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Mini-Workshop: The Escaping Set in Transcendental Dynamics

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Abstract. The escaping set of a transcendental entire or meromorphic function consists of all points which tend to infinity under iteration. Its importance in transcendental dynamics has increased significantly in recent years. The workshop focussed on a study of this set. The topics considered include the geometry of the escaping set, its Hausdorff dimension, its relation to the Julia set, and various subsets of the escaping set defined in terms of escape rates.

Mathematics Subject Classification (2000): 37F10, 30D05.

Introduction by the Organisers

Transcendental dynamics studies the behavior of transcendental entire functions under iteration. The main object considered has been the *Julia set* $J(f)$ of a function f which is defined as the set where the iterates f^n of f do not form a normal family. In recent years, however, it has become apparent that the *escaping set*

$$I(f) = \left\{ z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty \right\}$$

also plays a key role in transcendental dynamics. There is a close connection between the two sets since we always have $J(f) = \partial I(f)$.

The workshop focussed on the escaping set and its role in complex dynamics. It was attended by 16 participants who gave talks on different aspects of the subject. In addition, there was a problem session where open problems were presented (and discussed). The unique atmosphere of Oberwolfach also stimulated many mathematical discussions among the participants, and sufficient time for this was left as well.

In the first talk of the conference, L. Rempe reported on the most recent developments in connection with a question of Eremenko who asked whether every component of $I(f)$ is unbounded for transcendental entire f. While there has been considerable progress recently, the problem is still open. The talks by X. Jarque, J. Taixés and H. Mihaljević-Brandt also addressed questions concerning the topology of the escaping set and the Julia set.

In complex dynamics it is of particular interest to consider specific families of functions such at the exponential family λe^z and to study how the dynamics depend on the parameter. This was addressed in the talks by X. Jarque, A. Badenska, K. Barański, D. Schleicher and J. Peter.

Another topic of major importance in complex dynamics is the Hausdorff dimension of the escaping set, the Julia set, and related sets. These topics were addressed in the talks by A. Zdunik, B. Karpińska, M. Urbanski and J. Peter.

For quasiregular maps in higher dimensions, there is no obvious definition of the Julia set. The escaping set can still be defined, however. A. Eremenko and D. Nicks reported on recent results which show that some results about the escaping set can in fact be carried over from the plane to higher dimensions.

Components of the Fatou set which lie in the escaping set, as well as different escapes rates of points in the escaping set, were considered in the talks by W. Bergweiler, P. J. Rippon and G. M. Stallard.

Overall this was a very successful meeting with excellent talks, lively discussions and a fruitful exchange of ideas.

Mini-Workshop: The Escaping Set in Transcendental Dynamics

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The trouble with Eremenko's conjecture

Lasse Rempe

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function, and define the *escaping set* of f by

$$I(f) := \{z \in \mathbb{C} : f^n(z) \to \infty\}.$$

In 1989, Eremenko [E] proposed the following conjecture:

Conjecture. *Every connected component of the set $I(f)$ is unbounded.*

This problem, known as *Eremenko's conjecture* remains open (despite significant progress), and does not appear to be close to resolution. This lecture explored some of the reasons for its difficulty.

(We remark that Eremenko showed that every component of the *closure* $\overline{I(f)}$ is unbounded, and that Rippon and Stallard [RS] have shown that $I(f)$ always has an unbounded connected component.)

Recall that $\mathcal{B}$ denotes the *Eremenko-Lyubich class*, consisting of those transcendental entire functions for which the set $S(f)$ of singular values is bounded. Functions in this class satisfy certain expansion properties near infinity. For this reason, the escaping set is generally better understood when $f \in \mathcal{B}$; in particular, in this case the set $I(f)$ is contained in the Julia set $J(f)$.

Eremenko also asked the stronger question whether every point $z \in I(f)$ can be connected to infinity using a curve consisting entirely of escaping points. The answer to this question is negative, as shown in [R^{3}S]:

Theorem 1 (Rottenfußer, Rückert, R., Schleicher)**.** *There exists a function $f \in \mathcal{B}$ whose Julia set contains no nontrivial curve.*

This function can be chosen to be hyperbolic with connected Fatou set.

(Here an entire function $f \in \mathcal{B}$ is called *hyperbolic* if $S(f)$ is compactly contained in the union of attracting basins of f.)

Similarly, one can strengthen Eremenko's conjecture by asking whether every escaping point is contained in an unbounded connected set A on which the iterates escape to infinity *uniformly*. Using a similar construction as in [R^{3}S], it is possible to exhibit a hyperbolic counterexample to this property. (The proof was outlined in the lecture.)

On the other hand, it is shown in [R1] that Eremenko's conjecture is true for every hyperbolic $f \in \mathcal{B}$ – and more generally whenever the postsingular set of f is bounded. So the same methods *cannot* be used to construct a counterexample to the original conjecture.

Additionally, the study of connected components of escaping sets can be far more subtle than might be at first expected. Indeed, it has recently come to light that there are many cases where the escaping set is connected (and hence Eremenko's conjecture is trivially true). This was first discovered by Rippon and

Stallard [RS] for certain functions outside of the class $\mathcal{B}$, but happens even for the exponential map $f(z) = e^z$, where every *path-connected* component of $I(f)$ is nowhere dense and relatively closed [R2]. This result has been further generalized by Jarque [J]; we refer to his lecture for details.

In summary, the following difficulties currently stand in the way of a resolution of Eremenko's conjecture (even for the class $\mathcal{B}$):

- Eremenko's Conjecture cannot be proved using arguments that are "local near infinity". A proof would have to take the global dynamics into account.
- A counterexample cannot be dynamically simple: its postsingular set must be unbounded.
- A counterexample would have to be sufficiently well-controlled to avoid the kind of effects that cause escaping sets to be connected.

On the other hand, perhaps the methods used to prove connectivity of escaping sets could, if developed further, help to yield positive results in the direction of Eremenko's Conjecture.

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Hausdorff dimension of equilibrium measures for rational maps

Anna Zdunik

(joint work with Mariusz Urbański)

Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. We say that f is critically finite if the trajectories of all critical points are finite. To every critically finite map one can associate (in a canonical way) an orbifold. If the resulting orbifold is parabolic, we say that the map is critically finite with a parabolic orbifold (CFPO) (see [2])

The following theorem was proved in [1].

Theorem. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map. let μ be the unique maximal entropy measure for f. Let $\alpha = \dim_H(\mu)$, $h = \dim_H(J(f))$. Assume additionally that f is not CFPO. Then*

1. The measure μ is singular with respect to the Hausdorff measure H_α

2. $\dim_H(J(f)) = h > \alpha = \dim_H(\mu)$

The proof of this Theorem (and all its generalizations) was based on stochastic laws (in particular Central Limit Theorem) for some sequences of weakly dependent random variables. Quite surprisingly, it turned out that one can prove part 2. of the Theorem, avoiding the "stochastic laws approach" (while, apparently, the proof of part 1. still requires it). We present, therefore, an alternative proof of the following, generalized version of part 2. of the Theorem. It is known (see, e.g [3]) that, if $\phi : J(f) \to R$ is a Hölder continuous function such that $P(\phi) > \sup \phi$ then ϕ admits a unique equilibrium measure μ_ϕ (the case of maximal measure corresponds to the case $\phi = 0$).

Theorem. *Let f be rational map. Let $\phi : J(f) \to R$ be a Hölder continuous function such that $P(\phi) > \sup \phi$. Let μ_ϕ be the unique equilibrium measure for ϕ.*

Then either f is expanding on the Julia set and ϕ is cohomologous to $-h \log |f'|$ or

f is CFPO and ϕ is cohomologous to a constant

or $dim_H(\mu_\phi) < \dim_H(J(f))$.

The approach which we use here is based on the inducing procedure, which leads to an infinite conformal Iterated Function System. We then compare several equilibrium measures for this system. Using rigidity results for this induced system, we prove the rigidity of the original system (f, μ_ϕ).

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Dynamics of a three-dimensional analog of the sine function

Alexandre Eremenko

(joint work with Walter Bergweiler)

We construct a dynamical system in $\mathbf{R}^n$ which displays a strong form of the "Karpińska paradox" [2, 3].

A set $H \subset \mathbf{R}^n$ is called a *hair* if there exists a continuous injective map $\gamma : [0, \infty) \to \mathbf{R}^n$ such that $\gamma(t) \to \infty$ as $t \to \infty$, and $\gamma([0, \infty)) = H$. The point $E = \gamma(0)$ is called the *endpoint* of the hair.

Theorem. *The space $\mathbf{R}^n, n \geq 2$ can be represented as a union of hairs with the following properties. The intersection of any two hairs is either empty or consists of their common endpoint, and the union of hairs without endpoints has Hausdorff dimension one.*

Analogous decomposition of $\mathbf{R}^2$ was obtained by Schleicher [3] who used the dynamics of an entire function of the sine family.

To obtain the result in $\mathbf{R}^n$ we use a quasiregular map which generalizes the sine map of the complex plane in the similar way to Zorich's generalization of the exponential map [4].

To simplify notation, we describe our construction in $\mathbf{R}^3$.

