Mathematisches Forschungsinstitut Oberwolfach

Report No. 02/2010

DOI: 10.4171/OWR/2010/02

Moduli Spaces in Algebraic Geometry

Organised by
Dan Abramovich (Providence)
Gavril Farkas (Berlin)
Stefan Kebekus (Freiburg)

January 10th – January 16th, 2010

Abstract. The workshop on Moduli Spaces in Algebraic Geometry aimed to bring together researchers from all branches of moduli theory, in order to discuss moduli spaces from different points of view, and to give an overview of methods used in their respective fields. Highlights included a complete proof of Göttsche's conjecture, a proof of rationality of a moduli space constructed via GIT quotient using reduction modulo p, and a proof of a conjecture of Looijenga using the ideas of mirror symmetry.

Mathematics Subject Classification (2000): 14D22.

Introduction by the Organisers

The workshop *Moduli Spaces in Algebraic Geometry*, organised by Dan Abramovich (Brown), Gavril Farkas (HU Berlin), and Stefan Kebekus (Freiburg) was held January 10–14, 2010 and was attended by 52 participants from around the world. The participants ranged from senior leaders in the field to young post-doctoral fellows and even a few PhD students; their range of expertise covered areas from classical algebraic geometry to motivic Hall algebras.

Being central to a number of mathematical disciplines, moduli spaces are studied from many points of view, using a wide array of methods. Major progress has been achieved in virtually every branch of the field, and well-known questions have been answered lately. The workshop brought together researchers working on different aspects of moduli theory, to report on progress, discuss open problems, give overview, and in order to exchange methods and ideas. Lecture topics were chosen to cover many of the subject's disparate aspects, and most lectures were followed by lively discussions among participants, at times continuing well into the night.

For a flavor of the wide palate of subjects covered, a few of the talks are highlighted below.

Characteristic classes on surfaces. Proof of Göttsche's conjecture. Jun Li (Stanford) lectured on the solution of Göttsche's conjecture, obtained by his student Yu-jong Tzeng. Although the result was announced a few months ago, his talk in this workshop was the first time a complete proof was presented in a public lecture in Europe.

Given an algebraic surface X and a suitably general m-dimensional linear system V on X, the problem of counting the number of m-nodal elements of V can be traced back to the 19th century. Göttsche's conjecture predicts that, in a suitable range, this number is a universal function of four characteristic classes of X and V. Göttsche reduced his conjecture to a statement on intersection numbers on $Hilb(X)$. The key ideas in the proof of Tzeng are (a) a spectacular generalization of the work of Levine and Pandharipande on generators of the cobordism group of pairs (X, L) of a surface with line bundle, and (b) an equally spectacular proof of a degeneration formula showing that Göttsche's intersection numbers are cobordism invariants.

Understanding deformations using mirror symmetry. Paul Hacking (Amherst) presented a solution to a 28-year-old conjecture of Looijenga, using ideas that originate from mirror symmetry.

A surface cusp singularity has a cycle of rational curves as its exceptional configuration. In 1981 Looijenga conjectured that a cusp singularity is smoothable if and only if the exceptional set of the *dual* cusp lies on a rational surface as an anticanonical divisor. Gross, Hacking and Keel later recognized the appearance of the configuration on a rational surface as part of a construction coming from mirror symmetry. In this setting, mirror symmetry works perfectly: counting rational curves on the mirror dual, one obtains an explicit deformation of a given cusp. Looijenga's conjecture follows.

Topology of moduli spaces and their relative connectivity. Eduard Looijenga (Utrecht) discussed topological properties of moduli spaces, presenting a result of a very classical flavor. He reported on joint work with W. van der Kallen, proving the vanishing of the relative homology groups $H_k(\mathcal{A}_g, \mathcal{A}_{g,dec}; \mathbb{Q})$ for $k \leq g-2$, where $\mathcal{A}_g$ is the moduli space of principally polarized abelian varieties and $\mathcal{A}_{g,dec}$ is the locus of decomposable ones. The proof goes by a sequence of beautiful reductions, proving in particular that the corresponding decomposable locus on the Siegel space is homotopy equivalent to a bouquet of $(g-2)$-spheres, which in itself is reduced to a completely combinatorial problem.

An analogous result holds for moduli of curves of compact type: we have $H_k(\mathcal{M}_g^c, \Delta_g^c; \mathbb{Q}) = 0$ for $k \leq g-2$. Here, the result follows from a combinatorial discussion of the separating curve complex, which is shown to be $(g-3)$-connected.

Using computer algebra to prove rationality. Christian Böhning (Göttingen) reported on his joint work with H.-C. von Bothmer on rationality of the space of plane curves of sufficiently high degree $d \geq 0$. The starting point of the proof is rather classical and uses the Aronhold method of covariants which gives a map from the space of degree d plane curves to that of quartic curves. In order to show that a general fibre of this map is a vector bundle over a rational base, a certain matrix having entries polynomials in d, must have full rank. To achieve this, the authors introduce innovative techniques that rely on reduction to characteristic p and a computer check of the corresponding statement over a finite field. By semicontinuity, then rationality follows in characteristic 0 as well!

Tautological rings of the moduli space of curves. Carel Faber (Stockholm) gave the inaugural talk of the workshop and discussed developments about certain subrings of the cohomology of the moduli space M_g of curves of genus g. Around 1993, Faber formulated an amazing conjecture predicting that the tautological ring of the $(3g-3)$-dimensional moduli space M_g enjoys all the properties (vanishing, perfect pairing), of a smooth compact complex manifold of dimension $g-2$. Faber's Conjecture generated a great deal of interest in the last few years, and significant parts of it (vanishing, top degree predictions) have been confirmed. However, the part predicting the existence of a perfect pairing between complementary tautological rings has been more resistant to proofs. Quite surprisingly, it turns out that the Faber-Zagier method of producing enough tautological relations to verify this part of the conjecture, stops working exactly in genus 24! The occurrence of this genus in relation to Faber's Conjecture has caused quite a stir, especially since this is also the range when M_g starts to become a variety of general type.

Workshop: Moduli Spaces in Algebraic Geometry

Table of Contents

Abstracts

New developments regarding the tautological ring of the moduli space of curves

Carel Faber

The tautological rings $R^\bullet(\overline{M}_{g,n})$ are defined as the minimal system of $\mathbb{Q}$-subalgebras of the rational Chow rings closed under push-forward for the maps forgetting a marked point and the standard gluing maps. They contain the well-known ψ-, κ-, and λ-classes. For the partial compactifications $M_{g,n}^{ct}$ and (for $g \geq 2$) $M_{g,n}^{rt}$ (the moduli spaces of curves *of compact type* and *with rational tails*, respectively), the tautological rings are defined by restriction. The *Gorenstein conjectures* say that these rings are Gorenstein, with socle in degree $3g-3+n$ resp. $2g-3+n$ resp. $g-2+n$ for $\overline{M}_{g,n}$ resp. $M_{g,n}^{ct}$ resp. $M_{g,n}^{rt}$. Perhaps it is fair to say that the main evidence consists of the following results:

(1) The conjectures are true for $\overline{M}_{0,n}$ (Keel) and M_g , $2 \leq g \leq 23$ (Faber).
(2) In general, it is known that the degrees given above are the top degrees and that the top graded pieces are one-dimensional (Looijenga, Looijenga-Faber, Graber-Vakil and Faber-Pandharipande).
(3) The conjectures hold for $\overline{M}_{1,4}$, $\overline{M}_{2,3}$ and M_4^{ct}, each case requiring a new tautological relation (Getzler-Pandharipande, Belorousski-Pandharipande, Faber-Pandharipande).

For all these tautological rings, the product pairing into the top graded piece is completely determined by the integrals on $\overline{M}_{g,n}$ of monomials in the ψ-classes against 1 resp. λ_g resp. $\lambda_g\lambda_{g-1}$. These integrals are known, with several proofs for each set of integrals; it all begins with Witten and Kontsevich, of course. E.g., for the $\lambda_g\lambda_{g-1}$-integrals, the result follows from Givental's proof of the Virasoro conjecture (Eguchi-Hori-Xiong, S. Katz) for $\mathbb{P}^2$, as Getzler and Pandharipande observed; more direct proofs were given by Liu-Xu and (very recently) by Buryak-Shadrin (earlier, Goulden-Jackson-Vakil had obtained partial results for arbitrary g). Hence the Gorenstein quotients can be studied; Stephanie Yang has implemented an intersection number algorithm for arbitrary tautological classes (following Graber-Pandharipande) and has determined the dimensions of the graded pieces for approximately 70 of these rings.

In the first part of the talk, I reported on the following recent result of my student Mehdi Tavakol:

Theorem 1. (Tavakol.) *$R^\bullet(M_{1,n}^{ct})$ is Gorenstein with socle in degree $n-1$.*

Tavakol shows that $R^\bullet(M_{1,n}^{ct})$ is closely related to a naturally defined tautological ring $R^\bullet(C^{n-1})$, where (C, O) is a fixed elliptic curve; in fact, it is isomorphic to the tautological ring of $\overline{U}_{n-1}$, the Fulton-MacPherson compactification of $U_{n-1} = (C-O)^{n-1}$, where all n points stay apart. Particularly interesting is a certain block triangular structure found by Tavakol and the fact that some innocent-looking relations (that need to hold to obtain the Gorenstein property) are derived from the codimension 2 relation for $\overline{M}_{1,4}$ found by Getzler and shown to be algebraic by Pandharipande.

More precisely, the tautological ring $R^\bullet(C^n)$ (with C now a curve of genus g) is defined as the $\mathbb{Q}$-subalgebra of the rational Chow ring generated by the classes K_i (the pull-back of the canonical class via projection onto the ith factor) and $D_{i,j}$ (of the (i,j)th diagonal). Its image in cohomology (with the algebraic degree) is denoted $RH^\bullet(C^n)$. Approximately 10 years ago, Pandharipande and I determined the rings $RH^\bullet(C^n)$ completely. Denote by a_i the pull-back of the class of a point via projection onto the ith factor and write $b_{i,j}$ for $D_{i,j} - a_i - a_j$. Trivially, $a_i^2 = 0$ and $a_i b_{i,j} = 0$, and it is easy to see that $b_{i,j}^2 = -2g a_i a_j$ and $b_{i,j} b_{i,k} = a_i b_{j,k}$. Therefore, $RH^\bullet(C^n)$ is additively generated by the monomials with non-overlapping index sets. (For $g = 1$, include the a_i in $RH^\bullet(C^n)$). It is easy to check that there are equally many such monomials in degrees adding up to n. We analyzed the pairing and found that its nullspace is governed by pull-backs of the 'master relation'

$$b_{1,2}\, b_{3,4} \cdots b_{2g+1,2g+2} + \cdots = 0\,,$$

where the $(2g+1)!!$ terms in the relation correspond to the fixed point free involutions of $\{1,\ldots,2g+2\}$. (As we found out later, this result was obtained independently by Hanlon and Wales.) The master relation holds in cohomology and as a result, $RH^\bullet(C^n)$ is Gorenstein with socle in degree n. (To the embarrassment of the author, these results are not yet written up.)

In $R^\bullet(C^n)$, one can in general not replace K_i by $2g-2$ times the class of a point pulled back to the ith factor. Clearly, $R^\bullet(C^2)$ is Gorenstein if and only if the relation $K_1K_2 = (2g-2)K_1D_{1,2}$ holds. This can be shown for $g \le 3$, but Green and Griffiths proved that this relation doesn't hold for a generic complex curve if $g \ge 4$.

For an elliptic curve (C,O), one defines $a_i \in R^\bullet(C^n)$ as the pull-back of the class of O to the ith factor. Tavakol shows that the relation $b_{1,2}b_{1,3} = a_1b_{2,3}$ is obtained by restricting Getzler's relation for $\overline{M}_{1,4}$ to the fiber over $[C,O] \in M_{1,1}$ and that the master relation $b_{1,2}b_{3,4} + b_{1,3}b_{2,4} + b_{1,4}b_{2,3} = 0$ is obtained similarly from the pull-back of Getzler's relation to $\overline{M}_{1,5}$. Hence $R^\bullet(C^n)$ is Gorenstein in genus 1. The blocks referred to above correspond to copies of $R^\bullet(C^k)$ for various k. The nearly triangular structure is obtained by a careful analysis of the relations between the monomials in the a_i, the $b_{i,j}$, and the classes of the many exceptional divisors in $\overline{U}_{n-1}$. Tavakol first proves that $R^\bullet(\overline{U}_{n-1})$ is Gorenstein and then uses this to show that the natural map from $R^\bullet(M_{1,n}^{ct})$ onto this ring is an isomorphism.

In the second part of the talk, I reported on some recent calculations, done jointly with Pandharipande, in the tautological ring $R^\bullet(M_g)$, which is multiplicatively generated by $\kappa_1, \ldots, \kappa_{g-2}$ and has top degree $g-2$. (In fact, the first $\lfloor g/3 \rfloor$ kappa's suffice, as Morita and Ionel proved.) Some years ago, Zagier and I carefully studied the Gorenstein quotient of this ring and we obtained the following result. Let

$$\mathbf{p} = \{p_1, p_3, p_4, p_6, p_7, p_9, p_{10}, \ldots\}$$

be a collection of variables indexed by the positive integers not congruent to 2 modulo 3. Let $\Psi(t, \mathbf{p})$ be the following formal power series:

$$\Psi(t,\mathbf{p}) = \sum_{i=0}^{\infty} t^i p_{3i} \sum_{j=0}^{\infty} \frac{(6j)!}{(3j)!(2j)!} t^j + \sum_{i=0}^{\infty} t^i p_{3i+1} \sum_{j=0}^{\infty} \frac{(6j)!}{(3j)!(2j)!} \frac{6j+1}{6j-1} t^j ,$$

where $p_0 := 1$. Define rational numbers $C_r(\sigma)$, for σ any partition (of $|\sigma|$) with parts not congruent to 2 modulo 3, by the formula

$$\log(\Psi(t,\mathbf{p})) = \sum_{\sigma} \sum_{r=0}^{\infty} C_r(\sigma) t^r \mathbf{p}^{\sigma} ,$$

where $\mathbf{p}^\sigma$ denotes the monomial $p_1^{a_1} p_3^{a_3} p_4^{a_4} \ldots$ if σ is the partition $[1^{a_1} 3^{a_3} 4^{a_4} \ldots]$. Define

$$\gamma := \sum_{\sigma} \sum_{r=0}^{\infty} C_r(\sigma) \kappa_r t^r \mathbf{p}^{\sigma} ;$$

then the relation

$$[\exp(-\gamma)]_{t^r\,\mathbf{p}^\sigma} = 0$$

holds in the Gorenstein quotient when $g - 1 + |\sigma| < 3r$ and $g \equiv r + |\sigma| + 1 \pmod 2$. (Of course, $\kappa_0 = 2g - 2$.) Let me call these relations (in the Gorenstein quotient) the FZ-relations for brevity. Observe that this gives the expected number of relations in every codimension less than or equal to $\lfloor (g-2)/2 \rfloor$, although we didn't prove that the obtained relations in such a codimension are independent. Our goal was precisely to understand the relations 'until the middle'; it was clear to Zagier and me that these relations could never suffice for g large enough.

