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Mini-Workshop: Symmetric Varieties and Involutions of Algebraic Groups

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Abstract. This conference brought together experts from the areas of algebraic groups, Kac–Moody groups, Tits buildings, and symmetric varieties. The main theme presented and discussed during the workshop was the geometry of involutions of algebraic and Kac–Moody groups. In particular, symmetric varieties and the induced action of involutions on buildings. More specific topics that were covered include the Tits centre conjecture, compactifications of locally symmetric spaces and of buildings, Kac–Moody groups over ultrametric fields, and the structure of locally compact groups.

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Introduction by the Organisers

The topics of this conference all in some way evolved from the classical theory of real and complex Lie groups. Indeed, one of the important mathematical goals during the 1950's was to find analogs of the semisimple Lie groups of exceptional type over arbitrary fields. Chevalley completed the first crucial step by producing his famous basis theorem for simple complex Lie algebras, and later Steinberg succeeded in describing these analogs group-theoretically. An important development due to Tits was the theory of groups with a BN-pair and invented buildings; these buildings belong to arbitrary Chevalley groups as naturally as the projective spaces belong to the special linear groups.

Since then the theories of algebraic groups and of buildings developed into various directions. However, due to their common origin both theories often lead naturally to similar questions which were attacked by completely different means. In the context of this conference, the PhD thesis by Bernhard Mühlherr and the work by Aloysius Helminck and coauthors on involutions of algebraic groups illustrate this in a quite remarkable way. Both projects contributed strongly to the understanding of the geometry of involutions of algebraic groups, but surprisingly each one has gone unnoticed by the researchers of the other until recently.

One of the main objectives of this conference was to bring these two theories closer to each other. The first two lectures on Monday morning familiarised all participants with the concept of buildings; Pierre-Emmanuel Caprace explained the foundations of buildings from the simplicial point of view, while Bernhard Mühlherr introduced the chamber system approach to buildings and explained the power of filtrations when studying sub-geometries of (twin) buildings that arise from the action of certain subgroups of the isometry group of the (twin) building. As these two lectures were of an introductory nature and since their content is already well documented (we refer to the recently published book by Abramenko and Brown for the theory of buildings and to the contributions of Alice Devillers and of Hendrik Van Maldeghem to the Oberwolfach report 20/2008 for an account on filtrations and their powerful applications), we do not include acstracts of these lectures.

On Monday afternoon and on Friday morning Aloysius Helminck presented the theory of involutions of algebraic groups, while in Monday's final lecture Max Horn showed how to combine Helminck's theory with Mühlherr's PhD thesis in order to obtain general and powerful results on the geometry of involutions of groups with a root group datum, a class of groups that contains the semisimple algebraic groups, the split Kac–Moody groups, and the split finite groups of Lie type.

Most of Tuesday and part of Wednesday were focussed on the Tits centre conjecture. In a series of two lectures Gerhard Röhrle and Michael Bate presented an algebraic-group approach towards proving the conjecture, while on Tuesday afternoon Katrin Tent presented a combinatorial approach and on Wednesday morning Linus Kramer reported on metric considerations in the context of the Tits centre conjecture. It is our impression that these four lectures have triggered additional activity towards proving the centre conjecture, and that one or more of these approaches will be successful in the near future.

The fourth talk on Tuesday afternoon was given by Yiannis Sakellaridis on spherical varieties and automorphic forms, while the second talk on Wednesday by Lizhen Ji presented compactifications of locally symmetric spaces. Thursday's talks by Sergey Shpectorov and by Paul Levy concentrated on involutions of affine buildings, respectively automorphisms of finite order of semisimple Lie algebras.

The remaining three talks were more topologically in nature. Guy Rousseau presented his theory of microaffine buildings, hovels, and Kac–Moody groups over ultrametric fields on Thursday. Thursday's fourth talk was by Bertrand Rémy on Satake compactifications of buildings via Berkovich theory. The conference was concluded by Pierre-Emmanuel Caprace's report on aspects of the structure of locally compact groups.

We are particularly pleased by the lively interaction between the participants during the long afternoon breaks (each morning's lectures finished at 11.30 a.m. while the afternoon sessions only started at 4.20 p.m.) and during the evenings.

Mini-Workshop: Symmetric Varieties and Involutions of Algebraic Groups

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Abstracts

Involutions of algebraic groups I and II

Aloysius G. Helminck

Symmetric k-varieties were introduced in the late 1970's as a generalization of both real reductive symmetric spaces and symmetric varieties to homogenous spaces defined over general fields of characteristic not 2. The *real reductive symmetric spaces*, are the homogeneous spaces $G_{\mathbb{R}}/H_{\mathbb{R}}$, where $G_{\mathbb{R}}$ is a reductive Lie group of Harish Chandra class and $H_{\mathbb{R}}$ is an open subgroup of the fixed point group of an involution of $G_{\mathbb{R}}$. The representations associated with these real reductive symmetric spaces (i.e. a decomposition of $L^2(G_{\mathbb{R}}/H_{\mathbb{R}})$ into irreducible components) had been studied intensively by many prominent mathematicians starting with a study of compact groups and their representations by Cartan [7], to a study of Riemannian symmetric spaces and real Lie groups by Harish Chandra [15] to a more recent study of the non Riemannian symmetric spaces starting in the 1970's by work of Faraut [12], Flensted Jensen [13] and Oshima and Sekiguchi [26]. These were soon studied by many mathematicians, including Brylinski, Carmona, Delorme, Matsuki, Oshima, Schlichtkrull, van der Ban and many others (see for example [25, 5, 32, 6, 34, 33, 10]). In the mid 1980's a Plancherel formula for the general real reductive symmetric spaces was announced by Oshima, although a full proof was not published until 1996 by Delorme [10]. See also van der Ban and Schlichtkrull for a different approach [34, 33]. In the late 1980's it seemed natural to generalize the concept of these real reductive symmetric spaces to similar spaces over the p-adic numbers and study the representations associated with these spaces. At that same time generalizations of these real reductive symmetric spaces to other base fields started to play a role in other areas, for example in the study of arithmetic subgroups (see [31]), the study of character sheaves (see for example [22, 14]), geometry (see [8, 9] and [1]), singularity theory (see [23] and [21]), and the study of Harish Chandra modules (see [2] and [36, 35]). This prompted Helminck and Wang to commence a study of rationality properties of these homogenous spaces over general base fields, see [20] for some first results. For any field k of characteristic not 2 they defined a *symmetric k-variety* as the homogeneous space $X_k := G_k/H_k$, where G is a reductive algebraic group defined over k and $H = G^\sigma$ the fixed point group of a k-involution $\sigma : G \to G$ of G. Here we have used the notation H_k for the set of k-rational points of an affine algebraic group H defined over k. As in the real case the p-adic symmetric k-varieties are also called reductive p-adic symmetric spaces or simply p-adic symmetric spaces.

For k the p-adic numbers it is natural to study the harmonic analysis of these p-adic symmetric spaces, similar as in the real case. Namely, let dx be a G_k-invariant measure on the symmetric k-variety $X_k = G_k/H_k$. Given a complex vector bundle over these spaces we get a natural representation π_ρ of G_k on its space of global sections, where ρ is the representation of H_k on the fibers. If the representation ρ of H_k is unitary, then the G_k-action on the space of global sections that are square-integrable with respect to dx is a unitary representation $(\pi_\rho, L^2(G_k/H_k, \rho))$ of G_k. In particular for the trivial line bundle over G_k/H_k, this leads to the regular representation π_1 of G_k on the Hilbert space $L^2(G_k/H_k)$ of square integrable functions on G_k/H_k. Since the group G_k is of type I, any unitary representation R of G_k on a separable Hilbert space $\mathcal{H}_R$ has an abstract direct integral decomposition

$$R \simeq \int_{\hat{G}_k}^{\oplus} R^\pi \, d\mu_R(\pi), \tag{1}$$

where $\hat{G}_k$ is the unitary dual of $G_k, d\mu_R$ a Borel measure on $\hat{G}_k, (\pi, \mathcal{H}_\pi)$ is a representative of a class in $\hat{G}_k$ and R^π is a multiple of π, see [11]. This holds in particular for $(\pi_\rho, L^2(G_k/H_k, \rho))$. The main aim of harmonic analysis is to decompose this representation as explicitly as possible into irreducible components, what is also called finding the "Plancherel decomposition".

Most of the representations occurring in this decomposition are representations induced from a parabolic k-subgroup. So in order to study these representations it is essential to first have a thorough understanding of the orbits of parabolic k-subgroups acting on these symmetric k-varieties. Other decompositions which play an important role in the study of these symmetric k-varieties are orbits of symmetric subgroups, orbits of maximal k-anisotropic (compact) subgroups and in the p-adic case also orbits of parahoric subgroups.

While there are descriptions for some of these orbit decompositions, many properties of these symmetric k-varieties remain open. In these talks we will give a survey of known results and open results about these orbit decompositions for symmetric k-varieties, and illustrate this all with a number of examples.

We mainly focus on orbits of a parabolic k-subgroup P acting on the symmetric k-variety G_k/H_k. These orbits play a fundamental role in the study of representations associated with these symmetric k-varieties. For k algebraically closed and $P = B$ a Borel subgroup these orbits were characterized by Springer [30] and a characterization of these orbits for general parabolic subgroups was given by Brion and Helminck in [4, 18]. For $k = \mathbb{R}$ and P a minimal parabolic k-subgroup characterizations were given by Matsuki [24] and Rossmann [29] and for general fields these orbits were characterized by Helminck and Wang [20].

These orbits can be characterized in several equivalent ways. They can be characterized as the P_k-orbits acting on the symmetric k-variety G_k/H_k by σ-twisted conjugation, or as the H_k-orbits acting on the flag variety G_k/P_k by conjugation or also as the set $P_k\backslash G_k/H_k$ of (P_k, H_k)-double cosets in G_k. The last is the same as the set of $P_k \times H_k$-orbits on G_k. For the characterization as $P_k \times H_k$-orbits in G_k let A be a σ-stable maximal k-split torus of P, $N_{G_k}(A)$ resp $Z_{G_k}(A)$ the normalizer resp. centralizer of A in G_k and set $\mathcal{V}_k = \{x \in G_k \mid x\sigma(x)^{-1} \in N_{G_k}(A)\}$. The group $Z_{G_k}(A) \times H_k$ acts on $\mathcal{V}_k$ by $(x, z) \cdot y = xyz^{-1}$, $(x, z) \in Z_{G_k}(A) \times H_k$, $y \in \mathcal{V}_k$. Let V_k be the set of $(Z_{G_k}(A) \times H_k)$-orbits on $\mathcal{V}_k$. Then $P_k\backslash G_k/H_k \cong V_k$.

For general parabolic k-subgroups one can first consider the set of (P, H)-double cosets in G. Then the (P_k, H_k)-double cosets in G_k can be characterized by the

(P, H)-double cosets in G defined over k plus an additional invariant describing the decomposition of a (P, H)-double coset into (P_k, H_k)-double cosets. For this one considers first the natural map of the set of $P_k \times H_k$-orbits on G_k to the set of $P \times H$-orbits on G. In terms of the orbit set V_k this map can be described as follows. Let $\mathcal{V}_A = \{x \in G \mid x\sigma(x)^{-1} \in N_G(A)\}$. Then $Z_G(A) \times H$ acts on $\mathcal{V}_A$ by $(x, z) \cdot y = xyz^{-1}$, $(x, z) \in Z_G(A) \times H$, $y \in \mathcal{V}_A$. Denote the set of $(Z_G(A) \times H)$-orbits on $\mathcal{V}_A$ by V_A. The natural inclusion map $\mathcal{V}_k \to \mathcal{V}$ induces a map $\eta : V_k \to V_A$, where η maps the orbit $Z_{G_k}(A)gH_k$ onto $Z_G(A)gH$. The set V_A is finite, but in general the set V_k is infinite. In a number of cases one can show that there are only finitely many $(P_k \times H_k)$-orbits on G_k. If k is algebraically closed, the finiteness of V_k was proved by Springer [30]. The finiteness of the orbit decomposition for $k = \mathbb{R}$ was discussed by Wolf [37], Rossmann [29] and Matsuki [24]. For general local fields this result can be found in [20].

