:

(a) $\forall i, j \in \mathbb{Z} \quad \max(|i|, |j|) > K \Rightarrow s_{ij} \equiv 0.$

(b) For all nonempty subsets

$$A, B \subset \{(i,j) \in \{1,\ldots,N\}^2 \mid 1 \le i \le j \le N\}$$

such that

$$\min_{(i,j)\in A} \min_{(k,\ell)\in B} \max(|i-k|, |j-\ell|) > K,$$

the σ-fields

$$\sigma(\{X_{ij}^{(N)} \mid (i,j) \in A\}), \quad \sigma(\{X_{k\ell}^{(N)} \mid (k,\ell) \in B\})$$

are independent.

(c) $\forall i,j,k,\ell \in Q_K^{(N)}$ s.t. $\min(j-i, \ell-k) > K$
$EX_{ij}^{(N)}\overline{X}_{k\ell}^{(N)} = s_{i-k,\ell-j}(i/N, j/N)$.

2.2.2. *The empirical distribution of eigenvalues.* Let

$$\lambda_1^{(N)} \le \lambda_2^{(N)} \le \ldots \le \lambda_N^{(N)}$$

denote the eigenvalues of the hermitian matrix $X^{(N)}/\sqrt{N}$, and let

$$L^{(N)} = N^{-1} \sum_{i=1}^{N} \delta_{\lambda_i^{(N)}}$$

denote the corresponding empirical distribution of the eigenvalues.

2.3. **The measure μ.** For each positive integer N, let

$$C^{(N)} = \{c_1^{(N)}, \ldots, c_{N^2}^{(N)}\} \subset C$$

be the set of pairs $(x,\xi) \in C$ where $x \in [0,1) \cap \frac{1}{N}\mathbb{Z}$ and $\xi^N = 1$. Let $C^{(\infty)}$ be the union of the sets $C^{(N)}$. Let

$$\{\tilde{Y}_e\}_{e \subset C^{(\infty)} \text{ s.t. } |e|=1,2}$$

be an i.i.d. family of standard normal (mean 0 and variance 1) random variables. Let $\tilde{X}^{(N)}$ be the N^2-by-N^2 real symmetric random matrix with entries

$$\tilde{X}_{ij}^{(N)} = 2^{\delta_{ij}/2} \sqrt{s(c_i^{(N)}, c_j^{(N)})}\, \tilde{Y}_{\{c_i^{(N)}, c_j^{(N)}\}}.$$

Let $\tilde{\lambda}_1^{(N)} \le \cdots \le \tilde{\lambda}_{N^2}^{(N)}$ be the eigenvalues of $\tilde{X}^{(N)}/N$ and let $\tilde{L}^{(N)} = \frac{1}{N^2}\sum_{i=1}^{N^2} \delta_{\tilde{\lambda}_i^{(N)}}$ be the empirical distribution of the eigenvalues. By [1, Thm. 3.2] the empirical distribution $\tilde{L}^{(N)}$ tends weakly in probability as $N \to \infty$ to a limit μ with bounded support (explicit equations for the Stieltjes transform of μ are described in [2]).

Theorem 2.4. $L^{(N)}$ converges weakly in probability to μ.

Theorem 2.5. The Stieltjes transform $S(\lambda) = \int \frac{\mu(dx)}{\lambda - x}$ is an algebraic function of λ, i. e., there exists some not-identically-vanishing polynomial $F(X,Y)$ in two variables with complex coefficients such that $F(\lambda, S(\lambda))$ vanishes for all complex numbers λ not in the support of μ.

The algebraicity result applies also to the Stieltjes transforms of the limiting measures arising from the model of [1] in the case of a finite color space. It also applies to many other equations appearing in the Random Matrix literature.

2.6. **A regularity theorem.** Algebraicity implies a rather strong regularity property for μ. We state a theorem immediately below to explain this point in detail.

We declare a real-valued function h defined in a bounded open interval (a,b) to be of *rational beta type* under the following conditions:

- h is real-analytic and nonnegative on (a,b).
- There exist positive rational numbers c and d such that both limits
$$\lim_{x\downarrow a}(x-a)^{1-c}h(x), \quad \lim_{x\uparrow b}(b-x)^{1-d}h(x)$$
exist and are positive.

Now let μ be a probability measure on the real line with compact support K and algebraic Stieltjes transform $S(z) = \int \frac{\mu(dx)}{z-x}$. Let $F(X,Y)$ be a not-identically-vanishing polynomial such that $F(z,S(z)) = 0$ for all $z \in \mathbb{C}\setminus K$ and furthermore the discriminant $D(X)$ of $F(X,Y)$ with respect to Y is not-identically-vanishing. Let n be the degree of $F(X,Y)$ in Y and write $F(X,Y) = \sum_{i=0}^{n} F_i(X)Y^i$. Let A be the (finite) set of complex zeroes of $F_n(X)D(X)$.

Theorem 2.7. Notation and assumptions are as above. Let I be a connected component of $\mathbb{R}\setminus A$ and let $\mu|_I$ be the restriction of μ to (the Borel subsets of) I. If $I\setminus K$ is nonempty, then $\mu|_I$ vanishes identically. If I is bounded and $\mu|_I$ does not vanish identically, then $\mu|_I$ has density of rational beta type with respect to Lebesgue measure.

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Products of random transformations, Lyapunov exponents, and random operators

Ilya Goldsheid

We develop a unified approach to the theory of Lyapunov exponents. It allows one to control the behaviour of functionals responsible for the existence of distinct Lyapunov exponents as well as to estimate the difference between them in terms of norms of operators acting on Banach space of functions with suitably chosen norms.

Low energy properties of the random displacement model

Günter Stolz
(joint work with Jeff Baker, Michael Loss)

1. The displacement model

Consider a random Schrödinger operator

$$H_\omega = -\Delta + V_\omega, \tag{1}$$

where the random potential V_ω is given by displacing a single site potential q from the points of $\mathbb{Z}^d$,

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} q(x-i-\omega_i). \tag{2}$$

For the real valued single site potential q we assume $q \in L^\infty(\mathbb{R}^d)$ and $\operatorname{supp} q \subset [-r,r]^d$ for some $r < 1/2$. We also assume that q is reflection symmetric at each coordinate hyperplane, i.e. symmetric in each variable with the remaining variables fixed. The displacement configuration $\omega = (\omega_i)_{i\in\mathbb{Z}^d}$ consists of vectors $\omega_i \in [-d_{max}, d_{max}]^d$, where $r + d_{max} = 1/2$. The latter ensures that the displaced single site potentials in (2) are confined to the unit cube centered at i.

Most of our results will be deterministic, i.e. provide properties which hold for all configurations (ω_i). We will later consider applications to the case where the ω_i are i.i.d. random variables.

F. Klopp and S. Nakamura have recently found phenomena similar to the ones discussed here for Anderson models with sign-indefinite single site potentials [7].

2. Minimizing the ground state energy

Our main result provides a configuration ω^{min} which minimizes the bottom of the spectrum $\min \sigma(H_\omega)$ over all configurations ω. For this note that the operators H_ω are uniformly bounded below in ω and thus $E_0 := \inf_\omega \min \sigma(H_\omega)$ is finite.

Theorem 1 ([2]). *Let ω^{min} be given by*

$$\omega_i^{min} = ((-1)^{i_1} d_{max}, \ldots, (-1)^{i_d} d_{max}) \tag{3}$$

for all $i = (i_1, \ldots, i_d) \in \mathbb{Z}^d$. Then

$$E_0 = \min \sigma(H_{\omega^{min}}).$$

This configuration is 2-periodic in each coordinate, where in each period cell 2^d single sites cluster together in adjacent corners of unit cells.

3. Bubbles tend to the boundary

The following result is central to the proof of Theorem 1, but is also of interest by itself as a result in spectral geometry. Let $\Lambda_0 = (-\frac{1}{2}, \frac{1}{2})^d$ be the unit cube centered at 0 and q as above, i.e. bounded with support in $[-r, r]^d$, $r < 1/2$. For $d_{max} = \frac{1}{2} - r$ and $a \in [-d_{max}, d_{max}]^d$ consider $H_{\Lambda_0}^N(a) = -\Delta + q(x-a)$ with Neumann boundary condition on $\partial\Lambda_0$. By $E_0(a)$ we denote the ground state energy of $H_{\Lambda_0}^N(a)$.