We denote by $H_{\geq 0}$ and $H_{\leq 0}$ the upper and lower half-spaces in $\mathbf{R}^3$.

Let $h : Q \to U, \quad h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2), h_3(x_1, x_2))$ be a bi-Lipschitz map from the square

$$Q = [-1, 1] = \{(x_1, x_2) \in \mathbf{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}$$

onto the upper hemisphere

$$U = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \geq 0\}.$$

We first define our map F in the semi-cylinder

$$Q \times [1, \infty) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : |x_1| \leq 1, |x_2| \leq 1, x_3 \geq 1\}$$

by the formula

$$F(x_1, x_2, x_3) = (e^{x_3} h_1(x_1, x_2), e^{x_3} h_2(x_1, x_2), e^{x_3} h_3(x_1, x_2)) \, .$$

Then we extend F to the cube $Q \times [0, 1]$, so that extended function is bi-Lipschitz and maps this cube onto the upper half of the ball

$$\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq e^2, \ x_3 \geq 0\},$$

and the resulting map $F : Q \times [0, \infty) \to H_{\geq 0}$ is bi-Lipschitz on compact subsets. Then our map F is extended to a map $\mathbf{R}^3 \to \mathbf{R}^3$ by symmetry, using reflections in the faces of the semi-cylinders and in the plane $x_3 = 0$.

The resulting map is quasiregular, bi-Lipschitz on compact subsets, and the infinitesimal distortion tends to infinity as $x \to \infty$. Finally we multiply F by a positive constant so that the map $f = \lambda F$ is uniformly expanding.

The dynamics of this map f is studied using a symbolic dynamics similar to that introduced by Devaney and Krych [1] for the exponential map of the complex plane.

We put $S = \mathbf{Z} \times \mathbf{Z} \times \{-1, 1\}$ and for $r = (r_1, r_2, r_3) \in S$ define the set

$$T(r) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : |x_1 - 2r_1| \leq 1, |x_2 - 2r_2| \leq 1, r_3 x_3 \geq 0\}.$$

If $r_1 + r_2 + (r_3 - 1)/2$ is even then f maps $T(r)$ bijectively onto $H_{\geq 0}$. If $r_1 + r_2 + (r_3 - 1)/2$ is odd then f maps $T(r)$ bijectively onto $h_{\leq 0}$.

For a sequence $\underline{s} = (s_k)_{k \geq 0}$ of elements of S we put

$$H(\underline{s}) = \{x \in \mathbf{R}^3 : f^k(x) \in T(s_k) \quad \text{for all} \quad k \geq 0\}.$$

Evidently $\mathbf{R}^3 = \sum_{\underline{s} \in \underline{S}} H(\underline{s})$, where $\underline{S}$ is the set of all sequences $\underline{s}$ with elements in S for which $H(\underline{s})$ is not empty.

Proposition 1. *If $\underline{s} \in \underline{S}$ then $H(\underline{s})$ is a hair.*

For $\underline{s} \in \underline{S}$ we denote by $E(\underline{s})$ the endpoint of $H(\underline{s})$.

Proposition 2. *If* $\underline{s}' \neq \underline{s}''$ *then* $H(\underline{s}') \cap H(\underline{s}'') = \emptyset$ *or* $H(\underline{s}') \cap H(\underline{s}'') = E(\underline{s})$.

Proposition 3. *The Hausdorff dimension of the set*

$$\bigcup_{\underline{s}\in\underline{S}} H(\underline{s})\backslash\{E(\underline{s})\}$$

equals to one.

Theorem 1 follows from these three propositions.

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On the connectivity of the escaping set for complex exponential Misiurewicz parameters

Xavier Jarque

This talk is organized in two parts. In the first part we will give some basics in complex dynamics and introduce the main topics of the problem we want to address, that is, the connectivity of the escaping set for complex exponential maps. When this is done we state the theorem we want to prove and, finally, we give the main ingredients of how to prove it.

For an entire map, say f, there are two types of points for which the inverse map is not well defined, namely critical values and asymptotic values. A *critical value* is the image of a critical point, while we say that $z_0 \in \mathbb{C}$ is an *asymptotic value* if there is a curve $\alpha(t)$ satisfying $|\alpha(t)| \to \infty$ and $f(\alpha(t)) \to z_0$, as $t \to \infty$. The closure of the union of critical and asymptotic values is named *singular values*. It is well known that singular values plays an important role in determining the global dynamics associated to the iterates of the map. For instance, all maps belonging to the well known complex exponential family $E_\lambda(z) = \lambda\exp(z),\ \lambda \in \mathbb{C}$, have unique asymptotic value at $z = 0$ and no critical values. So, the structure and topology of the Fatou and Julia sets in the dynamical plane strongly depend on the asymptotic behavior of the iterates of the unique singular value at $z = 0$.

In contrast to the polynomial case, where all points which tend to infinity under iteration belong to the basin of attraction of $z = \infty$, and so belong to the Fatou set, the existence of an essential singularity at infinity and a unique finite singular value, implies that all points which tend to infinity under iteration, known as the *escaping set*

$$\mathcal{I}(E_\lambda) = \{z \in \mathbb{C} \mid E_\lambda^n(z) \to \infty\},$$

belong to the Julia set, $\mathcal{J}(E_\lambda)$. Indeed this inclusion is particularly interesting since it turns out that that $\mathcal{J}(E_\lambda) = \mathrm{cl}(\mathcal{I}(E_\lambda))$.

Since *Eremeko's conjecture* [2] about the boundedness (or not) of the connected components of the escaping set of general transcendental entire maps, there has been interest in understanding the topology of the connected components of $\mathcal{I}(f)$, and in particular of $\mathcal{I}(E_\lambda)$ for which it is known that the conjecture is true (see [5] for more general results about the conjecture).

It is easy to argue that for hyperbolic parameters like $\lambda \in (0, 1/e)$, each of the hairs in dynamical plane, is a distinct connected component of $\mathcal{I}(E_\lambda)$. Recently L. Rempe [4] used the construction of the Devaney's indecomposable continua [1] to show that when $\lambda > 1/e$ the escaping set is indeed a connected set of the plane. So, the infinitely many pairwise disjoint curves that extend to infinity (*hairs* or *dynamical rays*) are just a subset of a unique connected component.

These previous results seemed to show that having a singular value(s) in the Julia set made a crucial difference regarding the connectedness of the escaping set. However, H. Mihaljević-Brandt [3] proved that, for instance, for any Misiurewicz parameter of the sine family (parameters for which the two critical values are preperiodic), $S_\lambda(z) = \lambda \sin z$, the escaping set is not a connected subset of the plane. She obtained this result as a corollary of a much more general theorem about conjugacy of quasiconformal equivalent maps. We notice that the sine family has no finite asymptotic values but two critical values given by $\pm\lambda$. Of course, for Misiurewicz parameters we have that the Julia set is the whole plane, and so, both critical values belong to the Julia set.

Consequently a natural and interesting question in this setting is to study the connectivity of the escaping set for Misiurewicz parameters of the exponential family, that is parameters for which the singular value has a strictly preperiodic orbit. In this note we actually show that, in contrast to Misiurewicz parameters in the sine (or cosine) family, the escaping set is connected.

Theorem 1. *Let λ be a Misiurewicz parameter. Then $\mathcal{I}(E_\lambda)$ is a connected subset of the plane.*

The proof of this Theorem relies basically on two main considerations. The first one is that, for Misiurewicz parameters of the complex exponential family, the singular value is accessible *from infinity through the escaping set* [6]. That is, there exist a curve $\gamma : [0, \infty) \to \mathcal{J}(E_\lambda)$ with $\gamma(0) = 0$ and $\gamma(t) \in \mathcal{I}(E_\lambda)$ for all $t > 0$, and $\mathrm{Re}(\gamma(t)) \to \infty$ as $t \to \infty$. The second consideration is that if the escaping set were not a connected subset of the plane, so it was disconnected, then there would be an open connected set $U \subset \mathbb{C}$ such that the following three conditions were satisfied

$$\text{(a)}\quad \mathcal{I}(E_\lambda) \cap U \neq \emptyset$$
$$\text{(b)}\quad \mathcal{I}(E_\lambda) \cap \partial U = \emptyset$$
$$\text{(c)}\quad \mathcal{I}(E_\lambda) \not\subset U$$

From the curve γ we built a division of the plane in infinitely many strips of width 2π, such that the boundaries belong to the escaping set. The rest of the proof uses the symbolic dynamics associated to this partition and well known facts on complex exponentials.

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The set of Misiurewicz parameters in the transcendental case

Agnieszka Badenska

The notion of Misiurewicz maps derives from the paper [4] by Michal Misiurewicz, where the author considered e.g. the real quadratic family

$$g_a(x) = 1 - ax^2,$$

$a \in (0, 2)$, in the case when g_a is non-hyperbolic and the critical point 0 is non-recurrent. In 1998 D. Sands proved in [5] that these functions form a set of the Lebesgue measure zero in the parameter space, answering a question posed by Misiurewicz in his paper.