The conjugacy classes of σ-stable tori play a fundamental role in the description of the double cosets $P_k \backslash G_k / H_k$ and $P \backslash G / H$. In fact in many cases these conjugacy classes determine the double cosets modulo the action of the Weyl group. In the second talk we discuss some results about these tori. First we consider the natural action of the Weyl group W of A on the double cosets and the relation with the conjugacy classes of these σ-stable tori. Next we look at the natural map from V_A into W, induced by the natural map $\mathcal{V}_A \to N_G(A)$. This map can not only be used to give another characterization of some of the conjugacy classes of σ-stable tori (or equivamently W-orbits in V_k and V_A), but also plays a fundamental role in the study of the geometry and combinatorics of these double cosets. This map also enables us to port the natural combinatorial structure on the image (contained in the set of twisted involutions in W) to V_A and V_k. Finally, using all this, we discuss characterizations of the various conjugacy classes of σ-stable tori occurring in the characterization of the double cosets. We first consider the case of algebraically closed fields where the H-conjugacy classes of σ-stable maximal tori can be completely described by conjugacy classes of involutions in the Weyl group. After that we consider the H-conjugacy classes of σ-stable maximal k-split tori. We conclude with a discussion of some results about how these H-conjugacy classes split into H_k-conjugacy classes. These results come from [16, 17]. For $k = \mathbb{R}$ a characterization and a full classification of the H_k-conjugacy classes of σ-stable maximal k-split tori will be given in a forthcoming paper [19]. For other fields this problem is still open, except in some specific cases.

There are many other properties of these orbit decompositions that play an important role in the study of these symmetric k-varieties and their applications. For example, there is a natural geometry associated with these double cosets related to the Zariski closures of the double cosets in the case of algebraically closed fields or the topological closures of the double cosets in the case of fields with a topology. This geometry plays a fundamental role in representation theory. For example, in the case that $k = \mathbb{C}$ a reductive group G with an involution σ can be viewed as the complexification of a reductive real Lie group G_0 such that σ is the complexification of a Cartan involution of G_0. Then G/H is the complexification of the symmetric space defined by G_0 and the Cartan involution. The H-action on G/B here appears in connection with the infinite-dimensional representation theory of the Lie group G_0. In particular, the geometry of H-orbits on G/B plays a fundamental role in the classification of Harish-Chandra modules for G_0 (see [36]).

There is a partial order on the double cosets defined by the Zariski (or topological) closures. If $\mathcal{O}_{v_1}$ and $\mathcal{O}_{v_2}$ are orbits, then $\mathcal{O}_{v_1} \leq \mathcal{O}_{v_2}$ if and only if $\mathcal{O}_{v_1}$ is contained in the closure of $\mathcal{O}_{v_1}$. This order is called the *Bruhat order on* V_A (or V_k) and it generalizes the usual Bruhat order on a connected reductive algebraic group defined by the Bruhat decomposition. In the case of the Bruhat decomposition of the group, Chevalley showed that this geometric Bruhat order corresponds with the combinatorially defined Bruhat order on the Weyl group (see [3] for the first published proof of this). The combinatorial Bruhat order on the Weyl group has been studied by many mathematicians and much is known about the corresponding poset. For symmetric varieties over algebraically closed fields, Richardson and Springer [27, 28] showed that for k algebraically closed there is a similar combinatorial description of the Bruhat order on V_A. However, in this case the combinatorics is considerably more complicated. In this case the combinatorics of the orbit closures corresponds to a combinatorial order on the set of twisted involutions in a Weyl group. Several of Richardson and Springer's results can be generalized to describe the Bruhat order on the sets V_A (or V_k) of orbits of minimal parabolic k-subgroups acting on symmetric k-varieties, but a full combinatorial description of the Bruhat order is still open.

We give a survey of results about the geometry and combinatorics of these orbit decompositions that play an important role in the study of these symmetric k-varieties and their applications. We start with an overview of results about the combinatorics of the related set of twisted involutions in the Weyl group, which play an important role in all of this. The description of the combinatorics in [30, 27] for the case of groups defined over an algebraically closed field depends on the existence of a Borel subgroup invariant under the involution σ. For groups defined over non algebraically closed fields a σ-stable minimal parabolic k-subgroup does not need to exist. One can still obtain a similar combinatorial characterization, but one has to pass to another involution. We conclude these talks with a brief discussion of the combinatorics of the twisted involution poset related to the Bruhat order.

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Involutions of algebraic and Kac–Moody groups

Max Horn
(joint work with Ralf Gramlich and Bernhard Mühlherr)

Let G be a group with a saturated twin BN-pair (B_+, B_-, N) of type (W, S). (That is, (W, S) is a Coxeter system, and (B_+, N) and (B_-, N) are BN-pairs of type (W, S), and $B_+ \cap B_- = N \cap B_+ = N \cap B_-$. In particular, $W \cong N/(B_+ \cap B_-)$.) Examples include connected reductive algebraic groups, Kac–Moody groups, and finite groups of Lie type.

Definition 1. *We call an involutory automorphisms θ of G a quasi-flip if it maps the subgroup B_+ to a conjugate of B_-.*

Note that by [2], for the group of $\mathbb{F}$-rational points of a connected reductive algebraic $\mathbb{F}$-group (and also for finite groups of Lie type), *any* abstract involutory automorphism automatically satisfies this condition (as in these groups, B_+ and B_- are minimal parabolic $\mathbb{F}$-subgroups, hence conjugate, and their images are again minimal parabolic $\mathbb{F}$-subgroups).

Proposition 2. *If θ is a quasi-flip, then there exists $x \in G$ such that $\theta(B_\pm) = xB_\mp x^{-1}$, thus $\theta(T) = xTx^{-1}$; θ induces an automorphism of (W, S) or order at most two.*

It is well know that in a group with twin BN-pair, the Bruhat and Birkhoff decompositions hold, that is

$$G = \bigsqcup_{w \in W} B_+ w B_+ \qquad \text{and} \qquad G = \bigsqcup_{w \in W} B_+ w B_-.$$

Due to this, any group with a twin BN-pair is associated to a twin building of the same type, namely $(G/B_+, G/B_-, \delta^*)$, where the distance functions $\delta_\pm : G/B_\pm \times G/B_\pm \to W$ on the twin halves G/B_+ and G/B_- are given by

$$\delta_\pm(gB_\pm, hB_\pm) = w \Leftrightarrow B_\pm g^{-1} hB_\pm \in B_\pm wB_\pm.$$

Similarly, the codistance is defined via

$$\delta^*(gB_\pm, hB_\mp) = w \Leftrightarrow B_\pm g^{-1} hB_\mp \in B_\pm wB_\mp.$$

The following is now an easy consequence of the above and Proposition 2:

Proposition 3. *Let G be a group with twin BN-pair, let $\mathcal{B}$ be the associated twin building. Suppose θ is a quasi-flip of G. Then θ induces a unique permutation θ' of the chambers of the twin building with the following properties:*

(1) $\theta'^2 = \mathrm{Id}$, *and θ' swaps the two halves of the twin building.*
(2) *θ' preserves distances and codistances up to the the unique automorphism θ induces on (W, S). In particular, adjacency and opposition of chambers is preserved.*

We call the induced involution θ' a *building quasi-flip* or just a *quasi-flip*, and often drop the distinction between θ and θ'. Note that this generalizes the concept of a building flip, which was introduced as part of the *Phan program*, see [1]

Consequently, we can now apply tools from building theory to study quasi-flips. A first consequence is the following:

Proposition 4. *Let θ be a quasi-flip of a group with saturated twin BN-pair (root group datum). Assume we are "not in characteristic two" (= all root groups are uniquely two-divisible). Then any Borel group B contains a θ-stable torus (=a group conjugate of $T = B_+ \cap B_-$ and normalized by θ). Geometrically, every chamber is contained in a θ-stable twin apartment.*

This generalizes a lemma by Helminck-Wang [4]. The proof is purely building theoretic. Using this, the following can be proven:

Theorem 5. *Let G be a group with twin-BN-pair (B_+, B_-, N) of type (W, S) and "not in characteristic two". Let θ be a quasi-flip of G and denote by $G_\theta := \{g \in G \mid \theta(g) = g\}$. If $\{A_i \mid i \in I\}$ are representatives of the G_θ-conjugacy classes of θ-stable maximal tori in G, then:*

$$G_\theta \backslash G/B_+ \cong \bigsqcup_{i \in I} W_{G_\theta}(A_i) \backslash W_G(A_i).$$

Remark 6. *This generalizes previous work by Matsuki* [6] *and Rossmann* [7] *for connected reductive $\mathbb{R}$-groups; Springer* [8] *for connected reductive groups over algebraically closed fields (description given in different but equivalent form); Kac, Wang* [5] *for Kac–Moody-groups over algebraically closed field in characteristic 0; and Helminck, Wang* [4] *for $\mathbb{F}$-rational points of connected reductive algebraic groups.*

Another result we obtained with the help of building theory is the following (which generalizes previously known results for semi-linear flips):

Theorem 7. *Let G be a locally finite split KM-group over $\mathbb{F}_{q^2}$, $q \geq 5$ and odd, with two-spherical diagram (and no G_2 residue). Let θ be a flip. Then G_θ is finitely generated.*

Remark 8. *Let G be a locally finite Kac–Moody-group of type (W, S). Then G is always finitely generated. For G_θ to be finitely generated, we have to assume that G is two-spherical. In fact Pierre-Emmanuel Caprace, Ralf Gramlich, and Bernhard Mühlherr have recently observed that G_θ is not finitely generated if G is not two-spherical and q is larger than $|S|$.*

If G is two-spherical, then it is finitely presented. For G_θ to be finitely presented, we need G to be at least three-spherical. This "gap" between G and G_θ is believed to extend to higher finiteness properties.

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Complete Reducibility and the Tits Centre Conjecture

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(joint work with Benjamin Martin)

The following covers the material presented in two consecutive linked talks.

1. Introduction

Let G be a connected reductive linear algebraic group defined over an algebraically closed field k. Let $X = X(G)$ be the spherical Tits building of G, cf. [14]. Recall that the simplices in X correspond to the parabolic subgroups of G, [12, §3.1]; for a parabolic subgroup P of G, we let x_P denote the corresponding simplex of X. The conjugation action of G on itself naturally induces an action of G on the building X, so we can view G as a subgroup of the automorphism group of X. Given a subcomplex Y of X, let $N_G(Y)$ denote the subgroup of G consisting of elements which stabilize Y (in this induced action).

Recall the *geometric realization* of X as a bouquet of n-spheres. A subcomplex Y of X is called *convex* if whenever two points of Y (in the geometric realization) are not opposite in X, then Y contains the unique geodesic joining these points, [12, §2.1]. A convex subcomplex Y of X is *contractible* if it has the homotopy type of a point, [12, §2.2]. The following is a version due to J-P. Serre of the so-called "Centre Conjecture" by J. Tits, cf. [13, Lem. 1.2], [10, §4], [12, §2.4], [15]. This has been proved by B. Mühlherr and J. Tits for spherical buildings of classical type [8].

Conjecture 1. *Let Y be a convex contractible subcomplex of X. Then there is a simplex in Y which is fixed by all automorphisms of X which stabilize Y.*

A point whose existence is asserted in Conjecture 1 is sometimes referred to in the literature as a "natural centre" of Y. For an overview of special cases of Conjecture 1 that have been established, frequently relying on a case-by-case analysis, see [7, p. 64], [8], [9], [10, §4], [12, §2.4], [15].

For a subgroup H of G let X^H be the fixed point subcomplex of the action of H, i.e., X^H consists of the simplices $x_P \in X$ such that $H \subseteq P$. Thus, if $H \subseteq K \subseteq G$ are subgroups of G, then we have $X^K \subseteq X^H$; observe that X^H is always convex, [12, Prop. 3.1]. One of our results, Theorem 6, gives a short, conceptual proof of a special case of Conjecture 1 in case the subcomplex Y in question is of the form $Y = X^H$ for H a subgroup of G, and we consider automorphisms from $N_G(Y)$.

The initial motivation for Tits' Conjecture 1 was a question about the existence of a canonical parabolic subgroup associated with a unipotent subgroup of a Borel subgroup of G (cf. [10, §4.1], [12, §2.4]). This existence theorem was ultimately proved by other means, [4, §3]. In [2, Ex. 3.6] we show that this result is a special case of Theorem 6.

