OBERWOLFACH SEMINAR: EQUIDISTRIBUTION OF FINITE VOLUME ORBITS ON $\Gamma\backslash G$

M. EINSIEDLER, E. LINDENSTRAUSS, AND PH. MICHEL

Let $\mathbb{S}_{n-1} \subset \mathbb{R}^n$ be the unit sphere in the n-dimensional euclidean space. Let $d \geq 1$ be an integer and consider the set of point

$$R_n(d) = \mathbb{S}_{n-1} \cap \big(\frac{1}{\sqrt{d}}\mathbb{Z}^n\big);$$

i.e. the cardinality $|R_n(d)|$ is the number of ways of representing d as a sum of n squares of integers. A natural question is : what can we say about the distribution of the points $R_n(d)$ as a subset of $\mathbb{S}_{n-1}$ as $d \to \infty$ along a subsequence of integers for which $R_n(d)$ is non-empty ? When $n = 2$, $|R_n(d)|$ is essentially bounded. On the other hand, when $n \geq 3$, d odd, the cardinality $|R_n(d)|$ – when non zero – goes to infinity with d and, in fact, the sets $R_n(d)$ become equidistributed in $\mathbb{S}_{n-1}$ w.r.t. to the Lebesgue measure. However, although the conclusion is the same, the case $n = 3$ is significantly different from the $n > 3$ case. An objective of the seminar will be to explains these differences and to describe the methods available to reach this common conclusion.

This question, as well as many number theoretical problems, is part of a more general framework which we now quickly describe: let G be a Lie group, Γ a lattice in G and $H < G$ be a closed subgroup. We want to investigate the behavior of closed orbits xH inside the quotient $X = \Gamma\backslash G$ for suitable varying x's. For instance, in the problem considered above, one takes $G = \mathrm{SO}_n(\mathbb{R})$, $\Gamma = \mathrm{SO}_n(\mathbb{Z})$ and $H \simeq \mathrm{SO}_{n-1}(\mathbb{R})$ the stabilizer in G of a fixed non-zero vector in $\mathbb{R}^n$; other important examples are obtained by taking $\Gamma = \mathrm{SL}_n(\mathbb{Z}) \subset G = \mathrm{SL}_n(\mathbb{R})$ and H either the group of upper triangular unipotent matrices or the group of diagonal matrices. The courses outlined below are intended to give an introduction to this area of research with the goal to highlight the different aspects of the theory.

One basic but important class of questions concern the limiting behavior of orbits of finite volume (ie. *periodic orbits*) or families of such: for $x \in X$ and $H < G$ be closed subgroup.

- What does the closure of xH, the orbit closure, look like?
- If H is a one-parameter subgroup $\{h_t : t \in \mathbb{R}\}$, what is the limit of $\frac{1}{T}\int_0^T f(xh_t)dt$ as $T \to \infty$ for a continuous function f defined on X with compact support? I.e. does one have equidistribution of the orbit of a given point x? (In contrast the ergodic theorem only concerns typical points with respect to a given measure and will without further input never reveal the behavior of the above averages for a given point.)

Let $\mu_k,\ k \geq 1$ be a sequence of Haar probability measures associated to closed orbits x_kH of finite volume, viewed as measures on X.

- What are the possibly weak* limits of the sequence μ_k?

- Are all limit measures *homogeneous* (ie. Haar measures on orbits xL for some closed group containing H) ?
- Under which conditions do we have *equidistribution*, i.e. when does μ_k converge to the Haar measure on X?

To answer these questions one has to specify the structure of the subgroup H.

(1) If H equals the diagonal subgroup in $\mathrm{SL}_2(\mathbb{R})$ and $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$, then the situation is quite bad: this is precisely the geodesic flow on the modular surface. There is no reasonable description of orbit closures of a given point $x \in X$ — these can be quite arbitrary fractal sets. There is no convergence of the ergodic averages. However, the weak* limit of the natural measures on a certain collection of closed orbits (e.g. all closed orbits of the same length) is actually the Haar measure of X by a theorem of Duke (building on work of Iwaniec).
(2) If H equals the diagonal subgroup in $\mathrm{SL}_3(\mathbb{R})$ and $X = \mathrm{SL}_3(\mathbb{Z})\backslash\mathrm{SL}_3(\mathbb{R})$, it is conjectured that the bounded H-orbits are precisely the periodic H-orbits. Partial results towards this can be derived from a theorem of Einsiedler, Katok, and Lindenstrauss concerning the H-invariant probability measures on X.
(3) If H is generated by one-parameter unipotent subgroups, then the algebraicity of the orbit closure is a remarkable theorem of Ratner. An extension concerning the limiting behavior of measures on finite volume orbit was given by Mozes and Shah.

The seminar consists of the following 3 courses:

a) **Noncompact Semisimple groups.** This course includes a discussion of the theory of unipotent dynamics, a proof of a special case of Ratner's measure classification (which is the first step towards the equidistribution and orbit closure theorems) and the corresponding special case of the result due to Mozes and Shah. Finally the theorems will be applied to an equidistribution result for integer points on big spheres and ellipsoids, giving partial answers to the distribution problem on $\mathbb{S}_{n-1}$ mentioned above for $n \geq 4$ (and full answers for $n \geq 6$).
b) **Diagonalizable subgroups.** This course includes the definition and basic properties of measure-theoretic and topological entropy, statement of a partial measure classification theorem, and some number theoretic applications, e.g. to distribution of ideal classes.
c) **Dynamical properties of number field lattices** As pointed out earlier, the question of the distribution of $R_n(d)$ inside $\mathbb{S}_{n-1}$ has some special feature when $n = 3$: it is closely related to number fields (here quadratic number fields) and especially to the dynamical properties the lattices (in $\mathbb{R}^3$) constructed from of ideals in such fields and to the action of the ideal class group on such lattices. This lecture, which may be seen as a supplement to b), will focus on arithmetic aspects of the theory. This will be the occasion to discuss different (sometimes complementary) approaches and to introduce other tools such as L-functions, automorphic forms and (possibly) the language of adèles.

Prerequisites:[1]
Basic knowledge in Lie groups and Lie algebras (only the very basics); Haar measure on Lie groups; Poincare recurrence and pointwise ergodic theorem; p-adic numbers; Classical modular forms; basic properties of their associated L-functions.
Suggested reading:
An Introduction to Ergodic Theory, Walters, Chapters 0 and 1.
Alternatively the relevant portions of Chapters 2, 4, 8, 9 of
`http://www.mth.uea.ac.uk/ergodic/`
Analytic number theory, Iwaniec and Kowalski, Chapter 14.
Topics in Classical Automorphic Forms, Iwaniec, Chapter 11.
Some applications of modular forms, Sarnak, Chapter 1 and 4.

[1] I.e. the participant should familiarize himself with the relevant material before the workshop.