The recent calculations with Pandharipande revealed first of all that the FZ-relations give *all* relations whenever $g \leq 23$. However, for $g = 24$, one relation is missing, in codimension 12 (there is a quite unexpected syzygy). Further computations for higher genera revealed a few more such cases.

As to actual relations in $R^\bullet(M_g)$, recall the 'diagonal' relations introduced in my paper on the Gorenstein conjecture for $R^\bullet(M_g)$: the vanishing Chern class $c_g(\mathbb{F}_{2g-1} - \mathbb{E})$ is cut with some diagonals in the $(2g-1)$st fiber product of the universal curve over M_g; the push-down to M_g gives a tautological relation (the λ-classes are expressed in the κ-classes via Mumford's formula). The methods for actually computing such relations are by now quite good, and we find that the diagonal relations give all relations for $g \leq 23$. But for $g = 24$ and codimension 12 these relations don't seem to give a Gorenstein quotient. (Even with the current methods, it is difficult to compute all diagonal relations in this case.) For higher genus there seem to be a few more cases of similar nature. In fact, at this moment it is reasonable to think that the diagonal relations give exactly the same result as the FZ-relations. Whether the Gorenstein conjecture for $R^\bullet(M_{24})$ fails or not, is still open; there are many other geometric relations that have not yet been computed in this case. If one wishes to disprove the conjecture, one should probably look for a cycle of codimension 10 in M_{24} that is not tautological, but on which the product of $\lambda_{24}\lambda_{23}$ and an arbitrary κ-class of degree 12 can be evaluated.

Spherical objects and rational points

Daniel Huybrechts

In my talk I discussed the following special case of a conjecture of Bloch:

Conjecture 1. *Let X be a smooth projective K3 surface and $f : X \xrightarrow{\sim} X$ an automorphism which acts as the identity on $H^2(X, \mathcal{O}_X)$ (f is symplectic). Then f acts as the identity on the kernel of the cycle map* $\mathrm{CH}^*(X) \to H^*(X, \mathbb{Z})$.

The conjecture is still open even for symplectic involutions, but for non-generic Picard group derived techniques can be used to prove it under additional assumptions on the Picard group.

The structure of the Chow ring $\mathrm{CH}^*(X)$ of a K3 surface is very rich. Due to a result of Mumford, it is known to be of infinite dimension when X is a K3 surface over $\mathbb{C}$. More recently, Beauville and Voisin studied a natural subring $R(X) \subset \mathrm{CH}^*(X)$ on which the cycle map is injective. In contrast, the Bloch–Beilinson conjectures predict that the cycle map is injective for K3 surfaces over a number field. One way of attacking the latter conjecture would be via Bogomolov's 'logical possibility' suggesting that any rational point of a K3 surface over a number field might be contained in a (singular) rational curve. Another way of studying this question would be via the derived category of coherent sheaves $\mathrm{D^b}(X)$. Spherical objects play a central role (eg. their associated spherical twists generate the interesting part of the group of autoequivalences) and one might wonder whether over number fields, they generate $\mathrm{D^b}(X)$. These two logical possibilities seem related, but I am not able to make this precise.

Autoequivalences of derived categories can also be used to approach the original question on the action of symplectic automorphisms. Roughly, the idea is that under additional conditions symplectic automorphisms are contained in a bigger group that is generated by autoequivalences of $\mathrm{D^b}(X)$ whose action on $\mathrm{CH}^*(X)$ can be controlled.

The main results are:

Proposition 2. *Let X ba a smooth projective K3 surface of Picard rank at least two and let $E \in \mathrm{D^b}(X)$ be a spherical object. Then the Mukai vector $v(E)$ is contained in the Beauville–Voisin subring $R(X) \subset \mathrm{CH}^*(X)$ and the associated spherical twist acts as the identity on the homologically trivial part of* $\mathrm{CH}^2(X)$.

This result relies on techniques of Lazarsfeld showing that curves on K3 surfaces are Brill–Noether general.

The following result uses deformation theory and non-projective K3 surfaces as studied in [4]

Proposition 3. *If two autoequivalences Φ_1, Φ_2 of $\mathrm{D^b}(X)$ induce the same action on cohomology, their action on $\mathrm{CH}^*(X)$ coincides as well.*

The next result is a consequence of a result of Kneser:

Proposition 4. *If the* 2*-rank and* 3*-rank of* Pic(X) *is at least four resp. three, then for a symplectomorphism f the induced action f^* on $H^*(X,\mathbb{Z})$ is contained in the subgroup generated by reflection associated to algebraic* (-2)*-classes.*

Since reflections of the above type can be lifted to spherical twists, one obtains

Corollary 5. *Under the above assumptions on the* 2*- and* 3*-rank of* Pic(X) *one can show that any symplectic automorphism of X acts as the identity on* $\mathrm{CH}^2(X)$.

The result applies to many concrete examples (eg. Fermat quartics), but does not settle the general case (not even for symplectic involutions). Indeed the generic K3 surface endowed with a symplectic involution has Picard group (up to index two) of the form $\mathbb{Z} \oplus E_8(-2)$ whose 2-rank is zero (see [1]).

References

[1] B. van Geemen, A. Sarti *Nikulin involutions on K3 surfaces*, Math. Z. 255 (2007), 731–753.
[2] D. Huybrechts *Chow groups of K3 surfaces and spherical objects*, J. EMS. to appear.
[3] D. Huybrechts *Chow groups and derived categories of K3 surfaces*, submitted to Proc. Conf. Classical Alg. Geom. today. MSRI 2009.
[4] D. Huybrechts, E. Macrì, P. Stellari *Derived equivalences of K3 surfaces and orientation*, Duke Math. J. 149 (2009), 461–507.

Hall algebras and Donaldson-Thomas invariants

Tom Bridgeland

In the forthcoming paper [2] we use Joyce's theory of motivic Hall algebras to prove some basic properties of Donaldson-Thomas (DT) curve-counting invariants on Calabi-Yau threefolds. We prove that the reduced DT invariants coincide with the stable pair invariants introduced by Pandharipande and Thomas [9]

$$\mathrm{PT}_\beta(q) = \mathrm{DT}_\beta(q)/\,\mathrm{DT}_0(q),$$

and that the generating functions for these invariants are Laurent expansions of rational functions in q, invariant under the transformation $q \leftrightarrow q^{-1}$. Similar results have been obtained by Toda [10, 11].

The proof we give of these results is based on Joyce's theory of motivic Hall algebras [3, 4, 5, 6, 7, 8]. In this talk we explained some of this technology. In particular we defined the motivic Hall algebra $\mathrm{H}(M)$ of coherent sheaves on a complex variety M, and a certain subalgebra

$$\mathrm{H}_{\mathrm{reg}}(M) \subset \mathrm{H}(M)$$

of regular elements, having the structure of a Poisson algebra. In the case that M is a Calabi-Yau threefold we then constructed a Poisson algebra homomorphism

$$I\colon\ \mathrm{H}_{\mathrm{reg}}(M) \to \mathbb{C}[T]$$

to the ring of functions on an algebraic torus T equipped with a symplectic form. This material will be explained in detail in [1].

The results on Donaldson-Thomas invariants of [2] are obtained by first translating certain natural categorical statements (e.g. existence and uniqueness of Harder-Narasimhan filtrations) into identities in the motivic Hall algebra, and then applying (a completion of) the above map I to give the required identities of generating functions.

References

[1] T. Bridgeland, An introduction to motivic Hall algebras, to appear.

[2] T. Bridgeland, Hall algebras and curve-counting invariants, to appear.

[3] D. Joyce, Motivic invariants of Artin stacks and 'stack functions', Q. J. Math. 58 (2007), no. 3, 345–392.

[4] D. Joyce, Configurations in abelian categories I. Basic properties and moduli stacks, Adv. in Math. 203 (2006), 194-255.

[5] D. Joyce, Configurations in abelian categories. II. Ringel-Hall algebras, Adv. in Math. 210 (2007), no. 2, 635–706.

[6] D. Joyce, Configurations in abelian categories. III. Stability conditions and identities, Adv. in Math. 215 (2007), 153-219.

[7] D. Joyce, Configurations in abelian categories. IV. Invariants and changing stability conditions, Adv. in Math. 217 (2008)

[8] D. Joyce and Y. Song, A theory of generalized DT invariants, arXiv:0810.5645.

[9] R. Pandharipande and R.P. Thomas, Curve counting via stable pairs in the derived category, arxiv 0707.2348.

[10] Y. Toda, Generating functions of stable pair invariants via wall-crossings in derived categories, preprint.

[11] Y. Toda, Curve counting theories via stable objects I. DT/PT correspondence, preprint.

Differentials with real periods and subvarieties of the moduli space of curves

Samuel Grushevsky
(joint work with Igor Krichever)

In this talk we presented our joint work with Igor Krichever, obtaining a new proof of the Diaz' theorem: that any complex subvariety of the (uncompactified) moduli space of curves $\mathcal{M}_g$ has complex dimension at most $g-2$. Our proof is direct and uses no complicated machinery. We use differentials with real periods, which have seen much use in Whitham perturbation theory of integrable systems, appeared in other guises in Chas-Sullivan string topology, and are related to the study of geometric quadratic differentials by McMullen.

The constructions and the outline of the proof are as follows. We refer to [1] for all the details, motivation, and references. We work on the moduli space $\mathcal{M}_{g,2}$ of smooth (complex) curves X of genus g with two labeled marked points $p_\pm$. The basic tool of our method is the following elementary classical observation:

Lemma 1. *For any $(X,p_+,p_-)\in\mathcal{M}_{g,2}$ there exists a unique differential Ψ with simple poles at p_+,p_-, holomorphic on $X\setminus\{p_+,p_-\}$ (i.e. $\Psi\in|K_X+p_++p_-|$), with residues $\pm\sqrt{-1}$ at $p_\pm$, respectively, and all periods real.*

Sketch of proof. To prove uniqueness, we note that if two such existed, their difference would be a holomorphic differential with all periods real, which is impossible by Riemann's bilinear relations. To prove existence, note that $\dim_{\mathbb{C}} |K_X + p_+ + p_-| = g + 1$. The conditions of periods being real impose $2g$ real conditions on a differential; fixing the residues imposes one complex condition, and thus the expected dimension of the set of possible Ψ is zero. Thus the existence follows from uniqueness. □

One now uses this differential Ψ (a real-analytic section of the bundle of appropriate differentials over $\mathcal{M}_{g,2}$) to construct local real-analytic coordinates on the moduli space. This construction generalizes to the case of differentials the Lyashko-Looijenga map giving local coordinates on the Hurwitz space (the generalization is that in general for arbitrary order poles we replace the function by its differential, and then no longer assume the differential to be exact). These coordinates have actually been considered in Whitham theory, are essentially known in string topology, and are as follows:

Proposition 2. *The following give local real-analytic coordinates on $\mathcal{M}_{g,2}$: the set of $2g$ (real) periods $a_1, \ldots, a_{2g}$ of Ψ over a basis of cycles, and the set of "critical values", i.e. the integrals $\phi_i := \int_{p_0}^{q_i} \Psi$ from some point p_0 to all the $2g$ zeroes $\{q_1, \ldots, q_{2g}\}$ of Ψ. (More precisely, since one cannot label the zeroes of Ψ, the coordinates are symmetric functions of the critical values, and p_0 is chosen so that the sum of all critical values is zero, $\sum \phi_i = 0$.)*

The proof of this proposition is somewhat technical, see [1]. From the point of view of string topology it can be seen graphically by viewing the global well-defined harmonic function $f := \operatorname{Im} \int \Psi$ on X as a "height" map $f : X \to \mathbb{R}$, and gluing the surface at a given height from appropriate pieces. We note that, crucially, in general the imaginary parts $\operatorname{Im} \phi_i$ (more precisely, the symmetric functions of them) are *globally* well-defined on $\mathcal{M}_{g,2}$.

Using these coordinates, we define a foliation $\mathcal{L}$ of $\mathcal{M}_{g,2}$ by declaring locally a leaf $\mathcal{L}_{a_1,\ldots,a_{2g}}$ to be the locus where all periods of Ψ are locally constant and equal to a_i. Note that though Ψ depends on the point (X, p_+, p_-) real-analytically, the defining equations for a leaf (that there exists a differential with prescribed periods $a_1, \ldots, a_{2g}$) are holomorphic, and thus the leaves locally are complex subvarieties of $\mathcal{M}_{g,2}$. We now use this to bound the dimension of complete subvarieties of $\mathcal{M}_{g,2}$.

Sketch of the proof of Diaz' theorem. Suppose $Z \subset \mathcal{M}_g$ is a complete subvariety of complex dimension n; let $\tilde{Z} \subset \widetilde{\mathcal{M}}_{g,2}$ be its preimage in the partial compactification where the points are allowed to coincide — so $\tilde{Z}$ is a complete subvariety of dimension $n + 2$. Consider now a connected component Y of the intersection $\tilde{Z} \cap \mathcal{L}_{\underline{a}}$ of $\tilde{Z}$ with any leaf $\mathcal{L}_{\underline{a}}$ of the foliation $\mathcal{L}$. Any function $\operatorname{Im} \phi_i$ is a well-defined *global* harmonic function on the complete variety Y, and thus by the maximum principle is constant on it. Thus its conjugate harmonic function $\operatorname{Re} \phi_i$ (which a priori is only defined locally) is locally, and thus globally, constant on Y. Since all periods a_i are also constant on Y (since Y is contained in the leaf $\mathcal{L}_{\underline{a}}$), it means that all the coordinates that we have constructed on $\mathcal{M}_{g,2}$ are constant on Y, and thus Y is a point. Thus the intersection of $\tilde{Z}$ with any (complex codimension g) leaf $\mathcal{L}_{\underline{a}}$ is a point, and thus $\dim \tilde{Z} \leq g$, implying the Diaz' bound.

