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Mini-Workshop: Group Actions on Curves: Reduction and Lifting

Organised by
Irene I. Bouw, Ulm
Ariane Mezard, Versailles
Stefan Wewers, Hannover

November 16th – November 22nd, 2008

Abstract. Group actions on algebraic curves over local, global and finite fields play an important role in modern algebraic and arithmetic geometry. From the arithmetic perspective, the most difficult and interesting case occurs when a group with a nontrivial p-subgroup acts on a curve over a p-adic field or a finite field of characteristic p.

The goal of this workshop was to bring together a group of active and mostly young researchers working in this area, to discuss the latest developments and to stimulate further research.

Mathematics Subject Classification (2000): 11G20, 14D15, 14H30, 14L30.

Introduction by the Organisers

The mini-workshop *Group actions on curves: reduction and lifting*, organized by Irene I. Bouw (Ulm), Ariane Mézard (Versailles) and Stefan Wewers (Hannover) was held November 16th – November 22nd, 2008. There were 15 participants, 6 of whom were PhD-students or recent PhDs. There were 15 talks and 2 discussion sessions. There was a joint evening session with the mini-workshop *Symmetric Varieties and Involutions of Algebraic Groups.*

The talks discussed very recent progress on the following closely related topics:

- stable reduction of covers of curves,
- the local lifting problem, Oort's conjecture,
- reduction of group scheme actions and torsors, differential data,
- curves with many automorphisms,
- versal deformation rings.

The theme of the first discussion session was the local lifting problem. Several participants gave an informal presentation of possible approaches for solving it.

Afterwards, we discussed how the different approaches fit together. The main emphasis was the connection between differential data and torsors under group schemes.

The theme of the second discussion session was the structure of deformation spaces. We discussed several questions that came out of the talks. Among other things, given a family of lifts of a cover of formal germs of curves from positive characteristic to characteristic zero, is there a reasonable notion of 'moduli' of such lifts, and how does it relate to the position of the branch points? We formulated several conjectural statements and discussed their relevance and possible proofs.

Mini-Workshop: Group Actions on Curves: Reduction and Lifting

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Abstracts

Deformations of Artin-Schreier curves

José Bertin

Let $k = \overline{k}$ be an algebraically closed field of characteristic $p > 2$. We choose a primitive $(p-1)^2$ root of unity $\zeta \in k$. Let C be the smooth projective curve defined over k (indeed defined over $\mathbb{F}_p$) by the equation $Y^{p-1} = X^p - X$, the so called Artin-Schreier curve, i.e the smooth curve with function field $k(C) = k(x, y)$ with $y^{p-1} = x^p - x$.The genus of C is $g = \frac{(p-1)(p-2)}{2}$. Throughout $p \geq 5$. Let γ be the automorphism of C given by $\gamma(x) = \zeta^{p-1}x$, $\gamma(y) = \zeta^p y$. Then γ together with the Artin-Schreier order p automorphism $\sigma(x) = x + 1$, $\sigma(y) = y$ generate the group $\mathrm{Aut}(C)$.

The curve C is more conveniently seen as a cyclic cover of the projective line $\mathbb{P}^1$, viz. the covering $\eta : (x, y) \mapsto x$. The cyclic cover $\eta : C \to \mathbb{P}^1$ is of degree $p - 1$ and with group H the group generated by $\tau = \gamma^{p-1}$, where $\tau(x) = x, \tau(y) = \zeta^{p-1}y$. The branch locus of η is $\mathbb{P}^1_{\mathbb{F}_p} \subset \mathbb{P}^1_k$. Let $\mathcal{C}$ (resp. $\hat{\mathcal{C}}$) be the category of local artinian $W(k)$-algebras with residue field k (resp. local notherian complete). If $A \in \mathcal{C}$, denote $\mathrm{Def}_\eta(A)$ the set of deformations of the cover η to A [1]. Recall that by a deformation of the cover $\eta : C \to \mathbb{P}^1$ to $A \in \hat{C}$ we mean an equivalence class of liftings of the cover to A. The functor Def_η is known to be smooth and pro-representable. Note that in a deformation $\phi : \mathcal{C} \to \mathbb{P}^1_A$, the branching divisor $\mathcal{B} \subset \mathbb{P}^1_A$ is well-defined, and since A is complete it is a sum of disjoints sections that lift the branching points of η. Denote R_η the coordinate ring of the universal deformation. Likewise we may define the deformation functor of the branching divisor, i.e of the pair $(\mathbb{P}^1, B)$. This defines a morphism between the deformation functors [2]

$$\mathrm{Def}_\eta \longrightarrow \mathrm{Def}_{(\mathbb{P}^1, B)} \tag{1}$$

The principle that a deformation of a tame cover between smooth curves is controlled by the deformation of its branch divisor can be understood as the fact that (1) is an isomorphism. If we specialize to the cover η, this yields:

Proposition 1. *The functorial morphism (1) is an isomorphism. Moreover the universal deformation of the tame cover η is represented by the cover $(x, y) \mapsto x$, where*

$$y^{p-1} = x^p + u_1 x^{p-1} + \cdots + u_{p-2}x^2 + (-1 - \sum_{i=1}^{p-2} u_i)x \tag{2}$$

with base $R_\eta = W(k)[[u_1, \cdots, u_{p-2}]]$

The group $\mathrm{Aut}(C)$ acts the ring R_η. This action can be made explicit. Denote $\mathcal{M} = (p, u_1, \cdots, u_{p-2})$ the maximal ideal of R_η. Let us write

$$x^p + u_1 x^{p-1} + \cdots + u_{p-2}x^2 + (-1 - \sum_{i=1}^{p-2} u_i)x = \prod_{i=0}^{p-1}(x - e_i) \ (e_i \in R_\eta) \tag{3}$$

where $e_0 = 1, e_{p-1} = 0$ and for $1 \leq i \leq p-2$, $e_i \equiv i + 1 \pmod{\mathcal{M}}$. For any $j \in \mathbb{Z}$, we set $e_{i+pj} = e_i$, and also if $0 \leq i \leq p-1$, $\tilde{e}_i = e_i - i + 1$. The action of the automorphism τ is

$$\alpha_\tau(\tilde{e}_{j+1}) = -\frac{\tilde{e}_j}{e_{-2}} - (j+1)\left(\frac{1}{e_{-2}} + 1\right) \tag{4}$$

Likewise we can write the action of σ. The fixed point formal subscheme of K is defined by the equations $\alpha^*_\tau(\tilde{e}_j) - \tilde{e}_j$ $(0 \leq j \leq p-3)$.We know this subscheme pro-represents the deformation functor of the cover $C \to C/H \times K$. A short computation leads to the result:

Theorem 2. *The deformation functor of the curve C endowed with the action of $H \times K = \mathbb{Z}/p-1 \times \mathbb{Z}/p$ is $W(k)[X]/(P(X))$, with*

$$P(X) = X(X+p-1)^p - 2 + \sum_{k=1}^{p-2}(-1)^{k-1}(p-1-k)(X+p-1)^{p-2-k}(X+p) \tag{5}$$

Notice the constant term and the coefficient of X are 0 modulo p, which in turn says the tangent space is of dimension one. We can also extract from this result the local versal deformation of the $\mathbb{Z}/p(p-1)$-action at the ∞ point.

We now turn to the study of the universal deformation of the wildly ramified Artin-Schreier cover $\pi : C \to \mathbb{P}^1$ with one branch point at ∞. We know ([1], corollaire 3.4.5) that the universal deformation ring is not formally smooth, it takes the following form

$$R_\pi = R_\infty[[t_1, \cdots, t_{p-4}]] \tag{6}$$

at least if $p \geq 5$, and $R_\pi = R_\infty$ if $p = 3$. Here R_∞ denotes the local versal deformation ring at ∞, i.e of the inertia group acting on $\mathcal{O}_{C,\infty}$. The complete $W(k)$-algebra R_∞ is the formal deformation ring of the $K = \mathbb{Z}/p\mathbb{Z}$-action of conductor $m = p-1$ on $k[[t]]$. We know the dimension of the tangent space is (loc.cit., prop. 4.1.1) $p-1$, and the Krull dimension $\dim R_\infty = 2$ (loc.cit., Thm 5.3.3). If $N = \dim H^1(\mathbb{P}^1 = C/K, \pi^K_*(\Theta_C))$, the local-global principle yields $R_\pi = R_\infty[[t_1, \cdots, t_N]]$. A quick calculation gives us $N = p-4$ if $p \geq 5$ and $N = 0$ if $p = 3$. Finally since $\dim R_\infty = 2$, we get if $p \geq 5$

$$\dim R_\pi = p-2, \ \dim T_{R_\pi} = 2p-5 \tag{7}$$

and $\dim R_\pi = 2$, $\dim T_{R_\pi} = 2$ when $p = 3$. In the case $p = 3$, we have $R_\pi = R_\infty = W[[t_1, t_2]]/(\star)$. The ideal $(\star)$ is non zero, but the fact that $\dim H^2(K, \theta_C) = 2$ does not permit us to conclude if $(\star)$ is principal or not. This is open.

Now let $\eta : C \to \mathbb{P}^1$ be the tame cover as previously described. Let $\mathcal{C} \to \operatorname{Spec} A$ be a deformation over A of C together with the $H = \mathbb{Z}/(p-1)\mathbb{Z} = \langle\tau\rangle$ action (2), for instance the universal deformation. In that follows $\zeta \in \mathbb{F}_p^*$ stands for a primitive $(p-1)$- root of the unity. The A-module $\Omega_{\mathcal{C}/A} = \Gamma(\mathcal{C}, \Omega_{\mathcal{C}/A})$ is free of rank g, and is acted on by $H = \langle\tau\rangle$. We are interested in the representation of H in $\Omega_{\mathcal{C}/A}$. Let V_j be the simple ($\dim V_j = 1$) representation with weight ζ^j $(0 \le j \le p-2)$. The (H, A)-module $\Omega_{\mathcal{C}/A}$ decomposes into H-eigenspaces as follows

$$\Omega_{\mathcal{C}/A} = \bigoplus_{0 \le i \le j-1 \le p-3} V_{p-1-j}^{\oplus j} \tag{8}$$

Now let $\hat{\mathcal{J}}$ be the formal completion of the Jacobian $\mathcal{J} = \operatorname{Pic}^0_{\mathcal{C}/R_\eta}$. Recall $R = W[[u_1, \cdots, u_{p-2}]]$. Notice the R-curve is smooth, i.e with good reduction modulo $\mathcal{I} = (p, u_1, \cdots, u_{p-2})$. It is equipped with a section $\infty : \operatorname{Spec} R \to \mathcal{C}$, a specific ramification point. We are able to apply the Honda machinery [3] to identify the Manin one dimensional summand of $\hat{\mathcal{J}}$. The local parameter $z = \frac{x}{y}$ is defined over the formal completion of $\hat{\mathcal{C}}$ along the ∞ section.