2. Serre's notion of complete reducibility

Following Serre [12, Def. 2.2.1], we say that a convex subcomplex Y of X is X*-completely reducible* (X-cr) if for every simplex $y \in Y$ there exists a simplex $y' \in Y$ opposite to y in X. The following is part of a theorem due to Serre, [10, Thm. 2]; see also [12, §2] and [15].

Theorem 2. *Let Y be a convex subcomplex of X. Then Y is X-completely reducible if and only if Y is not contractible.*

Note that many subcomplexes which arise naturally in the building are fixed-point subcomplexes. For example, the apartments of X are the subcomplexes X^T for maximal tori T of G and, more generally, the convex hull of two simplices x_P and $x_{P'}$ is $X^{P \cap P'}$.

Following Serre [12], we say that a (closed) subgroup H of G is G*-completely reducible* (G-cr) provided that whenever H is contained in a parabolic subgroup

P of G, it is contained in a Levi subgroup of P; for an overview of this concept see for instance [11] and [12]. In the case $G = \mathrm{GL}(V)$ (V a finite-dimensional k-vector space) a subgroup H is G-cr exactly when V is a semisimple H-module, so this faithfully generalizes the notion of complete reducibility from representation theory. If H is a G-completely reducible subgroup of G, then H^0 is reductive, [11, Property 4]. Serre's proof uses the aforementioned construction due to Borel and Tits [4, §3].

Since X^H is a convex subcomplex of $X = X(G)$ for any subgroup H of G, Theorem 2 applies in this case, [12, §3]:

Theorem 3. *Let H be a subgroup of G. Then H is G-completely reducible if and only if the subcomplex X^H is X-completely reducible.*

Remark 4. *By convention, the empty subcomplex of X is not contractible.*

Our next result [1, Thm. 3.10] gives an affirmative answer to a question by Serre, [11, p. 24]. The special case when $G = \mathrm{GL}(V)$ is just a particular instance of Clifford Theory.

Theorem 5. *Let $N \subseteq H \subseteq G$ be subgroups of G with N normal in H. If H is G-completely reducible, then so is N.*

3. Tits' Centre Conjecture for fixed point subcomplexes

Here is the main result of the first talk.

Theorem 6. *Let Y be a convex, contractible subcomplex of X. Suppose that Y is of the form $Y = X^H$ for a subgroup H of G. Then there is a simplex in Y which is fixed by all elements in $N_G(Y)$.*

Proof. Let M be the intersection of all parabolic subgroups of G corresponding to simplices in Y. Since $H \subseteq M$, we have $X^M \subseteq X^H$. But every parabolic subgroup containing H contains M, by definition of M. Hence $X^M = X^H$. Set $K := N_G(Y)$. It is clear that M is normal in K. Since $X^K \subseteq X^M$, it suffices to show that $X^K \neq \varnothing$. Now $Y = X^M$ is contractible, so Theorem 3 implies that M is not G-cr. Thus, by Theorem 5, it follows that K is not G-cr and again by Theorem 3 that X^K is contractible. In particular, X^K is non-empty, by Remark 4. Thus K has a fixed point in X^M, as claimed. □

Remark 7. *(i). The special case of Theorem 6 when $G = \mathrm{GL}(V)$ generalizes the classical construction of upper and lower Loewy series, see* [2, Rem. 3.2(ii)].

(ii). In [12, Prop. 2.11]*, J-P. Serre showed that Theorem 5 is a consequence of Tits' Centre Conjecture 1. So, Theorem 6 is just the reverse implication of Serre's result* [12, Prop. 2.11] *in the special case when Theorem 5 applies.*

4. The theory of Kempf-Rousseau

Suppose the reductive group G acts on an affine variety V. Let $Y(G)$ denote the set of cocharacters of G, that is, the set of homomorphisms $\lambda : k^* \to G$. For each $v \in V$, $\lambda \in Y(G)$, we can define the morphism $\phi = \phi_{v,\lambda} : k^* \to V$ by

$\phi(a) = \lambda(a) \cdot v$ for each $a \in k^*$. If ϕ extends to a morphism $\overline{\phi} : k \to V$, then we say that $\lim_{a\to 0} \lambda(a) \cdot v$ *exists*, and set this limit to be $\overline{\phi}(0)$. Note that if such a limit exists, it is uniquely defined.

Taking limits along cocharacters in this way allows one to decide whether or not the G-orbit of a point $v \in V$ is closed:

Theorem 8 (Hilbert–Mumford Theorem)**.** *The orbit $G \cdot v$ of a point $v \in V$ is not closed if and only if there exists $\lambda \in Y(G)$ such that $\lim_{a\to 0} \lambda(a) \cdot v$ exists and lies outside $G \cdot v$.*

In this section we describe a refinement of this result due to Kempf [6] and Rousseau [9]. First, for each $v \in V$, set

$$\Lambda(v) = \{\lambda \in Y(G) \mid \lim_{a\to 0} \lambda(a) \cdot v \text{ exists}\}$$

and

$$\Lambda_0(v) = \{\lambda \in Y(G) \mid \lim_{a\to 0} \lambda(a) \cdot v \text{ exists and lies outside } G \cdot v\}.$$

Now recall that for each $\lambda \in Y(G)$, we have the parabolic subgroup $P_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$, and the Levi subgroup $L_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} = g\}$ of P_λ. We define the so-called *rational spherical building* of G (cf. [7, Ch. 2§2], [9, Sec. IV]). Let $\Delta = \Delta(G)$ be the set of nonzero cocharacters of G modulo the following equivalence relation:

$$\lambda_1 \sim \lambda_2$$
$$\Longleftrightarrow$$
$$\exists n_1, n_2 \in \mathbb{N}, g \in P_{\lambda_1} \text{ such that } \lambda_2(a^{n_2}) = g\lambda_1(a^{n_1})g^{-1} \ \forall\ a \in k^*.$$

Note that $\lambda_1 \sim \lambda_2$ implies $P_{\lambda_1} = P_{\lambda_2}$.

When G is semisimple, Δ can be viewed as the dense set of *rational points* in the building $X = X(G)$; basically, for each $\lambda \in Y(G)$, we get an equivalence class in Δ which corresponds to a point of the simplex x_{P_λ} in (the geometric realization of) X. (When G is reductive but not semisimple, then the situation is a tiny bit more complicated, see [9] for precise details). The notion of convexity also makes sense in Δ.

The subsets $\Lambda(v)$ and $\Lambda_0(v)$ defined above naturally correspond to subsets $C(v)$ and $C_0(v)$ in Δ. If $G \cdot v$ is closed, then $C_0(v)$ is empty, by the Hilbert–Mumford Theorem. On the other hand, Kempf and Rousseau showed that if $G \cdot v$ is not closed, the subset $C_0(v)$ is convex and has a natural centre in the sense of Conjecture 1 (note that $C_0(v)$ is not a subcomplex in general, but Tits' conjecture makes sense in this slightly more general setting, see [7], [9]). This centre corresponds to an equivalence class of "worst possible" cocharacters for the point $v \in V$; and all these cocharacters correspond to the same parabolic subgroup $P(v)$ of G. This parabolic subgroup is sometimes called a *canonical destabilizing parabolic* for v. It has many good properties: for example, $\mathrm{Stab}_G(v) \subseteq P(v)$.

The discussion above shows that a version of Conjecture 1 holds for convex subsets of X which arise from G-actions on affine varieties. This leads to the following question:

Question 9. *Given a convex contractible subcomplex Y of X, can we find a G-action on a variety V and a point $v \in V$ such that $C_0(v)$ is contained in Y and* $\mathrm{Stab}_{\mathrm{Aut}X} Y$ *stabilizes $C_0(v)$?*

If one could find such a construction for a given subcomplex Y, it would provide a natural centre for Y. In the next section we illustrate one case where this idea works.

5. Examples from G-complete reducibility

Consider the affine variety $V = G^n$, where G acts diagonally by simultaneous conjugation:

$$g \cdot (x_1, \ldots x_n) = (gx_1g^{-1}, \ldots, gx_ng^{-1}).$$

For a point $v = (x_1, \ldots, x_n) \in V$ let $H = H(v) = \overline{\langle x_1, \ldots, x_n \rangle}$, the algebraic subgroup of G generated by the elements $x_1, \ldots, x_n$. Then [1, Cor. 3.7] says that $G \cdot v$ is closed if and only if H is G-cr.

For each $\lambda \in Y(G)$, $\lim_{a \to 0} \lambda(a) \cdot v$ exists if and only if $H \subseteq P_\lambda$; moreover, this limit is outside $G \cdot v$ if and only if H is not contained in any Levi subgroup of P_λ. Thus, the subset $C(v)$ of the previous section simply consists of the rational points of the subcomplex X^H, and the subset $C_0(v)$ is empty if and only if H is G-cr. If H is not G-cr, then $C_0(v)$, and hence X^H, has a natural centre. These are essentially the tools used to prove Theorem 5, and we see an instance where Question 9 has a positive answer.

In the forthcoming paper [3] (with Rudolf Tange), we strengthen the results of Kempf–Rousseau by combining them with ideas of Hesselink [5]. One consequence of this strengthening is the following theorem:

Theorem 10. *Suppose $Y \subseteq X$ is a convex subcomplex. Set $H = \cap_{x_P \in Y} P$, and suppose Γ is a subgroup of G stabilizing Y. Suppose the following condition holds:*

(1) $\exists x_P \in Y$ *such that H is not contained in any Levi subgroup of P.*

Then there exists $x_{P_0} \in Y$ such that $\Gamma \subseteq P_0$.

This theorem provides a natural centre x_{P_0} for any subcomplex Y satisfying condition (1). In particular, subcomplexes of the form $Y = X^H$, where H is a non-G-cr subgroup of G, are covered by this result, so it is a strengthening of Theorem 6.

We finish by announcing another result which will appear in [3]. Suppose now that k is any field, and that G is defined over k. Say that a subgroup H of G is *G-cr over k* if whenever H is contained in a k-defined parabolic subgroup P of G, there exists a k-defined Levi subgroup L of P with $H \subseteq L$. Then Serre has made the following conjecture:

Conjecture 11 (Serre)**.** *Given a separable extension k_1/k, a subgroup H of G is G-cr over k if and only if it is G-cr over k_1.*

In [1, Thm. 5.8], we showed that the conjecture is true if k_1 and k are perfect. In [3], we show that H is G-cr over k_1 implies H is G-cr over k. The other direction is still open. There are examples which show that the conjecture fails in both directions if k_1/k is not a separable extension.

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Spherical varieties in automorphic forms

Yiannis Sakellaridis

1. Spherical varieties

Let $\mathbf{G}$ be a connected reductive group over a number field k, and let $\mathbf{X}$ be a homogeneous spherical variety for $\mathbf{G}$ over k. By definition, $\mathbf{X}$ carries an action of $\mathbf{G}$ such that, over the algebraic closure, the Borel subgroup has an open orbit. The stabilizer subgroup of a point on a spherical variety is called a spherical subgroup. This is a very interesting class of varieties which includes all symmetric (stabilizers are fixed point groups of involutions) and horospherical (stabilizers contain a maximal unipotent subgroup) ones. In the context of algebraic geometry and invariant theory, they have been studied extensively by Brion, Knop, Luna, Vust and many others, and a lot of interesting structure has been discovered. For the results mentioned in this talk, $\mathbf{G}$ is assumed split.

2. Harmonic analysis and automorphic forms

Given a space X with an action of group G, ($\mathbb{C}$-valued) functions on X are naturally a representation of G under translation: $(g \cdot f)(x) := f(x \cdot g)$, where we assume that G acts on the right. The purpose of harmonic analysis is to analyze this representation. According to the desired category of representations, we restrict ourselves to suitable subspaces of functions. For example, the irreducible (admissible) subrepresentations of $C^\infty(\mathbb{R})$ under the action of the group $\mathbb{R}$ are all one-dimensional spaces spanned by functions $t \mapsto \exp(st)$ (where $s \in \mathbb{C}$), but the unitary representation $L^2(\mathbb{R})$ decomposes as a direct integral of one-dimensional Hilbert spaces corresponding to imaginary values of s only.