To make this proof rigorous, one needs to argue that the boundary of $\widetilde{\mathcal{M}}_{g,2}$ causes no problems, and to deal with symmetric functions of ϕ_i instead of ϕ_i themselves — we refer to [1] for how this is done. The idea is as above but the construction becomes logically much more involved. □

The construction of differentials with real periods appears to also be useful for studying cycles on the moduli spaces of curves, and in [2] we will apply it to prove vanishing results for certain tautological classes on the moduli spaces of curves.

References

[1] S. Grushevsky, I. Krichever: *The universal Whitham hierarchy and geometry of the moduli space of pointed Riemann surfaces*, Surveys in Differential Geometry, to appear, arXiv:0810.2139, 17pp.

[2] S. Grushevsky, I. Krichever: *Differentials with real periods and vanishing in the tautological ring of the moduli space of curves*, in preparation.

The ring of invariants of n points on the projective line

Ravi Vakil

(joint work with Ben Howard, John Millson, Andrew Snowden)

(This extended abstract is extracted from the announcement [2], which contains more details.) We consider the ring of invariants of n points on the projective line, and the GIT quotient $(\mathbb{P}^1)^n//\mathrm{PGL}_2$. The quotient depends on a choice of n weights $\vec{w} := (w_1, \ldots, w_n) \in (\mathbb{Z}^+)^n$:

$$(\mathbb{P}^1)^n \dashrightarrow (\mathbb{P}^1)^n//_{\vec{w}}\mathrm{PGL}_2 := \mathrm{Proj}\left(\bigoplus_{\mathrm{k}} \mathrm{R}_{\mathrm{k}\tilde{\mathrm{w}}}\right)$$

where $R_{\vec{v}} = \Gamma((\mathbb{P}^1)^n, \mathcal{O}(v_1, \ldots, v_n))^{\mathrm{PGL}_2}$. Small cases ($n \leq 6$) yield familiar beautiful geometry. The case $n = 4$ gives the cross ratio $(\mathbb{P}^1)^4 \dashrightarrow \overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$. The case $n = 5$ yields the quintic del Pezzo surface $(\mathbb{P}^1)^5 \dashrightarrow \overline{\mathcal{M}}_{0,5} \hookrightarrow \mathbb{P}^5$. The case $n = 6$ is particularly beautiful, and is summarized in Figure 1. The case of $n = 8$, sketched in [2], turns out to be even more beautiful than the $n = 6$ case; the structure is shown in Figure 2.

Our main theorem describes the relations for any n and for any weighting. We describe the invariants in terms of a *graphical algebra*. To a directed graph Γ (with no loops) on n ordered vertices (in bijection with the n points), we associate $\prod_{\vec{ab}\in\Gamma}(x_a y_b - y_a x_b)$, an invariant element of $\mathcal{O}(\vec{v})$, where $\vec{v}$ is the n-tuple of valences of the vertices. The degree $\vec{w}$ invariants are generated (as a vector space or module) by these elements. This description can be used to show that the ring of invariants

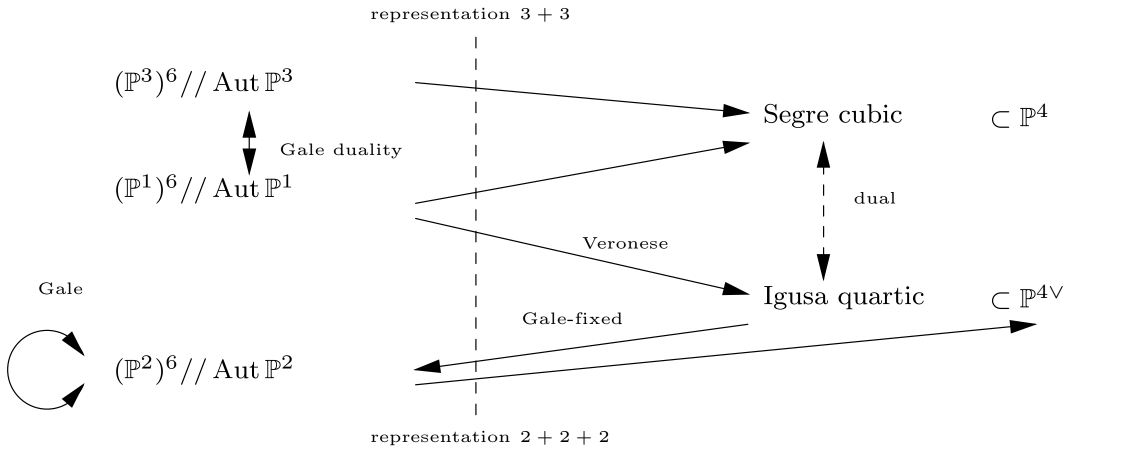

FIGURE 1. The classical geometry of six points in projective space

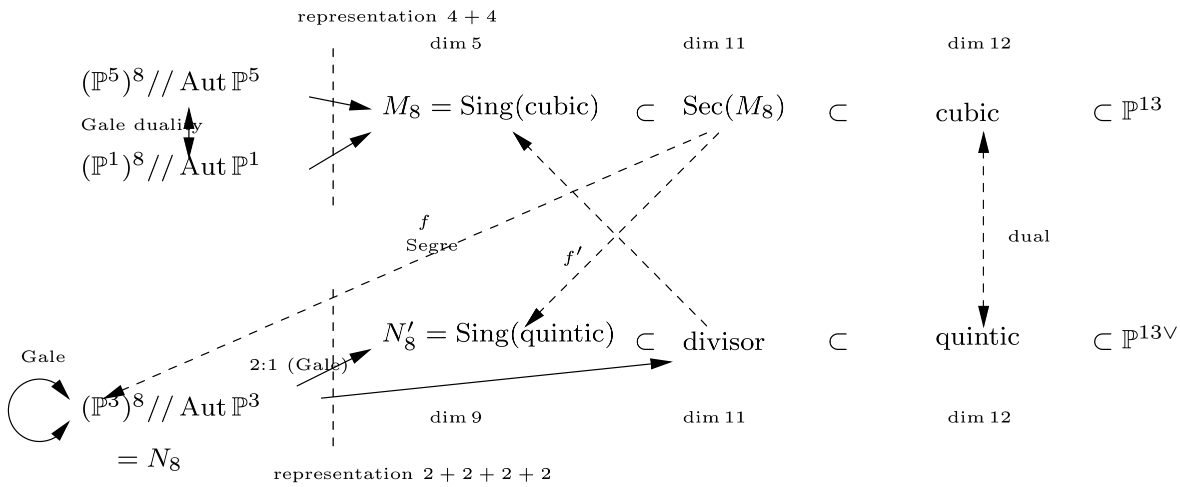

FIGURE 2. Relations among moduli spaces of eight points in projective space

for any $\vec{w}$ is generated in degree 1. In the unit weight case, this is Kempe's Theorem. We make a series of observations about this graphical algebra.

Multiplication. Multiplication of (elements associated to) graphs is by superposition. (See for example Figure 3(a). The vertex labels 1 through 4 are omitted for simplicity. In later figures, even the vertices will be left implicit.)

Sign (linear) relations. Changing the orientation of a single edge changes the sign of the invariant (e.g. Figure 3(b)).

Plücker (linear) relation. Direct calculation shows the relation of Figure 3(c).

Bigger relations from smaller ones. The "four-point" Plücker relation immediately "extends" to relations among more points, e.g. Figure 3(d) for 6 points. Any

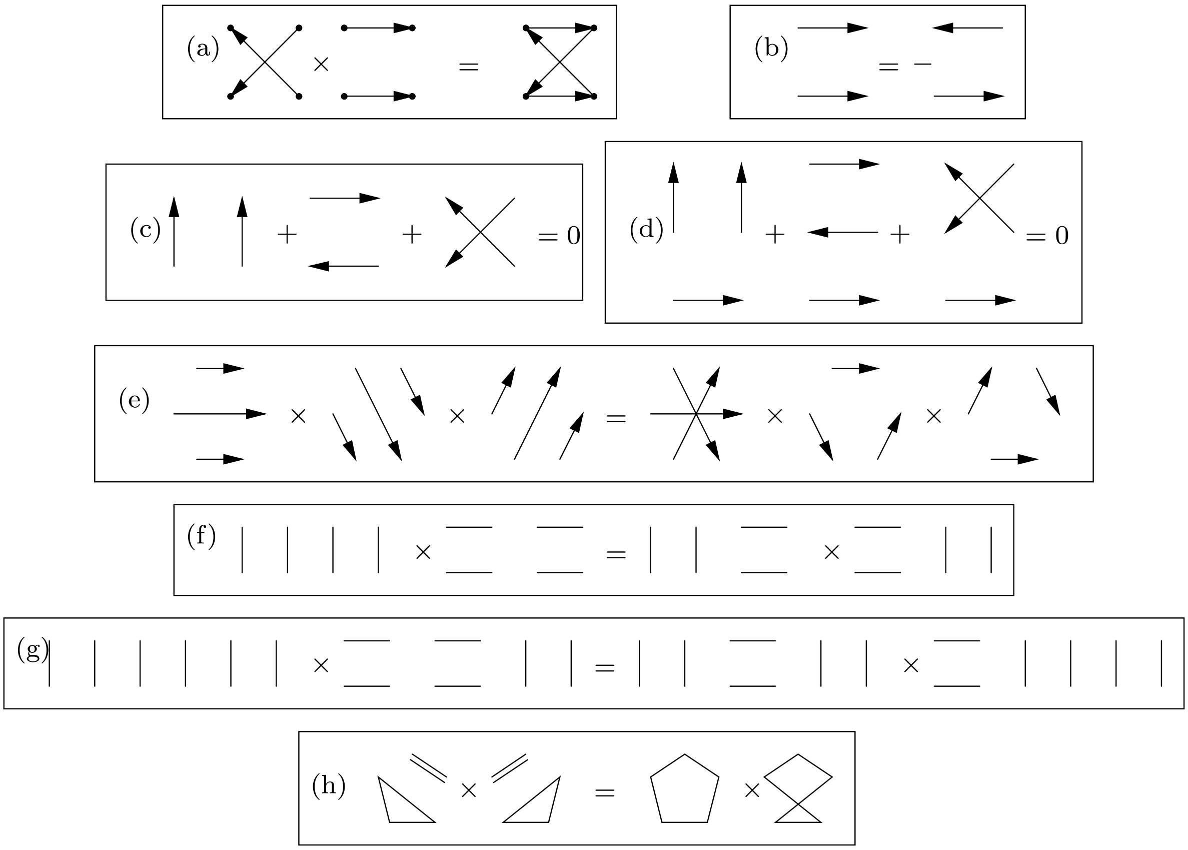

FIGURE 3. Relations in the graphical algebra

relation may be extended in this way. For example, the sign relation in general should be seen as an extension of the two-point sign relation.

Remark. The sign and (extended) Plücker relations generate all the linear relations, via a graphical version of the "straightening algorithm".

The Segre cubic. The relation of Figure 3(e) is patently true: the superposition of the three graphs on the left is the same as that of the three graphs on the right. This is a cubic relation on the six point space. It turns out to be nonzero, and is thus necessarily the Segre cubic relation. Of course, all that matters about the orientations of the edges is that they are the same on the both sides of the equation.

A simple (binomial) quadric on eight points. Figure 3(f) gives an obvious relation on 8 points. The arrowheads are omitted for simplicity; they should be chosen consistently on both sides, as in Figure 3(e).

Simple quadrics for at least eight points are obtained by "extending" the eight-point relations, e.g. Figure 3(g) is the extension to 12 points, where the same two edges are added to each graph in Figure 3(f).

Main Theorem of [3] **for the** n **even "unit weight" case** $\vec{w} = 1^n$ *If* $n \neq 6$, *the simple quadrics (i.e. the* S_n*-orbit of the quadric above) generate the ideal of relations.*

By [1, Thm. 1.2], the arbitrary weight case readily reduces to the "unit weight" case $\vec{w} = 1^n$ (n even), so this solves the problem for arbitrary weight. For example, an explicit description of the quadrics in the del Pezzo case of five points are as the five rotations of the patently true relation in Figure 3(h).

References

[1] B. Howard, J. Millson, A. Snowden and R. Vakil, *The equations for the moduli space of n points on the line*, Duke Math. J., **146** No. 2 (2009), 175-226.

[2] B. Howard, J. Millson, A. Snowden, R. Vakil, *The relations among invariant of points on the projective line*, C.R. Math. Acad. Sci. Paris, to appear.

[3] B. Howard, J. Millson, A. Snowden, R. Vakil, *The ideal of relations for the ring of invariants of n points on the line*, submitted.

Differential forms on singular varieties

Daniel Greb

(joint work with Stefan Kebekus, Sándor J. Kovács, Thomas Peternell)

1. Introduction

Differential forms are an important tool in the study of the geometry of (smooth) algebraic varieties. On singular varieties there are various approaches to define the right analogue of the sheaf of differential forms on a smooth variety. One candidate is the sheaf of reflexive differentials, i.e., the push-forward of the sheaf of differential forms on the smooth locus X_{smooth} of X, another candidate is the push-forward $\pi_*\Omega^p_{\widetilde{X}}$ of the sheaf of differential forms on a desingularisation $\widetilde{X}$ of X (which is in fact independent of the chosen resolution π). In general, these two sheaves do not coincide. It was observed by Grauert and Riemenschneider in [GR70] that on a normal variety Serre duality holds for the sheaf of reflexive n-forms while Kodaira vanishing holds for $\pi_*\Omega^n_{\widetilde{X}}$, $n = \dim X$.

It is hence natural consider those varieties on which both sheaves coincide. Assuming that the sheaf of reflexive n-forms is locally free, this is exactly the definition of canonical singularities. This class of singularities plays an important role in the classification theory of algebraic varieties known as the Minimal Model Program. In fact, for technical reasons it is often convenient to work in the class of pairs (X, Δ) with Kawamata log terminal (klt) singularities. These share many properties with canonical singularities. In particular, they are rational, see e.g. [KM98, Prop. 5.13].