Then if we formally integrate the differential $\omega_{0,1} = \frac{1}{p-1}\frac{dx}{y}$ $(\omega_{0,1} = df)$ of C, this yields the logarithm of a formal group law F of dimension one, height $p-1$, i.e $F \in R_\eta[[x, y]], F = f^{-1}(f(x) + f(y))$.

Let F_0 be the reduction of F modulo $\mathcal{M}$, i.e. the Manin summand of $\mathcal{J}(C)$. Using the previous fact and taking into account the action of the group $K = \mathbb{Z}/p\mathbb{Z}$, we can recover the following result (see also [3], [5]):

Theorem 3. *The formal group law F (equivalently G) over $W[[u_1, \cdots, u_{p-2}]]$ is the Lubin-Tate universal deformation of F_0 (resp. G_0).*

Question: Study along similar lines the formal completion of the Jacobian of the universal deformation of the wild cover π.

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Nonliftable generalized quaternion actions

Louis Hugo Brewis
(joint work with Stefan Wewers)

Let k be an algebraically closed field of characteristic p, and let R be a finite extension of $W(k)$, the Witt vectors of k. Let G be a finite group and let $\phi : G \hookrightarrow \mathrm{Aut}(k[\![t]\!])$ be a local action of G. We shall say that the action lifts to characteristic 0 if, after a finite extension of R, we can find an embedding $G \hookrightarrow \mathrm{Aut}(R[\![t]\!])$ reducing to ϕ. One method for proving the nonliftability of a local G-action is by the use of the Bertin-obstruction, introduced by Bertin in Bertin [1].

In our talk we shall study the local lifting problem for the case that G are the so-called generalized quaternion groups. The generalized quaternion group of order 2^{n+1}, $n \geq 2$, is defined as the finite group $Q_{2^{n+1}}$ with presentation

$$Q_{2^{n+1}} := \left\langle \tau, \sigma | \tau^{2^n} = 1, \quad \sigma^2 = \tau^{2^{n-1}}, \quad \sigma\tau\sigma^{-1} = \tau^{-1} \right\rangle.$$

In their paper Chinburg–Guralnick–Harbater [3], the authors proved that the Bertin-obstruction always vanishes for the generalized quaternion group of order 2^{n+1}, where $n \geq 3$. They asked the question what is to be expected for these groups in terms of liftability.

Let $F = k[\![t]\!]$ and let E/F be a $G := Q_{2^{n+1}}$-extension. We shall say that the extension E/F is simple if there exists a cyclic subgroup H of G with order 2^n, such that the local degree of different of E^H/F is exactly 2.

By studying the Hurwitz tree for a lift of a simple generalized quaternion extension E/F, we obtain a contradiction, thereby proving that these actions cannot be lifted to characteristic 0. More precisely, let $\mathcal{F} := R[\![t]\!]$ and assume that $\mathcal{E}/\mathcal{F}$ is a lift of the simple generalized quaternion extension E/F. Let $\mathcal{A}$ be a subset of the branching points $\mathcal{B}$ of $\mathrm{spec}(\mathcal{E}) \to \mathrm{spec}(\mathcal{F})$. For a point $a \in \mathcal{A}$, we introduce a notion of density $d(a, \mathcal{A})$ which, in some sense, measures the p-adic distances between the points of $\mathcal{A}$. One shows that if $\mathcal{A}_1 \subset \mathcal{A}_2$, then $d(a, \mathcal{A}_1) \leq d(a, \mathcal{A}_2)$.

The density $d(a, \mathcal{A})$ can be measured from the associated Hurwitz tree of the $Q_{2^{n+1}}$-action on $\mathcal{E}$ by means of the representation theory of $Q_{2^{n+1}}$ and the various depth characters associated to the vertices of the Hurwitz tree. We show therefore that the density $d(a, \mathcal{A})$ is intimately tied to the ramification theory associated to the Hurwitz tree.

Using this technique, we prove an exact formula for $d(a, \mathcal{B})$, where a is a specific branching point in $\mathcal{B}$. However, we identify a subset $\mathcal{A} \subset \mathcal{B}$ such that $d(a, \mathcal{A})$ violates this bound, thereby obtaining a contradiction. Our work therefore shows that the new obstruction introduced in Brewis–Wewers [2] is strictly stronger than the Bertin-obstruction.

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Computing Hurwitz numbers in positive characteristic

Irene I. Bouw
(joint work with Brian Osserman)

Let $f : Y \to \mathbb{P}^1$ be a cover of smooth projective curves branched at r ordered points $x_1, \ldots, x_r$ defined over an algebraically closed field k. The *ramification type* of f is a tuple $(d; C_1, \ldots, C_r)$, where $d = \deg(f)$ is the degree of f and $C_i = e_1(i)\text{-}\cdots\text{-}e_{n_i}(i)$ is a conjugacy class in S_d, where the $e_j(i)$ are the ramification indices of the points $f^{-1}(x_i)$.

Let $k = \mathbb{C}$. The *Hurwitz number* $h(d; C_1, \ldots, C_r)$ is the number of covers of ramification type $(d; C_1, \ldots, C_r)$ branched at fixed points $x_1, \ldots, x_r \in \mathbb{P}^1$, up to isomorphism. Riemann's Existence Theorem implies that the Hurwitz number is the cardinality of the set

$$\{(g_1, \ldots g_r) \mid g_i \in C_i,\ \langle g_i \rangle \subset S_d \,\text{transitive}\,,\ \prod_i g_i = 1\}/\sim,$$

where $\sim$ denotes uniform conjugacy by S_d. In particular, this number is finite and does not depend on the position of the branch points. Even though this allows one to compute the Hurwitz number in every given case, there are only few cases where a closed formula for the Hurwitz number is known.

Now we suppose that k is an algebraically closed field of characteristic $p > 0$. We suppose, moreover, that $p \not| e_j(i)$, for all i and j. In positive characteristic, the number of curves depends on the position of the branch points, in general. We define the p-Hurwitz number $h_p(d; C_1, \ldots, C_r)$ as the maximum over all branch points $(x_1, \ldots, x_r)$ of the number of covers of ramification type $(d; C_1, \ldots, C_r)$. This number is finite, since we assume that $p \not| e_j(i)$. There is no general approach to computing p-Hurwitz numbers in the case that the order of the Galois group of the Galois closure is divisible by p.

The following theorem is proved in [1]. The condition $\sum_{i=1}^4 e_i = 2p + 2$ implies that we consider covers $Y \to \mathbb{P}^1$ with $g(Y) = 0$.

Theorem 1. *Let $r = 4$ and $d = p$. We suppose that $C_i = e_i$ is the conjugacy class of a single cycle, and that $\sum_{i-1}^4 e_i = 2p + 2$. Then*

$$h_p(p; e_1, e_2, e_3, e_4) = h(p; e_1, e_2, e_3, e_4) - p.$$

In [3], we find a formula for the characteristic-0 Hurwitz number in this case. Moreover, in that paper, we find a description of the admissible covers of type $(p; e_1, e_2, e_3, e_4)$ in characteristic zero. These correspond to degenerate covers made up out of covers branched at three points, which need no longer have just single cycles as ramification type. As a first step we determine the number of such three-point covers with bad reduction using the results of Wewers [4]. Since every tame cover lifts to characteristic zero, we obtain a formula for the number of three-point covers in characteristic p, and hence also for the number of admissible covers. This yields a lower bound for the p-Hurwitz number $h_p(p; e_1, e_2, e_3, e_4)$.

To obtain an upper bound on $h_p(p; e_1, e_2, e_3, e_4)$, we use a partial generalization of the results of Wewers to the case of 4-point covers ([2]). We show that for general choice of the branch points $(x_1, \dots, x_4)$ the number of covers with bad reduction is nonzero and divisible by p.

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Dévissage for local deformation functors

Jakub Byszewski

We study the operations of restriction and induction on infinitesimal deformation functors of local G-actions. We compute how these operations act on their tangent spaces. Guided by the cohomological information we pose the following question: Does there exist a pro-representable functor F such that the map $D_G \to D_N^{G/N} \times_F D_{G/N}$ is smooth? Under additional assumptions of pro-representability, we compute the obstruction space to the morphism $D_G \to D_N^{G/N} \times D_{G/N}$, thus providing some evidence that the answer to the question is positive.

Let k be a perfect field with $\mathrm{char}(k) = p > 0$ and let G be a finite group. We call a *local G-action* an injective homomorphism $\rho\colon G \to \mathrm{Aut}_k k[[t]]$. A classical way to obtain local G-actions is to consider a finite group of automorphisms of a smooth curve X over k. Then the inertia group of a closed point $x \in X$ acts on the completed local ring $\widehat{\mathcal{O}}_{X,x} \simeq k[[t]]$.

Let $W(k)$ be the ring of Witt vectors over the field k and let $\mathcal{C}$ be a category of local artinian $W(k)$-algebras with residue field k. A lift of ρ to an object A of

$\mathcal{C}$ is a homomorphism $\rho_A\colon G \to \mathrm{Aut}_A A[[t]]$ which reduces to ρ in k. Two lifts are equivalent if they differ by a conjugation by an element $\beta \in \mathrm{Aut}_A A[[t]]$, $\beta \otimes k = \mathrm{id}$. We consider the infinitesimal deformation functor $D_G\colon \mathcal{C} \to \mathbf{Ens}$ which maps A to the equivalence classes of lifts of ρ to A.