In the complex case, there were various definitions of a Misiurewicz condition introduced. Probably the most common, referred as postcritically (postsingularly) finite, assumes that all singular values are eventually mapped onto repelling cycles in the Julia set. However, one can find many other versions and generalizations of this definition (see [1] for references). M. Aspenberg in [1] calls a rational function f Misiurewicz if it is a non-hyperbolic map without parabolic cycles and such that

$$\omega(c) \cap \mathrm{Crit}(f) = \emptyset \quad \text{for every} \quad c \in \mathrm{Crit}(f) \cap J(f),$$

where $\omega(c)$ denotes the omega limit set of c while $\mathrm{Crit}(f)$ is a set of all critical points of f. With this definition he proves that the set of Misiurewicz parameters has the Lebesgue measure zero in the space of rational functions of any fixed degree $d \geq 2$.

Our aim is to extend this result to the transcendental case, i.e. to prove that the set of Misiurewicz parameters has the Lebesgue measure zero for some parametrized families of transcendental maps. First, we consider the exponential family

$$f_\lambda(z) = \lambda e^z\ , \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

We say that λ_0 is a Misiurewicz parameter if the asymptotic value 0 (which is the only finite singular value) belongs to the Julia set $J(f_{\lambda_0})$ and has bounded trajectory under f_{λ_0}. Following Apenberg's idea, we prove that the set of Misiurewicz parameters for the exponential family has the Lebesgue measure zero in $\mathbb{C}$.

Next, we study two families of Weierstrass elliptic functions generated by triangle and square lattices. Recall that a lattice $\mathbf{L}$ is said to be triangular if $e^{2\pi i/3}\mathbf{L} = \mathbf{L}$ and it is called square if $i\mathbf{L} = \mathbf{L}$. Elliptic functions have no asymptotic values, thus the set of singularities consists only of critical values. Functions under consideration have three distinct critical values, moreover, there is a strong relationship between their trajectories (see [2, 3] for details), in particular if one of critical values lives in the Julia and has a bounded forward orbit, then the same is true for the remaining critical values (except for 0, which is always a pole in the quadratic case). This allows us to introduce the following definition: a Weierstrass elliptic function based on a triangular or a square lattice is called Misiurewicz if all critical values belong to the Julia set and either have bounded trajectories staying in a positive distance from critical points, or are eventually mapped onto infinite after finite number of iterates.

Again using Aspenberg's approach, we prove that a set of Misiurewicz parameters for which there exists a critical value which is not a prepole, has the Lebesgue measure zero in $\mathbb{C}$. The case when all critical values are prepoles has to be treated separately. Using different techniques we prove that parameters for which all critical values are mapped onto poles under iteration form a countable set. Therefore, the set of all Misiurewicz parameters in the families of Weierstrass elliptic functions generated by triangular and square lattices have the Lebesgue measure zero in $\mathbb{C}$.

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The Hausdorff dimension of Julia sets of entire functions with regular growth

Bogusława Karpińska
(joint work with Walter Bergweiler)

The Hausdorff dimension of the Julia sets of entire functions has been an object of study since 1987. The first result on this topic is due to McMullen, who proved that the Hausdorff dimension of the Julia set of the functions $f(z) = \lambda \exp(z)$, where $\lambda \in \mathbb{C} \setminus \{0\}$, is equal to 2 ([4]).

This result has been extended to various families of entire functions belonging to the Eremenko-Lyubich class B. This class consists of all entire functions for which the set of finite asymptotic values and critical values is bounded. Working in the class B allows to use the logarithmic change of variable which is a very powerful tool in transcendental dynamics. It was first introduced in this context by Eremenko and Lyubich in [2].

One of the extensions of McMullen's theorem is the general result proved independently by Barański and Schubert in [1] and [5] saying that if f belongs to the class B and f has finite order $\rho(f)$ which is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}, \quad \text{where} \quad M(r,f) = \max_{|z|=r} |f(z)|,$$

then the Hausdorff dimension of the Julia set of f is equal to 2. The assumption concerning the finite order cannot be omitted.

The purpose of the talk is to present a result on the Hausdorff dimension of the Julia set for entire functions which do not belong to the Eremenko-Lyubich class. We consider entire functions which grow sufficiently regularly. We prove the following:

Theorem 1. *Let f be an entire function for which there exist $A, B, C, r_0 > 1$ such that*

$$A \log M(r,f) \leq \log M(Cr,f) \leq B \log M(r,f) \quad \text{for } r \geq r_0. \tag{1}$$

Then the Julia set and the escaping set of f have Hausdorff dimension 2.

The escaping set $I(f)$ is defined as the set of all points z in the plane for which $f^n(z)$ tends to infinity with n.

Among the examples of functions satisfying our regularity condition (1) are the entire functions of completely regular growth described e.g. in [3].

The proof of Theorem 1 is based on some careful estimates of the logarithmic derivative of f. For $\alpha_1, \alpha_2, q, \lambda \geq 0$ we consider the set $T(f, \alpha_1, \alpha_2, q, \lambda)$ consisting of all $z \in \mathbb{C}$ for which

$$\alpha_1 \log M(|z|,f) \leq \left| \frac{zf'(z)}{f(z)} \right| \leq \alpha_2 \log M(|z|,f),$$

$$|f(z)| \geq |z|^q$$

and

$$\left| \frac{\zeta f'(\zeta)}{f(\zeta)} \right| \leq \alpha_2 \log M(|\zeta|,f) \quad \text{for} \quad |\zeta - z| \leq \lambda \frac{|z|}{\log M(|z|,f)}.$$

For $R > 0$ we denote $A(R) = \{z \in \mathbb{C} : R \leq |z| \leq 2R\}$.
We prove the following:

Theorem 2. *Let f be an entire function satisfying* (1)*. Then there exist* $\alpha_1, \alpha_2, \eta > 0$ *such that if* $q, \lambda \geq 0$*, then*

$$\frac{\operatorname{area}(T(f, \alpha_1, \alpha_2, q, \lambda) \cap A(R))}{\operatorname{area} A(R)} > \eta$$

for sufficiently large R.

We use Theorem 2 to estimate the Hausdorff dimension of the set $I(f)$. Our calculations are based on McMullen's technique from [4] which gives a lower bound on the Hausdorff dimension of the intersection of nested sets. We construct such a family of nested sets, whose intersection is contained in the escaping set $I(f)$, using Ahlfors three islands theorem. However our assumptions do not imply that $I(f)$ is a subset of the Julia set. To show that the set that we obtain is contained also in the Julia set we use Zheng's result from [6].

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Escaping set in the boundaries of exponential basins

Krzysztof Barański

(joint work with Bogusława Karpińska, Anna Zdunik)

We consider the dynamics of an exponential map

$$f(z) = \lambda \exp(z)$$

for $\lambda \in \mathbb{C} \setminus \{0\}$, $z \in \mathbb{C}$, such that f has an attracting periodic cycle of (minimal) period p (i.e. there exist points $z_0, z_1, \ldots, z_{p-1}, z_p = z_0$, such that $f^p(z_j) = z_j$ and $|(f^p)'(z_j)| > 1$, where f^p denotes the pth iterate of f). The Julia set $J(f)$ is defined as the set of points $z \in \mathbb{C}$ for which the family $\{f^n\}_{n>0}$ is not normal in any neighbourhood of z, while the Fatou set is the complement of $J(f)$. In our situation, the Julia set is equal to the boundary of the entire basin B of attraction of this cycle, where $B = \{z \in \mathbb{C} : f^{pn}(z) \to z_j \text{ for some } j \text{ as } n \to \infty\}$. We are interested in the dimensional properties of this boundary. We consider also the escaping set

$$I(f) = \{z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty\}$$

and the set of points in the Julia set with bounded forward trajectories

$$J_{\rm bd}(f) = \{z \in J(f) : \{f^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

In our context, both sets are contained in the boundary of B.

In the well-known case $p = 1$, the attracting basin consists of a unique simply connected component U. As proved in [5], the Hausdorff dimension $\dim_{\rm H}$ of its boundary $\partial U = J(f)$ is then equal to 2. In fact, it follows from the proof in [5] that $\dim_{\rm H}(\partial U) = \dim_{\rm H}(\partial U \cap I(f)) = 2$. On the other hand, we have $1 < \dim_{\rm H}(\partial U \cap J_{\rm bd}(f)) \leq \dim_{\rm H}(\partial U \setminus I(f)) < 2$ (see [3, 7]). Hence, it follows from the above results that in the case $p = 1$ the dimension of ∂U is carried by the set of escaping points.

Consider now the case $p > 1$. Then the basin B of the attracting cycle consists of infinitely many disjoint simply connected components. Again, the Hausdorff dimension of the boundary of B is equal to 2 (see [5]). However, this does not imply that the boundaries of components of B have Hausdorff dimension 2. In fact, we prove the following result.

Theorem A. *Let $p > 1$ and let U be a component of the basin of the attracting cycle for the map f. Then the boundary of U has Hausdorff dimension greater than 1 and less than 2.*

This shows that for $p > 1$ the dimension of ∂B is carried by points in the Julia set, which are not in the boundary of any component of the Fatou set (so-called buried points). By Theorem A, we immediately obtain:

Corollary A'. *If $p > 1$, then the set of buried points for the map f has Hausdorff dimension 2.*

Surprisingly, for $p > 1$ the role of escaping and non-escaping points in ∂U from a dimensional point of view is quite different than in the case $p = 1$. This is shown in the second result.