Requiring reflexive n-forms to extend with at worst simple poles to any resolution leads to the class of log canonical singularities, which in contrast to the situation in the case of klt singularities can display complicated cohomological behaviour.

2. Results

In [GKKP10] we prove the following extension result for log canonical pairs:

Theorem 1 (Extension Theorem)**.** *Let X be a normal complex algebraic variety of dimension n and $\Delta \subset X$ a $\mathbb{Q}$-divisor with coefficients in $[0,1] \cap \mathbb{Q}$. Assume that the pair (X, Δ) is log canonical. Let $\pi : \widetilde{X} \to X$ be a log resolution, and set*

$$\widetilde{\Delta} := \textit{largest reduced divisor contained in } \operatorname{supp} \pi^{-1}(\textit{non-klt locus}),$$

where the non-klt locus is the smallest closed subset $W \subset X$ such that (X, Δ) is klt away from W. If $1 \leq p \leq n$ is any index, then the sheaf $\pi_* \Omega^p_{\widetilde{X}}(\log \widetilde{\Delta})$ is reflexive.

This generalizes the results of [GKK08] to all values of p and to non-reduced log canonical pairs. Theorem 1 implies in particular that on a variety with klt singularities, the sheaf of reflexive p-forms and the sheaf $\pi_* \Omega^p_{\widetilde{X}}$ introduced above coincide. In other words, every differential form defined on the smooth locus of a variety X with at worst klt singularities extends to a regular differential form on any desingularisation of X; hence the name "Extension Theorem".

As corollaries of the result stated above we prove vanishing theorems of Kodaira-Akizuki-Nakano and Bogomolov-Sommese-type on log-canonical and klt varieties:

Furthermore, as part of the proof of Theorem 1 we generalize various techniques dealing with differential forms from the smooth to the singular case. For example, we establish the existence of a residue sequence and natural pull-back morphisms for reflexive differential forms on dlt spaces.

3. Sketch of the proof

The proof of Theorem 1 proceeds in two main steps:

First, we prove that differential forms defined on the smooth locus of a log canonical variety X extend with at worst logarithmic poles to any desingularisation of X. This part uses recent work of Kollár-Kovács [KK09] on cohomological properties of log canonical singularities. In particular, we deduce a generalized version of Steenbrink's vanishing theorem [Ste85] for log canonical singularities. This in turn leads to a vanishing theorem for local cohomology groups supported in fibres of resolutions $\pi : \widetilde{X} \to X$ from which extension with logarithmic poles follows.

In a second step, assuming that X is klt, we use the Minimal Model Program, residue sequences for reflexive differentials on dlt spaces, and Shokurov's Rational Connectedness Conjecture (as proven by Hacon and McKernan [HM07]) to deduce that in fact differential forms extend regularly to any resolution $\widetilde{X}$ of X.

4. Applications to moduli theory

For applications of these extension results to moduli theory, especially to Shafarevich's and Viehweg's Conjecture, as well as for further details concerning the Bogomolov and generalized Steenbrink vanishing theorem we refer the reader to Sándor Kovács' contribution in this report.

References

[GKK08] Daniel Greb, Stefan Kebekus, and Sándor J. Kovács, *Extension theorems for differential forms and Bogomolov-Sommese vanishing on log canonical varieties*, to appear in *Compos. Math.*, DOI:10.1112/S0010437X09004321, preprint arXiv:0808.3647, 2008.

[GKKP10] Daniel Greb, Stefan Kebekus, Sándor J. Kovács, and Thomas Peternell, *Differential forms on log canonical spaces*, in preparation, 2010.

[GR70] Hans Grauert and Oswald Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970), 263–292.

[HM07] Christopher D. Hacon and James Mckernan, *On Shokurov's rational connectedness conjecture*, Duke Math. J. **138** (2007), no. 1, 119–136.

[KK09] János Kollár and Sándor J. Kovács, *Log canonical singularities are Du Bois*, preprint arXiv:0902.0648, to appear in *J. Amer. Math. Soc.*, 2009.

[KM98] János Kollár and Shigefumi Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998.

[Ste85] J. H. M. Steenbrink, *Vanishing theorems on singular spaces*, Astérisque (1985), no. 130, 330–341, Differential systems and singularities (Luminy, 1983).

Vanishing theorems for log canonical pairs

Sándor J. Kovács

(joint work with Daniel Greb, Stefan Kebekus, Thomas Peternell)

1. Viehweg's conjecture

Let Y° be a quasi-projective manifold that admits a generically finite morphism $\mu : Y^\circ \to \mathscr{M}$ to a moduli stack of canonically polarized varieties. Generalizing Shafarevich' conjecture [Sha63], Viehweg conjectured [Vie01] that this can only happen if Y° is of log general type. Equivalently, if $f^\circ : X^\circ \to Y^\circ$ is a smooth family of canonically polarized varieties and the variation of f° is maximal, then Y° is of log general type, i.e., $\mathrm{Var}(f^\circ) = \dim Y^\circ$. This conjecture was refined in [KK08]:

Conjecture 1 (Refined Viehweg conjecture)**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized varieties, over a quasi-projective manifold Y°. Then either*

i) $\kappa(Y^\circ) = -\infty$ *and* $\mathrm{Var}(f^\circ) < \dim Y^\circ$, *or*
ii) $\kappa(Y^\circ) \geq 0$ *and* $\mathrm{Var}(f^\circ) \leq \kappa(Y^\circ)$. □

2. How to prove Viehweg's Conjecture

Conjecture 1 was confirmed for $\dim Y^\circ \leq 3$ in [KK08c]. Next we list the main ingredients of the proof.

Theorem 2 (Pluri-differentials on the base [VZ02])**.** *Let $f^\circ : X^\circ \to Y^\circ$ be a smooth projective family of canonically polarized varieties over a quasi-projective manifold Y°. Let Y be a smooth compactification of Y° such that $D := Y \setminus Y^\circ$*

is a divisor with simple normal crossings. Then there exists an $m \in \mathbb{N}$ and an invertible subsheaf

$$\mathscr{A} \subset \operatorname{Sym}^m \Omega^1_Y(\log D)$$

such that $\kappa(\mathscr{A}) \geq \operatorname{Var}(f^\circ)$.

Theorem 3 (Extension theorem [GKKP10])**.** *Let X be a complex variety of dimension n and $D \subset X$ a $\mathbb{Q}$-divisor. Assume that the pair (X, D) is log canonical. Let $\pi : \widetilde{X} \to X$ be a log resolution, and set*

$$\widetilde{D} := \textit{largest reduced divisor contained in } \operatorname{supp} \pi^{-1}(\textit{non-klt locus}),$$

where the non-klt locus is the smallest closed subset $W \subset X$ such that (X, D) is klt away from W. Let p be an integer such that $1 \leq p \leq n$. Then the sheaf $\pi_* \Omega^p_{\widetilde{X}}(\log \widetilde{D})$ is reflexive.

One corollary of Theorem 3 is the following generalization of the well-known Bogomolov-Sommese vanishing theorem for snc pairs, cf. [EV92].

Theorem 4 (Bogomolov-Sommese vanishing for log canonical pairs [GKKP10])**.** *Let (X, D) be a log canonical logarithmic pair, where X is projective. If $\mathscr{A} \subseteq \Omega^{[p]}_X(\log D)$ is a $\mathbb{Q}$-line bundle, then $\kappa(\mathscr{A}) \leq p$.*

The way these results combine is as follows: Assume (for instance) that the statement of (1.ii) is false, that is there exists a subsheaf $\mathscr{A} \subset \operatorname{Sym}^m \Omega^1_Y(\log D)$ with $\kappa(\mathscr{A}) > \kappa(Y^\circ)$. This may be used to prove that the tangent sheaf of a minimal model (Y_λ, D_λ) of the pair (Y, D) is unstable. Similarly, one may prove that the sheaf of reflexive differentials $\Omega^{[1]}_{Y_\lambda}(\log D_\lambda)$ is unstable. Let $\mathscr{B}$ be a maximal destabilizing subsheaf of $\Omega^{[1]}_{Y_\lambda}(\log D_\lambda)$ of rank p. Taking the determinant of $\mathscr{B}$ we obtain a subsheaf $\det \mathscr{B} \subset \Omega^{[p]}_{Y_\lambda}(\log D_\lambda)$ with $\kappa(\det \mathscr{B}) > \kappa(Y^\circ)$. At the same time by Theorem 4 we have that $\kappa(\det \mathscr{B}) \leq p$. This means that $\kappa(Y^\circ) < \dim Y^\circ$ implying Viehweg's conjecture. Further analysis yields the Refined Viehweg conjecture.

3. Inside the Bogomolov-Sommese vanishing theorem: Relative vanishing theorems for log canonical pairs

Theorem 5 (Steenbrink-type vanishing for log canonical pairs)**.** *Let (X, D) be a log canonical pair of dimension $n \geq 2$. If $\pi : \widetilde{X} \to X$ is a log resolution of (X, D) with π-exceptional set E, and if $\widetilde{D}$ is the reduced divisor*

$$\widetilde{D} := E \cup \pi^{-1}\big(\operatorname{supp}\lfloor D \rfloor\big),$$

then $R^{n-1}\pi_*\big(\Omega^p_{\widetilde{X}}(\log \widetilde{D}) \otimes \mathscr{O}_{\widetilde{X}}(-\widetilde{D})\big) = 0$ for all $0 \leq p \leq n$.

Remark 6. *For $p > 1$ the claim of Theorem 5 is proven in* [Ste85, Thm. 2(b)] *without any assumption on the nature of the singularities of X.*

Corollary 7 (Steenbrink-type vanishing for cohomology with supports)**.** *Let* (X, D) *be a log canonical pair of dimension* $n \geq 2$*. If* $\pi : \widetilde{X} \to X$ *is a log resolution of* (X, D) *with* π*-exceptional set* E*, if* $\widetilde{D} := E \cup \pi^{-1}\big(\operatorname{supp}\lfloor D \rfloor\big)$*, and if* $F_x = \pi^{-1}(x)$ *is the (reduced) fibre over a point* $x \in X$*, then we have*

$$H^1_{F_x}\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\big) = \{0\} \qquad \text{for } 0 \leq p \leq n.$$

Remark 8. *Using the standard exact sequence for cohomology with support, the conclusion of Corollary 7 can equivalently be reformulated as follows.*

1. *The restriction* $H^0\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\big) \to H^0\big(\widetilde{X} \setminus F_x,\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\big)$ *is surjective, and*
2. *The restriction* $H^1\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\big) \to H^1\big(\widetilde{X} \setminus F_x,\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\big)$ *is injective.*

Proof of Corollary 7. Duality for cohomology groups with support (cf. [GKK08, Appendix]) yields $H^1_{F_x}\big(\widetilde{X},\, \Omega^p_{\widetilde{X}}(\log \widetilde{D})\big) \overset{\text{dual}}{\sim} \big(R^{n-1}\pi_*\Omega^{n-p}_{\widetilde{X}}(\log \widetilde{D})(-\widetilde{D})_x\big)\widehat{\ }$, where $\widehat{\ }$ denotes completion with respect to the maximal ideal $\mathfrak{m}_x$ of the point $x \in X$. The latter group vanishes for the required range of p by Theorem 5. □

4. The case $p = 0$ of Theorem 5.

Theorem 9 (Vanishing for ideal sheaves on pairs of Du Bois spaces)**.** *Let* (X, D) *be a reduced pair such that* X *and* D *are both Du Bois, and let* $\pi : \widetilde{X} \to X$ *be a log resolution of* (X, D)*. If* $E := \operatorname{Exc}(\pi)$ *denotes the exceptional set and* $\widetilde{D} = E \cup \pi^{-1}(D)$*, both divisors considered with their reduced structure, then*

$$R^i\pi_*\mathscr{O}_{\widetilde{X}}(-\widetilde{D}) = 0 \qquad \text{for all} \quad i > \max\big(\dim \overline{\pi(E) \setminus D}, 0\big).$$

In particular, if X *is of dimension* $n \geq 2$*, then* $R^{n-1}\pi_*\mathscr{O}_{\widetilde{X}}(-\widetilde{D}) = 0$*.*

Corollary 10 (Vanishing for ideal sheaves on log canonical pairs)**.** *Let* (X, D) *be a log canonical pair of dimension* $n \geq 2$*. Let* $\pi : \widetilde{X} \to X$ *be a log resolution of* (X, D) *with* π*-exceptional set* E*. Then* $R^{n-1}\pi_*\,\mathscr{O}_{\widetilde{X}}(-\widetilde{D}) = 0$*, where* $\widetilde{D} := \operatorname{supp}\big(E + \pi^{-1}\lfloor D \rfloor\big)$*.*

Proof. Recall from [KK09, Theorem 1.4] that X is Du Bois, and that any finite union of log canonical centers is likewise Du Bois. Since the components of $\lfloor D \rfloor$ are log canonical centers, Theorem 9 applies to the reduced pair $\big(X, \lfloor D \rfloor\big)$ to prove the claim. □

The case $p = 1$ can be proved using this case, the fact that the result is known for $p > 1$ by [Ste85, Thm. 2(b)], and an argument using relative cohomology of the pair (X, D). For details, see [GKKP10].

References

[Ara71] S. J. Arakelov: *Families of algebraic curves with fixed degeneracies*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1269–1293. MR0321933 (48 #298)

[EV92] H. Esnault and E. Viehweg: *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser Verlag, Basel, 1992. MR1193913 (94a:14017)

[GKK08] D. Greb, S. Kebekus, and S. J. Kovács: *Extension theorems for differential forms and Bogomolov-Sommese vanishing on log canonical varieties*, Compos. Math., to appear, 2008. arXiv:0808.3647v1 [math.AG]

[GKKP10] D. Greb, S. Kebekus, S. J. Kovács, and T. Peternell: *Differential forms on log canonical spaces*, in preparation, 2010.