The object of the theory of automorphic forms is harmonic analysis on the space $\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A})$, where $\mathbf{G}, k$ are as in the first paragraph and $\mathbb{A}$ denotes the ring of adeles of k. Roughly speaking, an automorphic representation is an irreducible admissible representation π of $\mathbf{G}(\mathbb{A})$, together with an embedding $\nu : \pi \hookrightarrow C^\infty(\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}))$. Important invariants of automorphic representations are their L-functions $L(\pi, \rho, s)$, where ρ is an extra piece of data, an algebraic representation of the "L-group". As notation suggests, the definition of the L-function depends only on π as an abstract representation of $\mathbf{G}(\mathbb{A})$. More precisely, π is isomorphic to a "restricted tensor product" $\otimes'_v \pi_v$, where v runs over all completions ("places") k_v of k and π_v is an irreducible admissible representation of $G_v := \mathbf{G}(k_v)$. The L-function is defined in some region of convergence as an Euler product of local factors $L_v(\pi_v, \rho, s)$, and the factor at v depends only on π_v.

3. Local Langlands correspondence

To describe the local factor L_v, we recall that according to the local Langlands conjectures there should be a canonical way to attach to π_v a conjugacy class of "Langlands parameters", which are homomorphisms $\phi_v : \mathcal{L}_{k_v} \to {}^L G$. Here, $\mathcal{L}_{k_v}$ is the "local Langlands group", a version of the Galois group of $\bar{k}_v/k_v$.

The L-group ${}^L G$ is the semidirect product of a complex reductive group $\check{G}$ with $\mathrm{Gal}(\bar{k}_v/k_v)$, and the homomorphism ϕ_v is assumed to be over $\mathrm{Gal}(\bar{k}_v/k_v)$. Then the local Euler factor $L_v(\pi_v, \rho, s)$ is defined as the Artin L-factor of the "Galois representation" $\rho \circ \phi_v$.

4. Period integrals

Automorphic representations and their L-functions have long been studied with the help of period integrals: Let $\mathbf{H}$ be a reductive spherical subgroup of $\mathbf{G}$ and let $\mathcal{P}$ be the distribution: $\phi \mapsto \int_{\mathbf{H}(k)\backslash\mathbf{H}(\mathbb{A})} \phi(h) dh$ on $\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A})$, where dh denotes an invariant measure on $\mathbf{H}(k)\backslash\mathbf{H}(\mathbb{A})$. Let (π, ν) be an automorphic representation as above. Composing ν with $\mathcal{P}$ (assuming that this composition makes sense) we get an $\mathbf{H}(\mathbb{A})$-invariant functional on π. It has been observed that very often the non-vanishing of this functional reveals some qualitative information about the automorphic representation – namely, that it is a "functorial lift" from some other group. Also, if it is non-zero then it often is equal to a special value of an L-function, when evaluated on suitable vectors of π. The limited space of this abstract makes it impossible to even try to give a taste of the wealth of statements of this form which have been discovered. I only mention that a major source of examples and method of proof of such statements is the relative trace formula, devised by Jacquet and his collaborators. I also mention that the theory of Rankin-Selberg integrals was reinterpreted in [4] using non-reductive spherical subgroups.

5. Questions and results

The goal of this talk is to present some results and conjectures which attempt to create a general picture about the "qualitative information" that a period integral reveals and its relationship with L-functions. We start with some questions:

Question 1. *Which representations π of $\mathbf{G}(\mathbb{A})$ admit a non-zero $\mathbf{H}(\mathbb{A})$-invariant functional?*

This is essentially a local question, at least under the following result [5]:

Proposition 2 (Under additional assumptions)**.** *If* $\mathrm{Hom}_{H_v}(\pi_v, \mathbb{C}) \neq 0$ *for every* v *then* $\mathrm{Hom}_{\mathbf{H}(\mathbb{A})}(\pi, \mathbb{C}) \neq 0$.

Therefore, the question asks which representations π_v of G_v admit an H_v-invariant functional which, by Frobenius reciprocity, is equivalent to asking which representations admit an embedding: $\pi_v \hookrightarrow C^\infty(H_v\backslash G_v)$. Such representations are called H_v-distinguished. Not surprisingly, the correct setting is obtained by replacing $H_v\backslash G_v$ by $X_v := \mathbf{X}(k_v)$ where $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$. For the simplest representations, namely "unramified" ones, i.e. those which have a vector invariant under a "good" maximal compact subgroup K_v, a partial answer was given in [3], where a necessary condition for distinguished unramified representations was given in terms of their Langlands parameter, and also a formula for the dimension of the space $\mathrm{Hom}_{G_v}(\pi_v, C^\infty(X_v))$ for a generic π_v satisfying those conditions.

While this is the analog of the question "which are the irreducible representations of $\mathbb{R}$ in $C^\infty(\mathbb{R})$?", a nicer answer presents itself when we ask the L^2 question: Decompose $L^2(X_v)^{K_v}$ (we are assuming here a fixed invariant measure on X_v) as a direct integral of spaces belonging to irreducible representations.

Theorem 3 (Under additional assumptions, [5])**.** *The Hilbert space $L^2(X_v)^{K_v}$ admits an explicit direct integral decomposition over irreducible unramified representations with X-distinguished Arthur parameters, and with Plancherel density at the representation π_v related to an explicit quotient of local L-values $L_X(\pi_v)$.*

Here, Arthur parameters are a variant of Langlands parameters, of the form: $\mathrm{SL}_2(\mathbb{C}) \times \mathcal{L}_{k_v} \to {}^LG$. We say that an Arthur parameter is X-distinguished, if the image of $\mathcal{L}_{k_v}$ (is bounded and) lies in a certain subgroup $\check{G}_X \times \mathrm{Gal}(\bar{k}_v/k_v)$ of

${}^LG = \check{G} \times \mathrm{Gal}(\bar{k}_v/k_v)$ (recall that our results are for $\mathbf{G}$ split) and its restriction to SL_2 is the principal SL_2 of a distinguished Levi subgroup $\check{L}(X) \subset \check{G}$. The subgroup $\check{G}_X$ first appeared in the work of Gaitsgory and Nadler [1] in the context of the geometric Langlands program.

Question 4. *If (π, ν) is an automorphic representation and $\mathcal{P} \circ v$ is non-zero, can it be expressed as an Euler product, and how?*

In joint work with Akshay Venkatesh [6] we generalize a conjecture of Ichino and Ikeda [2], which was generalizing a theorem of Waldspurger. Under some multiplicity-one assumptions, the conjecture relates the functional $\mathcal{P} \circ v$ to an Euler product of local H_v-invariant functionals which appear in the Plancherel formula for $L^2(X_v)$. Moreover, we conjecture that the support of Plancherel measure for $L^2(X_v)$ is contained in the set of representations with X-distinguished Arthur parameters, and using a method of Joseph Bernstein we reduce the latter conjecture to the discrete spectra of spherical varieties:

Theorem 5 (Under additional assumptions)**.** *There is a direct sum decomposition: $L^2(X) = \oplus_\Theta L^2(X)_\Theta$, where Θ varies over all conjugacy classes of Levi subgroups of $\check{G}_X$ and $L^2(X)_\Theta$ is explicitly described in terms of the discrete (modulo the center) spectrum of a "relevant" spherical variety for a corresponding Levi subgroup L_Θ of G.*

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Tits' Center Conjecture

Katrin Tent

(joint work with Chris Parker)

Recall that the canonical example of a spherical building, $\Delta(G,k)$ for a reductive k-isotropic algebraic group $G(k)$,can be described as the simplicial complex

$$\Delta(G,k) = \{P \subseteq G \colon P\ k\text{-parabolic}, \supseteq\}.$$

Thus, vertices correspond to maximal parabolics and the incidence is given by reverse inclusion. Tits' Center Conjecture states that a convex subcomplex Ω

of a spherical building Δ either contains opposites or has a center. Considering buildings as simplicial complexes we can state the conjecture more precisely as follows: either every simplex of Ω has an opposite simplex in Ω, in this case Ω is called completely reducible, or there is a simplex which is fixed under $Aut(\Omega)$.

In [3], it was shown that for Ω to be completely reducible it suffices to show that every vertex of Ω has an opposite vertex in Ω and this result was used by Mühlherr and Tits [2] to prove the center conjecture for non-exceptional buildings.

Assuming that Ω is a subcomplex of maximal dimension, we can generalize Serre's result using the following lemma of Tits [4], p. 54:

Lemma 1. *Suppose s and s' are opposite simplices of a spherical building Δ and t_1, t_2 are simplices in the residue of s. Let t_1' denote the projections of t_1 onto s'. Then t_1 and t_2 are opposite in the residue of s if and only if t_2 and t_1' are opposite in Δ.*

Lemma 2. *Let Δ be an irreducible spherical building of type (W, I). Let Ω be a convex subcomplex of Δ of maximal dimension. If for some type of vertices every element of Ω has an opposite, then Ω is completely reducible.*

Proof. Suppose inductively that every simplex in Ω of type $J \subset I$ has an opposite in Ω. Let i be a neighbour of J in the Dynkin diagram.

Let C_0 be a chamber, $x_0 \subset C_0$ a simplex of type J, and let $l_0 \in C_0$ be of type i. We will construct an opposite for $z = x_0 \cup \{l_0\}$.

Let x_0^o be an opposite of x_0 (of type J^o). Put $C_0' = proj_{x_0^o} C_0$ and $C_1 = proj_{l_0} C_0'$. Let $x_1 \in C_1$ be of type J. So $x_1 \neq x_0$ and $y_0 = proj_{x_1} x_0 \supseteq x_1 \cup \{l_0\}$ (by Tits 2.30.1 and 2.30.5). We will first find an opposite of the simplex y_0.

Let $y_1 = proj_{x_1} x_0^o$, so y_1 and y_0 are opposite in the residue of x_1. Let x_1^o be opposite x_1, By Tits' lemma above, the projection y_2 of y_1 to x_1^o is opposite y_0.

In order to find an opposite for the simplex z, notice that $proj_{l_0} x_0 = z$. Let $z_1 = proj_{l_0} x_0^o$, so z_1 and z are opposite in the residue of l_0. The projection of z_1 to the opposite of l_0 in y_2 now yields the required opposite of z.

The claim now follows as (long as) the Dynkin diagram is connected. □

As an immediate corollary we can show:

Corollary 3. *The center conjecture holds for convex chamber subcomplexes of irreducible spherical buildings of classical type.*

Proof. For buildings of type A_n, B_n (or C_n) and D_n, consider the vertex type corresponding to 1-dimensional subspaces (for A_n) and 1-dimensional isotropic subspaces in the other cases. For simplicity we only consider the type preserving case.

A_n**:** If such a vertex in Ω does not have an opposite in Ω, then it is contained in all the hyperplanes of Ω. Thus the intersection of all hyperplanes of Ω is the required center.

B_n **or** D_n**:** A 1-dimensional isotropic subspace has no opposite if and only if it is collinear with every other vertex of this type in Ω. Hence the set of all vertices in Ω of this type having no opposite is contained in a totally isotropic subspace. □

Theorem 4. *The center conjecture holds for buildings of type F_4.*

The proof uses the geometry of metasymplectic spaces as described by Cohen [1]. The first lemma allows us to concentrate on points. Since the F_4-geometry is closely related to the geometries of E_6, E_7 and E_8, we hope to finish the remaining cases with similar arguments.

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Completely reducible sets in spherical buildings

Linus Kramer

In my lecture I explained the structure of completely reducible sets in spherical buildings from a 'Riemannian' viewpoint.

Let X be a (weak) spherical building, viewed as a simplicial complex [14]. Let $\bar{X}$ be a simplicial refinement of X, eg. $\bar{X} = sd\,X$ is the barycentric subdivision. We require that this subdivision in the apartments is invariant under the Coxeter group and under the opposition involution. We tacitly identify the complex $\bar{X}$ with its CAT(1) metric realization. A subcomplex $A \subseteq \bar{X}$ is called *convex* if for all points $x, y \in A$ with $d(x, y) < \pi$, the geodesic segment $[x, y]$ is also contained in A. (This is the usual notion of convexity in CAT(1) spaces.) A convex subcomplex A is called *completely reducible* if every vertex $p \in A^{(0)}$ has an opposite $q \in A^{(0)}$, i.e. $d(p, q) = \pi$. The terminology is due to Serre [13] and motivated as follows: a representation $\Gamma \to GL_nF$ is completely reducible if and only if the fixed point set $X(GL_nF)^\Gamma$ of Γ in the spherical building $X(GL_nF)$ is completely reducible.