It is very easy to verify that the functors D_G satisfy the conditions of Schlessinger [4], and hence have a versal hull. For our main result, we need a stronger condition of pro-representability. This condition has been positively verified in the case of weak ramification and $p \geq 3$ or $p = 2$ and G of order ≥ 8 [3].

The following two operations can be defined on local G-actions: one can restrict the action to a subgroup N; one can induce the action on the quotient group G/N acting on $k[[t]]^N$. In fact, both these operations are functorial and give morphisms $\mathrm{res}\colon D_G \to D_N$ and $\mathrm{ind}\colon D_G \to D_{G/N}$. In the global context, induction has been studied in [2]. To understand the tangent maps to these morphisms, we introduce the following notation: $\Theta = \mathrm{Der}_k(k[[t]])$, $\Theta^\sharp = \mathrm{Der}_k(k[[t]]^N)$. It is well-known that Θ is isomorphic to the different $\mathcal{D}(k[[t]]/k[[t]]^G)$ of the extension $k[[t]]^G \subseteq k[[t]]$, and similarly $\Theta^\sharp \simeq \mathcal{D}(k[[t]]^N/k[[t]]^G)$. Let $\Theta_1 = \mathcal{D}(k[[t]]^N/k[[t]]^G)k[[t]]$. We have $\Theta^\sharp = \Theta_1^N$ and we think of Θ_1 as of a module of relative derivations. The tangent spaces to D_G, D_N and $D_{G/N}$ are given by $\mathrm{H}^1(G,\Theta)$, $\mathrm{H}^1(N,\Theta)$ and $\mathrm{H}^1(G/N,\Theta^\sharp)$ respectively.

Proposition 1. *The tangent map to induction* $\mathrm{ind}\colon \mathrm{H}^1(G,\Theta) \to \mathrm{H}^1(G/N,\Theta^\sharp)$ *is such that the composite map*

$$\mathrm{H}^1(G,\Theta) \overset{\mathrm{ind}}{\to} \mathrm{H}^1(G/N,\Theta^\sharp) \overset{\mathrm{inf}}{\hookrightarrow} \mathrm{H}^1(G,\Theta_1)$$

is the natural map associated to the inclusion $\Theta \subseteq \Theta_1$.

In the case of the restriction morphism, the tangent map takes values in a subgroup $\mathrm{H}^1(N,\Theta)^{G/N}$. The action of G/N can in fact be lifted to an action of G/N on D_N in such a way that the restriction morphisms maps into $D_N^{G/N}$. (In the global context, this action was noticed in [1].)

We propose to continue our study by looking at the morphism $\psi\colon D_G \to D_N^{G/N} \times D_{G/N}$. As usual, we begin with computing the tangent map.

Proposition 2. *There is a (natural) morphism*

$$\gamma\colon \mathrm{H}^1(N,\Theta)^{G/N} \to \mathrm{H}^1(G/N,\Theta^\sharp/\Theta^N)$$

such that the image of the tangent map to ψ *is*

$$\mathrm{H}^1(N,\Theta)^{G/N} \times_{\mathrm{H}^1(G/N,\Theta^\sharp/\Theta^N)} \mathrm{H}^1(G/N,\Theta^\sharp).$$

We pose the following question: Does there exist a pro-representable functor F with a tangent space $\mathrm{H}^1(G/N,\Theta^\sharp/\Theta^N)$ and with morphisms $D_N^{G/N} \to F$ and $D_{G/N} \to F$ such that the morphism ψ maps into the fibered product $D_N^{G/N} \times_F D_{G/N}$ and the morphism $D_G \to D_N^{G/N} \times_F D_{G/N}$ is smooth?

We conclude by providing some evidence suggesting a positive answer. Existence of a suitable functor F would imply that an obstruction space to the morphism $D_G \to D_N^{G/N} \times D_{G/N}$ coincides with the tangent space to F. We provide an independent proof of this statement.

Theorem 3. *Assume that the functors D_N and $D_{G/N}$ are pro-representable. Then the vector space $\mathrm{H}^1(G/N, \Theta^\sharp/\Theta^N)$ is a complete obstruction space to the morphism*

$$D_G \to D_N^{G/N} \times D_{G/N}.$$

The main technical tool used in the proof is the Hochschild-Serre spectral sequence in group cohomology.

Corollary 4. *Assume that the functor D_N is pro-representable and that the order of G/N is prime to p. Then the map $D_G \to D_N^{G/N}$ is an isomorphism.*

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Wild monodromy singularities of stable models

Franz Király

Let X be a regular scheme of dimension 2, G a finite group and $\phi : G \hookrightarrow \mathrm{Aut}(X)$ a global action, let $q : X \to Y = X/G$ be the quotient morphism by ϕ. In this talk we shall assume that the singular locus $\mathrm{Sing}Y$ is finite and has codimension 2. For our purposes we may assume that $X = \mathrm{Spec}\, B$ and $Y = \mathrm{Spec}\, B^G$ with Y having only one singularity at the point y and $x = q^{-1}(y)$ with x the only fixed point of ϕ.

The main goal of this topic is to classify all possible singularities y up to analytic isomorphism. This is done by considering a minimal normal crossing resolution $\pi : Z \to Y$ of y and classifying the possible resolution graphs and combinatorial data associated to $\pi^{-1}(y)$.

One classical result is that if the residue field is $k(\mathcal{O}_{X,x}) = \mathbb{C}$, then the occurring singularities are exactly the ADE-singularities (see for example [1]). In the case of $p = \mathrm{char} k(\mathcal{O}_{X,x}) > 0$, one can extend this result to tame actions, i.e. $p \not| \#G$, by examining $\mathrm{SL}_2(k(\mathcal{O}_{X,x}))$. It remains an open problem to classify wild singularities, i.e. where $p | \#G$. Unlike in the tame case, many elementary techniques as local linearization fail in this case.

Now let L/K be a galois extension of complete discrete valuation fields, R/S the corresponding extension of integers, k the corresponding residue field. Let C be a smooth K-curve with potential good reduction over L, and let X be the stable model of C over R. Assume the monodromy group $G = \mathrm{Gal}(L/K)$ of this extension is $G \cong \mathbb{Z}/p\mathbb{Z}$. In this case, partial results have been obtained by Lorenzini [2] and Peskin [3]. In particular, Lorenzini shows that the resolution graph is rational and always a tree.

Without loss of generality one can consider the complete local germs $\hat{\mathcal{O}}_{X,x}$ under the germ of the action ϕ since one works over excellent rings. I.e. one can describe the singularity at x by resolving the local germs.

In the wild equicharacteristic case, one has $R \cong k[[x]]$ and $\hat{\mathcal{O}}_{X,x} \cong k[[x,y]]$. This case is treated in Peskin's thesis [3], where it is shown that for certain G-actions, the invariant ring is $k[[x,y]]^G \cong k[[x,y,z]]/I$ for some principal ideal I. This result is then used to calculate the resolution graphs explicitly. One can show that the invariant ring has a similar structure for all $\mathbb{Z}/p\mathbb{Z}-$actions, but unfortunately this cannot be used directly to classify the singularities since I is not known explicitly anymore.

Now if R is of mixed characteristic, one knows that $\hat{\mathcal{O}}_{X,x} \cong R[[x]]$. Then one has that $R[[x]]^G$ is the normalization of $R[[x,y]]/I$ where I is a principal ideal. This remains even true for all wild actions, but as above it is difficult to make the ring explicit.

In Lorenzini's recent work [2] it is proved that in the tame p-cyclic case the resolution graph is a rational chain and in the wild p-cyclic case the resolution graph is a tree which is no chain, i.e. contains a vertex with at least three neighbors. However, these results are obtained by making the Néron model of the Jacobian of C explicit which is very transcendental. This motivates efforts to find more elementary and general approaches to these results. In the tame case, this can be done as above. In the wild case, there is still much work to be done. Several critical examples and possible strategies were discussed in the talk.

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Deformations of curves with automorphisms and representations on Riemann-Roch spaces

Aristides Kontogeorgis

One of the main difficulties for understanding deformations of curves with automorphisms is to understand the local deformation functor or equivalently lifts of representations

$$\begin{array}{ccc} & & \mathrm{Aut}A[[t]] \\ & \nearrow & \downarrow \\ \rho : G & \longrightarrow & \mathrm{Aut}k[[t]]. \end{array} \tag{1}$$

Aim of my talk was the relation of this problem to a similar problem of deformations of matrix representations.

Namely, consider a nonsingular complete curve X of genus $g \geq 2$ together with a faithful action of a group G, and let P be a wild ramification point. Let m be the smallest pole number in the Weierstrass semigroup at P. Then there is a faithful representation [2]

$$\rho : G_1(P) \to \mathrm{GL}(L(mP)).$$

Question: Does every possible lift in (1) come from a matrix representation of $G_1(P)$ into a linear automorphism group of an A-module?

The answer is yes under some conditions we are going to describe. Every such lift gives rise to a local deformation which in term of the local-global principal of Bertin-Mézard [1] gives rise to a global deformation $\mathcal{X} \to \mathrm{Spec}A$.

Suppose that we can construct an effective $G_1(P)$-invariant horizontal Cartier divisor D, supported at the branch locus of $\mathcal{X} \to \mathcal{X}/G_1(P)$ so that $D \otimes_A k = mP$. Then we can prove that there is a lift

$$\begin{array}{ccc} & & \mathrm{GL}(H^0(\mathcal{X}, \mathcal{L}(D)) \\ & \nearrow & \downarrow \\ \rho : G_1(P) & \longrightarrow & \mathrm{GL}(L(mP)), \end{array} \tag{2}$$

and after selecting an appropriate basis for the free A-module $H^0(\mathcal{X}, \mathcal{L}(D))$ the lift of the matrix representation in (2) gives rise to the original lift in (1).