[KK05] S. Kebekus and S. J. Kovács: *Families of canonically polarized varieties over surfaces*, to appear in Invent. Math., November 2005. arXiv:math.AG/0511378

[KK08] S. Kebekus and S. J. Kovács: *Families of canonically polarized varieties over surfaces*, Invent. Math. **172** (2008), no. 3, 657–682. DOI: 10.1007/s00222-008-0128-8

[KK08c] S. Kebekus and S. J. Kovács: *The structure of surfaces and threefolds mapping to the moduli stack of canonically polarized varieties*, to appear in Duke Math. J. arXiv:0812.2305v1 [math.AG]

[KK09] J. Kollár and S. J. Kovács: *Log canonical singularities are Du Bois*, preprint arXiv:0902.0648, February 2009.

[Kov97] S. J. Kovács: *Families over a base with a birationally nef tangent bundle*, Math. Ann. **308** (1997), no. 2, 347–359. MR1464907 (98h:14039)

[Kov02] S. J. Kovács: *Logarithmic vanishing theorems and Arakelov-Parshin boundedness for singular varieties*, Compositio Math. **131** (2002), no. 3, 291–317. MR1905025 (2003a:14025)

[Kov03] S. J. Kovács: *Families of varieties of general type: the Shafarevich conjecture and related problems*, Higher dimensional varieties and rational points (Budapest, 2001), Bolyai Soc. Math. Stud., vol. 12, Springer, Berlin, 2003, pp. 133–167. MR2011746 (2004j:14041)

[Kov06] S. J. Kovács: *Subvarieties of moduli stacks of canonically polarized varieties*, Proceedings of the AMS Summer Research Institute held at the University of Washington, Seattle, WA, July 25–August 12, 2005, Proceedings of Symposia in Pure Mathematics, American Mathematical Society, Providence, RI, 2006, preprint, invited for submission.

[Par68] A. N. Parshin: *Algebraic curves over function fields. I*, Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 1191–1219. MR0257086 (41 #1740)

[Sha63] I. R. Shafarevich: *Algebraic number fields*, Proc. Internat. Congr. Mathematicians (Stockholm, 1962), Inst. Mittag-Leffler, Djursholm, 1963, pp. 163–176, English translation: Amer. Math. Soc. Transl. (2) **31** (1963), 25–39. MR0202709 (34 #2569)

[Ste85] J. H. M. Steenbrink: *Vanishing theorems on singular spaces*, Astérisque (1985), no. 130, 330–341, Differential systems and singularities (Luminy, 1983). MR804061 (87j:14026)

[Vie01] E. Viehweg: *Positivity of direct image sheaves and applications to families of higher dimensional manifolds*, School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), ICTP Lect. Notes, vol. 6, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001, pp. 249–284. MR1919460 (2003f:14024)

[VZ01] E. Viehweg and K. Zuo: *On the isotriviality of families of projective manifolds over curves*, J. Algebraic Geom. **10** (2001), no. 4, 781–799. MR1838979 (2002g:14012)

[VZ02] E. Viehweg and K. Zuo: *Base spaces of non-isotrivial families of smooth minimal models*, Complex geometry (Göttingen, 2000), Springer, Berlin, 2002, pp. 279–328. MR1922109 (2003h:14019)

Polarized K3 surfaces of genus 16

Shigeru Mukai

Let $\mathcal{T} = G(2,3;\mathbb{C}^4)$ be the EPS-moduli space of the twisted cubics in $\mathbb{P}^3$ constructed in [1]. $\mathcal{T}$ is the GIT-quotient of $\mathbb{C}^2\otimes\mathbb{C}^3\otimes\mathbb{C}^4$ by the action of $GL(2)\times GL(3)$ on the first and second factors. There exist two tautological vector bundles $\mathcal{E},\mathcal{F}$ of rank 3, 2 and the universal homomorphism $\mathcal{E}\otimes\mathbb{C}^4 \longrightarrow \mathcal{F}$ on $\mathcal{T}$. The vector bundle $\mathcal{E}$ embeds $\mathcal{T}$ into the 21-dimensional Grassmannian $G(S^2\mathbb{C}^4, 3)$.

Theorem 1. (1) *A general complete intersection S with respect to the rank* 10 *vector bundle $\mathcal{E}^{\oplus 2}\oplus\mathcal{F}^{\oplus 2}$ in the EPS-moduli space $\mathcal{T}$ is a K3 surface, and* $\det\mathcal{E}|_S$ *is a polarization of genus* 16*, that is, degree* 30*.*

(2) *Moreover, a moduli-theoretically general polarized K3 surface (S,h) of genus* 16 *is obtained in this way.*

Let $\mathcal{F}_g$ be the moduli space of primitively (quasi-)polarized K3 surfaces (S,h) of degree $2g-2$, and $\mathcal{S}_g$ be the (quasi-)universal family over it. The theorem yields a dominant rational map $P^{36} \dashrightarrow \mathcal{F}_{16}$ from a $G(2,12)$-bundle P^{36} over the 16-dimensional Grassmannian $G(2,S^2\mathbb{C}^4)$ of pencils of quadrics to $\mathcal{F}_{16}$.

Corollary 2. *The moduli space $\mathcal{F}_{16}$ is unirational.*

See [2] and [3] for the birational type of other $\mathcal{F}_g$'s.

Since $\mathcal{E}|_S$ is a stable semi-rigid vector bundle with Mukai vector $v=(3,h,5)$, the rational map factors through $\mathcal{S}_{16}$.

Conjecture 3. *The induced rational map $P^{36}//PGL(4) \dashrightarrow \mathcal{S}_{16}$ between 21-dimensional varieties is birational.*

References

[1] G. Ellingsrud, R. Piene, and S.A. Strømme, *On the variety of nets of quadrics defining twisted cubic curves.* In F. Ghione, C. Peskine and E. Sernesi, editors, *Space Curves*, Lecture Notes in Math. **1266** (1987), Springer-Verlag, pp. 84–96.

[2] V.A. Gritsenko, K. Hulek and G.A. Sankaran, *The Kodaira dimension of the moduli spaces of K3 surfaces*, Invent. math. **169** (2007), 519–567.

[3] S. Mukai, *Polarized K3 surfaces of genus thirteen*, In S. Mukai et. al, editors, *Moduli spaces and Arithmetic Geometry (Kyoto, 2004)*, Adv. Stud. Pure Math. **45** (2006), Math. Soc. Japan and Amer. Math. Soc., pp. 315–326.

Smoothing surface singularities via mirror symmetry

Paul Hacking

(joint work with Mark Gross and Sean Keel)

We construct deformations of surface singularities determined by counts of rational curves and holomorphic discs on a mirror surface. We prove a conjecture of Looijenga [5, III.2.11] on smoothability of cusp singularities.

Let Y be a rational surface (smooth and compact) and $B\subset Y$ a cycle of smooth rational curves of length n such that $K_Y+B=0$.

Let $P \in X$ be the reducible surface singularity

$$0 \in \mathbb{C}^2_{x_1,x_2} \cup \mathbb{C}^2_{x_2,x_3} \cup \cdots \cup \mathbb{C}^2_{x_n,x_1} \subset \mathbb{C}^n_{x_1,\ldots,x_n},$$

a cyclic union of coordinate planes in $\mathbb{C}^n$. We call $P \in X$ the *vertex of degree* n.

Let S be the affine toric variety associated to the closure of the Kähler cone $\overline{K} \subset H^2(Y,\mathbb{R})$. (Actually, $\overline{K}$ may not be rational polyhedral, in which case we consider rational polyhedral subcones of $\overline{K}$.)

Theorem 1. *The pair* (Y,B) *determines a natural deformation* $(X \subset \mathcal{X})/(0 \in S)$ *of the vertex of degree* n *over the germ* $(0 \in S)$ *with smooth general fibre.*

We view this as a version of local mirror symmetry, because we expect that $Y \setminus B$ and the general fibre $\mathcal{X}_t$ of $\mathcal{X}/S$ admit dual special Lagrangian torus fibrations (the Strominger–Yau–Zaslow interpretation of mirror symmetry) and $\mathcal{X}/S$ defines a map from the complexified Kähler cone of Y (the interior of S) to the moduli space of complex deformations of $\mathcal{X}_t$.

The construction uses the scattering diagram introduced by Kontsevich and Soibelman [4] in the algebraic setting developed by Gross and Siebert [1]. It has an enumerative description discovered by Gross, Pandharipande and Siebert [2] — we count rational curves $f\colon C \to Y$ such that $f^{-1}B$ is a single point.

A *cusp singularity* is a surface singularity such that its minimal resolution has exceptional locus a cycle of smooth rational curves. A cusp singularity admits an infinite cyclic quotient construction as follows [3, §2]. Let $P \in Z$ be a cusp singularity and $U = Z \setminus \{P\}$ the punctured singularity. Then U is the quotient of an open analytic subset of the torus $(\mathbb{C}^\times)^2$ by the action of a hyperbolic element of $\mathrm{SL}(2,\mathbb{Z})$. The *dual cusp* is obtained by the same construction applied to the induced action on the dual torus. The link of the dual cusp is diffeomorphic to that of the original cusp, but the orientation is reversed.

Corollary 2 (Looijenga's conjecture)**.** *Let* $P \in Z$ *be a cusp singularity. Then* Z *is smoothable iff the exceptional locus of the minimal resolution of the dual cusp lies on a rational surface as an anticanonical divisor.*

Looijenga's conjecture provides an effective algorithm to decide whether a given cusp is smoothable, because every rational surface with anticanonical cycle is obtained from a minimal surface with anticanonical cycle by a sequence of blowups of points of the boundary.

Sketch of proof of Corollary. Let Y be a rational surface with anticanonical boundary B which contracts to the dual cusp. Let $\mathcal{X}/S$ be the induced deformation of the vertex. Let $T \subset S$ be the toric stratum associated to the face

$$\langle B_1, .., B_n\rangle^\perp \cap \overline{K}(Y)$$

of $\overline{K}(Y)$. We show that the general fibre of $\mathcal{X}|_T$ is isomorphic to the cusp Z. So Z is smoothable. The converse was proved by Looijenga using Inoue surfaces. □

References

[1] M. Gross, B. Siebert, *From real affine geometry to complex geometry*, preprint arXiv:math/0703822v2 [math.AG] (2007).

[2] M. Gross, R. Pandharipande, B. Siebert, *The tropical vertex*, preprint arXiv:0902.0779v2 [math.AG] (2009).

[3] F. Hirzebruch, *Hilbert modular surfaces*, Enseignement Math. (2) **19** (1973), 183–281.

[4] M. Kontsevich, Y. Soibelman, *Affine structures and non-Archimedean analytic spaces*, in *The unity of mathematics*, 321–385, Progr. Math. 244, Birkhäuser, 2006.

[5] E. Looijenga, *Rational surfaces with an anticanonical cycle*, Ann. of Math. (2) **114** (1981), 267–322.

Tzeng's proof of the Goettsche-Yau-Zaslow formula on nodal curve counting

Jun Li

Let X be a smooth algebraic surface over $\mathbb{C}$ and L an ample line bundle on X. The generalized Severi problem asks for the number of r-nodal curves in a generic r-dimensional linear subsystem of $|L|$. This problem has been investigated for $\mathbb{P}^2$ and rational surfaces by many people, including Ran, Kontsevich-Manin, Harris-Pandharipande, Choi, Caporaso-Harris, Vakil, etc. For general surfaces, this problem has been investigated by Vainsencher, Kleiman and Piene, and others.

This problem took off after the work of Yau-Zaslow on enumerating the rational curves on K3 surfaces. Their work established that the generating function of the counting of rational curves in K3 surfaces is the dedekind η function. Inspired by Yau-Zaslow formula, Göttsche proposed several conjectures on the number of r-nodal curves in a general r-dimensional sublinear system in $|L|$ for sufficiently ample line bundles L on general surfaces.

The Göttsche conjecture for primitive classes on K3 surfaces was proved by Bryan-Leung. The full Göttsche conjecture was proved by A-K. Liu using symplectic technique. Recently, J-R. Tzeng in her thesis gave a nice algebro-geometric proof:

Theorem 1 (Göttsche's conjecture)**.** *For every integer $r \geq 0$, there exists a universal polynomial $T_r(x,y,z,t)$ of degree r with the following property: given a pair of a smooth projective surface X and a $(5r-1)$-very ample line bundle L on X, a general r-dimensional sublinear system in $|L|$ contains exactly $T_r(L^2, LK, c_1(X)^2, c_2(X))$ r-nodal curves.*

Let $G_2 = -\frac{1}{24} + \sum_{n>0}\left(\sum_{d|n} d\right) q^n$, $\Delta(q) = q\prod_{k>0}(1-q^k)^{24}$ and $D = q\frac{d}{dq}$. Write $q = e^{2\pi i\tau}$ then G_2, DG_2 and D^2G_2 are quasimodular forms and Δ is a modular form.

Theorem 2 (Göttsche-Yau-Zaslow formula)**.** *There exist universal power series B_1, B_2 in q such that*

$$\sum_{r\geq 0} T_r(L^2, LK, c_1(X)^2, c_2(X))(DG_2(\tau))^r = \frac{(DG_2(\tau)/q)^{\chi(L)} B_1(q)^{K_X^2} B_1(q)^{LK_X}}{(\Delta(\tau)D^2G_2(\tau)/q^2)^{\chi(\mathcal{O}_X)/2}}.$$

Outline of the proof. The proof of Tzeng consists of three main components. Let $d_r(X,L)$ be the number of r-nodal curves in a r-dimensional sublinear system in $|L|$ when L is sufficiently ample (relative to r). The first component is to express $d_r(X,L)$ in terms of enumerative number on the Hilbert scheme of points of X. This component was completed by Göttsche. Using this work of Göttsche, one can define $d_r(X,L)$ without reference to sufficiently ampleness of L, though the resulting number $d_r(X,L)$ is no longer enumerative. Nevertheless, this provides a homomorphism

$$\mathbb{Z}\{[X,L]\} \longrightarrow \mathbb{Q}[[t]]^\times, \quad [X,L] \mapsto \sum_{r\geq 0} d_r(X,L)\cdot x^r,\ d_0(X,L)=1.$$

Here $\mathbb{Z}\{[X,L]\}$ is the Abelian group generated by pairs $[X,L]$ of smooth algebraic surfaces and line bundles on them.

The second component of her proof is to find a structure result of the cobordism group of the pairs. Following the work of Levine-Pandharipande, one is led to the *algebraic cobordism group of surfaces and line bundles*

$$\omega_{2,1} = \mathbb{Z}\{[X,L]\}/R,$$

also introduced by Levine-Pandharipande, where $[X,L]$ is as before and R is the subgroup generated by double point relations.