Theorem B. *If $p > 1$ and U is a component of the basin of the attracting cycle for the map f, then*

$$1 < \dim_{\rm H}(\partial U \cap J_{\rm bd}(f)) \leq \dim_{\rm H}(\partial U \setminus I(f)) = \dim_{\rm H}(\partial U) < 2,$$

while

$$\dim_{\rm H}(\partial U \cap I(f)) = 1.$$

In fact, the proof of Theorem B shows that $\partial U \cap J_{\rm bd}(f)$ has hyperbolic dimension greater than 1. Therefore, in the case $p > 1$ the dimension of ∂U is carried by the set of non-escaping points.

Remark. *It is easy to check that for $z \in \partial U$ we have*

$$f^n(z) \xrightarrow[n\to\infty]{} \infty \iff f^{pn}(z) \xrightarrow[n\to\infty]{} \infty.$$

Topologically, the Julia set $J(f)$ consists of so-called hairs homeomorphic to the half-line $[0, \infty)$, which are curves tending to ∞ and composed of points with given symbolic itineraries. All points from a hair outside its endpoint (the point corresponding to 0 in $[0,\infty)$) are contained in $I(f)$. In the case $p=1$ the hairs are pairwise disjoint and the Julia set is homeomorphic to a so-called straight brush (see [1]), while in the case $p>1$ some hairs have common endpoints and the Julia set is a modified straight brush (see [2]). In both cases the Hausdorff dimension of the union of hairs without endpoints is equal to 1 (see [4, 6]).

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The escaping set in quasiregular dynamics

Daniel Nicks

(joint work with Alastair Fletcher)

Quasiregular mappings of $\mathbb{R}^n$ are a natural generalisation to higher dimensions of entire functions on the complex plane. See [8] for a formal definition and a description of their properties. While there has been some study of the dynamics of quasiregular mappings (see for example, [1, 4, 5, 6, 9]), for the most part this has been restricted to the case of uniformly quasiregular mappings; that is, those quasiregular mappings for which all the iterates obey a common bound on the distortion. In the uniformly quasiregular case, a version of Montel's theorem is available [7] and this allows one to carry over many of the concepts from entire dynamics, including the usual definitions of Fatou and Julia sets. However, these uniformly quasiregular mappings appear to be a fairly special case. If we wish to study the iterative behaviour of a general quasiregular map, then it is less clear how we should define a suitable Julia set.

Inspired by Eremenko's result that the Julia set of a non-linear entire function coincides with the boundary of its escaping set [2], we ask the following:

Question. *For quasiregular mappings, is the boundary of the escaping set analogous to a Julia set?*

In this context, the escaping set is defined to be

$$I(f) = \{x \in \mathbb{R}^n : f^k(x) \to \infty \text{ as } k \to \infty\}.$$

In the special case when f is a uniformly quasiregular mapping, it is shown in [3] that the Julia set $J(f)$ is equal to $\partial I(f)$, provided that f is not injective.

A quasiregular mapping is said to be of *polynomial type* if $f(x) \to \infty$ as $x \to \infty$. Otherwise, this limit does not exist and f has an *essential singularity at infinity.* A notion of the *degree* of a quasiregular mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ can be defined by

$$\deg(f) = \sup_{y \in \mathbb{R}^n} |f^{-1}(y)|.$$

It is known that a quasiregular mapping is of polynomial type if and only if it has finite degree. We present the following result for quasiregular mappings of polynomial type, which shows that $\partial I(f)$ has some of the properties typically associated with a Julia set.

Theorem 1 ([3])**.** *Let f be quasiregular of polynomial type, and such that the degree of f is greater than the inner dilatation $K_I(f)$. Then $I(f)$ contains a neighbourhood of infinity and is an open connected set. Moreover, for any $k \geq 2$ we have $I(f) = I(f^k)$. The boundary $\partial I(f)$ is a completely invariant set that is infinite, closed and perfect.*

The condition that the degree exceeds the inner dilatation compels such mappings to have an attracting fixed point at infinity, similar to that of a polynomial in the plane. The following example shows that this condition cannot be omitted from Theorem 1. Let $m \in \mathbb{N}$ and consider the quasiregular winding map given by

$$\Phi_m : (r, \theta, y) \mapsto (r, m\theta, y)$$

in n-dimensional cylindrical co-ordinates. Then the degree and inner dilatation of Φ_m are both equal to m, and yet the escaping set $I(\Phi_m)$ is empty.

We mention a result due to Bergweiler, Fletcher, Langley and Meyer [1].

Theorem 2 ([1])**.** *If f is a quasiregular mapping with an essential singularity at infinity, then $I(f)$ is non-empty and has an unbounded component.*

An example is also constructed in [1] of a quasiregular mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ for which the closure of $I(f)$ has a bounded component. This stands in contrast to the situation for entire functions, as discussed in [2].

There remain many questions about the suitability of $\partial I(f)$ as a substitute for the Julia set in the quasiregular setting. For example, do points of $\partial I(f)$ possess the 'expanding' property? That is, for any open set U containing a point of $\partial I(f)$, must $\bigcup_{k \geq 0} f^k(U)$ cover all but finitely many points of $\mathbb{R}^n$? For quasiregular mappings of polynomial type in two dimensions, the set of points with the expanding property has been investigated by Sun and Yang [10].

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Parameter rays in the exponential family

Dierk Schleicher
(joint work with Markus Förster)

The topic of this talk are exponential maps $z \mapsto E_\lambda(z) := \lambda \exp(z)$ with escaping singular orbits and their classification. The singular value is 0 (in this case, an asymptotic value), and the set of such parameters λ where the singular value escapes is classified in terms of *parameter rays* and their landing points [2].

To define the terms, we start with the dynamical plane of an exponential map E_λ. The set I_λ is the set of $z \in \mathbb{C}$ that converge to ∞ under iteration of E_λ. A *dynamic ray* is a maximal injective curve $g\colon (\tau, \infty) \to I_\lambda$ with $g(t) \to \infty$ as $t \to \infty$. An *endpoint* of a dynamic ray is the point $\lim_{t \to \tau} g(t)$ if the limit exists. The endpoint may or may not be in I_λ.

By [4], there is an explicit uncountable set $\mathcal{S} \subset \mathbb{Z}^{\mathbb{N}}$, independent of λ, so that every E_λ has the property that I_λ is the disjoint union, for $\underline{s} \in \mathcal{S}$, of dynamic rays $g_{\underline{s}}$ with or without their endpoints, provided that the singular value does not escape (if it does, then some straightforward modifications are necessary). Let R_λ be the union of the dynamic rays of E_λ. More precisely, there are $t_{\underline{s}} \in [0, \infty)$ (also explicit, independent of λ) so that for $X := \bigcup_{\underline{s} \in \mathcal{S}} \{\underline{s}\} \times (t_{\underline{s}}, \infty)$ there is a continuous bijection $g_\lambda\colon X \to R_\lambda$ with the property that $g_{\underline{s}}(t) = t - \log \lambda + 2\pi i s_1 + O(e^{-t})$ (where $\underline{s} = s_1 s_2 s_3 \dots$) and $E_\lambda^{\circ n}(g_{\underline{s}}(t)) = F^{\circ n}(t) - \log \lambda + 2\pi i s_{n+1} + O(1/F^{\circ(n+1)}(t))$ (where $F(t) = e^t - 1$); here we use the discrete topology in the $\underline{s}$-coordinate.

We are interested in the set R of parameters $\lambda \in \mathbb{C}$ for which the singular value is on a dynamic ray. This set plays an important role in the description of exponential parameter space.

Theorem [3]. *For every $\underline{s} \in \mathcal{S}$ and every $t > t_{\underline{s}}$ there is a unique $\lambda =: G_{\underline{s}}(t)$ so that $0 = g_{\underline{s}}(t)$ (where 0 is the singular value); the union over $\underline{s}$ and t is R.*

Moreover, each map $G_{\underline{s}}$ *is continuous in* t*, so that this yields a continuous bijection* $G\colon X \to R$ *with the property that* $G(\underline{s}, t) = G_{\underline{s}}(t) = t + 2\pi i s_1 + O(e^{-t})$*.*

We discuss three proof strategies for this result. The first, from [3], works very explicitly in exponential dynamical and parameter spaces, and is very difficult to generalize for larger classes of maps. The second, from [1], reformulates this problem in the context of spider theory, using spiders with infinitely many legs. This has the potential to generalize to larger classes of entire functions of finite type; it is a bare-hands approach to infinite-dimensional Teichmüller theory. The third approach is work in progress (following suggestions by Adam Epstein and others) and works directly using infinitely-dimensional Teichmüller theory.

