Arbeitsgemeinschaft mit aktuellem Thema:

Julia Sets of Positive Measure

Mathematatisches Forschungsinstitut Oberwolfach

March 30th - April 5th, 2008

Organizers:

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Introduction:

A polynomial $P : \mathbb{C} \to \mathbb{C}$ can be considered as a dynamical system: we are interested in the sequences (z_n) defined by induction:

$$z_0 \in \mathbb{C} \quad \text{and} \quad z_{n+1} = P(z_n).$$

The filled-in Julia set K_P is the set of points $z_0 \in \mathbb{C}$ for which the sequence (z_n) is bounded. This set is compact. The Julia set J_P is the boundary of K_P. In particular, it has empty interior.

There is a small collection of polynomials, for instance

$$P(z) = z^d \quad , \quad P(z) = z^2 - 2,$$

for which the Julia set can be fairly easily understood, but most exhibit "fractal" geometry and "chaotic" behavior, the analysis of which requires serious tools from complex analysis, dynamical systems, topology, combinatorics, ...

This subject has a fairly long history, with contributions by Koenigs [Koe84], Schröder [Sch71], Böttcher [Böt04] in the late 19th century, and the great memoirs of Fatou [Fat06] [Fat20a] [Fat20b] and Julia [Jul18] around 1920.

Followed a dormant period, with notable contributions by Cremer [Cre32] [Cre38] (1936) and Siegel [Sie42] (1942), and a rebirth in the 1960's (Brolin, Guckenheimer, Jakobson). Since the early 1980's, partly under the impetus of computer graphics, the subject has grown vigorously, with major contributions by Douady, Hubbard, Sullivan, Thurston, and more recently Lyubich, McMullen, Milnor, Shishikura, Yoccoz ...

Fatou suggested that one should apply to the Julia sets the methods of Borel-Lebesgue for the measure of sets. Since then, the question remained open.

Until the 1980's, the conjecture, reinforced by the analogy with Ahlfors's conjecture on the area of limit sets of Kleinian groups, was that no Julia set of a polynomial could have positive area.

Results in this direction were obtained by Douady and Hubbard in the case of hyperbolic or subhyperbolic maps, by Branner, Hubbard and McMullen in the case of non renormalizable cubic polynomials with an escaping critical point, by Lyubich and Shishikura in the case of finitely renormalizable quadratic polynomials without indifferent cycles, by Petersen in the case of quadratic polynomials having a Siegel disk with bounded type rotation number.

In the 1990's, Douady began to catch a glimpse of a method for Julia sets of positive area: in the family of degree 2 polynomials with an indifferent Cremer fixed point. Recently, we brought Douady's method to completion.

The present Arbeitsgemeinschaft will focus on the proof of existence of quadratic polynomials having a Julia set of positive area.

Main references: [BC] and [Yoc07].

Talks:

1. **Introductory talk.**

   The speaker will define the Julia and the Fatou set of a polynomial. One should focus on the case of quadratic polynomials, explain the dichotomy between Cantor Julia sets and connected Julia sets, show various pictures of Julia sets and of the Mandelbrot set. Upon request of the speaker, we might provide pictures. Standard references for this talk are [Bea91], [CG93], [Mil99] and [Ste93].

2. **Periodic Fatou components**

   The speaker will present the classification of periodic Fatou components of polynomials: attracting basins, parabolic basins and Siegel disks. The existence of Siegel disk will be obtained later. The non-wandering domain theorem will be mentioned. The proof will be given later. Reference: see for example [Mil99] Theorem 5.2.

3. **Does the Julia set depend continuously on the polynomial?**

   The speaker will study the semi-continuity of the Julia set and the filled-in Julia set with respect to the parameter. The discontinuity at polynomials having parabolic cycles will be explained later. In particular, the speaker will show that the area of the filled-in Julia set is upper semi-continuous with respect to the polynomial. The reference for this talk is [Dou94] part 1 (see also [Mil99] Appendix A, problem A-1).

4. **The dynamics is controlled by the behavior of critical points**

   The speaker will show that an attracting basin always contains a critical point and that the boundary of a Siegel disk is contained in the closure of the postcritical set. Reference: see for example [Mil99] Theorems 8.6 and 11.17

5. **Existence of Cremer points. Existence of Siegel disks**

   The speaker will show that in the family of quadratic polynomials $(P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2)_{\alpha\in\mathbb{R}}$, there is a Cremer point at 0 for a G_δ-dense set of α. The speaker will show that for almost every α, there is a Siegel disk. Reference: [Yoc95] or [Mil99] Theorem 11.2 and 11.14.