Suppose $[X_0,L_0]$, $[X_1,L_1]$ and $[X_2,L_2]$ are pairs of surfaces and line bundles. The *extended double point relation* is defined by

$$[X_0,L_0]-[X_1,L_1]-[X_2,L_2]+[\mathbf{P}(\pi),L_3] \tag{1}$$

with the assumption that there exists projective family $\pi:\mathcal{X}\to\mathbf{P}^1$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that:

(1) $\pi^{-1}(\infty)=X_1\cup_D X_2$ is a union of two irreducible smooth components that intersect transversally along a smooth divisor D;
(2) $\mathcal{X}$ is smooth and π is smooth away from a finite fibers of π;
(3) the fiber of $0\in\mathbf{P}^1$ equals $X_0=\pi^{-1}(0)$, which is a smooth surface;
(4) let $i_j: X_j\subset\mathcal{X}$ be the inclusion maps. Then $i_j^*(\mathcal{L})=L_j$;
(5) $\mathbf{P}(\pi):=\mathbf{P}(1_D\oplus N_{X_1/D})$, $\eta:\mathbf{P}(\pi)\to D$ is the projection and $L_3=\eta^*(\mathcal{L}|_D)$.

Tzeng proved a structure theorem of the cobordism group $\omega_{2,1}$:

Theorem 3. *As vector spaces, $\omega_{2,1}\otimes_{\mathbb{Z}}\mathbb{Q}\cong\mathbb{Q}^{\oplus 4}$. An integral generators are $[\mathbf{P}^2,\mathcal{O}]$, $[\mathbf{P}^2,\mathcal{O}(1)]$, $[K3,\mathcal{O}]$ and $[K3,\mathcal{O}(1)]$.*

The third component of her proof is the following factorization theorem

Theorem 4. *The homomorphism $\mathbb{Z}\{[X,L]\}\longrightarrow\mathbb{Q}[[t]]^\times$ defined earlier factors through the quotient homomorphism $\mathbb{Z}\{[X,L]\}\longrightarrow\omega_{2,1}$.*

The proof uses the degeneration of Hilbert scheme of points of surfaces constructed by B-S. Wu, following J. Li's work on degeneration of stable morphisms.

The Theorem 1 and 2 follow from Theorem 3 and 4, and Bryan-Leung's work on GW-invariants of K3 surfaces.

References

[1] . Göttsche,*A conjectural generating function for numbers of curves on surfaces.* Comm. Math. Phys. 196 (1998), no. 3, 523–533.
[2] M. Levine and R. Pandharipande, *Algebraic cobordism revisited*, Invent. Math. **176** (2009), no. 1, 63–130.
[3] J. Li and B.S. Wu, *Degeneration of Hilbert scheme of ideal sheaves*, in praparation.
[4] A.K Liu, *Family blowup formula, admissible graphs and the enumeration of singular curves. I*, J. Differential Geom. **56** (2000), no. 3, 381–579.
[5] Y.R. Tseng, *A proof of Goettsche-Yau-Zaslow formula*, in preparation.
[6] B.S. Wu, *The moduli stack of stable relative ideal sheaves*, math.AG/0701074.

Some recent progress on the rationality problem in invariant theory

Christian Böhning
(joint work with Hans-Christian Graf von Bothmer)

Let G be a connected linear algebraic group over $\mathbb{C}$ and V a finite dimensional complex linear representation of G. Denote by V/G any birational model of the field $\mathbb{C}(V)^G$ of invariant rational functions. The problem referred to in the title is whether V/G is rational, and under the hypotheses made, no counterexample is known. However, if G is not assumed to be connected, there exists examples where V/G is not even stably rational [Sa]. For more information we refer to [B09].

The talk was devoted to giving an overview of the proof of the following

Theorem 1. *Let G be $\mathrm{SL}_3(\mathbb{C})$ and put $V(d) = \mathrm{Sym}^d(\mathbb{C}^3)^\vee$ so that*

$$C(d) = \mathbb{P}(V(d))/G$$

is the moduli space of plane algebraic curves of degree d under projectivities. Then $C(d)$ is rational except possibly for one of the following values for which rationality remains unknown:

$$d = 6,\ 7,\ 8,\ 11,\ 12,\ 14,\ 15,\ 16,\ 18,\ 20,\ 23,\ 24,\ 26,\ 32,\ 48.$$

This is proven in [BvB1], [BvBK], and [Kat89] (the last reference supplies a proof for $d \equiv 0(\mathrm{mod}3)$, $d \geq 210$). However, the *method of covariants* used in [BvB1] appeared first in [Shep], and we learnt a lot from this source.

We give a brief sketch of the pattern of the argument for $d \equiv 1(\mathrm{mod}3)$, $d = 3n+1$. We construct a family of covariants

$$S_d \in \left(\mathrm{Sym}^4 V(d)^\vee \otimes V(4)\right)^G, \quad S_d : \mathbb{P}(V(d)) \dashrightarrow \mathbb{P}(V(4))$$

via the symbolical method of Aronhold and Clebsch [G-Y] and subspaces $L_d = x_1^{2n+3} \cdot \mathbb{C}[x_1, x_2, x_3]_{n-2} \subset V(d)$ (x_1, x_2, x_3 coordinates on $\mathbb{C}^3$) with the property that

$$I^3_{\mathbb{P}(L_d)} \supset I_{B_{S_d}}$$

where B_{S_d} is the base scheme of S_d. Via inner projection from L_d we may thus introduce a ruled structure for $S_d : \mathbb{P}(V(d)) \dashrightarrow \mathbb{P}(V(4))$, i.e. view $\mathbb{P}(V(d))$ birationally as a tower of Zariski-locally trivial projective bundles over $\mathbb{P}(V(4))$. Using a section of

$$\mathbb{P}(V(4)) \dashrightarrow \mathbb{P}(V(4))/G$$

we may then introduce a ruled structure also for $\bar{S}_d : \mathbb{P}(V(d))/G \dashrightarrow \mathbb{P}(V(4))/G$ and conclude by using the stable rationality of $\mathbb{P}(V(4))/G$.

This, however, so far hides the main technical problem which had to be addressed in [BvB1]: one needs the genericity statement that a general projection fibre $\mathbb{P}(L_d + \mathbb{C}g)$, $g \in V(d)$, is mapped surjectively to $\mathbb{P}(V(4))$ under S_d. The difficulty in checking this comes from the fact that L_d is defined in terms of monomials whereas S_d can be evaluated most conveniently on forms $f \in V(d)$ which are written as sums of powers of linear forms.

For the details of how this problem is resolved we have to refer to [BvB1], here we just list the main ingredients in the argument in the form of key words: interpolation polynomials, consideration of leading terms, reduction to finite fields $\mathbb{F}_p$, upper-semicontinuity over $\mathrm{Spec}(\mathbb{Z})$.

References

[Bo-Ka] Bogomolov, F. & Katsylo, P., *Rationality of some quotient varieties*, Mat. Sbornik **126** (1985), 584-589

[B09] Böhning, Chr., *The rationality problem in invariant theory*, (submitted, part of monograph in preparation) (2009) `arXiv:0904.0899v1`

[BvB1] Böhning, Chr. & Graf v. Bothmer, H.-Chr., *The rationality of the moduli spaces of plane curves of sufficiently large degree*, Inventiones Mathematicae (2010) **179**, p. 159-173, available via open access DOI 10.1007/s00222-009-0214-6

[BvBK] Böhning, Chr., Graf v. Bothmer, H.-Chr. & Kröker, Jakob, *Rationality of moduli spaces of plane curves of small degree*, (accepted for publication in Experimental Mathematics) (2009) `arXiv:0904.0890v1`

[G-Y] Grace, J.H. & Young, W.H., *The Algebra of Invariants*, Cambridge Univ. Press (1903); reprinted by Chelsea Publ. Co. New York (1965)

[Kat89] Katsylo, P.I., *Rationality of moduli varieties of plane curves of degree* $3k$, Math. USSR Sbornik, Vol. **64**, no. 2 (1989)

[Kat92/2] Katsylo, P.I., *On the birational geometry of the space of ternary quartics*, Advances in Soviet Math. **8** (1992), 95-103

[Kat96] Katsylo, P.I., *Rationality of the moduli variety of curves of genus* 3, Comment. Math. Helvetici **71** (1996), 507-524

[Sa] Saltman, D., *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984), 71-84

[Shep] Shepherd-Barron, N.I., *The rationality of some moduli spaces of plane curves*, Compositio Mathematica **67** (1988), 51-88

Geometric constructions of Enriques involutions and special families of Enriques surfaces

Klaus Hulek
(joint work with Matthias Schütt)

1. Introduction

The main purpose of this talk was to present a geometric construction of Enriques involutions on jacobian elliptic K3 surfaces and to relate this to familiar examples of families of Enriques surfaces with special geometric properties.

2. The construction

Let $S \to \mathbb{P}^1$ be a rational elliptic surface with a section. S is the blow-up of $\mathbb{P}^2$ in nine points (possibly infinitely near). If $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a degree 2 base change morphism which is ramified at points where the fibres of S are non-reduced, then the resulting surface $X \to \mathbb{P}^1$ is a jacobian elliptic K3 surface (which depends on 10 moduli). The generic such K3 surface has Néron-Severi group $\mathrm{NS}(X) = U \oplus E_8(-2)$ where U denotes the hyperbolic plane and $E_8(-2)$ is the unique even unimodular negative definite rank 8 lattice $E_8(-1)$ whose form has been multiplied by 2. Such a K3 surface X does not admit an Enriques involution since $\mathrm{NS}(X)$ does not contain the lattice $U(2) \oplus E_8(-2)$ as a primitive sublattice. The main point of this talk was to discuss a method to construct Enriques involutions on subfamilies (of dimension up to 9) where one can construct Enriques involutions geometrically.

Let ι be the deck transformstion on X and denote the hyperelliptic involution by (-1). Then $j = \iota \circ (-1)$ has 8 fixed points, namely the 2-torsion points on the fibres of X which lie over the ramification points of f. Thus j defines a Nikulin involution and the minimal model X' of the quotient $X/\langle j \rangle$ is again a K3 surface. We thus obtain the following diagram

$$\begin{array}{ccccc} & & X & & \\ & \overset{f_\iota}{\swarrow} & \downarrow & \overset{f_j}{\searrow} & \\ S & & \mathbb{P}^1 & & X' \\ \downarrow & \overset{f}{\swarrow} & & \overset{f}{\searrow} & \downarrow \\ \mathbb{P} & & = & & \mathbb{P}^1. \end{array}$$

Now assume that X' has a section P' different from the 0-section and let P be its pullback to X. By $\boxplus P$ we denote the addition on the elliptic fibration X given by the section P. Let

$$\tau = \boxplus P \circ \iota \in \mathrm{Aut}(X).$$

This is an involution which multiplies the 2-form on X by -1. If the fibres of X over the ramification points of f are smooth, then τ is fixed point free if and only if it does not intersect the 0-section of X on these fibres. This is equivalent to saying that P' intersects the corresponding fibres on X' at non-identity components (in general these fibres will be I_0^* fibres). In this case τ is an Enriques involution, i.e. the quotient surface $Y = X/\langle\tau\rangle$ is a smooth Enriques surface. We shall refer to this construction as an *Enriques involution of base change type.*

3. Special families

The above approach can be used to construct several interesting families of Enriques surfaces.

We first consider *special* Enriques surfaces, i.e. Enriques surfaces Y containing a smooth rational curve R (which is then nodal, i.e. $R^2 = -2$). The inverse image of R on the K3 cover X of Y splits into two disjoint curves. Cossec has shown that special Enriques surfaces admit elliptic fibrations which contain a smooth rational curve as bisection. Pulling this back to X this becomes a section P of X and one is exactly in the situation where the section P and the 0-section are disjoint.

In order to illustrate our method further we consider the lattice $U + 2E_8(-1) + \langle -2M\rangle$. This has a unique embedding into the K3 lattice $L_{\mathrm{K3}} = 3U + 2E_8(-1)$. Hence there is a 1-dimensional family of K3 surfaces whose generic element has this Néron-Severi group.

Proposition 1. *Let $M \in \mathbb{N}$ and X be a* K3 *surface with* $\mathrm{NS}(X) = U + 2E_8(-1) + \langle -2M\rangle$.

i) *If M is odd, then X does not admit an Enriques involution.*
ii) *If M is even, then X admits an Enriques involution of base change type.*

It is also possible to consider higher dimensional families with an Enriques involution. An example is given by the lattice $U + E_8(-2) + \langle -2M\rangle$ which also admits a unique embedding into the K3 lattice. In analogy to the result above we obtain

Proposition 2. *Let $M \in \mathbb{N}$ and X be a* K3 *surface with* $\mathrm{NS}(X) = U + E_8(-2) + \langle -2M\rangle$.

i) *If M is odd, then X does not admit an Enriques involution.*
ii) *If M is even, then X admits an Enriques involution of base change type.*

We note that Ohashi [3] has recently studied families of K3 surfaces which admit Enriques involutions from a lattice theoretic point of view and that the above 9-dimensional families appear in his classification. The case $M = 2$ is the case of special Enriques surfaces.

The *Barth-Peters family* is a 2-dimensional family of Enriques surfaces which admits a cohomologically trivial involution. This family was studied by Mukai [4], [5] and Mukai and Namikawa [6] showed that this is the only example of Enriques surfaces which admit a cohomologically trivial involution. One can show that this family also fits into our framework. In fact, it can be constructed by starting with the rational elliptic surface given by the Weierstrass equation

$$y^2 = x^3 + x^2 + sx.$$

Then $P = (0, 0)$ defines a 2-torsion section on S whose pullback to X defines the required Enriques involution, as P also descends to a section P' on the quotient

X'. The two parameters from the base change give rise to the two parameters of the Barth-Peters family. Moreover, the 1-dimensional families from Proposition 1 with $M = 2, 4$ can be identified as subfamilies of the Barth-Peters family.