The main problem we are facing, is that such a divisor D does not always exist. The existence of such divisor depends on combinatorial data of the action of $G_1(P)$ on the set of horizontal branch divisors that intersect the special fibre at P. Namely, to every horizontal branch divisor we attach an Artin representation. Collect all horizontal branch divisors P_i that collapse to the same point in the special fibre. The following equation must hold:

$$\mathrm{ar}_P(\sigma) = \sum_{i=1}^{s} \mathrm{ar}_{P_i}(\sigma).$$

The above equation is coming by an algebraic equivalence argument by interpreting the Artin representation of σ as the intersection of the graph of an automorphism with the diagonal.

If we suppose that the representation attached to the wild ramified point P is two dimensional, then the group $G_1(P)$ is elementary abelian, and we can write a compact formula for the action of $\mathrm{Aut}k[[t]]$, namely

$$\rho_\sigma(t) = \frac{t}{(1 + c(\sigma)t^m)^{1/m}}.$$

By selecting a $\tilde{f} \in H^0(\mathcal{X}, \mathcal{L}(D))$ so that $\tilde{f} = f \mod m_A$ we can give also a formula for the extension $\tilde{\rho}_\sigma$ of ρ_σ and eventually compute the image of $\tilde{\rho}_\sigma$ in the tangent space $H^1(G_1(P), \mathcal{T}_{k[[t]]})$. The dimension of this tangent space is computed [3] and the exact computation of representations that can be lifted gives us new obstructions of actions of elementary abelian groups. We hope that this approach will also allow us to compute the Krull dimension for the versal deformation ring for the local representation functor.

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Semistable reduction and maximal wild monodromy.

Michel Matignon

Let R be a discretely complete valuation ring with an algebraically closed residue field k of characteristic $p > 0$. We denote by K the fraction field of R.

Let C/K be a connected projective smooth curve over k with genus $g \geq 2$. There is a minimal Galois extension K'/K such that the curve $C \times K'$ has a unique stable model $\mathcal{C}$ over R', the valuation ring of K' ([De-Mu 69],[Des 81]). The Galois group $\mathrm{Gal}(K'/K)$ goes into $\mathrm{Aut}_k\mathcal{C} \times_R k$, it is called the *monodromy group* and its p-Sylow subgroup is the *wild monodromy group*.

In this talk we reported on the work by Silverberg-Zahrin ([Si-Za 05]) describing the possible monodromy groups in equal characteristic $p > 0$ and genus 2 by considering the corresponding question for abelian surfaces.

In the inequal characteristic case, we gave a report on the work by C. Lehr and the author ([Le-Ma 06]) describing the wild monodromy group in the case of p-cyclic covers of the projective line when the branch locus is étale over R (Raynaud's condition ([Ra 90])). Namely we give a polynomial whose splitting field is the monodromy extension K'/K. We then describe the wild monodromy groups when $p = 2$ and $g = 2$.

The slides of this talk are available at the following address:
http://www.math.u-bordeaux1.fr/˜matignon/preprints.html

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On the geometry of the versal $\mathbb{Z}/p\mathbb{Z}$ equivariant deformation space.

Sylvain Maugeais

Let k be an algebraically closed field of characteristic $p > 0$ and consider a faithful action of $\mathbb{Z}/p\mathbb{Z}$ on $X = \mathrm{Spec}k[[x]]$. According to Bertin and Mézard (cf. [1]) the $\mathbb{Z}/p\mathbb{Z}$-equivariant deformation functor admits a hull R_X. The purpose of this talk is to describe some properties of $\mathrm{Spec}R_X$.

By definition, the action of $\mathbb{Z}/p\mathbb{Z}$ on X lifts to an action on the versal deformation $R_X[[x]]$, this lifting being unique only up to conjugation by an automorphism of $R_X[[x]]$. We fix such a lifting for the rest of the talk.

Let m be the first (and only) jump in the ramification filtration of $\mathbb{Z}/p\mathbb{Z}$ relatively to the action on X. This integer is prime to p and is called the conductor of the action at the closed point of X.

Let σ be a generator of $\mathbb{Z}/p\mathbb{Z}$. By the Weierstrass preparation theorem, we can write $\sigma(x) - x = Pu$ where $u \in R_X[[x]]$ is invertible and P is a distinguished polynomial of degree $m + 1$. Let us write $P = x^{m+1} + a_m x^m + \ldots + a_0$ with $a_i \in R_X$ and let $W(k)$ denote the ring of Witt vectors of k.

Proposition 1. *The map* $\Phi : \mathrm{Spec}R_X \to \mathrm{Spec}W(k)[[A_0, \ldots, A_m]]$ *defined by* $A_i \mapsto a_i$ *is a closed immersion.*

This means that a deformation is uniquely determined by its divisor of fixed points (up to conjugation). The question is then to characterize the image of the map Φ, i.e. the possible set of fixed points.

To do this, it is actually more convenient to introduce the functor of equivariant deformations together with $m + 1$ ordered sections such that the divisor of fixed points is the product of this sections. This deformation functor has a hull (it can be seen as a fibred product using the morphism Φ), we will denote it by $\tilde{R}_X$. Forgetting the sections induces a morphism $\mathrm{Spec}\tilde{R}_X \to \mathrm{Spec}R_X$ which is finite flat.

Let us first describe the generic points of the space $\mathrm{Spec}\tilde{R}_X \otimes_{W(k)} k$.

Proposition 2. *A generic point of* $\mathrm{Spec}\tilde{R}_X \times_{W(k)} k$ *corresponds to a disk on which the action of* $\mathbb{Z}/p\mathbb{Z}$ *at the fixed points have a conductor* $< p$.

In particular, if we are looking at irreducible components of $\mathrm{Spec}\tilde{R}_X \otimes_{W(k)} k$ the corresponding divisor of fixed points should be of the form $\prod_i (x-\alpha_i)^{m_i}$ with $\alpha_i \neq \alpha_j$ if $i \neq j$ and $0 < m_i < p$. Using the local constancy of the different for the action on $\mathrm{Spec}\tilde{R}_X \otimes_{W(k)} k[[x]]$ and computing it at a generic point and at the closed point leads to

$$m + 1 = \sum_i (m_i + 1).$$

This gives a first necessary condition for $\prod_i (x-\alpha_i)^{m_i}$ to appear in the image of Φ.

Let us fix a sequence (m_i) satisfying this condition. We will now build some irreducible components of $\mathrm{Spec}\tilde{R}_X \otimes_{W(k)} k$ corresponding to this sequence.

Proposition 3. *Let* A *be an integral complete noetherian local* k*-algebra with residue field* k *and* $P = \prod_i (x-\alpha_i)^{m_i}$ *with* $0 < m_i < p$ *and* $\alpha_i \neq \alpha_j$ *if* $i \neq j$. *Suppose that* P^{p-1} *is exact (i.e. there exists a polynomial* $Q \in A[x]$ *such that* $\frac{d}{dx}Q = P^{p-1}$*) then there exists a* $\mathbb{Z}/p\mathbb{Z}$*-equivariant deformation of* X *over* A *whose divisor of fixed points is precisely* P.

This proposition is based on a local computation and the following theorem (cf. [3]).

Theorem 4. *Let* S *be a normal scheme and* $U \to S$ *be a smooth, affine and geometrically irreducible curve. For each* $s \in S$ *define* $g(s)$ *to be the genus of the unique smooth completion of* $U \times_S s$. *If* $s \mapsto g(s)$ *is locally constant there exists a unique proper and smooth curve* $C \to S$ *with an open embedding* $U \to C$.

For a sequence (m_i) as above, it is natural to consider the moduli space $Y_{(m_i)}$ of polynomials of the form $P = \prod_i (x-\alpha_i)^{m_i}$ such that P^{p-1} is exact. Using the proposition 3, we are able to construct a map $Y_{(m_i)} \to \mathrm{Spec}\tilde{R}_X \otimes_{W(k)} k$ which we prove to be finite and dominant over some irreducible components.

Theorem 5. *The map* $\coprod_{(m_i)} Y_{(m_i)} \to \mathrm{Spec}\tilde{R}_X \otimes_{W(k)} k$ *is surjective.*

We thus get a stratification of $\mathrm{Spec}\tilde{R}_X \otimes_{W(k)} k$ as the images of the $Y_{(m_i)}$ are distinct for two different sequences (m_i). We have thus reduced the problem to the study of each $Y_{(m_i)}$, for which we have an explicit set of equations.

Let us now look at the full spectrum $\mathrm{Spec}\tilde{R}_X$. First let us recall a result of Green and Matignon (see [2]).

Theorem 6. *Suppose that* $0 < m < p$. *Then* $\tilde{R}_X$ *is a finite flat* $W(k)$*-algebra and the geometric points of its generic fibre are in one to one correspondence with the solution of the system*

$$\begin{cases} h_0 a_0 + \ldots + h_m a_m = 0 \\ \qquad \vdots \\ h_0 a_0^m + \ldots + h_m a_m^m = 0 \end{cases} \quad \textit{with the condition} \begin{cases} a_0 = 0 \quad a_1 = 1 \\ \prod_{i\neq j}(a_i - a_j) \neq 0 \end{cases}$$

Let $Z \subset \mathrm{Spec}\tilde{R}_X$ be an irreducible component and $\eta \in Z \otimes_{W(k)} k$ be a generic point. As seen in the proposition 2, the point η corresponds to a disk D on which $\mathbb{Z}/p\mathbb{Z}$ acts and the conductor of this action at each fixed point a_i is $< p$. Using a local-global principle as in [1], we can prove that deforming the action on D amounts to deform each $\mathrm{Spec}\widehat{O}_{D,a_i}$ independently. As each such deformation is described by the theorem 6, we are able to get quite a good description of Z.

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On Breuil-Kisin modules

Ariane Mézard

Let k be a perfect field of characteristic $p > 0$, $W = W(k)$ ring of Witt vectors equipped with the Frobenius φ. Let $K_0 = W[1/p]$ and K/K_0 be a finite totally ramified extension with uniformizer π. Denote $q = E(u)$ the Eisenstein polynomial of π. Let $\mathfrak{S} = W(k)[[u]]$ equipped with a Frobenius operator φ which acts via the Frobenius on W and $\varphi(u) = u^p$. The object of this talk is to present some results of Kisin ([Ki1],[Ki2]) on classification of finite flat groups scheme over $\mathfrak{O}_K$ and using Caruso's computation, to rely with Tossici's results ([To]).

