Arbeitsgemeinschaft mit aktuellem Thema:
Optimal Transport and Geometry
Mathematisches Forschungsinstitut Oberwolfach
29.03. - 04.04.2009

Organizers:

Felix Otto	Karl-Theodor Sturm
Wegelerstrasse 10	Poppelsdorfer Allee 82
53115 Bonn	53115 Bonn
otto@iam.uni-bonn.de	sturm@uni-bonn.de

Introduction:

In its origin, optimal transportation is a variational problem where one minimizes a transportation cost when transporting one density into another (Monge). Via its relaxed version (Kantorovich), the solution of this problem (Brenier) connects with convex analysis. Entire classes of inequalities in analysis can be easily proven with this tool. Even for the simplest transportation cost, i. e. the square of the Euclidean distance, the regularity theory for the minimizers is subtle (Caffarelli and others): Its Euler-Lagrange equation is the role model for a fully nonlinear elliptic equation in non-divergence form, the Monge-Ampère equation. The existence and the elements of a theory for more subtle transportation costs, like the Euclidean distance itself or the square of a Riemannian distance, are areas of current research.

Optimal transportation can be used to introduce a metric (distance function) on the space of probability measures which metrizes the weak topology. If the transportation cost is the square of a Euclidean or Riemannian distance, this metric can be seen as induced from a formal, infinite-dimensional Riemannian structure on the space of probability measures (Otto). Loosely speaking this geometry is the "complement" (in the sense of polar decomposition of vector fields) of the one of the space of volume-preserving diffeomorphisms (Arnold), which is motivated from fluid mechanics. Like for the space of volume-preserving diffeomorphisms, the space of probability measures has interesting geometrical properties itself. For instance, in the Aleksandrov sense, this space has non-positive sectional curvature, if the underlying space has this property.

Certain entropy functionals (including the usual entropy) turn out to be convex with respect to this geometry (McCann). Moreover, the convexity properties of these functionals can be used to characterize lower bounds on the Ricci curvature and the dimension of the underlying space — and can be used to define Ricci curvature bounds in the absence of a smooth structure (Sturm, Lott-Villani). This relation between geodesic convexity and Ricci curvature can be assimilated to the longer-known relation between the logarithmic Sobolev inequality and Ricci curvature (Bakry-Emery). Closely related to this property is the fact that the gradient flow (steepest descent) of the entropy functional is a contraction if the Ricci curvature is non-negative. In fact it is always a contraction if the underlying geometry evolves by Ricci flow (McCann-Topping).

This brief tour d'horizon shows that over the past 15 years, many connections between optimal transportation and seemingly unrelated fields have been discovered. Three monographs [50, 51, 3] and several lecture notes address these recent developments.

Talks:

1. **Kantorovich Duality**

   - The first talk should begin with the formulation of the Monge and Kantorovich problems following [37], [50], [51]; the notions "transport maps" and "transport plans" should be introduced. The first result to be presented is the existence of solution to the Kantorovich problem; Prop 2.1 in [50] (cf. Thm 4.1 in [51]).
   - The explicit solution in the 1-dimensional case via monotone rearrangement should be given; Remark 2.19(iv) in [50].
   - For the particular case where the cost function is a metric $c(x,y) = d(x,y)$ the Kantorovich problem admits a well-known dual formulation ("Kantorovich–Rubinstein theorem"; e.g. Thm 1.14 in [50]) as follows

     $$\sup\{\int_X f d\mu - \int_X f d\nu : \ f : X \to \mathbb{R} \text{ with Lipschitz constant } \leq 1\}.$$

- The main results to be presented in this talk should be the Kantorovich duality for general cost functions ($c \geq 0$, lower semi-continuous) as stated e.g. as Thm 1.3 and Prop 1.22 in [50] (or as Thm 5.10(i) in [51]).

2. **Optimal transports and c-convex functions**

   - The concepts of "c-convex/c-concave functions" should be introduced for general cost functions c; the relation to the classical notion of convexity in the Euclidean with $c(x, y) = |x - y|^2/2$ should be explained. The talk should also include a discussion of the c-transform as a generalized Legendre transform and the trick of double convexification; [42], [37], Def 5.2, Prop 5.8 in [51] (cf. Lemma 2.10, Thm 2.9 in [50]).