4. Brauer groups

An Enriques surface Y has Brauer group $\mathrm{Br}(Y) = \mathbb{Z}/2\mathbb{Z}$. If $\pi : X \to Y$ is the K3 cover then one has the two possibilities that either $\pi^* \mathrm{Br}(Y) = \mathbb{Z}/2\mathbb{Z}$ or that $\pi^* \mathrm{Br}(Y) = 0$. The latter happens on a countable number of proper subvarieties in the moduli space on Enriques surfaces. Beauville has asked whether an example of such an Enriques surface can be found over the rationals. He has also shown the following

Theorem 3 (Beauville)**.** *In the above notation, the following statements are equivalent:*

i) $\pi^* \mathrm{Br}(Y) = \{0\} \subset \mathrm{Br}(X)$;
ii) *There is a divisor D on X such that $\tau^* D = -D$ in* $\mathrm{NS}(X)$ *and* $D^2 \equiv 2 \mod 4$ *(where τ is the Enriques involution).*

It is then easy to see that K3 surfaces with $\mathrm{NS}(X) = U + 2E_8(-1) + \langle -4M \rangle + \langle -2N \rangle$ for $N > 1$ odd admit an Enriques quotient Y with the property that the pullback of the Brauer group to the K3 cover is trivial. Since X is a singular K3 surface, these examples are defined over number fields (see also [2]). For $M = 1, N = 3$ one can show that X and Y have a model over the rationals, thus giving a positive answer to Beauville's question.

References

[1] K. Hulek, M. Schütt, *Enriques Surfaces and jacobian elliptic* K3 *surfaces*, arXiv:0912.0608.
[2] K. Hulek, M. Schütt, *The arithmetic of singular Enriques surfaces*, arXiv:1002.1598.
[3] H. Ohashi, *On the number of Enriques quotients of a* K3 *surface*, Publ. RIMS **43** (2007), 181–200.
[4] S. Mukai, *Numerically trivial involutions of an Enriques surface*, RIMS preprint **1544** (2006).
[5] S. Mukai, Addendum to *Numerically trivial involutions of an Enriques surface*, preprint (2009).
[6] S. Mukai, Y. Namikawa, *Automorphisms of Enriques surfaces which act trivially on the cohomology groups*, Invent. Math. **77** (1984), 383–397.

The Deligne-Mumford compactification of Hilbert modular varieties

Martin Möller
(joint work with Matt Bainbridge)

Each Hilbert modular surface has a beautiful minimal smooth compactification due to Hirzebruch. Higher-dimensional Hilbert modular varieties instead admit many toroidal compactifications none of which is clearly the best. In the talk, we describe a canonical compactifications of closely related varieties, namely the real multiplication locus $\mathcal{R}M_{\mathcal{O}}$ in the moduli space $\mathcal{M}_g$ of genus g Riemann surfaces.

$\mathcal{R}M_{\mathcal{O}}$ is the locus of Riemann surfaces, whose Jacobian has real multiplication by an order $\mathcal{O}$ in a totally real field of degree g over $\mathbb{Q}$. For $g = 2$ and $g = 3$ we give a complete characterization of the boundary components. The situation for $g \geq 4$ is more complicated due to the Schottky problem. We show a containment statement that yields a sharp upper bound for possible boundary components.

The original motivation was to understand (and classify) Teichmüller curves in genus three. The cross-ratio equation in the main theorem gives an enormous constraint for the existence of these Teichmüller curves. We refer to [1] for details on how far this classification problem has been pushed in $g = 3$. The main theorem also is likely to have applications to estimating the dimension of the intersection of Hilbert modular varieties with $\mathcal{M}_g$ for $g \geq 4$ as well as to the existence question of Shimura curves in $\mathcal{M}_g$ for large g.

The main idea to understand $\partial\mathcal{R}M_{\mathcal{O}}$ is to use not only the curve and its Jacobian with real multiplication but also the differential forms that are eigenforms for $\mathcal{O}$-multiplication. The number theory of the residues of the eigenforms at the nodes of a stable curve governs the question whether this stable curve lies in the boundary of $\mathcal{R}M_{\mathcal{O}}$. In order to state the main theorem, we thus do not work in $\mathcal{M}_g$ but rather in $\Omega\overline{\mathcal{M}}_g$, the total space of the relative dualizing sheaf over $\mathcal{M}_g$. We denote by $\mathcal{E}_{\mathcal{O}} \subset \Omega\mathcal{M}_g$ the space of eigenforms for real multiplication.

Consider the quadratic map $Q\colon F \to F$, defined by

$$Q(x) = N^F_{\mathbb{Q}}(x)/x. \tag{1}$$

We say that a finite subset $S \subset F$ satisfies the *no-half-space condition* if the interior of the convex hull of $Q(S)$ in the $\mathbb{R}$-span of $Q(S)$ in $F \otimes_{\mathbb{Q}} \mathbb{R}$ contains 0.

It is well known that every stable curve which is in the closure of the real multiplication locus $\mathcal{R}M_{\mathcal{O}} \subset \mathcal{M}_g$ has geometric genus 0 or g. Our description of the closure of the eigenform locus for the interesting special case $g = 3$ reduces to the following theorem.

Theorem 1. *A geometric genus* 0 *stable curve* X *together with a section* ω *of the dualizing sheaf of* X *lies in the boundary of the eigenform locus* $\Omega\mathcal{E}_{\mathcal{O}}$ *if and only if:*

- *The set of residues of* ω *is a multiple of* $\iota(S)$, *for some subset* $S \subset F$, *satisfying the no-half-plane condition and spanning an ideal* $\mathcal{I} \subset \mathcal{O}$, *and for some embedding* $\iota\colon F \to \mathbb{R}$.
- *If* $Q(S)$ *lies in a* $\mathbb{Q}$*-subspace of* F, *then an explicit additional equation, involving cross-ratios of the nodes of* X, *is satisfied.*

Existence questions and counting problems for the intersection of $\mathcal{R}M_{\mathcal{O}}$ with the various boundary components, depending on $\mathcal{O}$ (or the size of its discriminant) and the topology of the stable are very interesting and to a large extent open even for $g = 3$. We refer to [1] for details and the relation to counting the intersections of geodesic flats in the symmetric space $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.

References

[1] Bainbridge, M., Möller, M., *The Deligne-Mumford compactification of Hilbert modular varieties and Teichmüller curves in genus three*, preprint, math.AG/0911.4677

On the moduli space of spin curves

Alessandro Verra
(joint work with Gavril Farkas)

The purpose of this report is to describe some new results, jointly obtained by Gavril Farkas and the author, on the Kodaira dimension of the moduli space of spin curves; as well as on further global properties of these spaces in low genus, like uniruledness or unirationality.
As is well known an even (odd) spin curve of genus g is a pair (C, η) such that C is a smooth, irreducible projective curve and η is an even (odd) theta characteristic on C. We will assume that C is defined over $\mathbf{C}$.
For every $g \geq 1$ the moduli space of spin curves of genus g splits in two irreducible connected components $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$, which respectively parametrize even and odd spin curves. Suitable compactifications $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$ of the moduli $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$ are also well known, see [C].
Adding to the above mentioned new results some older ones, the picture on the Kodaira dimension of $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$, and on uniruledness / unirationality questions in low genus, appears quite complete. Such a picture can be summarized as follows.

Theorem 1. *$\overline{\mathcal{S}}_g^+$ has Kodaira dimension:*

- $3g-3$ *for* $g \geq 9$ [F],
- *zero for* $g = 8$ [FV1]
- *negative for* $g \leq 7$.

Moreover the following results are contained in [FV2]:

Theorem 2. *$\overline{\mathcal{S}}_g^-$ has Kodaira dimension:*

- $3g-3$ *for* $g \geq 12$,
- *negative for* $g \leq 7$.

Theorem 3.

- $\mathcal{S}_g^+$ *is uniruled for* $g \leq 7$,
- $\mathcal{S}_g^-$ *is uniruled for* $g \leq 11$.
- $\mathcal{S}_g^-$ *is unirational for* $g \leq 9$.

Among the previous results the case of $\overline{\mathcal{S}}_8^+$ is quite appealing. Here the transition from the uniruled/unirational case to the case where the moduli space is of general type has an intermediate step, because

$$kod(\overline{\mathcal{S}}_g^+) = 0$$

for $g = 8$. A sketch of the proof of this property goes as follows. It is due to Farkas that the canonical class of the canonical divisor $K_{\overline{\mathcal{S}}_8^+}$ of $\overline{\mathcal{S}}_8^+$ contains an effective divisor, [F]. Namely one has

$$K_{\overline{\mathcal{S}}_g^+} \equiv cM + 8\Theta_{null} + \sum_{i=1\dots4} (a_iA_i + b_iB_i)$$

with $a_i, b_i, c > 0$. Here $A_i, B_i, M, \Theta_{null}$ are the following divisors:
- A_i, B_i, $i = 1\dots4$, are the standard boundary divisors on $\overline{\mathcal{S}}_8^+$.
- Let $\pi : \overline{\mathcal{S}}_8^+ \to \overline{\mathcal{M}}_8$ be the forgetful map. Then M is the pull-back by π of the divisor in $\overline{\mathcal{M}}_8$ parametrizing plane septic curves of genus 8.
- Finally Θ_{null} parametrizes even spin curves (C, η) such that $h^0(\eta) > 0$.

To prove that the above effective canonical divisor of $\overline{\mathcal{S}}_8^+$ has Kodaira dimension zero, it suffices to apply to it the following elementary remark:

Remark 4. Let $D = D_1 + \cdots + D_m$ be a sum of effective, integral $\mathbf{Q}$-Cartier divisors $D_1, \dots, D_m$ on an integral variety X. Assume that each D_i is covered by a family of integral curves R_i such that $R_i \cdot D_j = 0$ for $i \neq j$ and $R_i \cdot D_i < 0$. Then D has Kodaira dimension zero.

To apply the remark to the divisors M, Θ_{null}, A_i and B_i one needs to exploit deeply the geometry of canonical curves of genus 8.
One of the steps is the construction of a family of covering curves R of Θ_{null} with the property prescribed by the remark. To this purpose the following theorems are proved in [FV1], which imply the existence of the required family of curves R in the divisor Θ_{null}:

Theorem 5. *Let (C, η) be a general even spin curve such that $h^0(\eta) = 2$. Then $C \subset S$, where S is a K3 surface of Picard number two such that*

$$Pic\ S \cong \mathbb{Z}[F_1] \oplus \mathbb{Z}[F_2]$$

and $F_i^2 = 0$, $i = 1, 2$, and $F_1F_2 = 7$. Moreover $F_1 + F_2$ is very ample and

$$C \in \mid F_1 + F_2 \mid .$$

Furthermore it holds

$$\eta \cong \mathcal{O}_C(F_1) \cong \mathcal{O}_C(F_2).$$

Note that the latter condition implies that $C \in \phi^* \mid \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,1) \mid$, where $\phi : S \to \mathbf{P}^1 \times \mathbf{P}^1$ is the morphism defined by the the product of the maps defined by the elliptic pencils $\mid F_1 \mid$ and $\mid F_2 \mid$. In particular C moves in a pencil

$$P \subset \phi^* \mid \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,1) \mid .$$

Let $D \in P$ be general, it is easy to see that then $\eta_D := \mathcal{O}_D(F_1) \cong \mathcal{O}_D(F_2)$ is an even theta characteristic such that $h^0(\eta_D) = 2$. Therefore P defines a family $\{(D, \eta_D),\ D \in P\}$ of even spin curves. The image of P in $\overline{\mathcal{S}}_8^+$ is a rational curve

$$R \subset \Theta_{null}$$

passing through the moduli point of (C, η). The conclusion is the following:

Theorem 6. $R \cdot \Theta_{null} = -1$ *and* $R \cap A_i = R \cap B_i = R \cap M = \emptyset$.

Similar results can be proved, with similar types of geometric constructions, for suitable family of integral rational curves covering A_i, B_i, M.
Hence $\overline{\mathcal{S}}_8^+$ has Kodaira dimension zero.

References

[C] M. Cornalba, *Moduli of curves and theta-characterstics*, in: Lectures on Riemann surfaces (Trieste, 1987), 560-589.

[F] G. Farkas, *The birational type of the moduli space of even spin curves*, Advances in Mathematics (2010), to appear.

[FV1] G. Farkas and A. Verra, *The intermediate type of certain moduli spaces of curves*, arXiv math.AG-0910.3905 (2009)

[FV2] G. Farkas and A. Verra, *The geometry of the odd spin moduli space*, preprint.

Functoriality of Gromov–Witten theory under crepant transformation

Yuan-Pin Lee

(joint work with H.-W. Lin, C.-L. Wang)

Let X be a nonsingular projective variety, and $\psi : X \to \bar{X}$ be a flopping contraction with $\bar{\psi} : Z \to S$ as the restriction to the exceptional loci. Assume the exceptional loci have the following structure: There are two rank $r+1$ bundles F, F' over S such that $Z = \mathbb{P}_S(F)$ and the normal bundle $N_{Z|X} \cong \mathcal{O}(-1) \otimes \bar{\psi}^* F'$. Then Mori theory tells us that there is a flop $X \dashrightarrow X'$, which is called an *ordinary* $\mathbb{P}^r$ *flop*.

In this project, we proved

Theorem 1. [1] *For an ordinary flop, the graph closure induces an equivalence of Chow motives of X and X'. In particular, the equivalence preserves the intersection pairing.*

However, the ring structure is not invariant under the above equivalence, as can be computed in simple examples. What is surprising about the next result is that the quantum ring structure becomes invariant after an analytic continuation.

Theorem 2. *The quantum ring (small or big) is invariant under the ordinary flops via the above identigication, after a (necessarily non-trivial) analytic continuation on the extended Kähler moduli space, "modelled" on the* Euler series

$$\sum_{d \in \mathbb{Z}} q^d = 0,$$

along the direction of the flopped curve class.

In fact, this result holds for higher genus as well.

Theorem 3. [1, 2, 3] *The full Gromov–Witten theory is invariant under the ordinary flops via the above identification, after an analytic continuation on the extended Kähler moduli space.*

A contraction is of *Mukai type* if $Z \cong \mathbb{P}_S(F)$ such that $N_{Z|X} \cong T^*_{Z|X}$. The corresponding flop $X \dashrightarrow X'$, whose existence is again guaranteed by the Mori theory, is called a *Mukai flop.*

Theorem 4. [1] *A Mukai flop is a slice of an ordinary flop. It preserves the diffeomorphism type, Hodge structure, and the full Gromov–Witten theory.*

In the literature, the *Crepant Transformation Conjecture* are usually established in the following two categories. The first category contains those examples where the global structure of X and X' are explicit and computable (toric, finite group quotients of $\mathbb{C}^n$ etc.), and the proof goes by more or less computing both sides and equating them. The second one is for those the Gromov–Witten invariants associated to the extremal ray vanishes (e.g. Mukai flops).