In [Ki1],[Ki2], Kisin proves that the crystalline representations with negative Hodge-Tate weight are faithfully embedded in the category of free Breuil-Kisin modules up to isogeny. Remind that crystalline representations of weights 0,-1 are equivalent to the category of p-divisible group schemes over $\mathfrak{O}_K$ ([Br]). Kisin obtains an equivalence of categories between finite group scheme over $\mathfrak{O}_K$ and the category of Breuil-Kisin modules killed by a power of p, that is finite ϕ_M module M over $\mathfrak{S}$ killed by p such that Coker ϕ_M is killed by a power of $E(u)$.

Using this equivalence of category, Caruso gives a complete classification of finite flat group schemes over $\mathfrak{O}_K$ killed by p^2. This gives an alternative proof of Tossici's result ([To]). Using Sekiguchi-Suwa's theory Tossici gives precise formula for the finite flat groups schemes over $\mathfrak{O}_K$ killed by p^2. To my knowledge, these formula can not be deduce directly from Kisin's equivalence of categories.

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Smooth curves with a large automorphism p-group in characteristic $p > 0$.

Magali Rocher

Let k be an algebraically closed field of characteristic $p \geq 0$ and C a connected nonsingular projective curve over k with genus $g \geq 2$. In characteristic zero, the full automorphism group $\operatorname{Aut} C$ of C is known to be finite and of order at most $84\,(g-1)$. An open question concerns the classification of such groups for a given genus g. This classification has been partially achieved for the Hurwitz groups, that is the groups whose order reaches the Hurwitz bound $84(g-1)$, but also for *large* automorphism groups G, *large* meaning that the order of G is greater than $4\,(g-1)$ ([Ku97]). Indeed, combined with the Hurwitz genus formula, this lower bound imposes restrictions on the genus of the quotient curve C/G, namely $g_{C/G} = 0$, and on the ramification of the cover $C \to C/G$.

In positive characteristic $p > 0$, the automorphism group $\operatorname{Aut}_k(C)$ is still finite but for groups whose order is not prime to p, the Hurwitz linear bound is replaced by a biquadratic one ([St73]). This is due to the appearance of wild ramification. To rigidify the situation as has been done in characteristic zero, an idea is to consider large automorphism p-groups. This motivates the following:

Definition 1. [LM05] *Let C/k be a connected nonsingular projective curve with genus $g \geq 1$. Let G be a finite subgroup of* $\operatorname{Aut}_k(C)$. *The pair (C, G) is called a* big action *if G is a p-group and*

$$|G| > \frac{2p}{p-1}\, g. \tag{1}$$

When (1) is satisfied, the Hurwitz and the Deuring-Shafarevitch formulas applied to $C \to C/G$ imply that $g_{C/G} = 0$ and that only one point $\infty \in C$ is ramified (and even totally ramified) in C/G. Call G_i the i-th lower ramification group of G at ∞. Then, $G = G_{-1} = G_0 = G_1$ and the quotient curve C/G_2 is isomorphic to the projective line. In particular, G_2 is nontrivial. The choice of the bound (1) is also necessary to get the strict inclusion $G_2 \subsetneq G_1 = G$. It follows that the quotient group G/G_2 acts as a group of translations of the affine line $C/G_2 - \{\infty\} = \operatorname{Spec} k[X]$, through $X \to X + y$, where y runs over a subgroup V of k. This gives the exact sequence

$$0 \longrightarrow G_2 \longrightarrow G = G_1 \xrightarrow{\pi} V \simeq (\mathbb{Z}/p\,\mathbb{Z})^v \longrightarrow 0, \tag{2}$$

where

$$\pi : \begin{cases} G \to V \\ g \to g(X) - X. \end{cases}$$

In the first part of this talk, we give necessary conditions on G_2 for (C, G) to be a big action.

Proposition 2. [MR08] *Let (C, G) be a big action with $g \geq 2$.*

(1) *Let H be a normal subgroup of G such that $H \subsetneq G_2$. Then $(C/H, G/H)$ is a big action with second ramification group $(G/H)_2 = G_2/H$.*
(2) *The group G_2 is equal to $D(G)$, the commutator subgroup of G. In particular, G cannot be abelian.*
(3) *The group G_2 cannot be cyclic unless G_2 has order p.*
(4) *If $\frac{|G|}{g^2} \geq \frac{4}{(p^2-1)^2}$, then G_2 is an elementary abelian p-group with order dividing p^3.*

A key idea in studying big actions is to use the first statement to go back to the situation of big actions with a p-cyclic G_2, a case fully characterized in [LM05]. Moreover, the last point is crucial in pursuing the classification of big actions (see [Ro08b]). The proof of this last statement requires two dual methods. The first one consists in studying the upper ramification filtration on G_2. When this method is not sufficient to conclude, one has to consider the *embedding problem* related to the exact sequence (2), that is to check the stability of the system of equations parameterizing $k(C)/k(X)$ under the translations by V.

The second part of this talk is devoted to examples of big actions with G_2 abelian with arbitrary large exponent (see [MR08]). Following [Au99] and [Lau99], we consider the maximal abelian extension K^m of $K := \mathbb{F}_q(X)$ (where $q = p^e$) that is unramified outside $X = \infty$, completely split over all finite rational places and whose conductor is smaller than $m\,\infty$, with $m \in \mathbb{N}$. It follows from the uniqueness and the maximality of K^m that the group of translations $\{X \to X + y,\ y \in \mathbb{F}_q\}$ extends to a p-group of $\mathbb{F}_q$-automorphisms of K^m, say G_m, with the exact sequence

$$0 \longrightarrow \mathrm{Gal}\,(K^m/K) \longrightarrow G_m \longrightarrow \mathbb{F}_q \longrightarrow 0.$$

Then, for a well-chosen conductor $m_2 := p^{\lceil e/2 \rceil + 1} + p + 1$, we get a big action whose second ramification group $\mathrm{Gal}\,(K^{m_2}/K)$ is abelian of exponent p^2. This example also relates the problem of big actions to the search of algebraic curves with many rational points. More precisely, if N_m denotes the number of $\mathbb{F}_q$-rational points of the nonsingular projective curve C_m with function field K^m, then $\frac{|G_m|}{g_{C_m}} \sim \frac{N_m}{g_{C_m}}$, when e grows large.

In the third part of this talk, we focus on the big actions (C, G) with a p-elementary abelian G_2.

Theorem 3. *(*[Ro08a]*) Let (C, G) be a big action with $g \geq 2$ such that $G_2 \simeq (\mathbb{Z}/p\,\mathbb{Z})^n$, $n \geq 1$. For all $t \geq 1$, let Σ_t be the k-vector subspace of $k[X]$ generated by 1 and the products of at most t additive polynomials. Then the function field of the curve can be parameterized by n Artin-Schreier equations:*

$$\forall\, i \in \{1, \ldots, n\}, \quad W_i^p - W_i = f_i(X) \in \Sigma_{i+1}.$$

This result generalizes the p-cyclic case described in [LM05] but, contrary to this case, the converse is no longer true for $n \geq 2$, which means that such a family $(f_i)_{1\leq i\leq n}$ does not necessary give birth to a big action. Nevertheless we give a group-theoretic characterization of the special case where each f_i lies in $\Sigma_{i+1} - \Sigma_i$. Then we construct a special family satisfying this condition and also give the parametrization of all big actions (C, G) such that each $f_i \in \Sigma_{i+1} - \Sigma_i$, for $p = 5$, $n \in \{2, 3\}$ and $\dim_{\mathbb{F}_p} V = 2$. This leads to discuss the corresponding deformation space.

To conclude, we give finiteness results for big actions (C, G) with $\frac{|G|}{g^2} \geq M > 0$.

Proposition 4. [Ro08b] *Let (C, G) be a big action such that $\frac{|G|}{g^2} \geq M > 0$. Let* $\mathrm{Fratt}(G_2)$ *be the Frattini subgroup of G_2 and $[G_2, G]$ be the commutator subgroup of G_2 and G.*

(1) *The order of G_2 only takes a finite number of values.*
(2) *If* $\mathrm{Fratt}\,(G_2) \subsetneq [G_2, G]$, $\frac{|G|}{g}$ *takes a finite number of values.*
(3) *If* $\mathrm{Fratt}\,(G_2) = [G_2, G]$, $\frac{|G|}{g}$ *is not necessarily bounded but*
    (a) *when* $\mathrm{Fratt}\,(G_2) = [G_2, G] = \{e\}$, $\frac{|G|}{g^2}$ *take a finite number of values.*
    (b) *when* $\mathrm{Fratt}\,(G_2) = [G_2, G] \neq \{e\}$ *and if $p > 2$, G_2 is non-abelian.*

Note that we do not know yet examples of big actions with a non-abelian G_2.