Theorem 2 ([2])**.** *The following alternative holds: Either*

(i) *$E_0(a)$ is strictly maximized at $a = 0$ and strictly minimized in the 2^d corners $(\pm d_{max}, \dots, \pm d_{max})$ of $[-d_{max}, d_{max}]^d$*

or

(ii) *$E_0(a)$ is identically zero. In this case the corresponding eigenfunction is constant outside of the support of q.*

A sufficient but far from necessary condition for case (i) to hold is that q has fixed sign and does not vanish identically, as in this case $E_0(a)$ never vanishes. It is interesting that the phenomenon "the bubble tends to the boundary" is independent of the sign of q. If one works with the Dirichlet problem instead, it is known that $E_0(a)$ is minimized for a in the corners if q is positive, while it is minimized for $a = 0$ if q is negative [5].

Bubbles tend to the boundary in much more general Neumann-domains than cubes. In [2] we also provide a result of this type for general strictly convex domains.

4. Uniqueness of the minimizer of $\min \sigma(H_\omega)$

Among *all* configurations $\omega \in ([-d_{max}, d_{max}]^d)^{\mathbb{Z}^d}$ the configuration ω^{min} is certainly not the unique minimizer of $\min \sigma(H_\omega)$. In fact, if the ω_i are i.i.d. random variables with, say, uniform distribution on $[-d_{max}, d_{max}]^d$, then it follows from a standard Weyl-sequence argument that $\min \sigma(H_\omega) = E_0$ for *almost every* ω. But it makes sense, and is useful for applications, to ask if ω^{min} is the unique minimizer among all *periodic* configurations.

First consider $d = 1$. For $L \in \mathbb{N}$ let S_L denote the set of all L-periodic configurations $(\omega_i)_{i \in \mathbb{Z}}$ such that $\omega_i = -d_{max}$ or $\omega_i = d_{max}$ for all i. Furthermore, let $n^\pm(\omega)$ be the number of $i \in \{1, \dots, L\}$ with $\omega_i = \pm d_{max}$.

Theorem 3 ([1])**.** *Let $d = 1$ and q such that alternative (i) of Theorem 2 holds. An L-periodic configuration ω satisfies $\min \sigma(H_\omega) = E_0$ if and only if L is even, $\omega \in S_L$ and*

$$n^-(\omega) = n^+(\omega). \tag{4}$$

Thus, in each period interval of V_ω, equally many of the single site potentials sit at the extreme right and the extreme left of their allowed range of positions. The *dimer configuration* ω^{min} is merely a special case of this situation.

Remarkably, for $d \geq 2$ one recovers uniqueness of the minimizer:

Theorem 4 (in preparation)**.** *Let $d \geq 2$ and q such that alternative (i) of Theorem 2 holds. Also assume that $r < \frac{1}{4}$. Then ω^{min} and its translates are the only periodic minimizers of* $\min \sigma(H_\omega)$*.*

We think that the assumption $r < 1/4$, guaranteeing that the bubble fits into half a unit-square, is of purely technical nature.

5. The integrated density of states

Now consider the random displacement model, i.e. the $(\omega_i)_{i\in\mathbb{Z}^d}$ are i.i.d. random variables with a given distribution μ on $[-d_{max}, d_{max}]^d$. For $d = 1$ Anderson localization at all energies, i.e. pure point spectrum with exponentially decaying eigenfunctions for almost every ω, follows from the results in [4]. Pioneering work on localization for the multi-dimensional random displacement model was done by Klopp [6] who studied a semiclassical version $-h^2\Delta + V_\omega$ of (1). For sufficiently small h he shows the existence of an Anderson localized region near the bottom of the spectrum. For the non-perturbative case $h = 1$ a good understanding of the nature of the bottom of the spectrum is crucial to get started. The main difficulty here is that, as opposed to the Anderson model, the random displacement model is non-monotone in the random parameters ω_i.

The random displacement model is ergodic. Thus the integrated density of states (IDS)

$$N(E) = \lim_{L\to\infty} \frac{1}{|\Lambda_L|}\mathbb{E}(\operatorname{tr}\chi_{(-\infty,E]}(H_\omega^L)) \tag{5}$$

exists. Here $\Lambda_L = (-L/2, L/2)^d$ and H_ω^L is the restriction of H_ω to $L^2(\Lambda_L)$, where all the standard boundary conditions can be used.

For Anderson models (at least those with single site potentials of fixed sign) the IDS has Lifshits tails at the bottom of the almost sure spectrum E_0, i.e., roughly,

$$N(E) \sim \exp\left(-C|E - E_0|^{-d/2}\right) \tag{6}$$

as $E \downarrow E_0$.

The one-dimensional random displacement model shows completely different behavior. The most extreme case is the "Bernoulli displacement model", where the distribution of the displacements is $\mu = \frac{1}{2}\delta_{-d_{max}} + \frac{1}{2}\delta_{d_{max}}$.

Theorem 5 ([1])**.** *For the one-dimensional Bernoulli displacement model with spectral minimum E_0 there exist $\varepsilon > 0$ and $C < \infty$ such that*

$$N(E) \geq \frac{C}{\log^2(E - E_0)} \quad \textit{for } E \in [E_0, E_0 + \varepsilon). \tag{7}$$

This might come as a surprise as the IDS vanishes at E_0 even more slowly than it does for the free Laplacian, where one has the van Hove law $N(E) \sim (E-E_0)^{d/2}$. In fact, Craig and Simon [3] have shown that the IDS of *arbitrary* one-dimensional ergodic Schrödinger operators is log-Hölder continuous at all energies, i.e. satisfies the upper bound

$$|N(E) - N(E')| \leq \frac{C}{|\log|E - E'||} \tag{8}$$

for all $E, E' \in \mathbb{R}$. The lower bound (7), with $E' = E_0$ where $N(E_0) = 0$, comes quite close to this, closer than any other known random potential (for Anderson models one has at least Hölder continuity of the IDS). Craig and Simon only had quasi-periodic examples showing that their bound is optimal.

The reason for the "fat tails" of the IDS at the bottom of the spectrum can be traced to the high degeneracy of the periodic minimizer in Theorem 3. In fact, not only are there $\binom{L}{L/2}$ configurations of even period L which satisfy (4), but, by the central limit theorem, most random configurations approximately satisfy this condition. Thus there are many configurations with a ground state energy near E_0.

Theorem 4 indicates to us that the IDS might have Lifshits tails at the bottom of the spectrum for $d \geq 2$, but currently we have no proof of this.

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Energy Level Statistics for Random Operators

Nariyuki Minami

1. The Anderson tight binding model

Let us consider the Anderson tight binding model $H_\omega = -\Delta + V_\omega$, where $(\Delta u)(x) = \sum_{|y-x|=1} u(y)$ $(x, y \in \mathbf{Z}^d)$, is the discretized Laplacian, and $V_\omega = \{V_x(\omega)\}_{x \in \mathbf{Z}^d}$: is the random potential consisting of i.i.d. random variables. For $L = 1, 2, \ldots$, let $\Lambda := \Lambda_L = [0, L]^d \cap \mathbf{Z}^d$, and consider $H_\omega^\Lambda := \chi_\Lambda H_\omega \chi_\Lambda$, the restriction of H_ω to the hypercube Λ with the Dirichlet boundary condition. Now let $E_1^\Lambda(\omega) \leq \cdots \leq E_n^\Lambda(\omega)$, $n = |\Lambda|$, be the eigenvalues of H_ω^Λ. At this point, we assume that each random variable $V_x(\omega)$ has absolutely continuous distribution with a bounded density $\rho(v)$. We also assume that its upperbound $\|\rho\|_\infty$ is small.

Smallnes of $\|\rho\|_\infty$ is assumed because it implies the exponential decay of the fractional moment of the Green's function $G_\omega^D(z;x,y) = (H_\omega^D - z)^{-1}(x,y)$. (See [1], [2] for the exact formulation.) We shall also assume that this estimate is valid for any $z \in \mathbf{C} \setminus \mathbf{R}$, so that the Anderson localization holds throughout the spectrum of H_ω.

We now consider the rescaled spectrum of H_ω^Λ expressed as a point process on $\mathbf{R}$: $\xi^\omega(\Lambda;E)(dx) = \sum_j \delta_{\xi_j^\omega(\Lambda;E)}(dx)$, with $\xi_j^\omega(\Lambda;E) = |\Lambda|(E_j^\Lambda(\omega) - E)$. It was proved in [3] that if the integrated density of states $N(E)$ of H_ω is differentiable at E with $n(E) = dN/dE$, then as $L \to \infty$, the probability law of $\xi^\omega(\Lambda;E)$ converges weakly to that of the stationary Poisson point process with mean density $n(E)$. Then a question arises: Can we compare the individual realizations of the spectrum $\{E_j^\Lambda(\omega)\}$ of H_ω^Λ, rather than their probability law, with the typical realization of a nice point process on $\mathbf{R}$?