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Hausdorff dimension of Julia sets of random transcendental meromorphic functions

Mariusz Urbański

Our goal is to understand the behavior of the Hausdorff dimension of the Julia sets under small random perturbations of transcendental meromorphic functions, or more generally, the Hausdorff dimension of random transcendental meromorphic functions. As a powerful tool we apply the concept of random conformal graph directed Markov systems as developed in the speaker's paper "Random graph directed Markov systems" written jointly with Mario Roy. We in fact use this paper in its simplest case of random iterated function systems. It is proved in this paper that the Hausdorff dimension of almost all limit sets is given as the infimum of all real values t such that the expected pressure function $EP(t)$ does not exceed zero. This number is shown to be greater than or equal to the finite parameter of the system, which we commonly denote by θ. As the first application of this result to random transcendental dynamics, we take an arbitrary non-constant elliptic function f and we consider its (sufficiently) small random perturbations of the form λf with complex numbers λ sufficiently close to 1. Guided by the deterministic construction described in the paper [1], we associate to such random elliptic functions a random conformal iterated function system consisting of holomorphic inverse branches of second iterates of the functions λf. It is then easy to check that this random system is cofinitely regular and that its θ number is equal to

$$\frac{2q}{q+1},$$

where $q \geq 2$ is the largest order of all poles of the function f. Consequently, this gives a strict lower bound for the Hausdorff dimension of almost random all random Julia sets. We can also show here that the Hausdorff dimension of the escaping set is $\leq \frac{2q}{q+1}$ for all random elliptic functions. If one starts of with an entire function f with a logarithmic tract, then it was shown by Barański, Karpińska and Zdunik that the Hausdorff dimension of the radial Julia set is strictly larger than 1. By forming a random iterated function systems consisting again of holomorphic inverse branches of second iterates of the function λf, and applying the results obtained with Mario Roy, we show that the Hausdorff dimension of radial random perturbations of f continues to be larger than 1.

As a different result, by constructing suitable quasi-conformal conjugacies, we show together with Volker Mayer and Bartek Skorulski that the Hausdorff dimension is stochastically stable under small random perturbations of topologically hyperbolic entire functions.

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Connectivity of Fatou components

Walter Bergweiler

The *Fatou set* $F(f)$ of an entire or rational function f is defined as the set where the iterates f^n of f form a normal family. If U_0 is a component of $F(f)$, then $f^n(U_0)$ is contained in a component U_n of $F(f)$. If all the U_n are different, then U_0 is called a *wandering domain* of f. Otherwise U_0 is called *preperiodic.* If $U_0 = U_n$ for some $n \in \mathbb{N}$, then U_0 is called *periodic.* It is classical that periodic Fatou components of a rational function are simply, doubly or infinitely connected. Baker, Kotus and Lü [5] showed that for rational functions the connectivity of a preperiodic component may take any value. Baker [1] showed that preperiodic Fatou components of entire functions are simply connected. Here we consider the connectivity of wandering domains.

While a famous theorem of Sullivan [15] says that rational functions do not have wandering domains, it had been shown already earlier by Baker [2] that such domains may exist for transcendental entire functions. The wandering domain in Baker's example was multiply connected. However, examples of simply connected wandering domains were given later by various authors; see [10, p. 106], [15, p. 414], [3, p. 564, p. 567] and [8, p. 222]. It is still not known whether the wandering domains in Baker's original example are finitely or infinitely connected. However, by suitably modifying his construction, Baker [4, Theorem 2] obtained wandering domains of infinite connectivity. Recently Kisaka and Shishikura [12] constructed an example with a multiply connected wandering domain of finite connectivity, thereby answering a question of Baker. In fact, they showed that the connectivity may take any preassigned value. Here we modify the construction of Kisaka and Shishikura to prove the following result.

Theorem 1. *There exists an entire function which has both a simply connected and a multiply connected wandering domain.*

The question whether an entire function with this property exists had been raised by Rippon and Stallard [14, p. 1125, Remark 3]. In the same paper, Rippon and Stallard also asked a question about the set $A(f)$ introduced in [7]. This is defined by

$$A(f) := \left\{z : \text{ there exists } L \in \mathbb{N} \text{ such that } |f^n(z)| > M\left(R, f^{n-L}\right) \text{ for } n > L\right\},$$

where $M(r,f) := \max_{|z|=r} |f(z)|$ and $R > \min_{z\in J(f)} |z|$. Roughly speaking, $A(f)$ consists of the points z where $f^n(z)$ tends to infinity "as fast as possible". Rippon and Stallard showed that $A(f)$ has no bounded components and that the closure of every multiply connected wandering domain is contained in $A(f)$. They also showed that if a simply connected wandering domain intersects $A(f)$, then it must lie entirely in $A(f)$, and they ask [14, p. 1126, Remark 4] whether an entire function f with such a simply connected wandering domain exists. It turns out that an example with this property is provided by the function constructed in Theorem 1.

Theorem 2. *There exists an entire function f for which $A(f)$ contains a simply connected wandering domain.*

The main tool used by Kisaka and Shishikura is quasiconformal surgery. They first construct a quasiregular map g which has the desired property and then show that g is quasiconformally conjugate to an entire function f.

More specifically, for certain sequences (P_n), (Q_n), (R_n), (S_n) and (T_n) tending to ∞ and satisfying $P_n < Q_n < R_n < S_n < T_n < P_{n+1}$ and for certain sequences (a_n) and (b_n) of complex numbers the map g is chosen such that

(i) $g(z) = a_n z^{n+1}$ for $T_n \le |z| \le P_{n+1}$ and
(ii) $g(z) = b_n(z - R_n)z^n$ for $Q_n \le |z| \le S_n$.

In the remaining annuli given by $P_n < |z| < Q_n$ and $S_n < |z| < T_n$ the map is defined by a certain interpolation. Kisaka and Shishikura then show that for suitably chosen sequences (P_n), (Q_n), (R_n), (S_n), (T_n), (a_n) and (b_n) the map g is uniformly quasiregular; that is, there exists $K > 1$ such that all iterates of g are K-quasiregular. Finally they show (see also [9, 11] besides [12]) that this implies that g is quasiconformally conjugate to an entire function f.

The annuli given by $T_n \le |z| \le P_{n+1}$ are contained in multiply connected wandering domains of g. To prove that these wandering domains are doubly connected, Kisaka and Shishikura show that they can be exhausted by an increasing sequence of doubly connected domains. In order to make this argument work, one has to control the critical points of g, which are given by $\xi_n = nR_n/(n+1)$. Kisaka and Shishikura choose b_n such that $g(\xi_n) = R_{n+1}$ and thus $g^2(\xi_n) = 0$. For the proof of Theorems 1 and 2 we instead choose b_n such t hat $g(\xi_n) = \xi_{n+1}$.

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Connectivity of Julia sets of transcendental meromorphic maps and weakly repelling fixed points

Jordi Taixés

(joint work with Núria Fagella and Xavier Jarque)

For a rational, transcendental entire or transcendental meromorphic map f, the *Fatou set* $\mathcal{F}(f)$ is defined as the set of points $z \in \widehat{\mathbb{C}}$ for which the family $\{f^k\}_{k\in\mathbb{N}}$ is defined and normal in a neighbourhood of z, and the *Julia set* is its complement, $\mathcal{J}(f) := \widehat{\mathbb{C}} \setminus \mathcal{F}(f)$.

It is known that the Julia set of the *Newton's method*

$$N_P(z) := z - \frac{P(z)}{P'(z)}$$

for a non-constant polynomial P is connected (M. Shishikura, [4]). This is, in fact, a consequence of a much more general result that establishes the relationship between simple connectivity of Fatou components of rational maps and *weakly* *repelling fixed points*, i.e., fixed points which are repelling or parabolic with multiplier 1.

Our aim is to extend this general result to transcendental meromorphic maps, which would give the connectedness of the Julia set of the Newton's method for a transcendental entire map as a consequence. More precisely, we would like to show the following: If the Julia set of a transcendental meromorphic map f is disconnected (or, equivalently, if $\mathcal{F}(f)$ is multiply connected), then f has at least one weakly repelling fixed point.

Using the classification of Fatou components for transcendental meromorphic maps, it suffices to prove the statement in a case-by-case fashion for the six distinct types of possible multiply-connected Fatou components — namely: (a) Multiply-connected immediate attractive basin; (b) multiply-connected parabolic basin; (c) Herman ring; (d) multiply-connected Baker domain; (e) multiply-connected Fatou component U such that $f(U)$ be simply connected; and (f) multiply-connected wandering domain.

So far we have proved Cases (a), (b), (c) and (e); in particular, our proofs for Cases (a) and (e) can be found in [3]. On the other hand, Case (f) was proved by W. Bergweiler and N. Terglane in [1] in the search of solutions of certain differential equations with no wandering domains. The remaining case is, therefore, that of the multiply-connected Baker domain.

In this talk we present our proof for Case (b), that is to say, we prove the following.

Theorem 1. *Let f be a transcendental meromorphic map with a multiply-connected parabolic basin. Then, f has at least one weakly repelling fixed point.*

The proof involves two quite different techniques. The first one is based upon M. Shishikura's quasi-conformal surgery construction for the rational case, and applies when preimages of certain sets behave not too wildly under the presence of the essential singularity. For the second technique, the assumption of a pole of the map f allows us to construct the precise setting to which a theorem of X. Buff (see [2]) can be applied in order to find a weakly repelling fixed point of f.