If G is a reductive Lie group and K is a finite or compact subgroup, then the fixed point set $X(G)^K$ is again completely reducible. This can be seen by looking at the noncompact symmetric space Z of G: the group K has a fixed point z in Z and so each fixed point ξ in the boundary $\partial Z = X$ has an antipodal point which is determined by the geodesic from z to ξ, extended in the opposite direction.

It is clear that the fixed point set of any group acting on a spherical building X is convex. This is one of the main motivations for studying such subsets. If the group action is not type preserving, then the fixed point set will in general not be a subcomplex of X, but rather of the barycentric subdivision $\bar{X} = sd\,X$. This is the main reason why we allow subtriangulations. One could also consider arbitrary closed convex subsets of X as in [2, 3]. Then most of the results presented here remain true, but one has to replace some combinatorial arguments by ultralimit constructions. The following observations can be found in [2, 3] and [13]. By a *round sphere* we mean a metric sphere of perimeter 2π.

Lemma 1. *Suppose that A is completely reducible and that $p \in A^{(0)}$. Then $lk(p) \cap A$ is completely reducible in $lk(p)$.*

Proposition 2. *Let A be convex and of dimension m. Then the following are equivalent.*

(1) *A is completely reducible.*
(2) *A is not contractible.*
(3) *A contains a round m-sphere.*
(4) *A contains a pair of antipodal vertices p, q and $lk(p) \cap A$ is completely reducible in $lk(p)$.*

It is also easy to see that a convex subcomplex $A \subseteq \bar{X}$ is always a pure chamber complex. Following [13] we call a round $\dim A$-sphere in A a *Levi sphere*. One observes that in the completely reducible case, A contains many Levi spheres. In fact, we have the following.

Lemma 3. *Suppose that A is completely reducible. The any two simplices $a, b \in A$ are contained in some Levi sphere.*

A simplex $b \in A$ of codimension 1 in A is called *singular* if it is contained in three maximal simplices in A. One shows that the set of singular simplices in a fixed Levi sphere S is invariant under the opposition involution and under the subgroup $W(S) \subseteq \mathrm{Isom}(S)$ generated by the reflections along singular simplices. The group $W(S)$ is a finite reflection group (and therefore a Coxeter group). The Levi sphere S splits naturally as a metric join $S = \mathbb{S}^k * \Sigma(W(S))$, where $\Sigma(W(S))$ is the Coxeter complex of $W(S)$. Furthermore, the singular structure in S can be transported from one Levi sphere to another along 'galleries' in A. This leads to the following structure result for completely reducible sets.

Theorem 4. *Let A be completely reducible in a thick spherical building X. Then there is a unique thick building Y such that A is isometric to a spherical join $\mathbb{S}^k * Y$.*

Note that this result says something even for $A = X$: every weak spherical building is the join of a sphere and a unique thick spherical building [12], [5]. The theorem above can also be easily deduced from the results in [6], but the proof we indicated is more direct.

We noted before that a convex set A which is not completely reducible is contractible.

Conjecture 5 (Center Conjecture)**.** *If A is convex and contractible, then $Aut(A)$ or $Aut(\bar{X}, A)$ has a fixed point in A.*

This conjecture, which was the topic of several lectures of this conference, is known to hold in the following cases:

a) $\bar{X} = X$ is a building of classical type [9] or of type F_4 [10].

b) $\dim A \leq 2$ [2, 3].

c) A is contained in a ball of radius $r \leq \pi/2$ [11], [2, 3]. (For $r < \pi/2$ this is the Bruhat-Tits fixed point theorem [4], the case $r = \pi/2$ is proved in [2, 3] using ultralimits.)

d) $\bar{X}$ is a thin building [15].

e) A contains a round sphere of codimension 1 [2, 3].

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Cofinite universal spaces for proper actions of arithmetic groups and mapping class groups

Lizhen Ji

This talk explains some constructions of natural cofinite universal spaces for proper actions of arithmetic groups Γ and mapping class groups $Mod_{g,n}$, connections with the spherical Tits buildings of algebraic groups and its analogue, the curve complex of surfaces, and some applications.

1. Definitions and preliminaries

Given a discrete group Γ, a Γ-space $\underline{E}\Gamma$ is called a *universal space* for proper actions of Γ if

(1) Γ acts properly on $\underline{E}\Gamma$, in particular, for every point $x \in \underline{E}\Gamma$, its stabilizer Γ_x is finite,
(2) for every finite subgroup $F \subset \Gamma$, the fixed-point set $(\underline{E}\Gamma)^F$ is nonempty and contractible.

If the quotient $\Gamma\backslash\underline{E}\Gamma$ is a finite CW-complex, then it is called a *cofinite universal space.*

If Γ is torsion-free, then $\underline{E}\Gamma$ is equal to the universal space for proper and fixed point free actions of Γ, which is denoted by $E\Gamma$, and the quotient $\Gamma\backslash E\Gamma$ is a classifying space $B\Gamma$, i.e., a $K(\Gamma, 1)$-space.

One difference between the spaces $\underline{E}\Gamma$ and $E\Gamma$ is that any group Γ containing nontrivial torsion elements does not admit finite dimensional $E\Gamma$, but many important examples of such groups do admit finite dimensional $\underline{E}\Gamma$ (for example, as the results of this talk show). It is perhaps important to point out that many natural groups such as the basic arithmetic group $SL(n, \mathbb{Z})$ and the mapping class groups $Mod_{g,n}$ of a surface $S_{g,n}$ of genus g with n-punctures contain nontrivial torsion elements.

An important and natural problem is to determine when a group Γ admits a cofinite universal $\underline{E}\Gamma$ space and find explicit models, for example, in studying the Baum-Connes conjecture and the Farrell-Jones conjecture (see the survey paper [Lu] for detailed discussion and references). Other important applications concern cohomological properties of Γ, as the following result shows (see [IJ]).

Proposition 1. (1) *If $\underline{E}\Gamma$ is assumed to be cofinite, then Γ contains only finitely many conjugacy classes of finite subgroups.*
(2) *Assume that Γ admits torsion-free subgroups of finite index. If $\dim \underline{E}\Gamma < +\infty$, then the virtual cohomological dimension $vcd\,(\Gamma)$ of Γ is bounded by $vcd\,(\Gamma) \leq \dim \underline{E}\Gamma$.*
(3) *If $\underline{E}\Gamma$ is further assumed to be a manifold with corners such that its boundary $\partial\underline{E}\Gamma$ is homotopic to a bouquet of spheres $\vee S^{r-1}$, then Γ is a virtual duality group of dimension equal to $\dim \underline{E}\Gamma - r$; furthermore, Γ is a virtual Poincaré duality group if and only if the bouquet $\vee S^{r-1}$ contains exactly one sphere S^{r-1}.*

2. Arithmetic groups

Let $\mathbf{G}$ be a semisimple linear algebraic group defined over $\mathbb{Q}$, $G = \mathbf{G}(\mathbb{R})$ be the real locus, which is a real semisimple Lie group of finitely many connected components, and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Let $K \subset G$ be a maximal compact subgroup. Then $X = G/K$ with an invariant metric is a symmetric space of noncompact type, and Γ acts isometrically and properly on X.

Since X is simply connected and nonpositively curved, the Cartan fixed point theorem implies that X is a $\underline{E}\Gamma$-space. It is known that the quotient $\Gamma\backslash X$ is compact if and only if the $\mathbb{Q}$-rank of $\mathbf{G}$, denoted by $r_{\mathbb{Q}}$, is equal to 0, which is in turn equivalent to that $\mathbf{G}$ does not contain any proper $\mathbb{Q}$-parabolic subgroup.

If $\Gamma\backslash X$ is compact, then the existence of equivariant triangulation of X implies that X is a cofinite $\underline{E}\Gamma$ space. On the other hand, if $\Gamma\backslash X$ is non-compact, then it is not a cofinite space.

Assume from now on that $r_{\mathbb{Q}} > 0$. There are two ways to overcome the non-compactness problem of $\Gamma\backslash X$. The first method is to construct a suitable compactification. In [BS], Borel and Serre constructed a partial compactification $\overline{X}^{BS}$ satisfying the following properties and hence obtained desired cohomological properties of Γ:

(1) $\overline{X}^{BS}$ is a real analytic manifold with corners whose interior is equal to X,
(2) the boundary components of $\overline{X}^{BS}$ are contractible and parametrized by the spherical Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$, and hence the Solomon-Tits Theorem on $\Delta_{\mathbb{Q}}(\mathbf{G})$ implies that the boundary $\partial\overline{X}^{BS}$ is homotopic to a bouquet of infinitely many spheres $S^{r_{\mathbb{Q}}-1}$,
(3) the Γ-action on X extends to a real analytic, proper action on $\overline{X}^{BS}$,
(4) for any torsion-free arithmetic group Γ, the quotient $\Gamma\backslash\overline{X}^{BS}$ is a compact manifold with corners,
(5) and consequently, Γ is a virtual duality group of dimension equal to $\dim X - r_{\mathbb{Q}}$, but is not a virtual Poincaré duality group.

It is natural to expect that for an arithmetic group contain torsion elements, $\overline{X}^{BS}$ is a cofinite universal space (see [Lu]). This is indeed true and was proved in [Ji].

Another natural method is to construct a compact deformation retract of $\Gamma\backslash X$ which is a finite CW-complex, for example, a submanifold with corners. Such compact deformation retracts have been constructed by the joint efforts of many people. See [Sa] for precise statements and references. Briefly, a truncated subspace X_T is obtained from X by removing a Γ-equivariant family of horoballs, where T is a suitable truncation parameter and is determined by the heights of the horoballs of associated with some fixed representatives of the finitely many Γ-conjugacy classes of maximal $\mathbb{Q}$-parabolic subgroups of $\mathbf{G}$.

In summary, we have obtained explicit models of cofinite $\underline{E}\Gamma$-spaces.

Proposition 2. *For a non-uniform arithmetic subgroup* Γ *as above, both the Borel-Serre partial compactification* $\overline{X}^{BS}$ *and the truncated subspaces* X_T *of the symmetric space* X *of noncompact type are cofinite* $\underline{E}\Gamma$*-spaces.*

3. Mapping class groups

Let $S_{g,n}$ be an orientable surface of genus g with n punctures. Let $\mathcal{M}_{g,n}$ be the moduli space of complex structures on $S_{g,n}$. It is a quasi-projective variety, and is one of the most important spaces in algebraic geometry.

Let $\mathcal{T}_{g,n}$ be the Teichmüller space of marked complex structures on $S_{g,n}$, where a marking on a Riemann surface $\Sigma_{g,n}$ is a homotopy equivalence class $[\varphi]$ of a diffeomorphism $\varphi : S_{g,n} \to \Sigma_{g,n}$. It is known that $\mathcal{T}_{g,n}$ is a contractible complex manifold of complex dimension $3g-3+n$, which can also be realized as a bounded contractible domain in $\mathbb{C}^{3g-3+n}$.

Let $Mod_{g,n} = \mathrm{Diff}^+(S_{g,n})/\mathrm{Diff}^0(S_{g,n})$ be the mapping class group of $S_{g,n}$. Then $Mod_{g,n}$ acts holomorphically and properly on $\mathcal{T}_{g,n}$, and the quotient $Mod_{g,n}\backslash\mathcal{T}_{g,n}$ is equal to $\mathcal{M}_{g,n}$.

The pair $(\mathcal{T}_{g,n}, Mod_{g,n})$ is similar to the pair (X, Γ) studied in the previous section, and many striking results have been obtained which are motivated by this similarity (see [Iv] and [Ha]).

Using either the Teichmüller metric or the Weil-Petersson metric of $\mathcal{T}_{g,n}$, we can show that $\mathcal{T}_{g,n}$ is a universal space $\underline{E}Mod_{g,n}$ (see [JW]).

Assume that $2g-2+n > 0$ from now on. Then each Riemann surface $\Sigma_{g,n}$ admits a unique complete hyperbolic metric of finite area, and hence $\mathcal{T}_{g,n}$ is the moduli space of marked hyperbolic complete metrics on $S_{g,n}$ of finite area.

