Arbeitsgemeinschaft mit aktuellem Thema:
Ricciflow and the Poincaré conjecture

Mathematisches Forschungsinstitut Oberwolfach
5 -10 October 2008

Organizers:

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Introduction:

The Ricciflow, introduced by Richard Hamilton [Ha1], is a geometric evolution equation which deforms the metric on a Riemannian manifold smoothly in the direction of its Ricci curvature. More precisely, the evolution equation for the family of metrics (g_{ij}) on a manifold M is given by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1}$$

where R_{ij} denotes the Ricci tensor corresponding to the metric. Written in suitable local coordinates this equation has the form of a nonlinear heat type equation for the metric symbols. Because of this one might naively expect that the equation will try to evolve the geometry on M to one which looks the same at every point on the manifold, a homogeneous geometry. This intuition is correct in dimension two, where the Ricciflow can be used ([Ha2], [Ch]) to conformally deform any metric on a closed surface to one of constant curvature, which provides a new proof of the famous uniformization theorem.

In higher dimensions however, the geometry will in general become singular in finite time, i.e. the norm of some of the sectional curvatures will tend to infinity at certain points on the manifold. In three dimensions, if the initial metric has positive Ricci curvature and M is closed and simply connected, Hamilton [Ha1] showed that, after suitable rescaling of the evolving metric, such as to keep the volume of the manifold constant, the metric tends smoothly to the metric on the standard S^3.

Soon after that, Hamilton [Ha3] set up a programme which had the aim of settling Thurstons geometrization conjecture using Ricciflow. This conjecture asks whether any closed 3-manifold can be decomposed along 2-spheres and incompressible tori in such a way that after capping of the 2-sphere boundaries by 3-balls, the resulting finitely many geodesically complete pieces would each admit one out of a list of eight homogeneous geometries formulated by Thurston [Th]. In particular, this would prove the famous Poincaré conjecture, that any simply connected orientable closed 3-manifold had to be topologically equivalent to S^3.

Hamilton himself, but also many others, completed many of the crucial steps in this programme (see [CLN]) but several severe technical difficulties remained unsettled for at least one decade. In 2002, Perelman ([P1] - [P3]) introduced a number of completely novel ideas and techniques which eventually led to the resolution of the geometrization and hence also the Poincaré conjecture.

It is the aim of this workshop, to introduce the participants to the basic concepts and techniques of Hamilton's Ricciflow programme and cover the main ideas of the proof of the Poincaré conjecture.

We will follow mainly the books [CLN] and [MT]. A more detailed description of the talks 10 - 16 will be given closer to or soon after the deadline. The material will be based on chapters 9 - 18 in [MT] which will be suitably portioned and divided among the speakers.

Talks:

1. **Geometry and topology of 3-manifolds**
   This talks will be given by one of the organisers. It reviews basi concepts of geometry and topology of 3-manifolds and introduces Thurston's

geometrisation conjecture and the Poincaré conjecture.

2. **Introduction to Ricciflow and basic examples**
   Def. of Ricciflow, evolution of volume, normalised Ricciflow, basic examples ($S^n, S^2 \times S^1, \mathbb{H}^n$), calculate $r(t)$ and $\rho(t)$ for $S^2_{r(t)} \times S^1_{\rho(t)}$ under normalised Ricciflow ($r(t) \to 0$ and $\rho(t) \to \infty$ in finite time!), Ricciflow of homogeneous spaces (statements only), self-shrinking solutions (mention examples), translating solutions (cigar Σ, $\Sigma \times \mathbb{R}$, Bryant soliton); References: [CLN] p.98, p. 128, Ch.4.

3. **Derivation of evolution equations**
   Short-time existence (statement only), Evolution equations for scalar curvature, Riemann tensor and curvature operator (Uhlenbeck trick). $\mathcal{R}^{\#}$ for $n = 3$ only. References: [CLN] p.108 up to and including Cor. 2.29, p.119 -123

4. **Maximum principles**
   Scalar and vector-valued version with proof sketches, state vector bundle version ([CLN], p.100-101, p.128-129 esp. Thm. 3.9.