For this purpose, we need to "unfold"the spectrum. (See [4] for a discussion on unfolding.) Let us assume that $N(E)$ is of C^1 and $n(E) > 0$ everywhere on $(\inf \Sigma, \sup \Sigma)$, where $\Sigma \subset \mathbf{R}$ is the closed set such that $spec(H_\omega) = \Sigma$ a.s.. Now let us call $e_j^\Lambda(\omega) := |\Lambda| \cdot N(E_j^\Lambda(\omega)) \in (0, |\Lambda|)$ the unfolded eigenvalues of H_ω^Λ. Then it is seen that the sequence $\{e_j^\Lambda(\omega)\}$ is asymptotically uniformly distributed on $(0,|\Lambda|)$ as $L \to \infty$ in the following sense: For P-a.a. ω,

$$\sharp\{j \geq 1 |\ e_j^\Lambda(\omega) \leq A|\Lambda|\} \ \sim \ |\Lambda|,\ L \to \infty$$

holds for all $A \in (0,1)$. Our conjecture is the following:

Conjecture 1 Let μ be the uniform distribution on $[a,b]$. For $\omega \in \Omega$ and $t \in [a,b]$, define a point process Ξ^Λ by letting

$$\Xi_{\omega,t}^\Lambda(dx) := \sum_j \delta_{e_j^\Lambda(\omega) - |\Lambda| t}(dx)\ .$$

Then for P-a.a. $\omega \in \Omega$, the probability law of $\Xi_{\omega,t}^\Lambda$ under $\mu(dt)$ converges weakly to that of the stationary Poisson point process with mean density 1.

We shall call "the weaker version of the conjecture"the assertion which require that the convergence holds in probability, rather than for P-a.a. ω. It can be shown that if we would be able to prove the following lemma, then one would obtain the weaker version of our conjecture:

Lemma(also a conjecture) For any finite intervals J and $E \neq E'$,

$$P\left(\eta^\omega(C_p;E)(J) \geq 1 \text{ and } \eta^\omega(C_p;E')(J) \geq 1\right) = o(N_L^{-d})$$

as $L \to \infty$. Here the cube Λ is divided into nearly equal smaller cubes C_p, $p = 1, \ldots, N_L^d = (L^\alpha)^d$ $(0 < \alpha < 1)$, and we have let $\eta^\omega(C_p;E)(dx) = \sum_j \delta_{\eta^\omega(C_p;E)}(dx)$, with $\eta^\omega(C_p;E) = |\Lambda|(E_j^{C_p}(\omega) - E)$.

As a by-product, this lemma would imply the asymptotic independence of two point processes $\xi^\omega(\Lambda;E)$ and $\xi^\omega(\Lambda;E')$ as $L \to \infty$ for each pair $E \neq E'$.

2. Other questions

Under the assumptions stated in the previous section, it is easy to prove the following assertion as a corollary of [3]:

Proposition Under the probability measure $P \times \mu$, $\Xi^{\Lambda}_{\omega,t}$ converges weakly to the stationary Poisson point process on $[0, \infty)$ with mean density 1.

This suggests us the following conjecture:

Conjecture 2 Let $\nu^{\Lambda}_{\omega}(dx) = \sum_{j=1}^{\infty} \delta_{e^{\Lambda}_j}(dx)$ be the point process made from the unfolded eigenvalues. Then under the probability measure P, ν^{Λ}_{ω} converges weakly to the stationary Poisson point process on $[0, \infty)$ with mean density 1.

If this statement were true, it would imply in particular that the law of $e^{\Lambda}_1(\omega) = |\Lambda| N(E^{\Lambda}_1(\omega))$, the first unfolded eigenvalue of H^{Λ}_{ω}, converges to the exponential distribution $e^{-x}dx$ $(x \geq 0)$. In [5], McKean actually proved this type of limit theorem for the one-dimensional Schrödinger operator $-d^2/dt^2 + B'_{\omega}(t)$ $(0 \leq t \leq L)$ with the Gaussian white noise potential and Dirichlet or Neumann boundary condition on $t = 0, L$. (See §5.4.4. of [4] for a discussion.)

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The random phase hypothesis for quasi-one-dimensional random Schrödinger

Hermann Schulz-Baldes
(joint work with Ch. Sadel)

For various quantities associated to random operators on a strip one needs to control Birkhoff averages which are associated to the random action of its transfer matrices on some Grassmannian. Examples are the Lyapunov exponents [3] and the integrated density of states [4] which both can be calculated in this way. Of particular interest is a perturbative evalution of the associated Birkhoff sums. This is possible under a certain coupling hypothesis similar to the one for discrete time Markov chains on a finite state space. Let us state the main new mathematical result in some detail. Comments on why the result is related to the well-known random phase hypothesis for quasi-one-dimensional media (or also maximal entropy ansatz) as well as proofs can be found in [2], which also contains a first application.

Suppose given a Lie group $\mathcal{G} \subset \mathrm{GL}(L, \mathbb{C})$, a compact, connected, symmetric space $\mathcal{M}$ given as a quotient of two compact Lie groups, and a smooth, transitive group action $\cdot : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$. Furthermore, let $\mathcal{T}_{\lambda,\sigma} \in \mathcal{G}$ be a family of group elements depending on a coupling constant $\lambda \geq 0$ and a parameter σ varying in some compact probability space $(\Sigma, \mathbf{p})$, which is of the following form:

$$\mathcal{T}_{\lambda,\sigma} \;=\; \mathcal{R}\, \exp\left(\lambda \mathcal{P}_\sigma\right)\;, \tag{1}$$

where $\mathcal{R} \in \mathcal{G}$ and $\mathcal{P}_\sigma$ is a measurable map on Σ with compact image in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. We suppose that $\mathbf{E}(\mathcal{P}_\sigma) = 0$. Let us consider the product probability space $(\Omega, \mathbf{P}) = (\Sigma^{\mathbb{N}}, \mathbf{p}^{\mathbb{N}})$. Associated to $\omega = (\sigma_n)_{n\in\mathbb{N}} \in \Omega$ there is a sequence $(\mathcal{T}_{\lambda,\sigma_n})_{n\in\mathbb{N}}$ of group elements. An $\mathcal{M}$-valued Markov process $x_n(\lambda, \omega)$ with starting point $x_0 \in \mathcal{M}$ is defined iteratively by

$$x_n(\lambda, \omega) \;=\; \mathcal{T}_{\lambda,\sigma_n} \cdot x_{n-1}(\lambda, \omega)\;. \tag{2}$$

The averaged Birkhoff sum of a complex function f on $\mathcal{M}$ is

$$I_{\lambda,N}(f) \;=\; \mathbf{E}_\omega\, \frac{1}{N} \sum_{n=0}^{N-1} f(x_n(\lambda, \omega))\;. \tag{3}$$

Next recall that an invariant measure ν_λ on $\mathcal{M}$ is defined by the property

$$\int \nu_\lambda(dx)\, f(x) = \int \nu_\lambda(dx)\, \mathbf{E}_\sigma f(\mathcal{T}_{\lambda,\sigma} \cdot x). \tag{4}$$

The operator ergodic theorem then states that $I_{\lambda,N}(f)$ converges almost surely (in x_0) w.r.t. any invariant measure ν_λ and for any integrable function f. If the group generated by $\mathcal{T}_{\lambda,\sigma}$, with σ varying in the support of $\mathbf{p}$, is non-compact and strongly irreducible, Furstenberg has proven that there is a unique invariant measure ν_λ which is, moreover, Hölder continuous [1]. To our best knowledge, little seems to be known in more general situations and also concerning the absolute continuity of ν_λ.

As final preparation before stating the result, let us introduce the measure $\overline{\mathbf{p}}$ on the Lie algebra $\mathfrak{g}$. For any measurable set $\mathfrak{b} \subset \mathfrak{g}$,

$$\overline{\mathbf{p}}(\mathfrak{b}) \;=\; \int_{\langle\mathcal{R}\rangle} \mathrm{d}R\; \mathbf{p}\big(\{\sigma \in \Sigma \;:\; R\mathcal{P}_\sigma R^{-1} \in \mathfrak{b}\}\big)\;.$$

where $\mathbf{p}$ is the distribution of the random variable $\mathcal{P}_\sigma$ on the Lie algebra $\mathfrak{g}$, *i.e.* for any measurable $\mathfrak{b} \subset \mathfrak{g}$ one has $\mathbf{p}(\mathfrak{b}) = \mathbf{p}(\{\mathcal{P}_\sigma \in \mathfrak{b}\})$. We are interested in a perturbative calculation of $I_{\lambda,N}(f)$ in λ for smooth functions f with rigorous control on the error terms.