The reason why the moduli space $\mathcal{M}_{g,n}$ is noncompact is that we can pinch the length of a simple closed geodesic on $\Sigma_{g,n}$ to 0 and hence obtain sequences of points of $\mathcal{M}_{g,n}$ which do not admit any accumulation point.

Since the quotient $Mod_{g,n}\backslash\mathcal{T}_{g,n}$ is noncompact, it is a natural problem to construct an analogue of Borel-Serre compactification and to prove that $Mod_{g,n}\backslash\mathcal{T}_{g,n}$ admits compact deformation retracts. It turns out that it is more difficult to construct compactifications (see [Iv]).

For a small and positive ε, define a truncated Teichmüller space $\mathcal{T}_{g,n}(\varepsilon)$ by $\mathcal{T}_{g,n}(\varepsilon) = \{(\Sigma_{g,n}, [\varphi]) \in \mathcal{T}_{g,n}\}$, where $\Sigma_{g,n}$ does admit a closed geodesic of length less than ε.

Clearly that $\mathcal{T}_{g,n}(\varepsilon)$ is stable under the action of $Mod_{g,n}$. It is also known that it is a real analytic manifold with corners and the quotient $Mod_{g,n}\backslash\mathcal{T}_{g,n}(\varepsilon)$ is compact. This is an analogue of the truncated subspace X_T of the symmetric space X in the previous section.

In [JW], the following result was proved.

Theorem 3. *When* $\varepsilon > 0$ *is sufficiently small, there exists a* $Mod_{g,n}$*-equivariant deformation retract of* $\mathcal{T}_{g,n}$ *to* $T_{g,n}(\varepsilon)$*. In particular,* $T_{g,n}(\varepsilon)$ *is a cofinite universal* $\underline{E}Mod_{g,n}$*-space.*

This result was claimed in [Ha, §3]. For torsion-free subgroups $\Gamma_{g,n}$ of $Mod_{g,n}$, the existence $\Gamma_{g,n}$-equivariant retraction of $\mathcal{T}_{g,n}$ to $T_{g,n}(\varepsilon)$ was proved in [Iv].

The boundary components of $T_{g,n}(\varepsilon)$ are described by truncated Teichmüller spaces of lower genus. By induction and combining results of [Ha] and [IJ], we can prove the following.

Corollary 4. *The boundary components of $T_{g,n}(\varepsilon)$ are parametrized by simplices of the curve complex $\mathcal{C}(S_{g,n})$ and the boundary $\partial T_{g,n}(\varepsilon)$ is homotopic to a bouquet of infinitely many spheres. Consequently, $Mod_{g,n}$ is a virtual duality group, but not a virtual Poincaré duality group.*

It is known that the curve complex $\mathcal{C}(S_{g,n})$ is an analogue of the Tits building $\Delta_{\mathbb{Q}}(\mathbf{G})$ and has played a fundamental role in the study of $Mod_{g,n}$ (see [Iv]).

The Novikov conjectures have played an important role in geometric topology (see [Lu]). Another corollary of Theorem 3 is the following.

Corollary 5. *The rational Novikov conjecture in algebraic K-theory holds for $Mod_{g,n}$.*

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From Phan's theorems to Phan theory

Sergey Shpectorov

In 1977 Phan published the following theorem. Suppose Γ is a simply laced Dynkin diagram, that is, $\Gamma = A_n$, D_n, or E_n. A group G is said to a be a *group of type* Γ if for some prime power q the group G is generated by subgroups $L_i \cong \mathrm{SU}(2, q^2)$ with a distinguished torus H_i of size $q + 1$, so that the following conditions hold:

(1) $[L_i, L_j] = 1$ if the vertices i and j are non-adjacent in Γ;

(2) $\langle L_i, L_j\rangle \cong \mathrm{SU}(3, q^2)$ and furthermore $\langle L_i, H_j\rangle \cong \mathrm{GU}(2, q^2)$, if i and j are adjacent in Γ;
(3) $\langle H_i, H_j\rangle = H_i \times H_j$ for all $i \neq j$.

Theorem 1. *Let G be a group of type A_n for some $q > 4$. Then G is isomorphic to a homomorphic image of* $\mathrm{SU}(n + 1, q^2)$

The second part of condition 2 and the entire condition 3 are only needed to ensure that L_i and L_j, as subgroups of $\langle L_i, L_j\rangle \cong \mathrm{SU}(3, q^2)$, correspond to the two-by-two blocks along the main diagonal.

The above was proven in [12]. In the second part, [13], Phan established similar results for the diagrams D_n and E_n, although in these cases he had to restrict himself to the case of odd $q > 3$. The above results found an immediate application in a pivotal paper of Aschbacher on Chevalley groups in odd characteristic, [1, 2], and thus became important for the overall classification of finite simple groups. The proof of Phan depends on rather delicate computations with matrices, which were largely omitted in the published text. With the revision of the classification under way, a similar revision of Phan's results aimed at a more conceptual proof was needed.

It was noticed early on that Phan's theorems are very similar to the corresponding cases of Curtis-Tits theorem. For example, if in the above theorem we substitute $\mathrm{SU}(k, q^2)$'s with $\mathrm{SL}(k, q)$'s (and take the tori H_i of size $q - 1$ instead of $q+1$) then the result is just the Curtis-Tits theorem for the diagram A_n. In his unpublished notes Aschbacher proposed the idea of a joint treatment of Curtis-Tits and Phan's theorems. He furthermore tied Phan's Theorem 1 with the geometry of nonsingular subspaces in the $n + 1$-dimensional unitary space. A complete treatment of Phan's theorem for A_n was achived by Bennett and the author in [5], where indeed it was deduced via Tits' Lemma from the simple connectedness of the above geometry. A large part of this paper was devoted to the amalgam uniqueness, which was entirely missing in Phan's paper.

With Gramlich and Hoffman joining the discussion of these results, the approach of [5] was gradually generalized to more and more cases, going well beyond the initial results of Phan. We will now proceed to describe the most general setup, see [4].

Let $\mathcal{B} = (B_+, B_-, \delta_*)$ be a twin building. The *opposites geometry* $\mathrm{Op}(\mathcal{B})$ of $\mathcal{B}$ is the geometry corresponding to the chamber system on all pairs (c, c'), where $c \in B_+$, $c' \in B_-$, and $\delta_*(c, c') = 1$. In the spherical case, every building becomes in a canonical way part of a twin building. The Curtis-Tits theorem for this diagram is then equivalent to the simple connectedness of the opposites geometry $\mathrm{Op}(\mathcal{B})$. The same idea works for arbitrary twin buildings. Thus, the result of Mühlherr in [11], establishing the simple connectedness of $\mathrm{Op}(\mathcal{B})$ for arbitrary 2-spherical diagrams (avoiding some small residues), yields a broad generalization of the Curtis-Tits theorem to arbitrary diagrams and fields.

Suppose now that $\mathcal{B}$ admits an involution σ such that σ interchanges B_+ and B_-, while preserving distances and codistance, that is, $\delta_\epsilon(c, c') = \delta_{-\epsilon}(c^\sigma, c'^\sigma)$ for all $c, c' \in B_\epsilon$, $\epsilon \in \{+, -\}$, and $\delta_*(c, c') = \delta_*(c^\sigma, c'^\sigma)$ for all $c \in B_+$ and $c' \in B_-$. The involution σ acts on $\mathrm{Op}(\mathcal{B})$ via $(c,c')^\sigma := (c'^\sigma, c^\sigma)$. We call σ a *flip* if $\delta_*(c,c^\sigma)=1$ for some $c \in B_+$, that is, if it takes some chamber to an opposite chamber. This condition guarantees that the *Phan geometry* $\mathrm{Op}(\mathcal{B})_\sigma$, associated with $\mathcal{B}$ and σ, is nonempty. Here $\mathrm{Op}(\mathcal{B})_\sigma$ denotes the set of elements of $\mathrm{Op}(\mathcal{B})$ fixed by σ.

When the flip σ is good, the corresponding Phan geometry is flag-transitive for the centralizer G_σ of σ in $G = \mathrm{Aut}(\mathcal{B})$. If furthermore $\mathrm{Op}(\mathcal{B})_\sigma$ is simply connected, Tits' Lemma yields a presentation for the group G_σ. For example, for the diagram A_n over the finite field $\mathrm{GF}(q^2)$ we can take σ to be the unitary involution (the contragredient automorphism times the field automorphism of order two), in which case the Phan geometry is exactly the geometry of the nonsingular subspaces of the unitary space and the resulting presentation of $G_\sigma = \mathrm{SU}(n+1, q^2)$ is equivalent via a brief reduction, as in [5], to the presentation from Phan's Theorem 1. Similarly, his results from [13] correspond to a particular choice of flip for the twin buildings of type D_n and E_n. Thus, the twin buildings and flips provide a broad generalization of Phan's theorems and a direct connection between Curtis-Tits Theorem and Phan's theorems.

Particular cases of these construction for diagrams B_n, C_n, and D_n were considered in a sequence of papers [3, 8, 9, 10]. In these papers the properties of the Phan geometry were studied on a case-by-case basis using a particular representation of the geometry via the natural module of the corresponding classical group. This approach would have been difficult or impossible for the exceptional spherical diagrams, and even more so for the non-spherical diagrams. Fortunately, Devillers and Mühlherr in [7] came up with an inductive (in terms of rank) approach which allows to prove simple connectedness of the Phan geometry for 3-spherical diagrams and sufficiently large fields, based on a more detailed information on the cases of rank two and three. These low rank cases can be dealt with using the classical group approach.

To conclude this brief survey, the following open problem seems to be very interesting.

Question 2. *In the nonspherical case, what are the groups G_σ that act on Phan geometries? Are they simple groups for good flips σ?*

In the spherical case, the group G_σ itself acts on a building, and so it can be easily identified with a suitable Chevalley group (for a finite field). For hyperbolic diagrams there is no such action and so G_σ may be new and interesting groups. Note also the recent result of Caprace and Remi [6] concerning the simplicity of the Kac–Moody groups $G = \mathrm{Aut}(\mathcal{B})$.

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Satake-Furstenberg compactifications of Bruhat-Tits buildings, via Berkovich techniques

Bertrand Rémy
(joint work with Amaury Thuillier, Annette Werner)

This is a short report on a joint work with A. Thuillier and A. Werner [1]. The general subject matter is a combination of the Bruhat-Tits theory of semisimple groups over valued fields and of the Berkovich theory of analytic spaces over complete fields of the same kind.

More precisely, the talk was intended to be a down-to-earth motivated introduction to a compactification procedure which provides all the analogues, in a non-archimedean context, of the compactifications of symmetric spaces due to H. Furstenberg and I. Satake in the case of real Lie groups [2], [3].

Let k be a locally compact non-archimedean valued field and let G be a reductive linear algebraic group over k. We denote by $\mathscr{B}(G, k)$ the Bruhat-Tits building of G over k [4], [5].

Euclidean buildings and group-theoretic compactification

One of the main arguments why $\mathscr{B}(G, k)$ is seen as a non-archimedean symmetric space is that it carries a non-positively curved complete metric, such that the natural $G(k)$-action on $\mathscr{B}(G, k)$ is by isometries. Moreover the Bruhat-Tits fixed-point lemma implies the existence of a $G(k)$-equivariant dictionary between vertices in $\mathscr{B}(G,k)$ – which is a polysimplicial complex – and maximal compact subgroups in $G(k)$. The map from the set of vertices $\mathscr{V}_{\mathscr{B}(G,k)}$ of $\mathscr{B}(G,k)$ to the set of maximal compact subgroups in $G(k)$ is $v \mapsto \mathrm{Stab}_{G(k)}(v)$, the inverse being the fixed-point set map $K \mapsto \mathscr{B}(G,k)^K$.