6. **Douady-Ghys's renormalization and Yoccoz's theorem on the Brjuno function**

   The speaker will present the proof of Yoccoz of the lower bound on the size of Siegel disks. One shall not go too deeply into the details of the proof but rather give the essential ideas. One will introduce the notations for continued fractions. A reference is [Yoc95] or [BC06] section 5.

7. **The Yoccoz inequality**

   The speaker will present the proof of the Yoccoz inequality on the size of the limbs of the Mandelbrot set. In particular, one will explain why for all p/q, and for all $\alpha \in D(p/q, 1/q^3) \setminus \{p/q\}$, the polynomial $P_\alpha : z \mapsto e^{2i\pi\alpha} z + z^2$ does not have a parabolic point of period dividing q. Reference: see for example [Hub93] and [BC04] section 3.

8. **Explosion functions**

   The speaker will show that as α varies in $D(p/q, 1/q^3)$, the polynomial $P_\alpha : z \mapsto e^{2i\pi\alpha} z + z^2$ has a cycle of period q that depends holomorphically on $\sqrt[q]{\alpha - p/q}$ which coalesces at $z = 0$ for $\alpha = p/q$. One will study the behavior of explosion functions associated to the approximants p_k/q_k of an irrational α such that P_α has a Siegel disk. Reference: see for example [BC04] section 3 and [ABC04] section 4.

9. **Digitated Siegel disks**

   The speaker will explain how one controls the shape of perturbed Siegel disks for well-chosen perturbations: the perturbed Siegel disk contain an amoeba which takes at least half of the area. Reference: [BC] Section 1.4 or [Ché].

10. **The measurable Riemann mapping theorem**

    The speaker will present the statement of the measurable Riemann mapping theorem. One should insist on the geometric interpretation of a Beltrami form as a measurable field of infinitesimal ellipses with bounded ellipticities. Reference: [CG93] or [Hub06].

11. **Sullivan's non-wandering theorem**

    The speaker will present the proof of the non-wandering theorem. In particular, one will show that a quadratic polynomial with a Cremer point has a filled-in Julia set with empty interior. Reference: see for example [Mil99] Appendix F.

12. **Herman-Światek's theorem and Siegel disks whose boundaries are quasicircles**

    The speaker will present Herman-Światek's theorem. One will show that the boundary of the Siegel disk of a quadratic polynomial $P_\alpha : z \mapsto e^{2i\pi\alpha} z + z^2$, with α a bounded type irrational, is a quasicircle. One will insist more on the quasiconformal surgery than on Herman-Swiatek's estimates. Reference: see for example [Dou87] and [Pet00].

13. **Bounded type Siegel disks and Lebesgue measure**

    The speaker will present Petersen's result that quadratic polynomials $P_\alpha : z \mapsto e^{2i\pi\alpha}z + z^2$, with α a bounded type irrational, have a Julia set of zero area. One will present the proof by McMullen that for all $\delta > 0$, the points of the boundary of the Siegel disk Δ are Lebesgue density points of the set of points whose orbit remain δ-close to Δ. Reference: [McM98].

14. **Fatou coordinates and perturbed Fatou coordinates**

    The speaker will present the theory of Fatou coordinates and perturbed Fatou coordinates. One will explain how one deduces a discontinuity of the filled-in Julia set (and its measure) at quadratic polynomials having a parabolic fixed point. Reference: [Dou94] part 2 and [Shi00].

15. **Inou-Shishikura's theorem**

    The speaker will present the result of Inou and Shishikura and the main lines of the proof. Reference: [IS].

16. **The control of the postcritical set**

    The speaker will explain how one uses Inou-Shishikura's theorem to control the explosion of the postcritical set of polynomials having an indifferent cycle with high bounded type. Reference: [BC] Section 1.5.

17. **The proof**

    The speaker will explain how one deduces the existence of quadratic polynomials with a Julia set of positive area from the previous talks. Reference: [BC] Section 1.7.

References

[ABC04] Artur Avila, Xavier Buff, and Arnaud Chéritat, *Siegel disks with smooth boundaries*, Acta Math. **193** (2004), 1–30.

[BC] Xavier Buff and Arnaud Chéritat, *Quadratic Julia sets with positive area*, `http://www.picard.ups-tlse.fr/~buff/Preprints/Area/Area.pdf`.

[BC04] Xavier Buff and Arnaud Chéritat, *Upper bound for the size of quadratic Siegel disks*, Invent. Math. **156** (2004), no. 1, 1–24.

[BC06] Xavier Buff and Arnaud Chéritat, *The Yoccoz function continuously estimates the size of Siegel disks*, Annals of Math **164** (2006), 265–312.