     To avoid confusion with different definitions of c-convex/c-concave functions (even by the same authors) we agree on the sign convention of [50] for c-concave functions and we say that φ is c-convex iff $-\varphi$ is c-concave.

   - Moreover, the concept of "cyclical monotonicity" should be introduced and as one of the basic results Rockafellar's theorem about the characterization of cyclically monotone sets on $\mathbb{R}^n$ (Thm 2.27 in [50]) as well as its generalization to general spaces and costs functions (Rüschendorf's theorem; [42]; Def 5.1 in [51]).

   - The main result to be presented in this talk is the characterization of optimal transports by the fact that they are supported on the graph of the subgradient of c-convex functions; "Knott-Smith Theorem" Thm 5.10 in [51] (cf. Thm 2.12, Thm 2.29, Cor 2.30, Thm 2.32 in [50]),

3. **Brenier's solution to the optimal transport problem in the Euclidean case, Polar factorization of vector fields**

   A crucial breakthrough in the understanding of optimal transports was the observation of Brenier [8] that in the case of absolutely continuous measures the solution to the Kantorovich problem with quadratic costs is always given in terms of a unique transport map. (In particular, it also yields a unique solution to the Monge problem.) The transport map ("Brenier map") is the gradient of a convex function. It is exactly this convex function which already showed up as one of the pair of conjugate convex functions in the dual problem [50, Theorem 2.12 (ii)-(iv)].

Various arguments presented in the previous talks will be used:

- Existence of solution π of the primal Monge-Kantorovich problem [50, Proposition 2.1].
- Existence of a solution (ϕ, ϕ^*) of the dual Monge-Kantorovich problem in the class of conjugate pairs of convex functions [8, Theorem 3.1, proof in Section 3.4] or [50, Theorem 2.9].
- Relation between π and (ϕ, ϕ^*) based on duality [50, Theorem 1.3].

The main new result is the uniqueness property in the absolutely continuous case [8, Proposition 3.1, proof in Section 3.2] or [50, Theorem 2.12].

All this uses Rademacher's theorem about the a. e. differentiability of convex functions.

An important aspect is the stability of the transport plans and transport maps under convergence of the target measures [51, Thm 5.20, Cor 5.23]. (It suffices to consider the Euclidean case with quadratic costs.)

As an application: Polar factorization of vector fields (i. e. the nonlinear version of Helmholtz decomposition) [8, Theorem 1.2].

4. **McCann's change of variable formula, validity of the Monge-Ampère equation, application to Sobolev inequality**

- Rigorous derivation of the Monge-Ampère equation for the potential ϕ of the optimal transport $\nabla\phi$ in the almost everywhere sense, see for instance [50, Theorem 4.8 iii)].
- Change of variable formula [28, Theorem 4.4] in [28, Section 4]. See also [50, Theorem 4.8 iv)], which makes use of [28, Proposition A.2].

All this uses Aleksandrov's theorem, i. e. the twice a. e. differentiability of convex functions ϕ, see [50, Subsection 2.1.3] for the necessary ingredients on convex functions. It also establishes delicate observations like the characterization of the determinant $\det D^2\phi$ of the Aleksandrov Hessian $D^2\phi$ of the convex potential ϕ as the Radon-Nikodym derivative of the Hessian measure $\partial\phi(A)$ w. r. t. the Lebesgue measure.

As an application: Elementary proof of the Sobolev inequality with optimal constant [15, Theorem 2], see also [50, Theorem 6.21].

5. **Caffarelli's regularity theory**

Caffarelli's counterexample: It shows that even for smooth densities, the optimal transport map can be discontinuous. A proof can be found in [50, Theorem 12.3] (the original paper [11, p.100] is very brief). It relies on the strong convergence of the optimal transport map under (weak) convergence of the densities [50, Corollary 5.23] and on the monotonicity of the optimal transport map [50, Lemma 12,2] & [50, Theorem 5.10]. In the Euclidean case, both ingredients have an elementary proof.

Caffarelli's regularity theory is one of the pinnacles of analysis and quite involved. A synopsis can be found in [50, 4.2.2]. The crucial assumption is the convex support of the target measure. The following first steps in the regularity theory should be presented:

- Validity of the Monge-Ampère equation for the potential ϕ of the optimal transport $\nabla\phi$ in the sense of Aleksandrov, see [11, Lemma 2] (also [50, Theorem 4.10]).
- Validity of the Monge-Ampère equation in the sense of viscosity solutions, [10, p. 129]. Two comparison lemmas [10, Lemma 1, Lemma 2]. For an overview of these different solution concepts (almost everywhere, Aleksandrov, viscosity) see [50, 4.1.4].
- Strict convexity of ϕ in a point implies differentiability in this point [10, Corollary 1].
- Strict convexity of ϕ [11, Lemma 3].