Theorem 1. [2] *Let $\mathcal{T}_{\lambda,\sigma}$ be of the form* (1) *and x_n the associated Markov process on $\mathcal{M}$ as given in* (2). *Let $\mathfrak{v} = \mathrm{Lie}\,(\mathrm{supp}(\overline{\mathbf{p}}))$ be the Lie subalgebra of $\mathfrak{g}$ generated by the support of $\overline{\mathbf{p}}$. Suppose that $\mathcal{U} \subset \mathcal{G}$ is a Lie subgroup of $\mathcal{G}$ acting transitively* *on $\mathcal{M}$ such that its Lie algebra $\mathfrak{u} \subset \mathfrak{g}$ is contained in $\mathfrak{v}$. Let us, moreover, suppose that $\mathcal{R}$ is a multi-dimensional diophantine. Furthermore let $f \in C^\infty(\mathcal{M})$*

Then there is a μ-almost surely positive function $\rho_0 \in C^\infty(\mathcal{M})$ normalized w.r.t. the Riemannian volume measure μ on $\mathcal{M}$, such that

$$I_{\lambda,N}(f) \;=\; \int_{\mathcal{M}} \mu(\mathrm{d}x)\, \rho_0(x)\, f(x) \;+\; \mathcal{O}\left(\frac{1}{N\lambda^2}, \lambda\right) .$$

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Open problems in random matrix theory

Percy A. Deift

We discussed the following open problems (for more information see [1] from which the numbering is taken)

(2.) **Universality for Random Matrix Theory (RMT).**
It is of considerable interest to prove universality for orthogonal and symplectic ensembles of $N \times N$ matrices, $N \to \infty$, with weights of the form $e^{-N \operatorname{tr} V(M)}\, dM$, where where $V(x) = x^{2m} + \ldots$ is a polynomial of degree $2m$.
For Wigner ensembles, universality at the edge (Soshnikov,...) is now well understood for a wide variety of distributions on the entries of the matrices. Universality in the bulk for Wigner matrices is a conjecture par excellence that digs deep into the structure of random matrices. Numerical experiments provide convincing evidence that it is true.

(3.) **Riemann-Hilbert Problem with non-analytic data.**
In many situations one is concerned with the asymptotic behavior of Riemann-Hilbert problems with exponentially varying data of the form $e^{in\phi(z)}r(z)$, $n \to \infty$. The Deift-Zhou nonlinear steepest descent method for such problems requires $\phi(z)$ to be analytic. It is of considerable theoretical and practical interest to extend the nonlinear steepest descent method to situations where ϕ is no longer analytic, and has, for example, only a finite number of derivatives.

(4.) **Painlevé equations.**
The six (nonlinear) Painlevé equations form the core of "modern special function theory." Increasingly, as nonlinear science develops, people are finding that the solutions to an extraordinarily broad array of scientific

problems, from neutron scattering theory, to PDEs, to transportation problems, to combinatorics,..., can be expressed in terms of Painlevé functions.

What is needed is a project, similar to the Bateman project, or a new volume of Abramowitz and Stegun, devoted to the Painlevé equations. A modern "Bateman Project: Painlevé equations" would not/should not provide tables for such solutions. Rather, it should provide reliable, easy to use software to compute the solutions.

(5.) **Multivariate analysis.** As a subject, RMT goes back to the work of statisticians at the beginning of the 20th century. Recently, advances in RMT have opened the way to the statistical analysis of data sets in cases where the number of variables is comparable to the number of samples, and both are large. At the technical level, one considers the statistics of the singular values of (appropriately centered and scaled) $p \times n$ matrices $M = (M_{ij})$, where $p \sim n \to \infty$. Here p is the number of variables and n is the sample size. More precisely, one centers the M_{ij}'s around their sample averages,

$$M_{ij} \to \hat{M}_{ij} = M_{ij} - \frac{1}{n}\sum_{k=1}^{n} M_{ik},$$

and considers the eigenvalues $l_1 \geq \cdots \geq l_p \geq 0$ and associated eigenvectors $w_1, \ldots, w_p$ of the $p \times p$ sample matrix $S = \frac{1}{n}\hat{M}\hat{M}^T$.

A common model for the variables M_{ij} is to assume that they follow a (real) p-variate Gaussian distribution $N_p(\mu, \Sigma)$ with mean μ and covariance matrix Σ. Using recent results from RMT, much has now been proved about the statistics of $l_1, l_2, \ldots$ as $p, n \to \infty$, $p/n \to \gamma \in (0, \infty)$, in the case $\Sigma = I$. In particular, we know that in the limit, l_1, appropriately centered and scaled, satisfies the Tracy-Widom distribution for the largest eigenvalues of a GOE matrix.

It is a major problem in multivariate analysis to analyze the statistics of the eigenvalues $l_1, l_2, \ldots$ as $p, n \to \infty$, $p/n \to \gamma \in (0, \infty)$ for spiked populations. In the spiked, complex case, i.e. when the columns $(M_{1j}, M_{2j}, \ldots, M_{pj})^T$ are sampled from the complex p-variate Gaussian distribution, much is known about the asymptotic distribution of the l_j's, as $p, n \to \infty$, $p/n \to \gamma \in (0, \infty)$. By contrast in the real case, apart from a.s. convergence of the l_i's, very little is known about their asymptotic distributions.

(6.) **β-ensembles.** Random point processes corresponding to β-ensembles, or, equivalently, log gases at inverse temperature β, are defined for arbitrary $\beta > 0$. The orthogonal, unitary, and symplectic ensembles corresponding to $\beta = 1, 2$, or 4, respectively, are now, of course, well understood, but other values of β are also believed to be relevant in applications, for example, in the statistical description of headway in freeway traffic. For certain rational values of β, β-ensembles are related to Jack polynomials, but for general β much less is known. The analysis of β-ensembles for general β using the recent results of Edelman-Sutton, Rider-Virag-Ramirez represents an interesting, and increasingly important, challenge.

(9.) **The parking problem.** A number of so-called "transportation" problems have now been analyzed in terms of RMT. These include: the "vicious" walker problem of M. Fisher, the bus problem in Cuernavaca, Mexico, the headway traffic problem on highways, and the airline boarding problem of Bachmat et al. It is a great challenge to develop a microscopic model for the parking problem.

(10.) **A Tracy-Widom Central Limit Theorem.** The situation for RMT analogous to the Central Limit Theorem is the following: take i.i.d.'s $(a_1, a_2, ...)$, perform an operation X on them,

$$(a_1, a_2, ...) \to (X_1, X_2, ...),$$

and as $n \to \infty$ the X_n's converge to the Gaudin distribution, or the Tracy-Widom distribution. The question is, "What is X?" Important progress towards answering this question has been made recently, and independently, by Baik-Suidan and Bodineau-Martin, but the full problem remains open and very challenging.

(15.) **Multi-matrix models and models with an external field.** There has been considerable progress (Kuijlaars,...) in understanding basic statistics such as the correlation functions for the 2-matrix random matrix model, and also matrix models with a source. The key element in these developments has been the successful extension by Kuijlaars et al of the Riemann-Hilbert/steepest descent method to 3×3 Riemann-Hilbert problems. So far only the simplest situations have been considered. In order to consider the generic situation, one must, in particular, extend the Riemann-Hilbert/steepest descent method to $n \times n$ Riemann-Hilbert problems. This is a challenging problem which would have important implications, not only for random matrix models, but also for problems in other areas.

(16.) **Poisson/Gaudin-Mehta transition.** On the appropriate scale, the bulk eigenvalues of a random GOE matrix M exhibit Gaudin-Mehta statistics. On the other hand, if $M = M^T$ has i.i.d. entries and bandwidth $W = 1$ (i.e. M is tridiagonal), then, on the appropriate scale, the bulk eigenvalues of M exhibit Poisson statistics. As the bandwidth W increases from 1 to $N - 1 = \dim(M) - 1$, the eigenvalue statistics must change from Poisson to Gaudin-Mehta. A back of the envelope calculation suggests that there should be a (sharp) transition in a narrow region around $W \approx \sqrt{N}$. This is a well-known, outstanding, open conjecture with many implications for theoretical physics, particularly wave propagation in random media.