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Images of finite schemes inside functors of homomorphisms

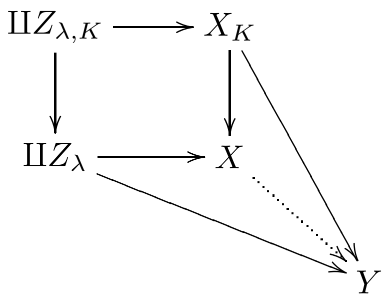
Matthieu Romagny

Let us fix a discrete valuation ring R with fraction field K. We consider a scheme X of finite type over R together with the action of a finite group G. Most often, in arithmetic geometry, such a situation arrives when one studies the reduction of a K-variety X_K with group action : if the variety has a model X satisfying some unicity property, then the action of G extends to X. Typical examples are the stable model of a smooth curve, or the Néron model of an abelian variety. In these cases, however, the action of G on the special fibre may fail to be faithful. The aim of the following result is to give a justification to the following naive idea : instead of G, one may look at the scheme-theoretic image of the morphism $G_R \to \mathrm{Aut}_R(X)$, where G_R is the finite flat R-group scheme associated to G. The problem that arises when one wants to do this is that the functor of automorphisms $\mathrm{Aut}_R(X)$, which is a sheaf for the fppf topology, is in general not representable by a scheme, in the absence of projective assumptions on X. However, it is possible to define the scheme-theoretic image in the setting of fppf sheaves, and our main result is :

Theorem 1. *Let X be an R-scheme locally of finite type, separated, flat and pure, of type* (FA). *Let G be a finite flat R-group scheme acting on X, faithfully on the generic fibre. Then the schematic image of G in* $\mathrm{Aut}_R(X)$ *is representable by a finite flat R-group scheme.*

In this statement, the most important notion is that of a *pure R-scheme.* (We leave aside the details concerning other undefined less important notions, such as the (FA) assumption.) If R is henselian, then roughly speaking, a scheme locally of finite type, flat and pure, of type (FA), is an R-scheme locally of finite that has a covering by open affine schemes whose function rings are free R-modules. Most schemes that one encounters in practical situations have these properties. The result above is based on some nice properties enjoyed by pure schemes. Especially important is the fact that for such a scheme X, the family of closed subschemes finite flat over R is schematically dense universally over R, and furthermore, X is the amalgamated sum of its generic fibre X_K and the family of all its closed subschemes finite flat over R. In fact, we prove :

Theorem 2. *Assume that R is henselian. Let X be an R-scheme locally of finite type, separated, flat and pure, of type* (FA). *Then, the family of all closed subschemes $Z_\lambda \subset X$ finite flat over R is R-universally schematically dense, and for all separated R-schemes Y and all diagrams in solid arrows*



there exists a unique morphism $X \to Y$ making the full diagram commutative.

Using theorem 2, we prove the following. Let X, Y be R-schemes locally of finite type, with X flat and pure, of type (FA) and Y separated. Let $\varphi : \Gamma \times X \to Y$ be an "action" of a finite flat R-scheme Γ, faithful on the generic fibre in an obvious sense. Then, the schematic image of Γ in $\mathrm{Hom}_R(X, Y)$ is representable by a finite flat R-scheme. Theorem 1 is an easy corollary of this.

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Computing orbits of inertia for the Galois action on curve covers

Björn Selander

Let $C \to D$ be a branched G-cover of the closed Riemann surface D, and let k be an algebraic closure of the field with p elements. If R is some suitable integrally closed finite extension of the Witt ring $W(k)$, then there is a stable model $C_R \to D_R$ (with an action of G on C_R) of $C \to D$. Let $\bar{f}\colon \bar{C} \to \bar{D}$ be the special fibre of this stable model.

Under the further assumptions that D is the Riemann sphere, $C \to D$ branches at three points, and $p \mid |G|$ but $p^2 \nmid |G|$, then in most cases $\bar{f}\colon \bar{C} \to \bar{D}$ is what is called a *special deformation datum* in [2]. In this case, the following formulas

$$L(\bar{f}) = \frac{p-1}{n'} \prod_{\text{tails of } \bar{D}} \frac{h_j}{|\operatorname{Aut}^{\eta_j}_{G_j}(\bar{C}_j)|}$$
$$N(\bar{f}) = \frac{p-1}{n'} \operatorname{lcm}_j \frac{h_j}{|\operatorname{Aut}^{\eta_j}_{G_j}(\bar{C}_j)|}$$

were proven in [ibid.]. Here $L(\bar{f})$ is the number of non-isomorphic characteristic zero G-covers which have $\bar{f}\colon \bar{C} \to \bar{D}$ as special fibre of their formal models, and $N(\bar{f})$ is the number of non-isomorphic characteristic zero G-covers which lie in the same orbit under the action of some inertia group I_p at p in $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

In my talk at the conference I explained the factors at the right hand side of the above-mentioned formulas. The denominators in the formulas come from counting the elements of the group $\mathrm{Aut}^0_G(\bar{f})$, consisting of the automorphisms of $\bar{C}$ which commute with the G-action and which induce identity on the 'central' component of $\bar{D}$. I briefly explained that there is a mistake in the combinatorial analysis of this group in [ibid.], which might lead to erroneous values for n' in the above formulas. A correct version of how to assemble the group $\mathrm{Aut}^0_G(\bar{f})$ from groups given on preimages of the different components of $\bar{D}$ is found in [1].

I also considered examples of families of characteristic zero G-covers with the same special fibres. Here the main goal was to show that it seems reasonable to believe that if we fix a prime p, then it is possible to recursively (on the degree of the permutation group G) compute almost all the I_p-orbits of characteristic zero G-covers for groups G satisfying our condition. The main tools here are the classification of special deformation data and the above-mentioned formulas. In addition, such a computation would produce interesting examples of *tail covers*, i.e., covers in characteristic p of the projective line which are branched at most at two points, and wildly branched at exactly one point.

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Weak and strong extension of torsors

Dajano Tossici

Let R be a discrete valuation ring of unequal characteristic with fraction field K. Let G be an abstract finite group, X a normal faithfully flat scheme over R and $Y_K \longrightarrow X_K$ a G_K-torsor. We study the problem of extending this torsor. There are two possible notions of extension of torsors. If, for some finite and flat R-group scheme $\mathcal{G}$, there exists a $\mathcal{G}$-torsor $Y' \longrightarrow X$ which coincides with the G-torsor $Y_K \longrightarrow X_K$ on the generic fiber we will say that $Y_K \longrightarrow X_K$ is *weakly extendible*. In the case Y' is also normal we will say that $Y_K \longrightarrow X_K$ is *strongly extendible*. We remark that, since X is normal, in such a case Y' coincides with the normalization Y of X in Y_K.

For any $m \in \mathbb{N}$ and any scheme Z, we define ${}_mPic(Z) := \ker(Pic(Z) \xrightarrow{m} Pic(Z))$. About the weak extension we have the following result.

Proposition 1. *Let G be an abelian group of order m and let us suppose that R contains a primitive m-th root of unity. Let X be a normal faithfully flat scheme over R with integral fibers and ${}_mPic(X) = {}_mPic(X_K)$. Let us consider a connected G-torsor $f_K : Y_K \longrightarrow X_K$ and let Y be the normalization of X in Y_K. Moreover, we assume that Y_k is reduced. Then there exists a (commutative) R-group-scheme G' and a G'-torsor $Y' \longrightarrow X$ over R which extends f_K.*

We now consider the problem of strong extension. We observe that the G-action on Y_K can be extended to a G-action $\mu : G \times_R Y \longrightarrow Y$. The action could be not faithful on the special fiber. The effective model, recently introduced by Romagny ([R]), solve this problem. An effective model for μ is a flat R-group scheme, dominated by G, with an action on Y compatible with μ, such that the action is also faithful. When it exists, it is unique. This implies that if there exists $\mathcal{G}$ as above, then $\mathcal{G}$ must be the effective model of μ. So in order to prove the strong extension problem one can try to answer a more precise question:

Question: *Which is the effective model $\mathcal{G}$ (if it exists) for the G-action? When is $Y \longrightarrow X$ a $\mathcal{G}$-torsor?*

We now briefly recall what is known. Let us suppose that $(|G|, p) = 1$. It is classical that, if X is regular with geometrically integral fibers then, up to an extension of R, any connected G-torsor can be strongly extended. This follows from the Theorem of Purity of Zariski-Nagata ([1, X 3.1]) and from the Lemma of Abhyankar ([1, X 3.6]).

Let us now consider the case when $p||G|$. The first case is $G = \mathbb{Z}/p\mathbb{Z}$. For this group the effective models of its actions have been calculated in some cases. For details see the papers of Raynaud ([5, 1.2.1]), when $X = \mathrm{Spec}(R)$ and R complete, of Green-Matignon ([2, III 1.1]), when X is the p-adic closed disc, of Henrio ([3, 1.6]), for factorial affine R-curves complete with respect to the π-adic topology, and of Saïdi ([7, 2.4]) for formal smooth curves of finite type. We remark that the above results are true under the further assumption that X_k and Y_k are integral. In all the above cases, the effective model induces a structure of torsor, i.e. the $\mathbb{Z}/p\mathbb{Z}$-torsor $Y_K \longrightarrow X_K$ is strongly extendible.

In [13] we consider the cases $G = \mathbb{Z}/p\mathbb{Z}$ and $G = \mathbb{Z}/p^2\mathbb{Z}$ in any dimension. Since we are looking for effective models of actions of $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p^2\mathbb{Z}$, in particular we are interested in finite and flat R-group schemes which are isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}/p^2\mathbb{Z}$ over K. The classification of models of $\mathbb{Z}/p\mathbb{Z}$ is well known. While the classification of models of $\mathbb{Z}/p^2\mathbb{Z}$ is the subject of [12]. We remark we need an explicit description of these models. More precisely we obtain these models as the kernel of an isogeny which generically coincides with (a slight generalization of) the Kummer sequence. This implies that we can easily describe torsors under these group schemes. The main tool used to obtain this classification is the Kummer-Artin-Schreier-Witt theory developed by Sekiguchi and Suwa (see [11] for the case of our interest, and [10] for the general case). We stress that it could be possible try to classify such models also using the recent approach of Breuil and Kisin, through the so called Breuil-Kisin modules (e.g. see [4]). But unfortunately in such a way it is not possible to describe explicitly such groups, which is indispensable for us.

We now come back to the strong extension problem. For $\mathbb{Z}/p\mathbb{Z}$-torsors we have the following result.