The two last points use as a black box the fact that convex sets don't differ much from ellipsoids.

6. **Optimal transports on Riemannian manifolds**

   - This will be the first talk in a Riemannian setting. Until now, the state space was either just a metric space or the Euclidean space. Basic concepts like Riemannian distance, volume, Laplacian should be presented; Rademacher's theorem should be mentioned (e.g. Thm 10.8 in [51]).
   - For the most important case of quadratic cost $c(x, y) = d^2(x, y)/2$ Lipschitz continuity and differentiability almost everywhere of c-convex functions; Lemma 2 and Lemma 4 in [29].
   - The main results here will be that in the case $c(x, y) = d(x, y)^2/2$, optimal transport maps ("Brenier maps") are given as exponential maps of gradients of c-convex function $T(x) = \exp_x(\nabla\varphi(x))$ Thm 8, Thm 9 and Cor 10 in [29]; Thm 10.41 in [51].
   - The extension to more general cost functions should be sketched; (it would be sufficient to consider he case $c(x, y) = f(d(x, y))$ for strictly convex f and the state space being Euclidean): Thm 13 in [29] or Thm 2.44 in [50].

7. **Monge's original problem**

   The first solution to the original Monge problem, formulated in 1781, was presented by Sudakov [45]; however, his proof was incomplete as pointed out (and corrected) by Ambrosio [2]. In the meantime, under quite restrictive assumptions on the measure to be transported, Evans and Gangbo [17] gave a rigorous answer.

   Results in full generality have been presented independently by Caffarelli, Feldman, McCann [12] and Trudinger, Wang [49]. In the talk, one of these complete solutions to the original Monge problem should be presented, cf. Thm 2.50 and Thm 2.52 in [50] (following either [2] or [12] or [49]).

   In order to construct a transport map one decomposes an optimal transport plan (on $\mathbf{R}^n$) via disintegration into transport plans on the $(n-1)$-dimensional family of transport rays. Optimal transports on each of the rays are well understood. Moreover, the cost function $|x-y|$ will be approximated by $|x-y|^p$ for $p > 1$ in order to obtain a unique transport map. The challenge is to keep control on Lipschitz constants in order to justify the change of variable formula on the whole configuration.

8. **The Wasserstein space $\mathcal{P}_p(M)$ as a metric space, Benamou-Brenier interpretation of the Wasserstein distance, Arnold's geometry of the diffeomorphism group**

   - The L^p-Wasserstein distance for the space of probability measures $\mathcal{P}_p(M)$ on a complete separable metric space M is a complete separable metric; [50, Theorem 7.3] (or [3, Section 7.1] or [51, Theorem 6.18]). The convergence in the Wasserstein metric amounts to weak convergence + convergence of moments [50, Theorem 7.12] (or [3, Proposition 7.1.5] or [51, Theorem 6.9]).
   - The L^2-Wasserstein distance has an "Eulerian" formulation [7, Proposition 1.1]. (The proof [7, Section 3] does not address the approximation issue, see [34, Proposition 4.3] for a detailed proof in the more complicated case where the underlying manifold M is Riemannian).
   - This Eulerian formulation of the L^2-Wasserstein distance is reminiscent of Arnold's observation that the Euler equations of an inviscid, incompressible fluid are the Eulerian formulation of the geodesic equation on the group of volume preserving diffeomorphisms. This is in fact an easy formal observation (the pressure is the Lagrange multiplier coming from the constraint of volume preservation).

     This motivates the study of the Lie group of volume-conserving diffeomorphisms (where the left-invariant metric tensor is the L^2-scalar product on the Lie algebra, i. e. the space of divergence-free vector fields), see [4, Appendix 2]. With this structure, one can express the curvature tensor in terms of the commutator bracket [4, Appendix 2, Theorem 10]. This can be formally used to argue that the sectional curvature on the Lie group of volume conserving diffeomorphisms is often negative [4, Appendix 2,Theorem 14], which clarifies the effective unpredictability of the Euler equations [4, Appendix 2, Section L]. See also [5, Chapter VI]. ¿From a metric point of view, this manifold has been studied in [44] where a couple of pathologies are worked out.