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Pinched Cantor bouquets in non-exponential dynamics

Helena Mihaljević-Brandt

It is well-known that the Julia set $\mathcal{J}(f)$ of a transcendental entire function f can be the whole complex plane. For instance, the exponential map $z \mapsto e^z$ has this property since its only singular value, the asymptotic value 0, escapes to ∞. Also, the Julia set of $z \mapsto \pi \sinh z$ equals $\mathbb{C}$ due to a different reason, namely the strict preperiodicity of its critical values πi and $-\pi i$. The topological dynamics of the given examples differ considerably: for both maps, the escaping set consists of curves to infinity, called *dynamic rays*, and while the Julia set of the exponential map contains infinitely many dynamic rays that accumulate everywhere upon themselves [4], it is known for the second example that every dynamic ray has a well-defined landing point and every point in the Julia set is either on such a ray or its landing point [6]. Both maps belong to the better understood examples of transcendental entire maps; for instance, the full description of the *combinatorial* dynamics of the latter map was previously given by Schleicher in [7]. Still, there seems to be no example of a transcendental entire map f with $\mathcal{J}(f) = \mathbb{C}$ for which the topological dynamics has been understood completely.

In the talk, a result will be presented [3] which enables us to give a complete description of the topological dynamics for a large class of maps, including examples such as $z \mapsto \pi \sinh z$. Let us introduce some definitions before we state the result in its full generality. A transcendental entire map f is called *subhyperbolic* if the intersection of the Fatou set $\mathcal{F}(f)$ and the postsingular set $P(f)$ is compact and the intersection of the Julia set $\mathcal{J}(f)$ and $P(f)$ is finite. A subhyperbolic map is called *hyperbolic* if $\mathcal{J}(f) \cap P(f) = \emptyset$. We are interested in the following class of subhyperbolic functions which includes all hyperbolic maps: A subhyperbolic transcendental entire map f is called *strongly subhyperbolic* if $\mathcal{J}(f)$ contains no asymptotic values of f and the local degree of f at the points in $\mathcal{J}(f)$ is bounded by some finite constant.

Note that the map $z \mapsto \pi \sinh z$ mentioned previously is strongly subhyperbolic, since $P(f) = P(f) \cap \mathcal{J}(f) = \{\pm\pi i, 0\}$ is finite, all critical points are simple and there are no asymptotic values.

Our main theorem describes the Julia set of any strongly subhyperbolic entire function as a quotient of the Julia set of a (particularly simple) hyperbolic function in the same parameter space. For many families, such as $z \mapsto \lambda \sinh z$, the dynamics of hyperbolic functions is well-understood, hence our main result will extend this understanding to all strongly subhyperbolic functions in these families.

Theorem 1. *Let f be strongly subhyperbolic, and let $\lambda \in \mathbb{C}$ be such that $g(z) := f(\lambda z)$ is hyperbolic with connected Fatou set. Then there exists a continuous surjection $\phi : \mathcal{J}(g) \to \mathcal{J}(f)$, such that*

$$f(\phi(z)) = \phi(g(z))$$

for all $z \in \mathcal{J}(g)$. Moreover, ϕ restricts to a homeomorphism between the escaping sets $I(g)$ and $I(f)$.

We obtain some interesting corollaries: Suppose that g is a hyperbolic map with connected Fatou set and that g has *finite order*, i.e., $\log\log|g(z)| = O(\log|z|)$ as $z \to \infty$, or, more generally, can be written as a finite composition of finite-order maps with bounded singular sets. Then it is known [2, 5] that $\mathcal{J}(g)$ is a *Cantor bouquet*, i.e., homeomorphic to a *straight brush* in the sense of [1].

Corollary 2. *Let $f = f_1 \circ \cdots \circ f_n$ be a strongly subhyperbolic map, where every f_i is an entire map with bounded set of singular values and with finite order of growth. Then $\mathcal{J}(f)$ is a* pinched Cantor bouquet*; that is, the quotient of a Cantor Bouquet by a closed equivalence relation defined on its endpoints.*

W. Bergweiler asked the question whether the escaping set of a cosine map with strictly preperiodic critical values (e.g., $z \mapsto \pi \sinh z$) is connected. Since the escaping set of a hyperbolic entire function with connected Fatou set is always disconnected, we also obtain the following corollary, settling this question for all strongly subhyperbolic maps.

Corollary 3. *The escaping set of a strongly subhyperbolic transcendental entire function is disconnected.*

The key idea in the proof of the main theorem is to show that the considered strongly subhyperbolic map is a uniform expansion with respect to some complete hyperbolic metric defined on a domain that contains $\mathcal{J}(f)$. We show that this can be done using *orbifold metrics*. As additional results, we will present certain global estimates for hyperbolic orbifold metrics that are interesting on their own.

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Hausdorff measure of escaping sets in the exponential family

Jörn Peter

We consider the exponential family, parametrized by $E_\lambda(z) := \lambda e^z, \lambda \in \mathbb{C} \setminus \{0\}$. Given some non-decreasing, continuous function $h : [0, \varepsilon) \to \mathbb{R}_{\geq 0}$ (which we call a *gauge function*), we define the *Hausdorff measure with respect to* h of a set $A \subset \mathbb{C}$ by

$$\mathcal{H}^h(A) := \lim_{\delta \to 0} \inf\{\sum_{i=1}^{\infty} h(A_i) | \bigcup A_i \supset A, \text{diam } A_i < \delta\}.$$

It can be shown that $\mathcal{H}^h$ is a metric outer measure, and it follows directly that $h_s(t) := t^s$ defines the s-dimensional Hausdorff measure.

McMullen proved in [1] that the escaping set

$$I(E_\lambda) := \{z \in \mathbb{C} | E_\lambda^n(z) \to \infty \text{ as } n \to \infty\}$$

of every map in the exponential family has Hausdorff dimension 2. Moreover, he remarked without proof that $I(E_\lambda)$ always has non-σ-finite $\mathcal{H}^h$-measure with respect to the function $h(t) = t^2 \log^k(1/t)$ (k arbitrary), where $\log^k$ denotes the k-th iterate of the logarithm.

On the other hand, Eremenko and Lyubich [2] showed that for every parameter λ, the escaping set $I(E_\lambda)$ has zero area, i.e. $\mathcal{H}^{h_2}(I(E_\lambda)) = 0$.

We generalize these two results. Let $\lambda_0 \in (0, 1/e)$ be arbitrary and let β be the unique real repelling fixed point of the function E_{λ_0}. Then it follows by standard arguments that there exists a real-analytic function $\Phi : (a, \infty) \to \mathbb{R}$ which satisfies the functional equation $\Phi(E_{\lambda_0}(x)) = \beta\Phi(x)$. It can be shown that Φ is strictly increasing and unbounded, but grows slower than any fixed iterate of the logarithm, i.e. $\lim_{x\to\infty} \Phi(x)/\log^k(x) = 0$ for all k.

We prove the following theorem:

Theorem 1. *Let $K := \log 2/\log\beta$ and let $h(t) = t^2 g(t)$ be a gauge function.*

(1) *Suppose that*

$$\liminf \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} > K.$$

Then $\mathcal{H}^h(I(E_\lambda)) = \infty$ for every parameter λ.

(2) *Suppose that*

$$\limsup \frac{\log g(t)}{\log \Phi_{\lambda_0}(1/t)} < K.$$

Then $\mathcal{H}^h(I(E_\lambda)) = 0$ for every parameter λ.

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Boundaries of escaping Fatou components

Phil Rippon
(joint work with Gwyneth Stallard)

Let f be a transcendental entire function and denote by f^n, $n = 0, 1, 2, \ldots$, the nth iterate of f. The *Fatou set* $F(f)$ is defined to be the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighborhood of z. The components of $F(f)$ are called *Fatou components*. The complement of $F(f)$ is called the *Julia set* $J(f)$. An introduction to the properties of these sets can be found in [1].

The set $F(f)$ is completely invariant, so for any component U of $F(f)$ there exists, for each $n = 0, 1, 2, \ldots$, a component of $F(f)$, which we call U_n, such that $f^n(U) \subset U_n$. If, for some $p \geq 1$, we have $U_p = U_0 = U$, then we say that U is a periodic component of *period* p, assuming p to be minimal. There are then five possible types of periodic components; see [1]. If U_n is not eventually periodic, then we say that U is a *wandering domain* of f.

The escaping set

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$$

was first studied for a general transcendental entire function f by Eremenko [4]. He proved that

$$I(f) \neq \emptyset, \quad I(f) \cap J(f) \neq \emptyset, \quad \partial I(f) = J(f),$$

and $\overline{I(f)}$ has no bounded components. Eremenko also asked if all the components of $I(f)$ are unbounded, a question that remains open in spite of much work on it and many partial results.

Any Fatou component that meets $I(f)$ must lie in $I(f)$ by normality; we call such components *escaping Fatou components*. This talk is about the relationship between an escaping Fatou component and its boundary.