Theorem 2. *Let us now suppose that R contains a primitive p-th root of unity. Moreover we suppose that $X = \mathrm{Spec}(A)$ is a normal faithfully flat R-scheme with integral fibers such that $\pi \in \mathcal{R}_A$, where $\mathcal{R}_A$ is the Jacobson radical of A, and ${}_pPic(X_K) = 0$ (e.g. A a local regular faithfully flat R-algebra with integral fibers). Then that any connected $\mathbb{Z}/p\mathbb{Z}$-torsor $Y_K \longrightarrow X_K$ is strongly extendible under the assumption that the special fiber of the normalization Y of X in Y_K is reduced.*

We now consider the case $G = \mathbb{Z}/p^2\mathbb{Z}$. Let us assume that R contains a primitive p^2-th root of unity, that $X = \mathrm{Spec}(A)$ has integral fibers and that $A \subseteq R^I$ for some set I. We moreover assume ${}_{p^2}Pic(X_K) = 0$. These conditions are for instance satisfied if R is complete with algebraically closed residue field and A is the local ring of a closed point of a smooth R-scheme of finite type. Let $Y_K \longrightarrow X_K$ a connected $\mathbb{Z}/p^2\mathbb{Z}$-torsor, Y be the normalization of X in Y_K, and assume that Y_k integral. We attach to any such $\mathbb{Z}/p^2\mathbb{Z}$-torsor a degeneration datum, which is an element of $\mathbb{N}^4$. We can prove that, for $p > 2$, this element determines its effective model $\mathcal{G}$, which we explicitly describe. Moreover, we give a criterion to see if Y is a $\mathcal{G}$-torsor. Finally, we can give an example of $\mathbb{Z}/p^2\mathbb{Z}$-torsor over X_K satisfying the above hypothesis but non-strongly extendible.

We recall that the case R of positive characteristic has been studied by Saïdi in [9]. He proved a result of strong extension of $\mathbb{Z}/p\mathbb{Z}$-torsors for formal normal schemes of finite type of any dimension ([9, 2.2.1]) and moreover he studied the case of $\mathbb{Z}/p^2\mathbb{Z}$-torsors. His approach is slightly different: he is not interested in effective models but only in explicit equations of the induced cover on the special fiber. Moreover, he gave an example of non strongly extendible $\mathbb{Z}/p^2\mathbb{Z}$-torsors in [8]. Another such example has been given by Romagny ([6, 2.2.2]). We remark that in equal characteristic there is no known criterion to determine if a $\mathbb{Z}/p^2\mathbb{Z}$-torsor is strongly extendible.

To study extensions of $\mathbb{Z}/p^n\mathbb{Z}$-torsors, with $n \geq 3$, it is necessary to have an explicit classification of models of $(\mathbb{Z}/p^n\mathbb{Z})_K$ as done in [12] for models of $(\mathbb{Z}/p^2\mathbb{Z})_K$. The strategy used in [12] could be used, in principle, also for models of $(\mathbb{Z}/p^n\mathbb{Z})_K$. However, this could lead to very complicated calculations. The right approach seems to be that one of combining the Theory of Sekiguchi-Suwa with that of Breuil-Kisin.

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Artin characters, Hurwitz trees, and the local lifting problem

Stefan Wewers
(joint work with Louis Hugo Brewis)

In my talk I explained how one can associate to a group of automorphisms of an open p-adic disk an essentially combinatorial object, called a *Hurwitz tree*. This is a generalization of a construction of Henrio [3], which itself is based on work of Green and Matignon [2]. We also use the theory of Huber [4]. Applications of our construction to the local lifting problem were given by Louis Brewis in his talk. Both talks are based on our recent preprint [1].

Let K be a nonarchimedian local field of characteristic zero, with residue field of characteristic p. Let $\mathsf{Y} = (\mathrm{Spf}\, R[[z]]) \otimes K$ be the rigid open unit disk over K, and let $G \subset \mathrm{Aut}_K(\mathsf{Y})$ denote a finite group of automorphisms. We can associate to (Y, G) a datum (T, G_v, a_e, δ_v), as follows.

Let $\mathsf{X} := \mathsf{Y}/G$ denote the quotient of Y by G. This is again an open disk over K. Let $\Delta \subset \mathsf{Y}$ and $B \subset \mathsf{X}$ denote the ramification resp. the branch locus of the G-Galois cover

$$f : \mathsf{Y} \longrightarrow \mathsf{X} = \mathsf{Y}/G.$$

Replacing K by some finite extension, we may assume that Δ and B consists of K-rational points. Then there exists a minimal semistable model $\mathcal{Y}$ of Y over R which separates the points of Δ. By uniqueness, the G-actions extends to $\mathcal{Y}$, and the quotient $\mathcal{X} := \mathcal{Y}/G$ is a semistable model of X which separates the points of B (in general, $\mathcal{X}$ is not the minimal model with this property). Reducing the G-cover $\mathcal{Y} \to \mathcal{X}$, we obtain a finite, G-equivariant map

$$f_k : Y_k \longrightarrow X_k$$

between semistable k-curves of genus zero. This map is in some sense the 'stable reduction' of the G-cover f. Whereas f is tamely ramified, the map f_k is in general inseparable, i.e. very badly ramified. With the Hurwitz tree (T, G_v, a_e, δ_v) we try to encode *combinatorial* and *metric* information related to this ramification.

It is well known how to associate to the pairs $(\mathcal{Y}, \Delta)$ and $(\mathcal{X}, B)$ metric oriented trees $\tilde{T}$ and T, respectively, and a G-invariant morphism $\tilde{T} \to T$ which identifies T with $\tilde{T}/G$. Very roughly: $\tilde{T}$ is the graph of components of the semistable curve Y_k, together with extra vertices for each element of Δ and one *root*, which represents the *boundary* of the disk Y. The metric on $\tilde{T}$ reflects the *thicknesses* of the singularities of Y_k inside $\mathcal{Y}$ (the metric tree T depends similarly on $(\mathcal{X}, B)$). From another point of view, a vertex v of T corresponds either to an element of $B \subset \mathsf{X}$, to a closed disk $D_v \subset \mathsf{X}$ or to the boundary of X.

The G-cover $\tilde{T} \to T$ of metric trees is equivalent to the datum (T, G_v), where $G_v \subset G$ is the stabilizer of a vertex of $\tilde{T}$ lying over a vertex v of T. It remains to construct the data (a_e, δ_v). Here a_e and δ_v are characters of virtual representations of G.

If v is a leaf of T (and thus corresponds to a ramification point $b \in B \subset \mathsf{X}$ of f), then $\delta_v = 0$. Moreover, if e is the unique edge of T with target v, then $a_e = u^*_{G_v}$ is the augmentation character of $G_v \subset G$, induced up to a character of G.

If v is a vertex of T which is not a leaf and not the root, then it corresponds to a closed disk $D_v \subset \mathsf{X}$. The character δ_v is essentially the Artin character of the G-cover

$$f^{-1}(D_v) \to D_v$$

with respect to the Gauss norm on the disk D_v (since the residue field of the Gauss norm is not perfect, one has to modify the definition of the Artin character slightly). Moreover, if e denotes the unique edge of T with target v, then a_e is a character related to the ramification of f over the boundary of D_v. Essentially because the inverse image $f^{-1}(D_v)$ is the disjoint union of disks, one has the following 'fix point formula':

$$a_e = \sum_{b \in B \cap D_v} u^*_{G_b}.$$

i.e. a_e is easily determined by the set of branch points of f lying inside D_v and their ramification groups.

The main point of this construction is that the various data occurring in a Hurwitz tree satisfy nontrivial compatibility conditions. For instance, for every edge e of the tree T we have the key formula

$$\delta_{t(e)} = \delta_{s(e)} + \epsilon_e \cdot (a_e - u_{G_{t(e)}}).$$

Here $\epsilon \in \mathbb{Q}_{\geq 0}$ is the 'length' of the edge e, which measures the width of the open annulus $D_{s(e)} \backslash D_{t(e)}$. These compatibility conditions impose strong restrictions on the possible Galois covers of p-adic disks. For instance, if we fix a finite groups G and conjugacy classes of cyclic subgroups $G_b \subset G$, indexed by a finite set B (the 'branch points'), our construction gives necessary conditions for the existence of a finite G-Galois cover $f : \mathsf{Y} \to \mathsf{X}$ of open p-adic disks with 'ramification type' $(G_b)_{b \in B}$.

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Existence of good deformation data

Leonardo Zapponi

Let k be an algebraically closed field of positive characteristic p. For any integer n, denote by $C_n = \langle\sigma\rangle$ the cyclic group of order n. Fix an integer m, a character $\chi : C_m \to \mathbf{F}_p^\times$ and consider the semi-direct product

$$G = C_p \rtimes_\chi C_m.$$

A *local G-action* is an injective homomorphism

$$\phi : G \to \mathrm{Aut}_k(k[\![t]\!]).$$

Setting $C_p = \langle\tau\rangle$, the integer

$$h = \mathrm{ord}_t\left(\frac{\tau(t)}{t} - 1\right)$$

is called *conductor* of the local G-action. The *local lifting problem* for G asks whether or not there exists a discrete valuation ring R of characteristic 0 with residue field k and a lift

$$\Phi : G \to \mathrm{Aut}_R(R[\![t]\!])$$

inducing ϕ under the natural specialization homomorphism from $\mathrm{Aut}_R(R[\![t]\!])$ to $\mathrm{Aut}_k(k[\![t]\!])$. In this case, we can view the homomorphism Φ as an action of G on the open p-adic disk $\mathbb{D}$ and the integer $h+1$ is just the cardinality of the set B consisting of the points of $\mathbb{D}$ fixed by the automorphism τ. If the set B (or, more precisely, the pointed curve $(\mathbf{P}^1_R, B)$ obtained by viewing $\mathbb{D}$ as a subset of the projective line over R) has good reduction then we say that the local lifting problem has an *equidistant* solution. We also say that the local G-action has an equidistant lift.

The notation and assumptions being as in the previous paragraph, a *deformation datum* with character χ is a logarithmic differential form ω on $\mathbf{P}^1_k$ for which there exists an injective homomorphism

$$\theta : C_m \to \mathrm{Aut}(\mathbf{P}^1_k)$$

such that, identifying an element of C_m with its image under θ, we have the identity

$$\sigma^*\omega = \chi(\sigma)\omega.$$

A deformation datum is *good* if it has a unique zero. If ω has n simple poles, we say that the integer $h = n-1$ is the *conductor* of the good deformation datum. Remark that in this case the unique zero of ω has order $h-1$. The connection between good deformation data and the local lifting problem is summarized in the following result:

Theorem 1. *A local G-action with conductor h has an equidistant lift if and only if there exists a good deformation datum with character χ and same conductor.*

We now give some necessary conditions for the existence of good deformation data, which can easily derived from the basic definitions of the previous paragraph. It turns out that these conditions are in fact sufficient for the existence of a lift (see for example [2], Theorem 2.1), which is nevertheless not equidistant in general.