9. **Formal Riemannian structure for space of probability measures, its sectional curvature**

In a similar spirit to Arnold's observation, the space of probability measures ρ on $M = \mathbb{R}^n$ carries a formal Riemannian structure [33, Section 1.3], which makes the Wasserstein distance its induced distance [33, Section 4.3] (see [36, Section 3, p.371-373] for a different heuristic argument, which can be made rigorous [34, Proposition 4.3]).

Due to an isometric submersion [33, Section 4.1], the sectional curvatures can be computed [33, Section 4.5]: they are non-negative. These observations have been extended to the space of probability measures on a Riemannian manifold M, see [25], where also the covariant derivative is identified.

This formal observation on non-negative sectional curvatures can be reformulated in a metric setting (via distances in a geodesic triangle, Aleksandrov) and has been established in [3, Theorem 7.3.2] in case of probability measures on $M = \mathbb{R}^n$. The proof is nice and relies on the characterization of displacement interpolations as "shortest paths" [3, Theorem 7.2.2] and the disintegration of measures [3, Lemma 5.3.4] (which is also used to prove the triangle inequality).

Another, even more elementary proof based on a different characterization of curvature in a metric context, is in [46, Proposition 2.10]. This proof also applies to the case where the underlying space M is a metric space with Aleksandrov curvature bounds.

10. **McCann's displacement convexity**

Functionals of the form $E(\rho) = \int_M e(\frac{d\rho}{d\text{vol}})d\text{vol}$ are convex the space of probability measures ρ on some (finite-dimensional) Riemannian manifold M endowed with Wasserstein distance under natural assumptions on the $\mathbb{R} \ni \rho \mapsto e(\rho)$ and on the Ricci curvature of M.

- In case of $M = \mathbb{R}^n$, this is McCann's celebrated observation of "displacement convexity", [28, Theorem 2.2] which is based on the change-of-variable formula from Talk 4. See also [50, 5.1.3] , [50, Theorem 5.15 i)].
- In [33, Section 4.4], this observation has been interpreted in terms of formal Riemannian structure on the space of probability measures, see Talk 9, by formally deriving the formula for the Hessian of E.

- In the same spirit, in [36, Section 3], the formula for the Hessian of E was derived in case of a general underlying Riemannian manifold, making the connection with Ricci curvature via Bochner's formula [36, p. 374].
- The fact that non-negative Ricci curvature is sufficient has been made rigorous [14, Theorem 6.2] by the same type of arguments as in the Euclidean case. This is quite technical and requires the analogue of McCann's result (change of variable) from Talk 4 [14, Theorem 4.2, Corollary 4.7].
- The fact that non-negative Ricci curvature is necessary has been made rigorous in [40, Theorem 1.1].

11. **Ricci bounds for metric measure space**

In 1951 A. D. Alexandrov presented his concept of generalized (upper and lower) bounds for the sectional curvature of metric spaces (M, d). For many results in geometric analysis, no explicit bounds on the sectional curvature are required but the crucial geometric quantity is a (lower) bound on the Ricci tensor. Various of these results also hold true for limits of such spaces, cf. [13]. One of the challenges in metric geometry and geometric analysis, therefore, was to develop a synthetic concept of generalized lower Ricci bounds for singular spaces. It was clear that such bounds on the Ricci curvature should be formulated in the framework of metric measure spaces (M, d, m).

In two independent but quite similar approaches [46], [26], the K-convexity of the relative entropy (considered as a function on the L^2-Wasserstein space) was used as a defining property for "Ricci curvature being bounded from below by K", briefly denoted by $CD(K, \infty)$ or $\underline{\text{Curv}} \geq K$.

For the equivalence of $CD(K, \infty)$ with lower Ricci curvature bounds for Riemannian manifolds, see Talk 10 (or Thm 4.9 in [46], Cor 17.19 (i) in [51]).

A natural metric on the space of (equivalence classes of) metric measure spaces is given by the transportation distance $\mathbb{D}$. It can be regarded as a combination of L^2-Wasserstein distance (based on coupling of the measures) with Gromov-Hausdorff distance (based on coupling of the metrics). Its topology is closely related to the topology of measured

Gromov-Hausdorff convergence; Sections 3.1 and 3.4 in [46] and/or Chapter 27 in [51].