In a previous work with Y. Guivarc'h [6], the first map is used together with the well-known fact that for any locally compact group H, such as $G(k)$, the space Σ_H of all closed subgroups in H has a natural compact topology [7, VIII.5], sometimes referred to as the *Chabauty topology.*

Theorem 1. *The map $\mathscr{V}_{\mathscr{B}(G,k)} \to \Sigma_{G(k)}$, attaching to each vertex v its stabilizer $\mathrm{Stab}_{G(k)}(v)$ in $G(k)$, defines a natural $G(k)$-equivariant compactification of $\mathscr{V}_{\mathscr{B}(G,k)}$: this compactification is simply the closure of the image of the map.*

The so-obtained compactification, denoted by $\overline{\mathscr{V}}^{\mathrm{gp}}_{\mathscr{B}(G,k)}$, is called the *group-theoretic compactification* of $\mathscr{B}(G,k)$ not only because the techniques used are relevant to topological group theory, but also because it allows one to formulate some extensions of the above dictionary {vertices} $\leftrightarrow$ {maximal compact subgroups}. Indeed, by taking stabilizers in $G(k)$ of points in $\overline{\mathscr{V}}^{\mathrm{gp}}_{\mathscr{B}(G,k)}$, one obtains, up to finite index, a $G(k)$-equivariant dictionary between maximal (Zariski-connected) amenable subgroups of $G(k)$ and $\overline{\mathscr{V}}^{\mathrm{gp}}_{\mathscr{B}(G,k)}$. But one problem is that, strictly speaking, the group-theoretic compactification doesn't take into account the full building $\mathscr{B}(G,k)$. Moreover, in the analogy with Satake compactifications, it only corresponds to the maximal one.

Analytic geometry and fillings of the compactifications

Berkovich geometry, as presented in [8], is a version of analytic geometry over complete non-archimedean valued fields, in which the spaces have nice local connectivity properties. Roughly speaking, in algebraic geometry on uses as building blocks (algebraic) spectra $\mathrm{Spec}(A)$ consisting of prime ideals of commutative rings A endowed with the Zariski topology, while in Berkovich geometry one uses (analytic) spectra $\mathscr{M}(A)$ of Banach k-algebras, consisting of multiplicative bounded seminorms $A \to \mathbf{R}_+$.

To each variety V over k is attached a Berkovich analytic space over k, which is denoted by V^{an}. The attachment $V \mapsto V^{\mathrm{an}}$ is functorial and moreover satisfies:

(i) if V is affine with coordinate ring $k[V]$, then V^{an} consists of all the multiplicative seminorms $k[V] \to \mathbf{R}_+$ extending the absolute value of k;
(ii) if V is projective, then V^{an} is compact.

One further feature is the possibility to work with any complete extension K of k; at the geometric level of the buildings, this corresponds to the fact each point in the building $\mathscr{B}(G,k)$ can be seen as a special point in the (usually much) bigger building $\mathscr{B}(G,K)$ of $G(K)$ (the latter group acting on $\mathscr{B}(G,K)$ usually "more transitively"). Together with some faithfully flat descent argument (also available in this context), one obtains the possibility to attach to each point x of $\mathscr{B}(G,k)$ a Berkovich analytic subgroup G_x, and the assignment $x \mapsto G_x$ is injective (in particular it takes distinct values for any two distinct points, even in the same facet). More precisely:

Theorem 2. (i) *For any $x \in \mathscr{B}(G,k)$, there is an analytic subgroup G_x of G^{an} defined over k such that for any non-archimedean extension K/k, we have: $G_x(K) = \mathrm{Stab}_{G(K)}(x)$.*

(ii) *For any $x \in \mathscr{B}(G,k)$, there is a unique point $\vartheta(x) \in G^{\mathrm{an}}$ such that:*
$$G_x = \{g \in G^{\mathrm{an}} : |f(g)| \leq |f(\vartheta(x))| \text{ for any } f \in k[G]\}.$$

(iii) *The so-obtained map $x \mapsto \vartheta(x)$ is a $G(k)$-equivariant embedding of $\mathscr{B}(G,k)$ into G^{an} with closed image.*

In order to define a compactification of $\mathscr{B}(G,k)$, it remains to compose ϑ with the analytification of the quotient map $G \to G/P$, where P is a parabolic k-subgroup of G. When P varies over all the (conjugacy classes of) parabolic k-subgroups of G, one obtains all the expected analogues of the Furstenberg compactifications. In [1], we also describe the boundary structure of these compactifications and prove some extensions of decompositions from Bruhat-Tits theory.

We note that this family of compactifications had already been obtained by A. Werner by means of a gluing procedure [9]: one compactifies first the model of an apartment by means of suitable root-theoretic considerations, and then one extends the equivalence (gluing) relation used in the construction of Bruhat-Tits buildings, by replacing the apartments by their compactifications.

We finally note that the idea to combine Bruhat-Tits theory and Berkovich geometry in order to realize buildings and to compactify them, already appears in V. G. Berkovich's book [8, §5], in the case when the algebraic group G is split over k.

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Kac–Moody groups over ultrametric fields

Guy Rousseau

The Kac–Moody groups studied here are the minimal (=algebraic) and split ones, as introduced by J. Tits in [8]. When they are defined over an ultrametric field, it seems natural to associate to them some analogues of the Bruhat-Tits buildings.

Actually I came to this problem when I was trying to build new buildings of non-discrete type. If G is a Kac–Moody group over an ultrametric field K, I was able to build a *microaffine* building $\mathcal{I}^\mu$ on which $G(K)$ acts [5]. This building is an union of apartments in one to one correspondence with the maximal split tori and the usual axioms of buildings are satisfied, among them the fundamental axiom: any two points are in a same apartment. It is closely related to the Satake (or polyhedral) compactification of the Bruhat-Tits building of a semi-simple group over K. One knows that this compactification is the disjoint union of the Bruhat-Tits buildings of the semi-simple quotients of all parabolic subgroups of this semi-simple group. For a Kac–Moody group the same definition gives the microaffine building, but now the parabolic subgroups give something in $\mathcal{I}^\mu$ only when they are of finite type, so G itself gives nothing. We just have to define the apartments and prove the usual axioms of buildings, see [5].

Unfortunately $\mathcal{I}^\mu$ seems to give only a few informations about the structure of $G(K)$. Moreover P. Littelmann asked me whether it could be used to generalize his results with S. Gaussent in the semi-simple case [2]: they proved in particular that a LS-path may be seen (in an apartment of a Bruhat-Tits building over the field of Laurent series $\mathbb{C}((t))$) as an image of a segment of the building under some fixed retraction (with center a sector-germ), satisfying also some numerical condition. It was soon clear that, in the Kac–Moody case, $\mathcal{I}^\mu$ is not suitable. One has to mimic more closely the Bruhat-Tits construction. The normalizer of the standard maximal split torus in $G(K)$ acts on the corresponding apartment $\mathbb{A}$ by a group of affine transformations, generated by reflections on walls. But there is a lot of walls (infinitely many directions), moreover in the loop group situation, H. Garland in [1] had proved that there is no Cartan decomposition, so the expected building would not satisfy the fundamental axiom of buildings: it seemed at first too ugly.

Nevertheless it is possible to build this close analogue to Bruhat-Tits buildings for some split Kac–Moody groups (joint work with Stéphane Gaussent):

Theorem 1 ([3]). *Suppose that $K = \mathbb{C}((t))$ (or more generally that $\mathbb{C}$ is in the residue field of K) and moreover that G is symmetrizable. Then there exists a set $\mathcal{I}$, with an action of $G(K)$, containing a subset identified with $\mathbb{A}$. The stabilizer of $\mathbb{A}$ is the normalizer of the standard maximal split torus and the induced action on*

$\mathbb{A}$ is as described above. The set $\mathcal{I}$ is covered by the apartments i.e. the conjugates of $\mathbb{A}$ by elements of $G(K)$.

The unfortunate restriction on K is due to heavy technical complications; more general cases should be proved in the near future.

The space $\mathcal{I}$ doesn't satisfy the fundamental axiom of buildings, so this ugly $\mathcal{I}$ is called a *hovel*. But Iwasawa decomposition is still verified in the Kac–Moody group acting on $\mathcal{I}$, so any point and any sector-germ in $\mathcal{I}$ are always in a same apartment. This enables us to define a retraction ρ of $\mathcal{I}$ onto an apartment with center a sector-germ in this apartment. With this retraction it is possible to prove, for Kac–Moody groups, the above quoted result of S. Gaussent and P. Littelmann about LS-paths and to associate to such a path a quasi-affine toric variety (complex and finite dimensional) which is a reasonable generalization of the Mirkovic-Vilonen cycles, see [3].

Actually this hovel is not so ugly: we proved in [3] that there is on $\mathcal{I}$ a preorder relation which induces on each apartment the preorder given by the Tits cone. Moreover the sets of increasing (resp. decreasing) segment-germs of origin $x \in \mathcal{I}$ are twin buildings: the residue of $\mathcal{I}$ in x.

There is also an abstract definition of affine hovels in the spirit of the abstract definition of affine buildings given by J. Tits in [7]: to be short the fundamental axiom is now that any point and any sector-germ or any two sector-germs have to be in a same apartment. This abstract definition is satisfied by the hovels constructed for Kac–Moody groups and there are interesting consequences. These affine hovels look like affine buildings, but the spherical buildings associated to affine buildings are replaced by twin buildings. More precisely we recover at infinity some buildings: the parallel classes of sector-faces are the faces of a twin building and the germs of these sector-faces are the points of two microaffine buildings. In the case of the hovel associated to a Kac–Moody group G, these buildings are the, now well known, twin building of $G(K)$ [4] and two microaffine buildings as in [5], one for each of the two possible choices (positive or negative) of the Tits cone in an apartment. The preorder relation and the structure of the residues are also consequences of this abstract definition. See [6].

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Invariant Theory of Periodic Automorphisms of Semisimple Lie Algebras

Paul Levy

1. Classical Lie invariant theory

Let G be a reductive algebraic group over the algebraically closed field k and let $\mathfrak{g}$ be the Lie algebra of G. Classical results of invariant theory relate the geometry of the adjoint representation to familiar algebraic properties of $\mathfrak{g}$. To put the later sections into context, we give here an outline of some of these results.

First we recall Mumford's categorical quotient. If H is an algebraic group whose connected component is reductive, and V is an affine variety on which H acts via a morphism $H \times V \to V$, then H also acts on $k[V]$. The ring of invariants $k[V]^H$ is finitely generated and hence is the coordinate ring of an affine variety $V/\!\!/H = \mathrm{Spec} k[V]^H$, the *categorical quotient*. The embedding $k[V]^H \hookrightarrow k[V]$ induces a morphism $\pi : V \to V/\!\!/H$ called the quotient morphism; each fibre of π contains a unique closed orbit, and therefore π induces a bijection between closed H-orbits in V and points of $V/\!\!/H$.

Let T be a maximal torus of G, let $\mathfrak{t} = \mathrm{Lie}(T)$ be the corresponding Cartan subalgebra of $\mathfrak{g}$ and let $W = N_G(T)/T$ be the Weyl group of G with respect to T. Then:

- $x \in \mathfrak{g}$ is semisimple if and only if it is conjugate to an element of $\mathfrak{t}$,
- the orbit $\mathrm{Ad}G(x)$ is closed if and only if x is semisimple,
- $x \in \mathfrak{g}$ is G-unstable (that is, 0 is in the closure of the G-orbit through x) if and only if it is nilpotent,
- restriction to $\mathfrak{t}$ induces an isomorphism $k[\mathfrak{g}]^G \to k[\mathfrak{t}]^W$ (and hence an isomorphism $\mathfrak{t}/W \to \mathfrak{g}/\!\!/G$),
- $k[\mathfrak{t}]^W$ is a polynomial ring, that is, there exist $r = \dim \mathfrak{t}$ algebraically independent homogeneous polynomials $f_1, \ldots, f_r$ such that $k[\mathfrak{t}]^W = k[f_1, \ldots, f_r]$,
- let $\{h, e, f\} \subset \mathfrak{g}$ be an $\mathfrak{sl}(2)$-triple such that e is a regular nilpotent element of $\mathfrak{g}$ and let $\mathfrak{v} = e + \mathfrak{z}_{\mathfrak{g}}(f)$, where $\mathfrak{z}_{\mathfrak{g}}(f)$ denotes the centralizer of f in $\mathfrak{g}$. Then the composition of the embedding $\mathfrak{v} \hookrightarrow \mathfrak{g}$ with the quotient morphism gives an isomorphism $\mathfrak{v} \to \mathfrak{g}/\!\!/G$.