The *fast escaping set* $A(f)$, introduced by Bergweiler and Hinkkanen in [3], can be defined as follows (see [6]):

$$A(f) = \{z : \text{ there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| \geq M^n(R), \text{ for } n \in \mathbb{N}\},$$

where $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$ and R can be taken to be any value such that $R > \min_{z\in J(f)} |z|$. The set $A(f)$ has some properties similar to those of $I(f)$ (see [3]):

$$A(f) \neq \emptyset, \quad A(f) \cap J(f) \neq \emptyset, \quad \partial A(f) = J(f). \tag{1}$$

But $A(f)$ also has some properties that are much stronger than those of $I(f)$, namely (see [5] and [6]):

(2) all the components of $A(f)$ are unbounded,

and

(3) if U is a Fatou component of f that meets $A(f)$, then $\overline{U} \subset A(f)$.

All multiply connected Fatou components of a transcendental entire function lie in $A(f)$ (see [5]). Bergweiler has constructed a transcendental entire function with both simply and multiply connected Fatou components in $A(f)$ (see [2] and also the talk by Walter Bergweiler at this workshop).

The Fatou function $f(z) = z+1+e^{-z}$ shows that an escaping Fatou component may have many boundary points that are *not* in $I(f)$. It is natural to ask therefore whether every escaping Fatou component must have at least one boundary point in $I(f)$. We have the following partial result.

Theorem 1. *Let f be a transcendental entire function and let U be a wandering domain of f such that $U \subset I(f)$. Then*

$$\partial U \cap I(f) \neq \emptyset.$$

Indeed, the set $\partial U \cap I(f)^c$ has zero harmonic measure relative to U.

It remains open whether $\partial U \cap I(f) \neq \emptyset$ when U is a *periodic* escaping Fatou component; that is, a *Baker domain* of f.

The proof of Theorem 1 uses harmonic majorisation and Löwner's lemma. This theorem has several corollaries, including the following.

Corollary 1. *Let f be a transcendental entire function and let G be a bounded simply connected domain. If $I(f) \cap G \neq \emptyset$, then $I(f) \cap \partial G \neq \emptyset$.*

In addition to Theorem 1, the proof of Corollary 1 uses the facts that $J(f) = \partial A(f)$ and that all components of $A(f)$ are unbounded.

Lasse Rempe has pointed out that Corollary 1 implies that for any transcendental entire function f the set $I(f) \cup \{\infty\}$ is connected.

Corollary 2. *Let f be a transcendental entire function and suppose there is a set $E \subset I(f)$ such that $\overline{E} = J(f)$ and E is contained in the union of a finite number of components of $I(f)$. Then all the components of $I(f)$ are unbounded.*

More precisely,

(a) *there is a single component I_1 say, of $I(f)$ that contains $I(f) \cap J(f)$ and also contains any escaping wandering domains and completely invariant Baker domains of f,*
(b) *any other component of $I(f)$ is either a Baker domain of f or one of its infinitely many preimage components.*

The proof of Corollary 2 uses Theorem 1, the blowing-up property of $J(f)$, and the fact that the boundary of any completely invariant Fatou component of f is equal to $J(f)$.

Some particular cases of Corollary 2 are (a) that if $I(f)$ has only finitely many components, then $I(f)$ is connected, and (b) if $A(f)$ is contained in a finite union of components of $I(f)$, then all the components of $I(f)$ are unbounded.

In the opposite direction to Theorem 1, it is natural to ask whether a Fatou component must be escaping if a large enough subset of its boundary is escaping. Functions of the form $f(z) = \lambda e^z$, $0 < \lambda < 1/e$, show that it is not sufficient to require that $\partial U \cap I(f)$ has Hausdorff dimension 2, since these functions have a completely invariant attracting basin U while the Hausdorff dimension of $\partial U \cap I(f)$ is 2. We have the following result.

Theorem 2. *Let f be a transcendental entire function and let U be a Fatou component of f.*

(a) *If $\partial U \cap I(f)$ has positive harmonic measure relative to U, then $U \subset I(f)$.*
(b) *If $\partial U \cap A(f)$ has positive harmonic measure relative to U, then $U \subset A(f)$.*

The proof of Theorem 2 uses harmonic majorisation and Egorov's theorem.

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Multiply connected Fatou components of meromorphic functions

Gwyneth Stallard
(joint work with Walter Bergweiler and Phil Rippon)

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function, and denote by f^n, $n = 0, 1, 2, \ldots$, the nth iterate of f. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}$ such that $(f^n)_{n\in\mathbb{N}}$ is well-defined and forms a normal family in some neighborhood of z. The complement of $F(f)$ is called the *Julia set* $J(f)$ of f. An introduction to the properties of these sets can be found in [4].

The set $F(f)$ is completely invariant under f in the sense that $z \in F(f)$ if and only if $f(z) \in F(f)$ whenever $f(z)$ is defined. Therefore, if U is a component of $F(f)$, then there exists, for each $n = 0, 1, 2, \ldots$, a component of $F(f)$, which we call U_n, such that $f^n(U) \subset U_n$. If, for some $p \geq 1$, we have $U_p = U_0 = U$, then we say that U is a *periodic* component of *period* p, assuming p to be minimal. If U_n is not eventually periodic, then we say that U is a *wandering domain* of f.

The first example of a transcendental entire function with a wandering domain was given by Baker in [1]. This function has a multiply connected wandering domain whose forward orbit of Fatou components is nested and tends to ∞. Indeed, it follows from a result of Baker [2, Theorem 3.2] that if f is any transcendental entire function with a multiply connected Fatou component U, then U is a wandering domain such that

(a) each U_n is bounded,
(b) there exists $N \in \mathbb{N}$ such that U_{n+1} surrounds U_n for $n \geq N$,
(c) $U_n \to \infty$ as $n \to \infty$.

If f is meromorphic, then there are many different types of multiply connected Fatou components. We say that U is a *nested wandering domain* if U is a wandering domain and the associated components U_n have properties (a), (b) and (c). Note that such Fatou components belong to the escaping set

$$I(f) = \{z : f^n(z) \text{ is defined for } n \in \mathbb{N} \text{ and } f^n(z) \to \infty \text{ as } n \to \infty\}.$$

The main purpose of the talk is to discuss the possible sizes of such wandering domains.

It was shown by Zheng [7, Theorem] that, if f is a meromorphic function with finitely many poles, U is a nested wandering domain, and A is a domain in U such that each $f^n(A)$, $n \in \mathbb{N}$, contains a closed curve that is not null-homotopic in U_n, then there exists $N \in \mathbb{N}$ such that $f^n(A) \supset A(r_n, R_n)$, for $n \geq N$, where $R_n/r_n \to \infty$ as $n \to \infty$. Here, $A(r, R) = \{z : r < |z| < R\}$, $0 < r < R$.

Zheng's result actually holds for any meromorphic function f with a *direct tract* D; that is, D is an unbounded domain such that f is analytic in $\overline{D}$ and, for some $R > 0$, $|f(z)| = R$ for $z \in \partial D$ and $|f(z)| > R$ for $z \in D$. Note that any transcendental meromorphic function with a finite number of poles has at least one direct tract. However, a transcendental meromorphic function with infinitely many poles may or may not have a direct tract; see [5] for properties of functions with direct tracts and examples.

We prove the following result which shows that forward images of a nested wandering domain actually contain much larger annuli than those given by Zheng's result. Further we show how these large annuli relate to each other.

Theorem 1. *Let f be a transcendental meromorphic function with a direct tract and a nested wandering domain U, and let A be a domain in U such that each $f^n(A)$, $n \in \mathbb{N}$, contains a closed curve that is not null-homotopic in U_n. Then there exist $c > 1$, $N \in \mathbb{N}$ and a positive sequence $(r_n)_{n \geq N}$ such that $r_n \to \infty$ as $n \to \infty$, and*

$$f^n(A) \supset A(r_n, r_n^c), \quad \text{for } n \geq N. \tag{1}$$

In addition, we can choose c, N and $(r_n)_{n \geq N}$ so that

$$f^n(A(r_N, r_N^c)) \supset A(r_{n+N}, r_{n+N}^c) \supset f^n(A(r_N^a, r_N^b)), \quad \text{for } n \in \mathbb{N}, \tag{2}$$

for some a, b, with $1 < a < b < c$.

Theorem 1 is false without the hypothesis that f has a direct tract. In [3] Baker, Kotus and Lü used techniques from approximation theory to construct several examples of meromorphic functions, each with infinitely many poles, having multiply connected wandering domains of various types. As pointed out in [6], their method can be used to construct a transcendental meromorphic function with a doubly connected nested wandering domain U such that $U_n \subset A(r_n, Cr_n)$, for $n \in \mathbb{N}$, where $C > 1$ is an absolute constant.

It might be hoped that the conclusion of Theorem 1 could be strengthened by replacing the constant c with constants c_n with $c_n \to \infty$ as $n \to \infty$, since all existing examples of nested wandering domains have this property. We give a new example which shows that this is not possible.

More precisely, we let $c \in (1, \infty)$ and consider the transcendental entire function defined by

$$f(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z}{r_n^c}\right)\left(1 - \frac{z}{r_{n^2}}\right), \tag{3}$$

where $r_{n+1} = r_n^{n+[\sqrt{n}]}$.