The main result to be presented in this talk is the stability of the curvature bound $CD(K,\infty)$ under convergence of metric measure spaces; Thm 4.20 in [46]; Thm 5.19 in [26].

A simple proof for the lower semicontinuity of the relative entropy (based on a duality argument) can be found or in Thm B.33 in [26] or in Lemma 9.4.3 in [3].

Another important property of the curvature bound $CD(K,\infty)$ is the local-to-global property (Thm 4.17 in [46]).

It might be worthwhile to mention that $CD(K,\infty)$ implies a Talagrand and a logarithmic Sobolev inequality (with the same constant K), Chapter 6 in [26] and/or Thm 30.22, Thm 30.28 in [51]. For more details, see discussion in Talk 15.

12. **The curvature-dimension condition** $CD(K,N)$

The condition $CD(K,\infty)$ presented in the previous talk is the weakest in a family of conditions $CD(K,N)$ where the parameter $N \in [1,\infty]$ plays the role of a generalized upper bound for the dimension.

The formulation of $CD(K,N)$ is quite involved. It gives a precise estimate for the deformation (under the influence of curvature of the underlying space) of volume elements along geodesics of optimal transports. The proof that $CD(K,N)$ holds true on Riemannian manifolds with given bounds for dimension and Ricci curvature is a modification of the proof for $CD(K,\infty)$, presented in Talk 10. It is worked out (and discussed) in great detail in Chapters 14, 16 and 17 of [51]. At least, the basic ideas should be sketched.

For weighted Riemannian manifolds $(M,d,e^{-V}m)$ of dimension n, the condition $CD(K,N)$ is equivalent to boundedness from below by K for the weighted N-Ricci tensor

$$\mathrm{Ric} + \mathrm{Hess} V - \frac{1}{N-n}\left(\nabla V \otimes \nabla V\right).$$

See also [24] for interpretations and applications of this tensor, e.g. in Bochner's inequality (14.51) in [51]. In the case $N=\infty$ it yields the so-called Bakry-Emery tensor.

One of the main results here is that for metric measure spaces, the condition $CD(K,N)$ implies various functional analytic and geometric inequalities, e.g. inequalities of Brunn-Minkowski, Bishop-Gromov, Bonnet-Myers: Chapter 2 in [46] and/or Sections 5.4, Chapter 6 in [26] and/or Chapter 30 in [51].

Another main result is that the condition $CD(K,N)$ is stable under convergence. The proof is technically more involved but similar to that of $CD(K,\infty)$ presented in the previous talk. Therefore, it can be omitted.

A crucial consequence is the compactness of the family of metric measure spaces satisfying CD(K,N) and diameter $\leq L$ (analogous to Gromov's compactness result under sectional curvature bounds); Chapter 3 in [46] and Chapter 29 in [51].

The "measure contraction property" $MCP(K,N)$ is slightly weaker than $CD(K,N)$: here one considers only optimal transports where one of the endpoints is a Dirac measure. Particular results for the following non-Riemannian spaces might be of interest:

- On the Heisenberg group, MCP holds and CD does not hold [20].
- On Alexandrov spaces of lower bounded curvature, MCP holds and CD is conjectured.
- On Finsler spaces, CD is equivalent to a lower bound for the weighted N-Ricci tensor (based on so-called flag curvature) [31].

13. **Diffusions are gradient flows of entropy w. r. t. Wasserstein metric**

    - Formally, a (nonlinear) diffusion equation can be seen as the gradient flow of an entropy functional $E(\rho)$ (see Talk 10) on the space of probability measures endowed with the Riemannian structure (see Talk 9), cf. [33, Sections 1.2, 1.3].
    - This connection can be given a rigorous sense via time-discretization, which just involves E and the Wasserstein distance, see [33, Section 4.6]. The convergence of the time discretization has been established in many cases by standard methods in PDE, see e. g. [19, Theorem 5.1] for the case of linear diffusion or [32, Theorem 3] for the case of nonlinear diffusion.