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On the localization of a matrix valued Anderson Hamiltonian

Alexander Elgart

We show the Anderson localization for a random spin Hamiltonian with a disorder generated by a random magnetic field acting on the spin degrees of freedom. In particular, we present a proof of the regularity of the corresponding Green functions, which is a central component of the Aizenman-Molchanov theory of localization.

About the behavior of the IDS of random divergence operators

Hatem Najar

Spectral theory of selfadjoint operators, differential and finite-difference operators in particular, is an important branch of mathematics and of mathematical physics, having numerous and diverse applications.

In recent decades the considerable progress in the filed has been achieved via synthesis of the spectral analytic and probalistic ideas. The result of this progress is rather detailed description of spectral properties of the finite-difference and the differential operators with random coefficients in the one dimensional case, and serval approaches to spectral analysis of multidimensional operators in the strong localization regime (neighborhoods of spectrum edges and /or strong coupling) and a variety of results obtained by these methods. An indispensable ingredient of these studies for a self adjoint operator H_ω, is $N(E)$, the Integrated Density of states (IDS). It is defined as follows:

$$N(E) = \lim_{L\to+\infty} \frac{1}{|\Lambda_L|}\mathbb{E}\Big(tr(\chi_{(-\infty,E]}(H^L_\omega))\Big),$$

$\Lambda_L = [-L,L]^d$ and H^L_ω is the restrection of H_ω to $L^2(\Lambda_L)$. The IDS, controls a number of important spectral properties and has serval interesting regime.

In 1964, Lifshitz [6] argued that, for a random Schrödinger operator of the form $H_\omega = -\Delta + V_\omega$, there exists $c_1, c_2 > 0$ such that $N(E)$ satisfies the asymptotic:

$$N(E) \simeq c_1 \exp(-c_2(E-E_0)^{-\alpha}), \quad E \to E_0. \tag{1}$$

Here E_0 is the bottom of the spectrum of H_ω and $\alpha > 0$. The behavior (1) is known as **Lifshitz tails** (for more details see part IV.9.A of [14]). Lifshitz predicted (1) also at fluctuating edges inside the spectrum. The latter are those parts of the spectrum which are determined by rather rare events.

The principal results known on Lifshitz tails are mainly shown for Schrödinger operators (L. Pastur, W. Kirsch and F. Martinelli, B. Simon, F. Klopp,...) for continuous and discrete cases. (See [1, 2, 3, 13, 15] and others). The first results known in this context are given for the bottom of the spectrum. While opening the

way in this domain, the work of F. Klopp is the first that allowed crossing technical obstacles that marked the research in the internal edges of the spectrum for a long time (for the continuous case) [3]. It should be noted that even in the case of Schrödinger operators spectral analysis gaps is farfrom being complete. Besides, other classes of random operators, operators of divergent form, the Maxwell and acoustic operators in particular, are also of considerable interest from serval points of views.
We studied the behavior of the IDS and the spectrum itself near an edge of a spectral gap of a version of the multidimensional divergent differential operator, known in mathematical physics as acoustic operators. The random coefficient of the operator is the product of a bounded and strictly positive periodic function and a bounded strictly positive random function of "alloy type", whose lattice of the period coincides with that of periodic coefficient and whose random amplitudes are independent identically distributed random variables with common probability law that doses not vanish too fast at the upper edge of its support. Precisely we consider operator in the following form:

$$H_\omega = -\nabla \varrho_\omega^{-1} \nabla. \tag{2}$$

Where

$$\varrho_\omega = \varrho_0 \Big(1 + \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma u_\gamma \Big).$$

It is assumed that H_ω an internal spectral gap and the goal of the study is the asymptotic form of the IDS and the spectral type of the spectrum of the operator near of an edge of the gap.
In [7], we prove that the integrated density of states of H_ω has a Lifshitz behavior at the edges of internal spectral gaps if and only if the integrated density of states of a well chosen periodic operator is non degenerate at the same edges.
Among the technic used to prove the result is the approach proposed by F. Klopp and based on the idea of periodic approximations of ergodic random coefficients of an operator, variational principle of the spectral theory and large deviation argument. Motivated by [5], in [10] we study Lifshitz tails for acoustic operators in quantum wave guide.
In [8] we prove that when ϱ_ω is considered as an Anderson type long range perturbations of some periodic function, the behavior of the integrated density of states of H_ω in the vicinity of the internal spectral edges is given without any assumption on the behavior of the IDS of the background operator.
In [9] we study internal band edges localization of acoustic waves in 2-dimensional space obtained by random perturbation of some periodic media. Our results rely on the study of Lifshitz tails for the IDS. Localization is then deduced by the standard multiscale argument.
As a continuation with the investigation of the behavior of the IDS of operators of the H_ω. In in [11] we are interested in its behavior at 0, the bottom of the spectrum of H_ω. We prove that it converges exponentially fast to that of some periodic operator $\overline{H}$. This result relates to the result of S.M. Kozlov and improve

it [4].
In [12], we consider a more general form of divergence operators. Indeed, we replace ϱ_ω by A_ω; an elliptic, $d \times d$-matrix valued, $\mathbb{Z}^d$-ergodic random field. The main improvement is that the behavior of the random variables is linked up to the lifshitz exponent, and determines if one is located in a classical regime or in a quantum one. One concludes that the disorder is responsible for the transition between those two regimes.

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Limit theorems for random matrix ensembles associated to symmetric spaces

Michael Stolz

The classical objects of physics-inspired random matrix theory are probability measures on what Freeman Dyson in 1962 called the "threefold way", namely, the spaces of hermitian, real symmetric, and quaternion real matrices, along with their "circular", compact counterparts. Nevertheless, it has emerged in theoretical condensed matter physics ([1]) that matrix descriptions of systems such as mesoscopic normal-superconducting hybrid structures belong to a more general "tenfold way", consisting of matrix versions of the classical symmetric spaces.

This talk presents two limit theorems for random matrices that have been established in this broader framework. The first, joint work with Katrin Hofmann-Credner (Bochum) [5], is a generalization of Wigner's theorem to the full tenfold way, which allows for a certain amount of dependence between the matrix entries, in the spirit of Schenker and Schulz-Baldes [6]. It turns out that in the case of symmetric spaces of type A, AI, AII, B/D, DIII, C, CI, the mean empirical eigenvalue measure converges weakly to the semicircle distribution. On the other hand, for the "chiral classes" AIII, BDI, CII, which are related to sample covariance matrices, one obtains weak convergence to a variant of the Marčenko–Pastur distribution.

Joint work with Benoît Collins (Lyon/Ottawa) [4] is devoted to random vectors of the form $(\mathrm{Tr}(A^{(1)}V), \dots, \mathrm{Tr}(A^{(r)}V))$, where V is a uniformly distributed element of a matrix version of a classical compact symmetric space, and the $A^{(\nu)}$ are deterministic parameter matrices. Under a growth condition on the $A^{(\nu)}$, it is proven that for increasing matrix sizes these random vectors converge to a joint Gaussian limit, and one obtains a formula for the covariance structure. This generalizes work of D'Aristotile, Diaconis and Newman [2] on the compact classical groups, which in turn generalizes a classical result of É. Borel. The proof uses integration formulae of Collins and Śniady [3], which are rooted in classical invariant theory.

This motivates a final remark on another instance of the interplay between matrix integrals and invariant theory, based on joint work with Tatsuya Tate (Nagoya) [7]. A stationary phase analysis of the integral

$$\int_G (\mathrm{Tr}\,\rho_\lambda(g))^{a_1}(\mathrm{Tr}\,\rho_\lambda(g^2))^{a_2}\dots(\mathrm{Tr}\,\rho_\lambda(g^r))^{a_r}(\overline{\mathrm{Tr}\,\rho_\lambda(g)})^{b_1}\dots(\overline{\mathrm{Tr}\,\rho_\lambda(g^r)})^{b_r}\,\omega_G(dg),$$

where G is a connected compact semisimple Lie group, $(V_\lambda, \rho_\lambda)$ a regular irreducible representation of G with highest weight λ, and ω_G normalized Haar measure, yields an asymptotic formula for the trace of permutation operators on the space of invariants of G in a growing tensor power of V_λ.

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Characteristic Polynomials of Random Matrices

Holger Kösters
(joint work with Friedrich Götze)

Following the discovery by Keating and Snaith [9] that the moments of the Riemann zeta-function along the critical line exhibit some striking similarities to the moments of the characteristic polynomial of a random matrix from the Circular Unitary Ensemble (CUE), the characteristic polynomials of random matrices have found considerable interest [4, 5, 12, 7, 13, 2, 3]. A major question behind these investigations is that of universality, i.e. whether the results for the "classical" random matrix ensembles (see e.g. [6, 11]) continue to hold for more general random matrix ensembles.