If H is a reductive group acting linearly on a vector space V, then a linear subvariety $\mathfrak{v} \subset V$ which maps isomorphically onto $V/\!\!/H$ is called a Weierstrass slice or a Weierstrass section.

2. Symmetric Spaces

A well-known generalization of the above results is the case of a symmetric space. Thus suppose that $\mathrm{char}(k) \neq 2$ and that $\theta : G \to G$ is a (rational) involutive automorphism of G. Then there is a direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the $(+1)$ eigenspace, and $\mathfrak{p}$ the (-1) eigenspace for $d\theta$. In the seminal work of Kostant and Rallis [1] it was established that the action of K on $\mathfrak{p}$ shares many similar invariant-theoretic features with the adjoint representation. The role of Cartan subalgebra is here played by a Cartan sub*space*: $\mathfrak{a} \subset \mathfrak{p}$ is a Cartan subspace if it is a maximal commutative subspace consisting of semisimple elements.

- $x \in \mathfrak{p}$ is semisimple if and only if it is contained in a Cartan subspace,
- any two Cartan subspaces of $\mathfrak{p}$ are K-conjugate,
- $\mathrm{Ad}K(x)$ is closed if and only if x is semisimple,
- $x \in \mathfrak{p}$ is K-unstable if and only if it is nilpotent,
- Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p}$. Then the "baby Weyl group" $W_{\mathfrak{a}} = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is a Weyl group and restriction to $\mathfrak{a}$ induces an isomorphism $k[\mathfrak{p}]^K \to k[\mathfrak{a}]^{W_{\mathfrak{a}}}$. Moreover, $k[\mathfrak{a}]^{W_{\mathfrak{a}}}$ is a polynomial ring.
- let $\{h, e, f\} \subset \mathfrak{g}$ be an $\mathfrak{sl}(2)$-triple such that $h \in \mathfrak{k}$, $e, f \in \mathfrak{p}$ and e is a 'θ-regular' element of $\mathfrak{p}$, that is $\dim \mathfrak{z}_{\mathfrak{g}}(e) \leq \dim \mathfrak{z}_{\mathfrak{g}}(x)$ for every $x \in \mathfrak{p}$. Then $\mathfrak{v} = e + \mathfrak{z}_{\mathfrak{p}}(f)$ is a Weierstrass slice for the action of K on $\mathfrak{p}$.

The above results for $k = \mathbb{C}$ appeared in [1]. In positive characteristic they were established by the author in [2], although some of these results could be deduced relatively easily from the work of Richardson [6].

3. Vinberg's θ-groups

By work of Vinberg, the above set-up extends to the much more general case of an arbitrary semisimple (rational) automorphism of G of finite order. Thus let $\theta : G \to G$ be a semisimple automorphism of order m (that is, such that m is not divisible by p if $\mathrm{char}(k) = p > 0$) and let ζ be a fixed primitive m-th root of unity in k. Then there is a direct sum decompositiion

$$\mathfrak{g} = \mathfrak{g}(0) \oplus \ldots \oplus \mathfrak{g}(m-1)$$

where $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid d\theta(x) = \zeta x\}$. Moreover, if $G(0)$ is the connected component of G^θ then $G(0)$ is reductive, $\mathrm{Lie}(G(0)) = \mathfrak{g}(0)$ and $G(0)$ normalizes each $\mathfrak{g}(i)$. In [7], Vinberg generalized (for $k = \mathbb{C}$) that broad outline of Kostant-Rallis's results to the action of $G(0)$ on $\mathfrak{g}(1)$. Extending the $m = 2$ case, we define a Cartan subspace to be a maximal commutative subspace of $\mathfrak{g}(1)$ consisting of semisimple elements. Then we have:

- if $x \in \mathfrak{g}(1)$ then $\mathrm{Ad}G(0)(x)$ is closed if and only if x is semisimple,
- x is $G(0)$-unstable if and only if x is nilpotent,
- any semisimple element of $\mathfrak{g}(1)$ is contained in a Cartan subspace, and any two Cartan subspaces are $G(0)$-conjugate,
- let $\mathfrak{c}$ be a Cartan subspace and let $W_{\mathfrak{c}} = N_{G(0)}(\mathfrak{c})/Z_{G(0)}(\mathfrak{c})$. Then the embedding $\mathfrak{c} \hookrightarrow \mathfrak{g}(1)$ induces an isomorphism $\mathfrak{c}/W_{\mathfrak{c}} \to \mathfrak{g}(1)/\!\!/G(0)$,

- The group $W_\mathfrak{c}$ is generated by *complex* reflections and therefore $k[\mathfrak{g}(1)]^{G(0)}$ is a polynomial ring.

The major difference with the involution case is that $W_\mathfrak{c}$ is no longer necessarily a Weyl group, but is generated by complex, or pseudo-reflections. The Shephard-Todd theorem states (in characteristic zero) that the ring of invariants for a complex reflection group is a polynomial ring. In general the Shephard-Todd theorem fails when the order of the group is divisible by the characteristic of the ground field.

The main aim of our research was to generalize Vinberg's results to positive characteristic. The biggest problem concerns the description of $W_\mathfrak{c}$ and the failure of the Shephard-Todd theorem in dividing characteristic. In [7], Vinberg described $W_\mathfrak{c}$ in all classical cases and used an induction argument to show that $W_\mathfrak{c}$ is a complex reflection group in the remaining types. In [3] we also described $W_\mathfrak{c}$ in the classical cases, although by a different approach which relates $W_\mathfrak{c}$ to the centralizer of an element in the Weyl group. This approach also showed that if G is of exceptional type and $m > 2$, then $W_\mathfrak{c}$ has order coprime to p. We then applied a theorem of Panyushev on Chevalley-type isomorphisms to extend the results of [7] to positive characteristic.

4. KW-sections

The existence of a Weierstrass slice for the action of $G(0)$ on $\mathfrak{g}(1)$ was conjectured (for $k = \mathbb{C}$) by Popov in 1976. Because of the similarity with Kostant's slice in $\mathfrak{g}$, a Weierstrass slice in $\mathfrak{g}(1)$ is sometimes called a Kostant-Weierstrass slice, or KW-section. The existence of KW-sections for θ-groups was proved by Panyushev in two cases: that $G(0)$ is semisimple [4]; and in the 'N-regular' case (when $\mathfrak{g}(1)$ contains a regular nilpotent element of $\mathfrak{g}$) [5]. A major benefit of our approach to describing the little Weyl group is that it provides a way to construct a 'minimal' θ-stable subgroup L of G such that $\mathfrak{c} \subset \mathrm{Lie}(L)$ and each element of $W_\mathfrak{c}$ has a representative in L^θ. When G is classical, it can then be checked that the restriction of θ to L is N-regular. Generalizing Panyushev's theorem to positive characteristic, we therefore show (for $\mathrm{char}(k) \neq 2$) [3]:

Theorem 1. *Let G be one of* $\mathrm{GL}(n,k), \mathrm{SL}(n,k), \mathrm{Sp}(n,k), \mathrm{SO}(n,k)$. *Then there is a KW-section for the action of $G(0)$ on $\mathfrak{g}(1)$.*

A detailed case-by-case analysis shows that this result extends to the case where G is of type F_4, G_2 or D_4 (including the triality automorphisms which do not exist for SO(8)). It seems likely that it will be possible to solve the conjecture in the remaining cases via a similar approach.

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Decomposing locally compact groups into simple pieces

Pierre-Emmanuel Caprace
(joint work with Nicolas Monod)

Any finite group may be decomposed along a subnormal series with every subquotient simple. This basic observation lies at the basis of finite group theory and provides the first obvious motivation for a comprehensive study of the finite simple groups. A similar picture depicts the category of connected locally compact groups. Indeed, as a consequence of the solution to Hilbert's fifth problem, any connected locally compact group G admits a finite subnormal series

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n,$$

where every subquotient G_i/G_{i-1} is either compact, or isomorphic to $\mathbb{R}$ or $\mathbb{Z}$, or isomorphic to a connected non-compact simple Lie group with trivial centre. A spectacular feature in the finite and the connected case is that the simple groups that appear are classified, and virtually all of them are (forms of) algebraic groups. One may wonder whether this feature extends to a more general class of groups. A moment's thought shows that this is very doubtful since it is not even clear *a priori* why simple groups should play any role at all. A clear illustration of this fact is provided by residually finite groups, and even more strikingly by residually finite groups all of whose proper quotients are compact. Two well-known examples of such groups are $\mathrm{SL}(n, \mathbb{Z})$ with $n \geq 3$ and the Grigorchuk group.

The results reported here are established in [CM] and provide some evidence that simple groups indeed play a role in the structure theory of locally compact groups beyond the almost connected case. The groups we shall consider will satisfy various subsets of the following set of conditions on a locally compact group G:

(a) G has no infinite discrete quotient[1].
(b) G is compactly generated.
(c) G is **(topologically) Noetherian**, *i.e.* G satisfies an ascending chain condition on open subgroups.

[1] Here and in what follows, it is understood that the terminology qualifying the structure of a group G has to be interpreted in the category of locally compact groups. In particular, quotients are meant to be continuous; normal subgroups are meant to be closed, etc.

Conditions (a), (b) and (c) are logically independent, except for the fact that (c) implies (b). More generally, one has the following.

Lemma 1. *Let G be a Noetherian locally compact group. Then every closed normal subgroup of G is compactly generated.*

All finite and connected locally compact groups satisfy (a), (b) and (c). The following characterises the groups admitting a decomposition similar to the aforementioned decomposition of connected groups.

Theorem 2. *Let G be a locally compact group satisfying (a). Then the following assertions are equivalent.*

(1) *G admits a finite subnormal series*

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n,$$

where every subquotient G_i/G_{i-1} is either compact, or isomorphic to $\mathbb{R}$ or $\mathbb{Z}$, or non-compact compactly generated and (topologically) simple.

(2) *Every closed normal subgroup of G is compactly generated.*

(3) *Every closed characteristic subgroup of G is compactly generated.*

The conclusion of course fails without the assumption that (a) holds. Standard arguments allow to deduce a Jordan–Hölder type theorem in the Noetherian case:

Corollary 3. *Let G be a locally compact group satisfying (a) and (c). Then G admits a subnormal series with every subquotient compact, or compactly generated Abelian, or compactly generated simple. Furthermore the set of non-compact simple subquotients of G depends only on the group G and not on the chosen decomposition, and is non-empty provided G is not {connected soluble}-by-compact.*

The next result shows that simple subquotients arise even for groups which are not assumed to be Noetherian.

Theorem 4. *Let G be a locally compact group satisfying (a) and (b). Then G possesses characteristic subgroups $F < D < G$ satisfying the following.*

- *G/D is compact and D has no nontrivial discrete quotient; in particular every compact quotient of D is connected.*
- *H/D splits as a direct product of the form $S_1 \times \cdots \times S_p \times \mathbb{R}^n$, where each S_i is non-compact compactly generated and (topologically) simple. Moreover $p + n > 0$ provided G is not compact.*

The characteristic subgroup D is defined as the **discrete residual** of G, *i.e.* the intersection of all normal open subgroups. The subgroup F is called the **Frattini radical** of D; by definition, the Frattini radical of a topological group H is the intersection of the (possibly empty) collection of all closed normal subgroups which are *not* cocompact and *maximal* for this property. It was observed by R. Grigorchuk and G. Willis that every non-compact compactly generated locally compact group admits such a normal subgroup; the Frattini radical is thus non-trivial in this case.

For any topological group G, the quotient G/D of G by its discrete residual is residually discrete. In view of the fact that a compactly generated locally compact residually discrete group is compact-by-discrete (see [CM, Corollary 4.1]), Theorem 4 has the following consequence.

Corollary 5. *Let G be a compactly generated characteristically simple group. Then either G is discrete, or compact, or isomorphic to $\mathbb{R}^n$, or isomorphic to a finite direct product of pairwise isomorphic non-compact (topologically) simple groups.*

The above results call for a better understanding of compactly generated simple locally compact groups. Numerous specimens are known, including many examples of non-linear groups. However, although a general investigation of non-discrete compactly generated simple groups has already been undertaken (see [Ws] and [BEW]), there is not yet even a conjectural exhaustive description. We hope that this fascinating topic will attract much attention in the forthcoming times.

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