- Using the displacement convexity of $E(\rho)$, see Talk 10, a suitable "metric" notion of gradient flow can be given and its existence be proven via the above time-discretization, see [43, Theorem 7] and [3, Section 4]. The general theory is complicated by the fact that the space of probability measures as non-negative curvature, see Talk 9. This is kind of compensated by the robust convexity property of E ("convexity along generalized geodesics" [3, Definition 9.2.4, Lemma 9.2.7, Theorem 9.4.12]) and motivates the key assumption [3, Assumption 4.0.1] of this abstract existence theory.

14. **Contraction properties of Wasserstein metric under diffusions**

In a finite-dimensional smooth Riemannian case, the convexity of $E(\rho)$ is equivalent to the fact that the distance between any two trajectories of the gradient flow $\frac{d\rho}{dt} = -\text{grad}E_{|}\rho$ cannot increase, see for instance [34, Section 3.1].

This principle is applied to a displacement convex entropy functional $E(\rho)$ on the space of probability measures on $M = \mathbb{R}^n$ endowed with the Wasserstein in [33]. As an application, the time asymptotics for a specific nonlinear diffusion equation, the porous medium equation, is derived [33, Theorem 1]. The crucial statement on the contraction property is (114) in Proposition 1, generalized to two arbitrary solutions in (133). The proof is on pp. 145-149 and follows closely the formal arguments in Section 3.5. It relies on the results from Talks 4 and 10.

In case of a *linear* diffusion equation on a manifold M, the connection between Ricci curvature and contraction in Wasserstein metric can also be established by other methods, for instance the construction of a coupled Brownian motion with pathwise contraction, see x) in [40, Corollary 1.4] using geometric information xii) in [40, Theorem 1.5].

In the nonlinear case, there is another, more self-contained, rigorous argument, which also is valid on general Riemannian manifolds M [34, Theorem 4.1]. It relies on an infinitesimal contraction property [34, Proposition 4.2] and a characterization of the Wasserstein distance as infimal energy of connecting *smooth* curves [34, Proposition 4.3] (the approximation argument is technical on general M's but much easier for $M = \mathbb{R}^n$).

The relation between convexity and contraction is also used on an abstract "metric" level in [3, Section 4].

15. **Sobolev inequality, Talagrand inequality and applications**

   - The logarithmic Sobolev inequality (LSI) is a non-linear version of the spectral gap estimate (which is a Poincaré estimate) for the generator of a semigroup coming from a drift-diffusion process. As for the usual Sobolev inequality, it expresses a gain in integrability — which is only logarithmic but therefore dimension-independent and thus of use to study spin systems. LSI yields hypercontractivity of the semigroup [18, Theorem 4.1].
   - The most famous criterion for LSI is due to Bakry & Emery. It can be elegantly proved using the semigroup, see for instance [22, Corollary 1.6]. This is related to the more general Γ_2-calculus [1, Théorème 5.4.7], which in turn in related with Talk 12.
   - The Bakry-Emery criterion has a simple formal interpretation in terms of the geometry introduced in Talk 9 and the Hessian of the entropy calculated in Talk 10 and the interpretation of the (linear) diffusion equation as gradient flow from Talk 13 [36, Section 3]. Also the Talagrand inequality, which is a concentration property of the stationary measure (see [21, Introduction] for these phenomena), can be interpreted in this context. It becomes clear that the Talagrand inequality is a consequence of LSI, which can also be rigorously proved using the semigroup [36, Theorem 1].
   - Talagrand's inequality is related to a covariance estimate [35, Lemma 5]. This can be used to derive a criterion for LSI [35, Theorem 1] which in its form resembles the Bakry-Emery criterion. It allows for a simple proof of LSI for a weakly interacting spin system [35, Application I].

16. **Optimal transport and Ricci flow** One of the main tools in Perelman's approach to the Poincaré conjecture is the monotonicity of various functionals, e.g. of his $\mathcal{W}$-functional. Given a family of compact Riemannian manifolds $(M, g(\tau))$ evolving under backward Ricci flow, the manifold
$$\tilde{M} = M \times S^N \times \mathbb{R}_+$$
with metric $\tilde{g} = g(\tau) + 2N\tau g_{S^N} + (N/(2\tau) + R)d\tau^2$ will have "approximatively" nonnegative Ricci curvature. Hence, Bishop-Gromov volume comparison will imply monotonicity of $r^{-\tilde{n}} \cdot |\tilde{B}_r(x)|$.

Optimal transport provides additional characterization of nonnegative Ricci curvature, e.g. convexity of the entropy (cf. Talks 10 and 11) or monotonicity of the Wasserstein distance between two solutions of the heat equation (cf. Talk 14).