We consider the case of Hermitian Wigner matrices. Let Q be a probability measure on the real line such that

$$\int x\ Q(dx) = 0\,,\ \int x^2\ Q(dx) = 1/2\,,\ b := \int x^4\ Q(dx) < \infty\,,$$

and let $X_N = (X_{ij})_{1 \le i,j \le N}$ be the associated Hermitian Wigner matrix, i.e. $(\Re X_{ij})_{1 \le i < j \le N}$, $(\Im X_{ij})_{1 \le i < j \le N}$, and $(X_{ii}/\sqrt{2})_{1 \le i \le N}$ are independent families of independent random variables with distribution Q, and the remaining matrix elements are determined by Hermiticity. Furthermore, let

$$D_N(\xi,\eta) := \det\left(X_N - \sqrt{N}\xi - \frac{\eta}{\sqrt{N}\varrho(\xi)}\right),\quad \widetilde{D}_N(\xi,\eta) := D_N(\xi,\eta) - \mathbb{E}D_N(\xi,\eta)\,,$$

where $\varrho(\xi) := \frac{1}{2\pi}\sqrt{4-\xi^2}$ denotes the density of the semi-circle law. Then we have the following result for the second-order correlation function of the characteristic polynomial:

Proposition 1. ([8]) *For any $\xi \in (-2,+2)$ and any $\mu,\nu \in \mathbb{R}$,*

$$\lim_{N\to\infty} \sqrt{\frac{1}{2\pi N}}\,\frac{1}{N!}\,e^{-N\xi^2/2}\,\mathbb{E}\big(D_N(\xi,\mu)\cdot D_N(\xi,\nu)\big)$$
$$= \exp\left(b-\tfrac{3}{4}\right)\,\exp\left(\tfrac{1}{2}\xi(\mu+\nu)\,/\,\varrho(\xi)\right)\varrho(\xi)\,\frac{\sin\pi(\mu-\nu)}{\pi(\mu-\nu)}\,.$$

In the special case where Q is the Gaussian distribution, this result reduces to the known result for the Gaussian Unitary Ensemble (GUE) [4]. In general, the underlying distribution Q enters into the asymptotics only as a multiplicative factor depending on the fourth cumulant; the remaining factors are universal. Moreover, we have the following universal result for the correlation coefficient of the characteristic polynomial:

Proposition 2. ([8]) *For any $\xi \in (-2, +2)$ and any $\mu, \nu \in \mathbb{R}$,*

$$\lim_{N\to\infty} \frac{\mathbb{E}\big(\widetilde{D}_N(\xi,\mu)\cdot \widetilde{D}_N(\xi,\nu)\big)}{\sqrt{\mathbb{E}\widetilde{D}_N(\xi,\mu)^2}\cdot\sqrt{\mathbb{E}\widetilde{D}_N(\xi,\nu)^2}} = \frac{\sin\pi(\mu-\nu)}{\pi(\mu-\nu)}\,.$$

Similar results hold for real-symmetric Wigner matrices [10]. The most notable difference is that the sine kernel $S(x) := \sin x/x$ is replaced with the function $T(x) := \frac{1}{2}\left(\sin x/x^3 - \cos x/x^2\right)$ in this case. Again, in the special case where Q is the Gaussian distribution, this result reduces to the known result for the Gaussian Orthogonal Ensemble (GOE) [5].

The proofs of our results are based on an explicit expression for the exponential generating function of the second-order correlation function of the characteristic polynomial.

Finally, it seems interesting to note that the sine kernel also shows up in connection with the shifted second moments of the Riemann zeta-function, as already observed by Atkinson [1].

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Asymptotic analysis of eigenvalue correlations in a coupled random matrix model

Arno Kuijlaars
(joint work with Maurice Duits)

Statistical properties of eigenvalues of random matrices taken from a probability measure $Z_n^{-1}\exp(-n\mathrm{Tr}\,V(M))dM$ defined on the space of $n\times n$ Hermitian matrices M can be fully analyzed using orthogonal polynomials. In this way an almost complete picture has arisen about the possible limiting eigenvalue behaviors as $n\to\infty$, both in the macroscopic and microscopic regimes [9].

The coupled random matrix model is a probability measure

$$Z_n^{-1}\exp(-n\mathrm{Tr}\,(V(M_1)+W(M_2)-\tau M_1M_2))dM_1dM_2$$

defined on pairs (M_1, M_2) of $n\times n$ Hermitian matrices. Here V and W are two polynomial potentials and $\tau>0$ is a coupling constant. The model is of interest in 2D quantum gravity where it is used to construct generating functions for the number of bicolored graphs on surfaces. The role of orthogonal polynomials is taken over by two sequences of polynomials $p_j^{(n)}$ and $q_k^{(n)}$ that satisfy the biorthogonality condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_j^{(n)}(x)q_k^{(n)}(y)e^{-n(V(x)+W(y)-\tau xy)}dxdy=0, \qquad \text{if } j\neq k.$$

Statistical properties of the eigenvalues of M_1 and M_2 are described by kernels built out of the biorthogonal polynomials and certain transformed functions, [2, 12, 13, 16]

Despite many contributions in the physics and mathematics literature (see e.g. [3, 12, 14] for mathematical papers), the limiting behavior as $n\to\infty$ is not fully understood. The case $V(x)=\frac{1}{2}x^2$, $W(y)=\frac{1}{2}ay^2$ was analyzed in detail in [12]. In the talk I reported on an approach to the case $W(y)=\frac{1}{4}y^4$ and V an even polynomial, which involves the following steps.

- A characterization of the polynomials $p_j^{(n)}$ as multiple orthogonal polynomials, which leads to the formulation of a 4×4 matrix valued Riemann-Hilbert problem [15, 19].
- An equilibrium problem for a triple of measures (μ_1,μ_2,μ_3) where μ_1 is the asymptotic zero distribution of the polynomials $p_n^{(n)}$ as $n\to\infty$, as well as the limiting mean eigenvalue distribution of M_1.
- The steepest descent analysis of the Riemann-Hilbert problem in which the equilibrium measures play a prominent role.

More precisely, the equilibrium problem consists of minimizing the energy functional

$$\sum_{j=1}^{3} \iint \log \frac{1}{|x-y|} d\mu_j(x) d\mu_j(y)$$
$$- \sum_{j=1}^{2} \iint \log \frac{1}{|x-y|} d\mu_j(x) d\mu_{j+1}(y) + \int \left(V(x) - \frac{3}{4} |\tau x|^{4/3} \right) d\mu_1(x)$$

among all vectors of measures (μ_1, μ_2, μ_3) satisfying

- μ_1 and μ_3 are supported on $\mathbb{R}$ and μ_2 is supported on $i\mathbb{R}$ with total masses $\mu_1(\mathbb{R}) = 1$, $\mu_2(i\mathbb{R}) = 2/3$, $\mu_3(\mathbb{R}) = 1/3$,
- μ_2 has the upper constraint $\mu_2 \leq \sigma$, where σ is the measure on $i\mathbb{R}$ given by $d\sigma(z) = \frac{\sqrt{3}}{2\pi} |\tau y|^{1/3} dy$ for $z = iy \in i\mathbb{R}$.

Equilibrium problems with external field [7, 18] are common in the theory of random matrices and orthogonal polynomials with varying weights. Upper constraints appeared before in the context of discrete orthogonal polynomials, see. e.g. [1, 10, 17].

We prove in [11] that a unique minimizer exists and that the effective external field

$$V(x) - \frac{3}{4} \tau^{4/3} |x|^{4/3} + \int \log |x-y| d\mu_2(y)$$

acting on μ_1 is real analytic on $\mathbb{R}$. Then by results of [8] we have that μ_1 is supported on a finite union of compact intervals with a density that is real analytic on the interior of each interval and vanishes with an exponent $2k+1/2$, $k \in \mathbb{N} \cup \{0\}$ at each endpoint. Furthermore, we prove that the constraint $\mu_2 \leq \sigma$ is active on an interval $[-ic, ic]$ with $c > 0$ and μ_2 is supported on the full imaginary axis with an analytic density on $i\mathbb{R} \setminus [-ic, ic]$. The measure μ_3 is supported on he full real line.

The equilibrium problem is used in the steepest descent analysis of the Riemann-Hilbert problem. A first step, suggested to us by [4], uses Pearcey integrals to transform the Riemann-Hilbert problem of [15] into a form comparable to the one in [3]. The analysis can be completely carried out in the case that μ_1 is supported on one interval (the one-cut case).