Proposition 2. *Let ω be a good deformation datum with character χ and conductor h. We then have the following properties:*

1. *The integer h is prime to p.*
2. *The character χ is either trivial or injective. In the first case m divides h while in the latter case m divides $p-1$ and $h+1$.*

As an immediate consequence of the above proposition, the conductor h cannot be equal to p. We then distinguish between the case of *small* conductor, namely when $h < p$, and the case of *large* conductor, for which $h > p$. These two cases are intrinsically different. For example, we will see that there exists finitely many (isomorphism classes of) good deformation data with small conductor while there exist positive dimensional moduli spaces for good deformation data with big conductor.

When the character χ is trivial, which means that the group G is cyclic of order mp, the existence of good deformation data is ensured for any conductor.

Proposition 3. *For any integer h divisible by m (cf. Proposition 2), the differential form*

$$\omega = \frac{dx}{x^{h+1} - x}$$

is a good deformation datum with trivial character and conductor h.

In view of Theorem 1, we then obtain the following consequence for the equidistant local lifting problem:

Corollary 4. *If G is cyclic of order mp (with m prime to p) then any local G-action has an equidistant lift.*

Lemma 5. *A differential form on the projective line over k is logarithmic if and only if all its poles are simple, with residues belonging to $\mathbf{F}_p$.*

We then have the following existence result (cf. [2], Proposition 1.4); we include its simple and constructive proof.

Proposition 6. *Suppose that the integer m divides $p-1$ and h, with $h < p$. Then there exists a good deformation datum with character χ and conductor h.*

Proof. Set $h = mr - 1$. Since $h < p$, there exist elements $x_1, \ldots, x_r \in \mathbf{F}_p$ such that $x_i^m \neq x_j^m$ for any $i \neq j$. In this case, it is easily checked that the differential form

$$\omega = \frac{dx}{\prod_{i=1}^{r}(x^m - x_i^m)}$$

is a good deformation datum with character χ and conductor h. □

As we will see in the next sections, there exist positive dimensional moduli spaces for good deformation data with large conductor. This is not the case for small conductors. Indeed, we have the following result:

Proposition 7. *There exist finitely many isomorphism classes of good deformation data with fixed conductor* $h < p$.

Here, two deformation data are isomorphic if one is a $\mathbf{F}_p$-multiple of the pull-back of the other under an automorphism of the projective line. There are many possible proofs of the above result. For example, in [2], it is shown that such isomorphism classes bijectively correspond to equivalence classes of particular covers of the projective line (in characteristic 0) unramified outside three points having bounded degree.

We start with an example which was the initial motivation for the construction of the next paragraph. From now on, we restrict to the case $m = 2$, so that the group G is dihedral of order $2p$. Denote by $\mathcal{M}$ the moduli space of good deformation data over $k = \bar{\mathbf{F}}_3$ with conductor $h = 5$. Using an explicit construction of good deformation data, it is easily shown that $\mathcal{M}$ is isomorphic to the projective line minus two points,

$$\mathcal{M} = \mathrm{Spec}(k[t, t^{-1}]).$$

More precisely, for any $t \in k^{\times}$, we can consider the good deformation datum

$$\omega_t = \frac{t^2}{(x^3 + tx + t)(x^3 + tx - t)} dx$$

and, up to isomorphism, any good deformation datum with conductor $h = 5$ arises in this way. The next task is to find the good notion of degeneration of the good deformation data considered here. In other words, we want to study the compactification $\bar{\mathcal{M}}$ of $\mathcal{M}$. As a k-variety, we clearly have $\bar{\mathcal{M}} = \mathbf{P}^1_k$. We just have to understand how the differential form ω_t degenerates at the points $t = 0$ and $t = \infty$. The idea is to consider $\mathcal{M}$ as a (closed) subset of the moduli space $\mathcal{M}_{0,[7]}$ of (unordered) 7-pointed curves of genus 0 (the seven points being the 6 poles and the unique zero of ω_t). From a practical point of view, we have to compute the semi-stable models corresponding to $t = 0$ and $t = \infty$. For $t = 0$, as it is shown in the following figure, we find a Hurwitz tree consisting in three projective lines (see [3, 4, 1] for a general introduction to Hurwitz trees). The six poles specialize in the tails (three of them in each tail) while the zero specializes in the central component. In the central component ω_0 induces an exact differential form having two triple poles at the singular points and a unique zero (of order 4). In the tails, we obtain two good deformation data with no group action ($m = 1$) and conductor $h = 2$.

For $t = \infty$, the situation is quite similar. Here, as it is shown in the next figure, we obtain a Hurwitz tree with three tails and one central component. The six poles specialize once again in the tails, two on each, and the zero specializes in the central component. As above, the differential form ω_∞ induces an exact

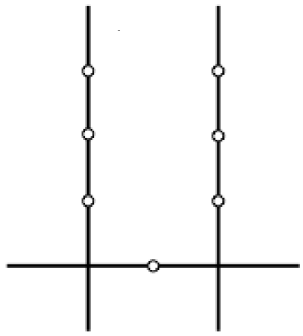

FIGURE 1. The degeneration at $t = 0$.

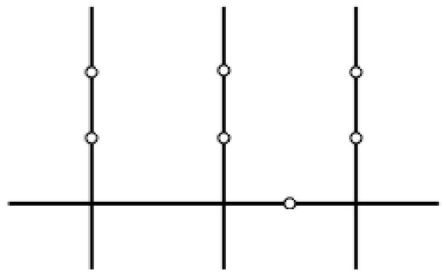

FIGURE 2. The degeneration at $t = \infty$.

differential form on the central component and three good deformation data on the tails, one with $m = 2$, while the others with $m = 1$.

The strategy adopted here for the construction of good deformation data with large conductor is inspired from the previous paragraph, and more specially from the case $t = 0$. In other words, we want to deform a 'singular' good deformation datum, corresponding to a point of the compactification of the moduli space of good deformation data, in order to obtain an ordinary good deformation datum. This is a quite classical approach in the theory of covers of curves. We now state the result and then sketch its proof.

Theorem 8. *Suppose that G is dihedral of order $2p$ (that is, $m = 2$ and χ injective). Then there exist infinitely many good deformation data with conductor $h = p + 2$.*

The proof essentially goes as follows: first of all, a direct and explicit approach to the problem leads to some explicit equations defining the moduli space $\mathcal{M}$ for good deformation data of the desired type. It turns out that $\mathcal{M}$ is an affine k-scheme of finite type. We just have to check that it is nonempty. To prove this last assertion it is sufficient to construct a point of $\mathcal{M}$ defined over $k[\![t]\!]$ (specializing 'outside' $\mathcal{M}$ at $t = 0$). From a geometric point of view, we start with a Hurwitz tree similar to the one in figure 1; one central component and two tails permuted by the group C_2. Here, we want $r = (p+3)/2$ points specializing in each tail and defining a good deformation datum on it, with conductor $h = (p+1)/2 < p$ (and $m = 1$), which exists from Proposition 6. One essential fact in the case $t = 0$ of the previous paragraph is that the good deformation datum of one of the tails completely determines the differential data of the Hurwitz tree. In fact, in this more general situation, the same phenomenon occurs and we are moreover able to deform the Hurwitz tree to a good deformation datum defined over $k[\![t]\!]$. From a practical point of view, we work with explicit equations and, just using Hensel's Lemma, we can lift a solution of the system of equations defining a good deformation datum with conductor $h = (p+1)/2$ over k to a solution of the system of equations defining a good deformation datum with conductor $h = p+2$ over $k[\![t]\!]$. The scheme $\mathcal{M}$ being non-empty and the field k being algebraically closed, we then deduce that $\mathcal{M}(k)$ is infinite. In terms of local lifting problem, the consequence is now clear.

Corollary 9. *If G is dihedral of order $2p$ then any local G-action with conductor $p+2$ has an equidistant lift.*

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Reporter: Franz Kiraly

Participants

Prof. Dr. Jose Bertin
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Prof. Dr. Irene Bouw
Institut f. Reine Mathematik
Universität Ulm
Helmholtzstr. 18
89081 Ulm

Louis H. Brewis
Institut für Reine Mathematik
Universität Ulm
89069 Ulm

Jakub Byszewski
Mathematisch Instituut
Universiteit Utrecht
Budapestlaan 6
P. O. Box 80.010
NL-3508 TA Utrecht

Franz Kiraly
Institut für Reine Mathematik
Universität Ulm
89069 Ulm

Prof. Dr. Aristides Kontogeorgis
K. Versi Street
Papagou
15669 Athens
GREECE

Prof. Dr. Michel Matignon
Mathematiques et Informatique
Universite Bordeaux I
351, cours de la Liberation
F-33405 Talence Cedex

Dr. Sylvain Maugeais
Faculte des Sciences
Universite du Maine
B.P. 535
Avenue Olivier Messiaen
F-72017 Le Mans Cedex

Prof. Dr. Ariane Mezard
Departement de Mathematiques
Universite de Versailles Saint
Quentin
45, av. des Etats-Unis
F-78035 Versailles Cedex

Dr. Magali Rocher
Mathematiques et Informatique
Universite Bordeaux I
351, cours de la Liberation
F-33405 Talence Cedex

Dr. Matthieu Romagny
Inst. de Mathematiques de Jussieu
Universite Paris VI
175 rue du Chevaleret
F-75013 Paris

Dr. Björn Selander
SFB 478
Geom. Strukturen in der Mathematik
Universität Münster
Hittorfstr. 27
48149 Münster

Dr. Dajano Tossici
Dipartimento di Matematica
Universita degli Studi Roma Tre
Largo S. L. Murialdo, 1
I-00146 Roma

Prof. Dr. Stefan Wewers
Institut für Algebra, Zahlentheorie
und Diskrete Mathematik
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover

Dr. Leonardo Zapponi
Institut de Mathematiques
175, rue du Chevaleret
F-75013 Paris