We show that there exist sequences (a_n), (b_n) with $1 \leq a_n \leq b_n \leq c$, for $n \in \mathbb{N}$, such that

$$f(A_n) \subset A_{n+1}, \quad \text{where } A_n = A(r_n^{a_n}, r_n^{b_n}). \tag{4}$$

This is sufficient to show that there exists a multiply connected Fatou component U_1 such that $A_1 \subset U_1$. Because of the positions of the zeros (which do not belong to the Fatou components U_n such that $A_n \subset U_n$), we see that, for $n \in \mathbb{N}$, the largest annulus in U_{n^2} that contains A_{n^2} must be contained in $A(r_{n^2}, r_{n^2}^c)$.

Moreover, in this example, for any $c' > c$ there are only finitely many n such that U_n contains some annulus of the form $A(s_n, s_n^{c'})$, where $s_n > 0$.

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Some unsolved problems in holomorphic dynamics

Alexandre Eremenko

Problem 1. (A. Eremenko)

In the very last paragraph of [2] Fatou writes:

"In nous resterait à étudier les courbes analytiques invariantes par une transformation rationnelle et dont l'étude est intimement liée á celle des fonctions étudiées dans ce Chapitre. Nous espérons y revenir bientôt.

As far as I know, Fatou never returned to this question in his published work.

Which Jordan analytic curves γ in the Riemann sphere can be invariant under rational functions? Of course, γ can be a circle, and it is easy to describe all rational functions which leave a given circle invariant. I know only two classes of examples where γ is not a circle.

1. Let D be a rotation domain (a Siegel disc or an Herman ring) of a rational function f and ϕ the linearizing map from D onto a disc or onto a round ring centered at the origin. Then the ϕ-preimage of any circle centered at the origin is an analytic invariant curve.

If f is a polynomial then the only possible invariant analytic curves are either circles of preimages of circles under the linearizing function of a Siegel disc [1].

2. Let f be a Lattés function satisfying

$$\wp(2z) = f \circ \wp(z),$$

where $\wp$ is the Weierstrass function with periods 1 and i. Let L be the line $\operatorname{Re} z = 1/3$. It is easy to see that $\wp(L)$ is an analytic Jordan curve. As $2L \equiv -L$ we conclude that $\wp(L)$ is invariant under f.

Are there any other examples, not coming from the rotation domains or Lattés functions?

The question is related to factorization theory of rational and meromorphic functions, namely to classifying all triples of rational functions that satisfy the equation $f \circ h = h \circ g$.

About this equation, the question is the following: Suppose that f has an invariant circle C. Is it true that $h(C)$ is contained in a circle, unless f and g are Lattés examples?

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Problem 2. (W. Bergweiler)

This is related to the previous problem. Let f be a rational function and γ an analytic f-periodic Jordan curve such that the restriction of some iterate f^n on γ is a homeomorphism with an irrational rotation number and such that γ is not in the closure of a Herman ring. It is natural to call such γ a degenerate Herman ring.

1. Do there exist degenerate Herman rings other than circles?

2. Can one estimate the number of degenerate Hermann rings in terms of the degree of f ?

Problem 3. (W. Bergweiler).

Let f be a transcendental entire function having a completely invariant component D of the Fatou set. Baker [1] proved that all critical values of f must be in D. Eremenko and Lyubich [4] proved that all logarithmic singularities of f must be in D. Bergweiler and Eremenko [3] proved that an omitted value of f (if it exists) must be in D. On the other hand, Bergweiler [2] constructed an example of an entire function f having a completely invariant domain D and an asymptotic value a such that $a \notin D$.

The following question remains unsolved. Is it true that every *locally omitted value* of an entire function belongs to D ? We say that a complex number a is locally omitted by an entire function f if there exists a component G of the set

$$\{z \in \mathbf{C} : |f(z) - a| < \epsilon\}$$

such that $f(z) \neq a$ for $z \in G$.

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Problem 4. (W. Bergweiler)

Let f be an entire function and D an immediate basin of an attracting fixed point. Suppose that D contains all critical and asymptotic values of f. Does it follow that D is completely invariant ?

A similar question can be asked in non-dynamical context. Let f be an entire function and suppose that some region D contains all critical and asymptotic values of f. Does it follow that the full preimage $f^{-1}(D)$ is connected ?

Problem 5. (Xavier Jarque)

Conjecture. Let $f(z) = exp(z) + \kappa$ be an exponential map such that all periodic points of f are repelling. Then the escaping set $I(f)$ is connected.

Open problem. Let f be a transcendental entire function for which the postsingular set is finite and contained in the Julia set. (In particular, f belongs to the class $\mathcal{S}$). Under which hypotheses is the escaping set $I(f)$ connected?

Remark. Under the additional hypothesis that the critical points of f have uniformly bounded degree and there are no asymptotic values, Mihaljevic-Brandt has shown that $I(f)$ is disconnected. On the other hand, Jarque has shown that when f is an exponential map (i.e., if there is a single asymptotic value and no critical values), then the escaping set is connected.

Problem 6. (L. Rempe)

Let $p : \mathbf{R}^2 \to \mathbf{R}^2$ be a K-quasiregular map of polynomial type, such that $K < \deg p$. Can p have a wandering component of Fatou set ?

Problem 7. (L. Rempe)

Adam Epstein introduced the concept of *Ahlfors islands maps* as a generalization of the dynamics of entire and meromorphic functions. Let X be a compact Riemann surface and let W be a nonempty open subset of a compact Riemann surface Y. Roughly speaking, a nonconstant analytic map $f : W \to X$ is called an Ahlfors islands map (AIM) if it satisfies a version of the Ahlfors islands theorem near every point of the boundary of W. An AIM is called *elementary* if f is not a conformal isomorphism.

When $X = Y$, one can consider f as a dynamical system, and all basic results of one-dimensional complex dynamics (such as the density of repelling periodic points in the Julia set) hold in this setting. For more background and precise definitions, see the manuscript "Iteration of Ahlfors and Picard functions which overflow their domains" by Epstein and Oudkerk, or the article "Hyperbolic dimension and radial Julia sets of transcendental functions", Proc. Amer. Math. Soc. 137 (2009), 1411-1420.

In the case where X is the Riemann sphere or a torus, one can construct AIM on any prescribed domain W, and these can be chosen (if $X = Y$) to have Baker and/or wandering domains. ("Exotic Baker and wandering domains of Ahlfors islands functions" by Rempe and Rippon, in preparation.) However, there are only few known constructions of AIM on hyperbolic surfaces.

Question 1 (Epstein): Let X and Y be compact Riemann surfaces, let $W \subset Y$ be a domain and suppose that X is hyperbolic. When does there exist an Ahlfors islands map $f : W \to X$? Does the answer depend on the Riemann surface X?

Remark. Note that such maps cannot exist for *every* W; for example, W cannot have isolated boundary points.

Question 2: Construct an AIM $f : W \to X$ with X hyperbolic and $W \subset X$ such that f has a Baker domain or a wandering domain.

Problem 8. (P. Rippon)

It is known that the escaping set $I(f)$ of every transcendental entire function f is of the type $F_{\sigma\delta}$. Moreover, $I(f)$ is never open, never closed and never G_δ. Can $I(f)$ be F_σ ?

Problem 9. (P. Rippon and G. Stallard)

One way to prove that the escaping set of a transcendental entire function is connected is to show that there is a *loop* in $I(f)$; that is, for some bounded domain G we have $\partial G \subset I(f)$ and $G \cap J(f) \neq \emptyset$; see [1, Theorem 2].

(a) The Fatou function $f(z) = z + 1 + e^{-z}$ has a connected escaping set, because it has a Baker domain whose closure is $\mathbb{C}$. But is there a loop in $I(f)$ in this case?

(b) This is related to the question: is there a function with a loop in $I(f)$ that is not a loop in the fast escaping set $A(f)$? (All known examples of loops in $I(f)$ are in $A(f)$.)

Reference

[1] P.J. Rippon and G.M. Stallard, Escaping points of entire functions of small growth, *Math. Z.*, 261 (2009), 557–570.

Problem 10. (M. Urbański)

Suppose that $f : \mathbf{C} \to \mathbf{C}$ is an elliptic function whose all critical points are attracted either by attracting periodic points or rationally indifferent periodic (parabolic) points. Assume also that at least one parabolic exists. Let h be the Hausdorff dimension of the Julia set $J(f)$ of f. Then (*J. Kotus, M. Urbanński " Geometry and ergodic theory of non-recurrent elliptic functions", J. d'Analyse Math. 93 (2004), 35-102*) the h-dimensional Hausdorff measure of $J(f)$ vanishes while packing measure is locally infinite at every point of $J(f)$. However, it was also proved in the same paper that an atomless h-conformal measure m on $J(f)$ exists. The problem is whether there exists a gauge function g such that the conformal measure m is equal to the Hausdorff measure H_g on $J(f)$.

Problem 11. (M. Urbański)

Let $f(z) = \lambda e^z$ be a hyperbolic exponential function. Let h be the Hausdorff dimension of the radial Julia set. It was proved in [*M. Urbański, A. Zdunik, " The finer geometry and dynamics of exponential family" Michigan Math. J. 51 (2003), 227-250*] that the Hausdorff measure H_h of every horizontal strip of finite width is finite. The question is whether this measure is extremal in the sense of Kleinbock, Lindenstrauss and Barak Weiss. Extremality of a measure means that the very well approximable points form a set of measure zero.

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