In that case, we show in [11] that the correlation kernel for the eigenvalues of M_1 has the usual local scaling limits that are known from unitary one-matrix models, namely the sine kernel in the bulk and the Airy kernel at regular edge points. Critical phenomena occur at singular interior points where the density of μ_1 vanishes, and at singular edge points where the density vanishes to higher order. These critical phenomena are already present in the one-matrix model [5, 6], and we find no new ones in the coupled random matrix model with $W(y) = \frac{1}{4} y^4$.

In addition, we also obtain uniform Plancherel-Rotach type asymptotics for the biorthogonal polynomials $p_n^{(n)}$ as $n \to \infty$, confirming earlier results in [13].

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Lifshitz tails for non monotonous alloy type models

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(joint work with Frédéric Klopp)

Here we discuss Lifshitz tails for alloy type random Schrödinger operators with non sign-definite local potential. Under the assumption of symmetry of the potentials, we show the location of the infimum of the almost sure spectrum, and we then discuss the existence and the non existence of Lifshitz tails.

1. The model

We consider the standard alloy type random Schrödinger operator:

$$H_\omega = -\triangle + V_0 + V_\omega, \quad V_\omega = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma V(x-\gamma), \quad \text{on } L^2(\mathbb{R}^d),\ d \geq 1,$$

where $\{\omega_\gamma\}_{\gamma\in\mathbb{Z}^d}$ are i.i.d. random variables and V_0 is a periodic background potential. We are interested in the Lifshitz tail behavior of the integrated density of states (IDS) near the bottom of the spectrum. Such results has been studied extensively by many people, e.g., Pastur, Kirsch, Martinelli, Simon, Klopp, etc., but mostly in the case V is nonnegative (see, e.g., a review by W. Kirsch [3]). Here we consider the case V changes sign. Then the sectrum of H_ω is not necessarily monotonous in the random parameter ω_γ, and we cannot apply many techniques used in these works. This is an example of the non monotonous random perturbations of Schrödinger operaters, e.g.,

- V changes sign (Veselić's talk in the same meeting; our model);
- random displacement models (Stolz's talk in the same meeting, see [1]);
- random magnetic Schrödnger operators (N, Ueki, Klopp - Nakano - Nomura - N, etc.).

In fact, we have found that there are a lot in common in these problems, and we are investigating the applicability of our methods to other problems.
In this talk, we suppose

(1) $V \in C_c^0(C_1(0);\mathbb{R})$, where $C_\ell(x)$ denotes the cube: $C_\ell(x) = \{y \in \mathbb{R}^d \,|\, 0 \leq y_j - x_j \leq \ell, j = 1,\dots,d\}$.
(2) $\mathrm{supp}(\omega_\gamma) \subset [a,b]$ and $a, b \in \mathrm{supp}(\omega_\gamma)$, where $\mathrm{supp}(\omega_\gamma)$ denotes the support of the distribution of (ω_γ).
(3) V is symmetric about $x_j = 1/2$, $j = 1,\dots,d$, i.e.,

$$V(x_1 - \tfrac{1}{2},\dots,x_d - \tfrac{1}{2}) = V(\sigma_1(x_1 - \tfrac{1}{2}),\dots,\sigma_d(x_d - \tfrac{1}{2}))$$

for any $\sigma_j \in \{\pm 1\}$. We assume V_0 is also symmetric about $x_j = 1/2$, $j = 1,\dots,d$.

2. Location of the bottom of the spectrum

At first we decide $\inf\Sigma$, where Σ is the almost sure spectrum of H_ω, i.e., $\Sigma = \sigma(H_\omega)$ almost surely. Let

$$H_\lambda^N = -\triangle + V_0 + \lambda V \quad \text{on } L^2(C_1(0))$$

with Neumann boundary conditions, and set

$$E_-(\lambda) = \inf \sigma(H_\lambda^N).$$

Theorem 1: $\inf\Sigma = \min(E_-(a), E_-(b))$.

Remark: Najar showed this if $\int V > 0$ and λ is small, without the symmetry condition ($\inf\Sigma = E_-(a)$ in this case, [4]).

Sketch of Proof: We note $E_-(\lambda)$ is a concave function of λ, and hence

$$\inf_{\lambda\in[a,b]} E_-(\lambda) = \min(E_-(a), E_-(b)).$$

By the Neumann decoupling, we easily see

$$H_\omega \geq \bigoplus_{\gamma\in\mathbb{Z}^d} H_{\omega_\gamma}^N \geq \min(E_-(a), E_-(b)) \quad \text{on } L^2(\mathbb{R}^d) = \bigoplus_{\gamma\in\mathbb{Z}^d} L^2(C_1(\gamma)).$$

Let us suppose $E_-(a) \leq E_-(b)$. We note by the assumption (3), the ground state of H_a^N is symmetric. Using this fact, we can show $\inf\sigma(H_{\overline{a}}) = E_-(a)$, where $\overline{a}_\gamma = a$ for all γ. By the standard argument, this implies $E_-(a) \in \Sigma$ and Theorem 1 follows. $\square$

Here we have used concavity of $\inf\sigma(H_\lambda^N)$ and the symmetry of the ground state. These observations are extensively used in the proof of the Lifshitz tail.

3. The integrated density of states

We set $H_{\omega,L}^N$ be the operator $-\triangle + V_0 + V_\omega$ on $L^2(C_L(0))$ with Neumann boundary conditions. Then the integrated density of states (IDS) is defined by

$$N(E) = \lim_{L\to\infty} L^{-d}\#\{\text{e.v. of } H_{\omega,L}^N \leq E\},$$

and it is well-known that $N(E)$ is well-defined. Then we have

Theorem 2: Assume (1) – (3).

(1) if $E_-(a) \neq E_-(b)$, then

$$\limsup_{E\to E_-+0} \frac{\log|\log N(E)|}{\log(E-E_-)} \leq -\frac{d}{2}.$$

(2) If $E_-(a) = E_-(b)$ and if (ω_γ) is not Bernoulli, i.e., if

$$\mathbb{P}(\omega_\gamma = a) + \mathbb{P}(\omega_\gamma = b) < 1,$$

then

$$\limsup_{E\to E_-+0} \frac{\log|\log N(E)|}{\log(E-E_-)} \leq -\frac{1}{2}.$$

Remark: The estimate in (2) is weaker for technical reasons, and we expect the same bound as in (1) in this case also. The estimate in (1) is optimal. If we assume, for example, $\mathbb{P}(|\omega_\gamma - c| < \varepsilon) \geq \alpha\varepsilon^N$ for $\varepsilon > 0$ with some $\alpha, N > 0$ and $c = a, b$, then we can show

$$\liminf_{E\to E_-+0} \frac{\log|\log N(E)|}{\log(E-E_-)} \geq -\frac{d}{2}.$$

by standard argument.

If $E_-(a) = E_-(b)$ and the distribution of (ω_γ) is Bernoulli, then the situation looks more complicated. We denote $\mathbf{e}_j = (\delta_{ij})_{i=1}^d \in \mathbb{R}^d$, $\omega_j = C_1(0) \cup C_1(\mathbf{e}_j)$, and we set

$$H^N_{abj} = \begin{cases} -\triangle + V_0(x) + aV(x) & \text{on } C_1(0) \\ -\triangle + V_0(x) + bV(x) & \text{on } C_1(\mathbf{e}_j) \end{cases}$$

with Neumann boundary conditions. Then we have

Theorem 3: Suppose $E_- = E_-(a) = E_-(b)$ and the distribution of (ω_γ) is Bernoulli, i.e., $\mathbb{P}(\omega_\gamma = a) + \mathbb{P}(\omega_\gamma = b) = 1$. Then

(1) If $\inf\sigma(H^N_{abj}) > E_-$ for some j, then

$$\limsup_{E\to E_-+0} \frac{\log|\log N(E)|}{\log(E-E_-)} \leq -\frac{1}{2}.$$

(2) If $\inf\sigma(H^N_{abj}) = E_-$ for all $j = 1,\dots,d$, then

$$\limsup_{E\to E_-+0} \frac{\log N(E)}{\log(E-E_-)} = \frac{d}{2}.$$

Combining these with the Wegner estimate obtained by Hislop and Klopp [2], we can prove the Anderson localization near the bottom of the spectrum (with additional assumptions on the distribution of (ω_γ)).