5. **Preserving curvature inequalities; Hamilton-Ivey theorem**
   Restate Thm 3.9 in [CLN], p.132 -135, p. 240-245; state Harnack inequality for scalar curvature [CLN], Corollary 10.32, mention that this also follows from the maximum principle

6. **Shi's local estimates**
   Local estimates for the derivatives of curvature when the curvature is locally bounded. References: [CLN], Ch. 6, Thm 6.6 (statement only), state Thm 6.9 but prove only Thm. 6.15 (m=1), Proof of Lemma 6.16; summary of results on pp. 221 - 227, especially definition of barrier function H; sketch of Step 4 p.227; state Cor. 6.23

7. **Compactness theorems for Ricciflow**
   Compactness theorems for Ricciflow solutions with certain assumptions such as curvature bounds and lower injectivity radius estimate [MT] p. 107-113, blow-up limits [MT] p.121-122

8. **Perelman's ℓ- distance**
   Definition and basic properties; order of results as in [P1], Ch.7; state all essential differential inequalities for ℓ; details are in [MT] p.125-156

9. **Monotonicity of reduced volume; local non-collapsing**
   Outline in [P1] (7.12) up to monotonicity of reduced volume (details [MT] p. 157-162; local non-collapsing [P1] Ch.8, Thm 8.2; (details in [MT] Ch.8)

10. **Properties of κ-solutions**
    [MT], Ch. 9, more detailed selection of material to be covered will be given later

11. **Bounded curvature at bounded distance**
    [MT], Ch.10, more detailed selection of material to be covered will be given later

12. **Standard solution**
    [MT], Ch.12, more detailed selection of material to be covered will be given later

13. **Surgery**
    [MT], Ch.13, more detailed selection of material to be covered will be given later

14. **Canonical neighbourhood theorem**
    [MT], Ch.17, section 1, more detailed selection of material to be covered will be given later

15. **Non-accumulation of surgery times**
    [MT], Ch.17, section 2, more detailed selection of material to be covered will be given later

16. **Finite extinction time for simply connected 3-manifolds**
    [MT], Ch.18, more detailed selection of material to be covered will be given later

17. **Proof of the Poincaré conjecture**
    This talk will be given by one of the organisers. It will survey how the ideas and the techniques outlined in the previous talks are combined to yield a proof of the Poincaré conjecture.

References

[Ch] B. Chow, The Ricci flow on the 2-sphere, J. Differ. Geom. **33** (1991), 325-334

[CLN] B. Chow, P. Lu, L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, Volume 77, AMS (2006)

[CM] T.H. Colding, W.P. Minicozzi, Estimates for the extinction time for the Ricci flow on certain three-manifolds and a question of Perelman, Journal of the AMS, **318** (2005), 561-569

[CZ] H.D. Cao, X.P.Zhu, A complete proof of the Poicaré and Geometrization conjectures - Application of the Hamilton-Perelman theory of Ricci flow, Asian J. of Math, **10** (2006), 169-492

[Ha1] R.S. Hamilton, Three-manifolds with positive Ricci curvature, J. Differ. Geom. **17**, no. 2 (1982), 255-306

[Ha2] R.S. Hamilton, The Ricci flow on surfaces, Contemp. Math. **71**, Amer. Math. Soc., Providence RI, 1988

[Ha3] R.S. Hamilton, The formation of singularities in the Ricci flow, Surveys in Differential Geometry, Vol II, Cambridge MA (1995) 7-136

[KL] B. Kleiner, J. Lott, Notes on Perelmans papers, arxiv:math/0605667v2

[MT] J.W. Morgan, Gang Tian, Ricciflow and the Poincaré conjecture, arxiv:math/0607607v2

[P1] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159v1 11Nov2002

[P2] G. Perelman, Ricci flow with surgery on three-manifolds, axXiv:math.DG/0303109, 2003

[P3] G. Perelman, Finite extinction time for solutions to the Ricci flow on certain three-manifolds, arXiv:math.DG/0307245, 2003

[Po] H. Poincaré, Analysis Situs, Cinquième complément à l'analysis Situs, Rend. Circ. mat. Palermo **18** (1904), 45-110

[Th] W.P. Thurston, *Three-dimensional Geometry and Topology*, Volume 1, Princeton University Press, Princeton, New Jersey, 1997

Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`ecker@mi.fu-berlin.de`

by 29 August 2008 at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.