[Bea91] Alan F. Beardon, *Iteration of rational functions*, Graduate Texts in Mathematics, vol. 132, Springer-Verlag, New York, 1991, Complex analytic dynamical systems.

[Böt04] L. E. Böttcher, *The principle laws of convergence of iterates and their applications to analysis (russian)*, Izv. Kazan. Fiz.-Mat. Obshch. **14** (1904), 155–234.

[CG93] Lennart Carleson and Theodore W. Gamelin, *Complex dynamics*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.

[Ché] Arnaud Chéritat, *Introverted Siegel disks*, Presentation given at a conference in Toronto. `http://av.fields.utoronto.ca/slides/05-06/holodynamics_workshop/cheritat/download.pdf`.

[Cre32] Hubert Cremer, *über die schrödersche funktionalgleichung und das schwartsche eckenabbildungsproblem*, Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl. **84** (1932).

[Cre38] ———, *über die häufigkeit der nichtzentren*, Math. Ann. **115** (1938), 573–580.

[Dou87] Adrien Douady, *Disques de Siegel et anneaux de Herman*, Astérisque (1987), no. 152-153, 4, 151–172 (1988), Séminaire Bourbaki, Vol. 1986/87.

[Dou94] ———, *Does a Julia set depend continuously on the polynomial?*, Complex dynamical systems (Cincinnati, OH, 1994), Proc. Sympos. Appl. Math., vol. 49, Amer. Math. Soc., Providence, RI, 1994, pp. 91–138.

[Fat06] Pierre Fatou, *Sur les solutions uniformes de certaines équations fonctionnelles*, C.R. Acad. Sci. Paris **143** (1906), 546–548.

[Fat20a] ———, *Sur les équations fonctionnelles (deuxième mémoire)*, Bull. Soc. Math. France **48** (1920), 33–94.

[Fat20b] ———, *Sur les équations fonctionnelles (troisième mémoire)*, Bull. Soc. Math. France **48** (1920), 208–314.

[Hub93] J. H. Hubbard, *Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz*, Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, pp. 467–511.

[Hub06] John Hamal Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*, Matrix Editions, Ithaca, NY, 2006, Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle.

[IS] Hiroyuki Inou and Mitsuhiro Shishikura, *The renormalization for parabolic fixed points and their perturbation*, `http://www.math.kyoto-u.ac.jp/~mitsu/pararenorm/`.

[Jul18] Gaston Julia, *Mémoire sur l'itération des fonctions rationelles*, J. Math. Pures Appl. **8** (1918), 47–245.

[Koe84] G. Koenigs, *Recherches sur les intégrales de certaines équations fonctionnelles*, Ann. Sci. École Norm. Sup. (3) **1** (1884), 1–41.

[McM98] Curtis T. McMullen, *Self-similarity of Siegel disks and Hausdorff dimension of Julia sets*, Acta Math. **180** (1998), no. 2, 247–292.

[Mil99] John Milnor, *Dynamics in one complex variable*, Friedr. Vieweg & Sohn, Braunschweig, 1999, Introductory lectures.

[Pet00] Carsten Lunde Petersen, *The Herman-Światek theorems with applications*, The Mandelbrot set, theme and variations, London Math. Soc. Lecture Note Ser., vol. 274, Cambridge Univ. Press, Cambridge, 2000, pp. 211–225.

[Sch71] E. Schröder, *über iterirte functionen*, Math. Ann. **3** (1871).

[Shi00] Mitsuhiro Shishikura, *Bifurcation of parabolic fixed points*, The Mandelbrot set, theme and variations, London Math. Soc. Lecture Note Ser., vol. 274, Cambridge Univ. Press, Cambridge, 2000, pp. 325–363.

[Sie42] Carl Ludwig Siegel, *Iteration of analytic functions*, Ann. of Math. (2) **43** (1942), 607–612.

[Ste93] Norbert Steinmetz, *Rational iteration*, de Gruyter Studies in Mathematics, vol. 16, Walter de Gruyter & Co., Berlin, 1993, Complex analytic dynamical systems.

[Yoc95] Jean-Christophe Yoccoz, *Théorème de Siegel, nombres de Bruno et polynômes quadratiques*, Astérisque (1995), no. 231, 3–88, Petits diviseurs en dimension 1.

[Yoc07] ———, *Ensembles de Julia de mesure positive et disques de Siegel des polynômes quadratiques (d'après X. Buff et A. Chéritat)*, Astérisque (2007), no. 311, 385–401, `http://www.bourbaki.ens.fr/TEXTES/966.pdf`.

Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`buff@picard.ups-tlse.fr`

by February 20th at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.