Thus under the assumption of the reflection symmetry, we now have the necessary and sufficient condition for the existence and the absence of Lifshitz tails. General case (without the symmetry condition) is open at present, and we expect more complicated phenomena. In general, though, we have the following conjecture:

Conjecture: Unless the distribution is Bernoulli and $E_-(a) = E_-(b)$, we have Lifshitz tails at the bottom of the spectrum.

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The Caccioppoli inequality for general Schrödinger operators

Peter Stollmann
(joint work with Anne Boutet de Monvel, Daniel Lenz, Ivan Veselić)

Dirichlet forms. Throughout we work with a locally compact, separable metric space X endowed with a positive Radon measure m with supp$m = X$. We refer to [7] as the classical standard reference as well as [3, 8, 10, 6] for literature on Dirichlet forms. Our disuccion of the intrinsic metric goes pretty much along the same lines as those in [2, 12]. Let us emphasize that in contrast to most of the work done on Dirichlet forms, we treat real and complex function spaces at the same time and write $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

The central object of our studies is a regular Dirichlet form $\mathcal{E}$ with domain $\mathcal{D}$ in $L^2(X)$ and the selfadjoint operator H_0 associated with $\mathcal{E}$. This means that $\mathcal{D} \subset L^2(X, m)$ is a dense subspace, $\mathcal{E}\colon \mathcal{D} \times \mathcal{D} \to \mathbb{K}$ is sesquilinear and $\mathcal{D}$ is closed with respect to the energy norm $\| \cdot \|_{\mathcal{E}}$, given by

$$\|u\|_{\mathcal{E}}^2 = \mathcal{E}(u, u) + \|u\|_{L^2(X,m)}^2,$$

in which case one speaks of a *closed form* in $L^2(X, m)$. In the sequel we will write

$$\mathcal{E}(u) := \mathcal{E}(u, u).$$

The unique operator H_0 associated with $\mathcal{E}$ is then characterized by

$$D(H_0) \subset \mathcal{D} \text{ and } \mathcal{E}(f, v) = (H_0 f \mid v) \quad (f \in D(H_0), v \in \mathcal{D}).$$

Such a closed form is said to be a *Dirichlet form* if $\mathcal{D}$ is stable under certain pointwise operations; more precisely, $T : \mathbb{K} \to \mathbb{K}$ is called a *normal contraction* if $T(0) = 0$ and $|T(\xi) - T(\zeta)| \leq |\xi - \zeta|$ for any $\xi, \zeta \in \mathbb{K}$ and we require that for any $u \in \mathcal{D}$ also

$$T \circ u \in \mathcal{D} \text{ and } \mathcal{E}(T \circ u) \leq \mathcal{E}(u).$$

Here we used the original condition from [1] that applies in the real and the complex case at the same time. Today, particularly in the real case, it is mostly expressed in an equivalent but formally weaker statement involving $u \vee 0$ and $u \wedge 1$, see [7], Thm. 1.4.1 and [10], Section I.4.

A Dirichlet form is called *regular* if $\mathcal{D} \cap C_c(X)$ is dense both in $(\mathcal{D}, \| \cdot \|_{\mathcal{E}})$ and $(C_c(X), \| \cdot \|_\infty)$, where $C_c(X)$ denotes the space of continuous functions with compact support.

Strong locality and the energy measure. $\mathcal{E}$ is called *strongly local* if

$$\mathcal{E}(u, v) = 0$$

whenever u is constant a.s. on the support of v.

The typical example one should keep in mind is the Laplacian

$$H_0 = -\Delta \text{ on } L^2(\Omega), \quad \Omega \subset \mathbb{R}^d \text{ open},$$

in which case

$$\mathcal{D} = W_0^{1,2}(\Omega) \text{ and } \mathcal{E}(u, v) = \int_\Omega (\nabla u | \nabla v) dx.$$

Now we turn to an important notion generalizing the measure $(\nabla u|\nabla v)dx$ appearing above.

In fact, every strongly local, regular Dirichlet form $\mathcal{E}$ can be represented in the form

$$\mathcal{E}(u,v) = \int_X d\Gamma(u,v)$$

where Γ is a nonnegative sesquilinear mapping from $\mathcal{D}\times\mathcal{D}$ to the set of $\mathbb{K}$-valued Radon measures on X. It is determined by

$$\int_X \phi\, d\Gamma(u,u) = \mathcal{E}(u,\phi u) - \frac{1}{2}\mathcal{E}(u^2,\phi)$$

and called *energy measure*; see also [3]. The energy measure satisfies the Leibniz rule,

$$d\Gamma(u\cdot v,w) = ud\Gamma(v,w) + vd\Gamma(u,w),$$

as well as the chain rule

$$d\Gamma(\eta(u),w) = \eta'(u)d\Gamma(u,w).$$

One can even insert functions from $\mathcal{D}_{\mathrm{loc}}$ into $d\Gamma$, where

$$\mathcal{D}_{\mathrm{loc}} := \{u \in L^2_{\mathrm{loc}} \text{ such that } \phi u \in \mathcal{D} \text{ for all } \phi \in \mathcal{D}\cap C_c(X)\},$$

as is readily seen from the following important property of the energy measure, **strong locality**:

Let U be an open set in X on which the function $\eta \in \mathcal{D}_{\mathrm{loc}}$ is constant, then

$$\chi_U d\Gamma(\eta,u) = 0, \tag{1}$$

for any $u \in \mathcal{D}$. This, in turn, is a consequence of the strong locality of $\mathcal{E}$ and in fact equivalent to the validity of the Leibniz rule.

We write $d\Gamma(u) := d\Gamma(u,u)$ and note that the energy measure satisfies the **Cauchy-Schwarz inequality**:

$$\begin{aligned}\int_X |fg|d|\Gamma(u,v)| &\leq \left(\int_X |f|^2 d\Gamma(u)\right)^{\frac{1}{2}}\left(\int_X |g|^2 d\Gamma(v)\right)^{\frac{1}{2}}\\ &\leq \frac{1}{2}\int_X |f|^2 d\Gamma(u) + \frac{1}{2}\int_X |g|^2 d\Gamma(v).\end{aligned}$$

The intrinsic metric. Using the energy measure one can define the *intrinsic metric* ρ by

$$\rho(x,y) = \sup\{|u(x)-u(y)|\ |u \in \mathcal{D}_{\mathrm{loc}}\cap C(X) \text{ and } d\Gamma(u)\leq dm\}$$

where the latter condition signifies that $\Gamma(u)$ is absolutely continuous with respect to m and the Radon-Nikodym derivative is bounded by 1 on X. Note that, in general, ρ need not be a metric. (See the Appendix for a discussion of the finiteness of the sup.) However, here we will mostly rely on the following

Assumption 1. The intrinsic metric ρ induces the original topology on X.

We denote the intrinsic balls by

$$B(x,r) := \{y \in X | \rho(x,y) \leq r\}.$$

An important consequence of the latter assumption is that the distance function $\rho_x(\cdot) := \rho(x,\cdot)$ itself is a function in $\mathcal{D}_{\mathrm{loc}}$ with $d\Gamma(\rho_x) \leq dm$, see [12]. This easily extends to the fact that for every closed $E \subset X$ the function $\rho_E(x) := \inf\{\rho(x,y) | y \in E\}$ enjoys the same properties.

We have the following version of the Caccioppoli inequality that is slightly stronger than the result from [4]. We need the following notation: For $E \in X$ and $b > 0$ we define the b-neighborhood of E as

$$B_b(E) := \{y \in X : \rho(y,E) \leq b\}.$$

Theorem 2. *Let $\mathcal{E}$ be a strongly local regular Dirichlet form satisfying Assumption 1. Let $\mu_+ \in \mathcal{M}_0$ and $\mu_- \in \mathcal{M}_1$ be given. Let $\lambda_0 \in \mathbb{R}$ be given. Then, there exists a $C = C(\lambda_0, \mu_-)$ such that for any generalized eigenfunctions u to an eigenvalue $\lambda \leq \lambda_0$ of $H_0 + \mu$ the inequality*

$$\int_E d\Gamma(u) \leq C \left(\frac{1}{b^2}\right) \int_{B_b(E)\setminus E} |u|^2 dm + \int_E |u|^2 dm$$

holds for any closed $E \subset X$ and any $b > 0$.

Remark 3. (I) For $\mu = 0, \lambda$, the result can be found in [2].

(II) In [4] the result is used to prove Sch'nol's theorem which says that solutions $u \neq$ as above with sufficiently moderate growth imply that λ lies in the spectrum of H.

(III) In [9] we prove results of Allegretto-Piepenbrink type stating that one doesn't find positive solutions for λ above the ground state energy. See [11] for a survey on this topic.

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