McCann and Topping extended the latter from a fixed manifold to a time-dependent family of manifolds evolving under backward Ricci flow. They proved that the L^2-Wasserstein distance between two solutions of the heat equation never increases, – independent of any curvature assumption; Cor 1 in [30].

It might be interesting to remark that the latter also admits a pathwise probabilistic interpretation via stochastic parallel transport, see [6].

In a more general approach, Topping [48] introduces the notion of $\mathcal{L}$-optimal transports, based on Perelman's $\mathcal{L}$-length and associated with some time-dependent $\mathcal{L}$-Wasserstein distance. And he used it to deduce various important monotonicity formulas of Perelman.

The above concept of $\mathcal{L}$-optimal transports and its application to monotonicity formulas should be presented as one of the main topics of this talk.

An alternative approach, but similar in spirit, was proposed by Lott [23]. See also [16].

References

[1] C. Ané et. al. , *Sur les inégalités de Sobolev logarithmiques, Panormas et Synthèses*, Société Mathématiques de France (2000)

[2] L. Ambrosio, *Lecture notes on optimal transport problems*, Notes from a CIME summer scholl in Madeira, 2-9 July 2000. To appear in proceedings of the CIME Course held in Marina Franca (Italy), *Otimal transportation and application*, 2-9 September 2001, Lecture Notes in Mathematics.

[3] L. Ambrosio, N. Gigli and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel (2005).

[4] VI. Arnold, *Mathematical methods of classical mechanics*, Springer (1978)

[5] VI. Arnold, B. Khesin, *Topological methods in hydrodynamics*, Springer (1998)

[6] M. Arnaudon, K. A. Coulibaly and A. Thalmaier, *Brownian motion with respect to a metric depending on time; definition, existence and applications to Ricci flow*, C. R. Acad. Sci. Paris 346 (2008) 773-778.

[7] JD. Benamou, Y. Brenier, *A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem*, Numer. Math. **84** (2000), no. 3, 375–393

[8] Y. Brenier, *Polar factorization and monotone arrangement of vector-valued functions*, Comm. Pure Applied Math **XLIV=44** p.375-417 (1991)

[9] Y. Brenier, *Décomposition polaire et réarrangement monotone des champs de vecteurs*, C. R. Acad. Sci. Paris Sér. I Math. **305** no.19 (1987), 805–808.

[10] L. Caffarelli, *A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity*, Annals of Math **131** p.129-134 (1990)

[11] L. Caffarelli, *The regularity of mappings with convex potential*, Journal of the AMS **5 (1)** p. 99-104 (1997)

[12] L. A. Caffarelli,M. Feldman,R. J. McCann *Constructing optimal maps for Monge's transport problem as a limit of strictly convex costs*, J. Amer. Math. Soc. 15, 1 (2002)

[13] J. Cheeger,T. H. Colding, *On the structure of spaces with Ricci curvature bounded below. I,II,III*, J. Differential Geom., 46, 3 (1997)

[14] D. Cordero-Erausquin, R.J. McCann and M. Schmuckenschläger, *A Riemannian interpolation inequality à la Borell, Brascamp and Lieb*, Invent. Math. **146** no.2 (2001), 219–257.

[15] D. Cordero-Erausquin, B. Nazaret, C. Villani, *A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities*, Advances in Math **182** p.307-332 (2004)

[16] E. Cabezas-Rivas, P. Topping, *The canonical shrinking soliton associated to a Ricci flow.* Preprint (2008)

[17] L. C. Evans, W. Gangbo, *Differential equations methods for the Monge-Kantorovich mass transfer problem*, Mem. Amer. Math. Soc., 137, 653 (1999)

[18] A. Guionnet, B. Zegarlinski, *Lecture notes on logarithmic Sobolev inequalities*, Séminaire de Probabilités, XXXVI, Lecture Notes in Math 1801, Springer (2003), pp. 1-134

[19] R. Jordan, D. Kinderlehrer, F. Otto, *The variational formulation of the Fokker-Planck equation*, SIAM J. Math. Anal. **29** (1998) 1–17

[20] N. Juillet, *Geometric Inequalities and Generalized Ricci Bounds in the Heisenberg Group*, SFB-Preprint (2006)

[21] M. Ledoux, *The concentration of measure phenomenon*, Math. Surveys and monographs 89, AMS (2001)

