Arbeitsgemeinschaft mit aktuellem Thema:
Ricciflow and the Poincaré conjecture

Mathematisches Forschungsinstitut Oberwolfach
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Organizers:

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Talks:

We will follow mainly the books [CLN] and [MT].

!! Talk 2 has been newly added. Therefore the numbering of talks has been shifted by one. More detailed descriptions of the later talks have been added!!

The page numbers in [MT] refer to the electronic copy on the Clay Institute website; the numbering in the book may be different.

Recommendation: For talks 10 - 17 follow [P1] and [P2] regarding the order of statements. Find details of proofs in [MT]. The statements in here often appear in different order to [P1] and [P2].

1. **Geometry and topology of 3-manifolds**
   This talks will be given by one of the organisers. It reviews basic concepts of geometry and topology of 3-manifolds and introduces Thurston's geometrisation conjecture and the Poincaré conjecture

2. **Topology of canonical neighbourhoods**
   This talks presents a topological classification of the canonical neighbourhoods defined by Perelman; [MT], Ch.19 Appendix

3. **Introduction to Ricciflow and basic examples**
   Def. of Ricciflow, evolution of volume, normalised Ricciflow, basic examples ($S^n, S^2 \times S^1, \mathbb{H}^n$), calculate $r(t)$ and $\rho(t)$ for $S^2_{r(t)} \times S^1_{\rho(t)}$ under normalised Ricciflow ($r(t) \to 0$ and $\rho(t) \to \infty$ in finite time!), Ricciflow of homogeneous spaces (statements only), self-shrinking solutions (mention examples), translating solutions (cigar Σ, $\Sigma \times \mathbb{R}$, Bryant soliton); References: [CLN] p.98, p. 128, Ch.4.

4. **Derivation of evolution equations**
   Short-time existence (statement only), Evolution equations for scalar curvature, Riemann tensor and curvature operator (Uhlenbeck trick). $\mathcal{R}^{\#}$ for $n = 3$ only. References: [CLN] p.108 up to and including Cor. 2.29, p.119 -123

5. **Maximum principles**
   Scalar and vector-valued version with proof sketches, state vector bundle version ([CLN], p.100-101, p.128-129 esp. Thm. 3.9.

6. **Preserving curvature inequalities; Hamilton-Ivey theorem**
   Restate Thm 3.9 in [CLN], p.132 -135, p. 240-245; state Harnack inequality for scalar curvature [CLN], Corollary 10.32, mention that this also follows from the maximum principle

7. **Shi's local estimates**
   Local estimates for the derivatives of curvature when the curvature is locally bounded. References: [CLN], Ch. 6, Thm 6.6 (statement only), state Thm 6.9 but prove only Thm. 6.15 (m=1), Proof of Lemma 6.16; summary of results on pp. 221 - 227, especially definition of barrier function H; sketch of Step 4 p.227; state Cor. 6.23

8. **Compactness theorems for Ricciflow**
   Compactness theorems for Ricciflow solutions with certain assumptions such as curvature bounds and lower injectivity radius estimate [MT] p. 107-113, blow-up limits [MT] p.121-122

9. **Perelman's ℓ- distance**
   Definition and basic properties; order of results as in [P1], Ch.7; state all essential differential inequalities for ℓ; details are in [MT] p.125-156

10. **Monotonicity of reduced volume; local non-collapsing**
    Outline in [P1] (7.12) up to monotonicity of reduced volume (details [MT] p. 157-162; local non-collapsing [P1] Ch.8, Thm 8.2; (details in [MT] Ch.8)

11. **Properties of κ-solutions**
    [P1], Ch.11 and [P2], Ch.1 (details [MT], Ch. 9)

    [P1] Def. 11.1 ([MT] p. 191, 192; present examples 1.2)

    [P1] 11.2 ([MT], Thm 9.11, details p.193-212)

    [P1] 11.4 ([MT] Thm.9.59; proof uses p.213-215 ($\mathcal{R} = \infty$ case), p.226-228 ($0 \leq \mathcal{R} < \infty$ not possible)

    [P1] 11.5 and 11.6 (volume comparison) ([MT] p.230-232)

    [P1] Cor.11.3, first paragraph of 11.7 (statement only), [P2] 1.1. and Lemma 1.2 (without proof) ([MT] p.215-226)

    [P1] Thm. 11.7 (compactness of κ -solutions), remark 11.9 (universal κ)([MT] Ch. 9.5, Ch.9.7)

    [P1] 11.8 and [P2] 1.3 1.5 (qualitative description of κ - solutions) ([MT] Cor. 9.71 and Ch. 9.8)

12. **Bounded curvature at bounded distance**
    If possible present ideas without introducing the concept of a generalized Ricciflow used in [MT] (used there mainly to put Perelmans arguments on a formal basis but does not add any new geometric ideas)

    [P1], Ch. 12.1 ([MT], Ch.10, Thm. 10.2, 11.3, Thm. 11.9)

    [P2], Sect 3 ([MT], Ch.11), [P2] Sect 4 in particular Lemma 4.3! ([MT] Thm. 11.31)

13. **Standard solution**
[P2] ([MT], Ch.12; order and selection of statements as in [P2]; leave out proof of existence, uniqueness and radial symmetry)

14. **Surgery**
[MT], Ch.13, selection of statements and proof sketches as appropriate for 1 hour talk

15. **Canonical neighbourhood theorem**
[MT], Ch.17, section 1, uses [MT], Ch.14 16 as prerequisite. Only state results of these chapters where needed. Try to avoid formalities of definitions in [MT] relating to RF with surgery. Stay closer to outline fo argument given in [P2].

16. **Non-accumulation of surgery times**
[MT], Ch.17, section 2, comments as for previous talk; you could also use one talk to cover Ch. 14 -16 in [MT] and the second talk for ch.17 [MT]

17. **Finite extinction time for simply connected 3-manifolds**
Follow [CM], not [MT], Ch.18!! Concentrate on calculations in Ch.1 and finite extinction time conclusion in Ch.2. mention results about harmonic maps and minimal surfaces only very briefly.

18. **Proof of the Poincaré conjecture**
This talk will be given by one of the organisers. It will survey how the ideas and the techniques outlined in the previous talks are combined to yield a proof of the Poincaré conjecture.

References

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[CLN] B. Chow, P. Lu, L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, Volume 77, AMS (2006)

[CM] T.H. Colding, W.P. Minicozzi, Estimates for the extinction time for the Ricci flow on certain three-manifolds and a question of Perelman, Journal of the AMS, **318** (2005), 561-569; see also Clay Institute web-site and on arxiv

[CZ] H.D. Cao, X.P.Zhu, A complete proof of the Poicaré and Geometrization conjectures - Application of the Hamilton-Perelman theory of Ricci flow, Asian J. of Math, **10** (2006), 169-492

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[Ha2] R.S. Hamilton, The Ricci flow on surfaces, Contemp. Math. **71**, Amer. Math. Soc., Providence RI, 1988

[Ha3] R.S. Hamilton, The formation of singularities in the Ricci flow, Surveys in Differential Geometry, Vol II, Cambridge MA (1995) 7-136

[KL] B. Kleiner, J. Lott, Notes on Perelmans papers, arxiv:math/0605667v2

[MT] J.W. Morgan, Gang Tian, Ricciflow and the Poincaré conjecture, arxiv:math/0607607v2

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[P3] G. Perelman, Finite extinction time for solutions to the Ricci flow on certain three-manifolds, arXiv:math.DG/0307245, 2003

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