[22] M. Ledoux, *Logarithmic Sobolev inequalities for unbounded spin systems revisited*, Séminaire de Probabilités, XXXV, 167–194, Lecture Notes in Math., 1755, Springer, Berlin, (2001)

[23] J. Lott, *Optimal transport and Perelman's reduced volume*

[24] J. Lott, *Some geometric properties of Bakry-Émery-Ricci tensor*, Comment. Math. Helv. 78, 4 (2003)

[25] J. Lott, *Some geometric calculations on Wasserstein space*, Comm Math Phys **277** 423-437 (2008)

[26] J. Lott, C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. Math. (2008).

[27] J. Lott, C. Villani, *Weak curvature conditions and functional inequalities*, Funct. Anal. **245** no.1 (2007), 311–333.

[28] R. McCann, *A convexity principle for interacting gases*, Advances in Mathematics **128** p. 153-179 (1997)

[29] R.J. McCann, *Polar factorization of maps on Riemannian manifolds*, Geom. Funct. Anal. **11** no.3 (2001), 589–608.

[30] R.J. McCann, P. Topping, *Ricci flow, entropy and optimal transportation*, Preprint (2007)

[31] S. Ohta, *Finsler interpolation inequalities*, Preprint (2008).

[32] F. Otto, *Dynamics of labyrinthine pattern formation in magnetic fluids: a mean-field approach*, Arch Rational Mechanics Anal. **141** 63-103 (1998)

[33] F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations **26** no.1-2 (2001), 101–174.

[34] F. Otto, M. Westdickenberg, *Eulerian calculus for the contraction in the Wasserstein distance*, SIAM J. Math. Anal. **37** (2006) 1227-1255

[35] F. Otto, M. Reznikoff, *A new criterion for the logarithmic Sobolev inequality and two applications*, J Functional Analysis **243** (2007) 121-157

[36] F. Otto, C. Villani, *Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality*, J. Funct. Anal. **173** no.2 (2000), 361–400.

[37] S.T. Rachev, L. Rüschendorf, *Mass Transportation Problems, Vol. I: Theory*, Springer (1998).

[38] S.T. Rachev, L. Rüschendorf, *Mass Transportation Problems, Vol. II: Applications*, Springer (1998).

[39] R.T. Rockafellar, *Convex analysis*, Princeton Mathematical Series **28** (1970).

[40] M.K. von Renesse and K.T. Sturm, *Transport inequalities, gradient estimates, entropy, and Ricci curvature*, *Comm Pure Appl Math* **58** (2005) 923-940.

[41] M.K. von Renesse and K.T. Sturm, *Entropic measure and Wasserstein Diffusion*, To appear in Ann. Probab. (2008).

[42] L. Rüschendorf, *On c-optimal random variables*, Statist. Probab. Lett. **27** no.3 (1996), 267–270.

[43] G. Savaré, Gradient flows and diffusion semigroups in metric spaces under lower curvature bounds, *C. R. Acad. Sci Paris I*, **345**, 151-154 (2007)

[44] AI Shnirelman, *The geometry of the group of diffeomorphisms and the dynamics of an ideal incompressible fluid*, Math USSR-Sbornik **56** (1987) 79-105

[45] V. N. Sudakov, *Geometric problems of the theory of infinite-dimensional probability distributions*, Trudy Mat. Inst. Steklov 141 (1976).

[46] K.T. Sturm, *On the geometry of metric measure spaces I*, Acta Math. **196** no.1 (2006), 65–131.

[47] K.T. Sturm, *On the geometry of metric measure spaces II*, Acta Math. **196** no.1 (2006), 133–177.

[48] P.M. Topping, *L-optimal transportation for Ricci ow.* Preprint (2007) to appear in Journal für die reine und angewandte Mathematik (Crelle's journal).

[49] N. S. Trudinger, X.-J. Wang, *On the Monge mass transfer problem*, Calc. Var. Partial Differential Equations, 13 (2001)

[50] C. Villani, *Topics in Mass Transportation*, Graduate Studies in Mathematics, American Mathematical Society (2003).

[51] C. Villani, *Optimal Transport - Old and new*, Grundlehren, Springer, Berlin (2008).

Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`otto@iam.uni-bonn.de`,   `sturm@uni-bonn.de`

by February 28th, 2009 at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.