In [1, 2], we establish a class of crepant transformation (i.e. K-equivalence) where the global structure of the varieties are non-explicit. In [3], we generalize this to the cases where even the local structure of exceptional loci are non-explicit. Note that Gromov–Witten invariants are invariant under (symplectic) deformation, and the above results naturally generalize to those cases.

The main ingredients in the proof are

- Explicit computation of Chow rings of projective bundles under a flop.
- Degeneration formula and Virtual localization.
- Classification of *algebraic cobordism* of vector bundles on varieties [4].
- Gamma function regularization and analytic continuation.

These consist of parts of a joint project with H.-W. Lin and C.-L. Wang from National Taiwan University.

References

[1] Y.-P. Lee, H.-W. Lin, C.-L. Wang, *Flops, motives and invariance of quantum rings*, Annals of Math., to appear.
[2] Y. Iwao, Y.-P. Lee, H.-W. Lin, C.-L. Wang, *Invariance of Gromov–Witten theory under simples flops*, preprint, 2007.
[3] Y.-P. Lee, H.-W. Lin, C.-L. Wang, in preparation.
[4] Y.-P. Lee, R. Pandharipande, *Algebraic Cobordism of Bundles on Varieties*, preprint, 2010.

Virtual push-forwards

Cristina Manolache

Ideally, we would like to give an answer to the following question:

If we are given a morphism of smooth projective varieties $p : X \to Y$ and we know the Gromov-Witten invariants of Y can we compute (some of) the Gromov-Witten invariants of X?

One way of attacking this problem is to try to compare the virtual classes (see [1], [6]) of the moduli spaces of stable maps $\bar{M}_{g,n}(X,\beta)$ and $\bar{M}_{g,n}(X,p_*\beta)$. One of the

easiest examples of such a comparison is the following:
Let $\tilde{\mathbb{P}^n}$ be a blow up of a projective space $\mathbb{P}^n$ in a smooth subvariety. Then, the projection $p : \tilde{\mathbb{P}^n} \to \mathbb{P}^n$ induces a map between the moduli spaces of stable maps

$$\bar{p} : \bar{M}_{0,n}(\tilde{\mathbb{P}^n}, \tilde{d}) \to \bar{M}_{0,n}(\mathbb{P}^n, d)$$

where $\tilde{d}$ is the class of a strict transform of a general line in $\mathbb{P}^n$ of degree d. Using the fact that $\bar{M}_{0,n}(\mathbb{P}^n, d)$ is smooth of the expected dimension, one can easily see that

$$\bar{p}_*([\bar{M}_{0,n}(\tilde{\mathbb{P}^n}]^{\text{virt}}\tilde{d}) = [\bar{M}_{0,n}(\mathbb{P}^n, d)]^{\text{virt}}.$$

Using the projection formula (see [4]) one obtains that for any $\gamma \in A^*(\mathbb{P}^n)$ we have

$$ev_i^* p^* \gamma \cdot \bar{p}_*([\bar{M}_{0,n}(\tilde{\mathbb{P}^n}]^{\text{virt}}\tilde{d}) = ev_i^* \gamma \cdot [\bar{M}_{0,n}(\mathbb{P}^n, d)]^{\text{virt}}.$$

Having this example in mind, we can move to a more general context. Let us slightly change a definition of Gathmann (see [2])

Definition 1. *Let $p : F \to G$ be a proper morphism of stacks possessing virtual classes $[F]^{\text{virt}} \in A_{k_1}(F)$ and $[G]^{\text{virt}} \in A_{k_2}(G)$ with $k_1 \geq k_2$ and let $[G]_1^{\text{virt}}, ..., [G]_s^{\text{virt}} \in A_{k_2}(G)$ be irreducible cycles such that $[G]^{\text{virt}} = m_1[G]_1^{\text{virt}} + ... + m_s[G]_s^{\text{virt}}$ for some $m_1, ..., m_s \in \mathbb{Q}$. Let $\gamma \in A^{k_3}(F)$, with $k_3 \leq k_1 - k_2$ be a cohomology class. We say that p satisfies the virtual push-forward property for $[F]^{\text{virt}}$ and $[G]^{\text{virt}}$ if the following two conditions hold:*
1. If the dimension of the cycle $\gamma \cdot [F]^{\text{virt}}$ is bigger than the virtual dimension of G then $p_*(\gamma \cdot [F]^{\text{virt}}) = 0$.
2. If the dimension of the cycle $\gamma \cdot [F]^{\text{virt}}$ is equal to the virtual dimension of G then $p_*(\gamma \cdot [F]^{\text{virt}}) = n_1[G]_1^{\text{virt}} + ... + n_s[G]_s^{\text{virt}}$ for some $n_1, ..., n_s \in \mathbb{Q}$.
If moreover, the following condition holds, we say that p satisfies the strong *virtual push forward property for $[F]^{\text{virt}}$ and $[G]^{\text{virt}}$:*
2′. If the dimension of the cycle $\gamma \cdot [F]^{\text{virt}}$ is equal to the virtual dimension of G then $p_*(\gamma \cdot [F]^{\text{virt}})$ is a scalar multiple of $[G]^{\text{virt}}$.

We are interested in finding conditions for a morphism $p : F \to G$ to satisfy the (strong) virtual push-forward property. For this let us first give a definition.

Definition 2. *Let $p : F \to G$ be a proper morphism of stacks possessing virtual classes of virtual dimensions k_1 respectively k_2 with $k_1 \geq k_2$ and let us assume that we have a morphism of obstruction theories $\varphi : p^* E_G^\bullet \to E_F^\bullet$. If the relative obstruction theory induced by φ (see* [7]*, Construction 2) is perfect (in the sense of* [1]*), then we call p a virtually smooth morphism.*

Using the properties of virtual classes in [3] and [7] we obtain the following results.

Lemma 3. *Let $p : F \to G$ be a proper virtually smooth morphism of Deligne-Mumford stacks. Then p satisfies the virtual push-forward property.*

Theorem 4. *Let $p : F \to G$ be a virtually smooth morphism. If G is connected, then p satisfies the strong virtual push-forward property (in homology).*

These results have applications in Gromov-Witten theory: blow-ups (see [5], [7]), smooth fibrations $p : X \to Y$). One question that arises naturally is: "When is a moduli space of stable maps connected?" and "Are there cases in which we can replace the connectivity by a weaker condition?"

References

[1] K. Behrend, B. Fantechi, The intrinsic normal cone. Invent. Math. **127** (1997), no.1, 45–88.

[2] A. Gathmann, Gromov-Witten invariants of hypersurfaces, Habilitation thesis, University of Kaiserslautern, Germany (2003)

[3] B. Kim, A. Kresch and T. Pantev, Functoriality in intersection theory and a conjecture of Cox, Katz and Lee, J. Pure and Appl. Algebra, **179** (2003), no. 1-2, 127–146.

[4] A. Kresch, Cycle groups for Artin stacks. Invent. Math., **138** (1999), no. 3, 495–536.

[5] Hsin-Hong Lai, Gromov-Witten invariants of blow-ups along manifolds with convex normal bundle, Geom. Topol. 13 (2009), no. 1, 1–48.

[6] J Li, G. Tian, Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties, J. Amer. Math. Soc. 11 (1998) 119–174.

[7] C. Manolache, Virtual pull-backs, arXiv:0805.2065v1 [math.AG]

Hodge classes on families of Calabi-Yau manifolds

Stefan Müller-Stach

(joint work with Pedro Luis del Angel, Duco van Straten and Kang Zuo)

Zucker has developed a Dolbeault version of L^2-cohomology for variations of Hodge structures over curves. This was later extended by Jost, Yang and Zuo to compactifiable Kähler manifolds. The construction uses the monodromy weight filtration. In the talk smooth families $f : X \to S$ of Calabi-Yau 3-folds over a (non-compact) curve S are discussed. In joint work with del Angel, van Straten and Zuo we have computed formulas to obtain L^2-Hodge numbers $h^{p,q}$ $(p + q = 4)$ of $H^1_{L^2}(S, V)$, where $V = R^3 f_* \mathbb{C}$. Such formulas are interesting because the Hodge number $h^{2,2}$ allows to predict the existence of algebraic cycles in $CH^2(X)$ which have non-trivial Abel-Jacobi map on each fibre. Such classes occur naturally in open string theory. Our results are published in Acta Vietnamica Vol. 35, pp. 1-16 (2010). If the fibers are elliptic curves or K3 surfaces such formulas are also interesting and can be obtained in the same way.

Cohomology of Moduli Spaces and Modular Forms

Gerard van der Geer

(joint work with Jonas Bergström and Carel Faber)

On the one hand we are interested in the cohomology of moduli spaces, such as the moduli M_g of curves of genus g or the moduli A_g of principally polarized abelian varieties of dimension g for small values of g, and on the other hand in modular forms on $\mathrm{SL}(2, \mathbb{Z})$ or on $\mathrm{Sp}(2g, \mathbb{Z})$ for small values of g. There is an intimate relation between the two that can be used to let information flow both ways. These moduli spaces are defined over $\mathbb{Z}$ and the idea is that one can study the cohomology over $\mathbb{Q}$ by looking at the fibre $M_g \otimes \mathbb{F}_p$ with $\mathbb{F}_p$ a finite field and using comparison theorems; we get information about the ℓ-adic étale cohomology ($\ell \neq p$) of $M_g \otimes \overline{\mathbb{F}}_p$ by counting points over finite fields.

Let us start with $g = 1$. The space S_k of cusp forms of weight k on $\mathrm{SL}(2, \mathbb{Z})$ has a cohomological interpretation: consider the universal elliptic curve $\pi : \mathcal{X}_1 \to A_1$ and the local system $V = R^1\pi_*\mathbb{Q}$ of rank 2. For $a \in \mathbb{Z}_{\geq 1}$ we have the local system $V_a = \mathrm{Sym}^a(V)$ of rank $a + 1$. We look at the Euler characteristic

$$e_c(A_1, V_a) = \sum_{i=0}^{2}(-1)^i[H_c^i(A_1, V_a)]\,,$$

where the subindex c refers to compactly supported cohomology and the square brackets indicate that we consider the cohomology in an appropriate Grothendieck group of mixed Hodge modules or Galois representations (for the ℓ-adic counterpart $V_a^{(\ell)}$). Note that the cohomology vanishes for a odd.

Then we have $e_c(A_1, V_a) = -S[a + 2] - 1$ for even $a \geq 2$ with $S[k]$ the motive associated to the space of cusp forms S_k as constructed by Scholl. The Eichler-Shimura congruence relation then implies that the trace of Frobenius on $H_c^1(A_1 \otimes \overline{\mathbb{F}}_p, V_a^{(\ell)})$ equals $1 + \mathrm{tr}(T(p), S_{a+2})$, that is, 1 plus the trace of the Hecke operator $T(p)$ on S_{a+2}. After enumerating elliptic curves over $\mathbb{F}_p$ (with the order of their automorphism groups) up to isomorphism over $\mathbb{F}_p$ and counting their number of rational points one can thus calculate the trace of the Hecke operator $T(p)$ on S_k for all $k \geq 4$. Of course, there are other ways to calculate these.

We applied this approach to genus 2 by looking at the universal abelian surface $\pi : \mathcal{X}_2 \to A_2$, the local system $V = R^1\pi_*\mathbb{Q}$ and the symplectic local systems V_λ with $\lambda = (a, b)$ associated to a representation of $\mathrm{Sp}(4, \mathbb{Q})$ of highest weight $a - b, b$. We write $e_c(A_2, V_\lambda) = \sum_i(-1)^i[H_c^i(A_2, V_\lambda)]$ for the Euler characteristic. By a beautiful formula of Getzler the cohomology of $M_{2,n}$ can be expressed in the cohomology of such local systems on M_2, see [7]. So we cover the spaces $M_{2,n}$ as well.

Note that the cohomology vanishes if $a + b$ is odd. A result of Faltings tells us that $H^i(A_2, V_\lambda)$ and H_c^i have mixed Hodge structures and $H_!^i = \mathrm{Im}(H_c^i \to H^i)$ has a pure Hodge structure. Moreover, if λ is regular, i.e., $a > b > 0$, then if $H_!^i(A_2, V_\lambda) \neq (0)$ we have $i = 3$. The first step in the Hodge filtration $F^{a+b+3} \subset F^{a+2} \subset F^{b+1} \subset F^0 = H_!^3(A_2, V_\lambda)$ can be interpreted as a space of vector-valued Siegel modular cusp forms:

$$F^{a+b+3} \cong S_{a-b,b+3},$$

with the factor of automorphy being $\mathrm{Sym}^{a-b}(C\tau + D)\det(C\tau + D)^{b+3}$ for a matrix $\tau = (A, B; C, D) \in \mathrm{Sp}(2g, \mathbb{Z})$.

If we want to use the traces of Frobenius obtained by counting over finite fields to calculate the traces of the Hecke operators as we did for $g = 1$ we face for $g = 2$ two problems. First we must calculate the Eisenstein cohomology, that is, the kernel $\sum(-1)^i \ker(H_c^i \to H^i)$; this we did in [6, 4]. Second, there are contributions that do not see the first and the last part of the Hodge filtration (endoscopy). We gave a conjectural formula for this in [4]. In [9], Weissauer shows that the conjecture (in the case of a regular weight) can be deduced from earlier work of his. Assuming this the formula for the trace of the Hecke operator $T(p)$ on $S_{a-b,b+3}$ is

$$-\text{trace of } F \text{ on } e_c(A_2 \otimes \mathbb{F}_p, V^{\ell}_{a,b}) + \text{trace of } F \text{ on } e_{2,\text{extra}}(a,b)$$

with F Frobenius and $e_{2,\text{extra}}(a,b)$ given by

$$s_{a-b+2} - s_{a+b+4}(S[a-b+2]+1)L^{b+1} + \begin{cases} S[b+2]+1 & a \equiv 0(\text{mod}2) \\ -S[a+3] & a \equiv 1(\text{mod}2), \end{cases}$$

and L the Lefschetz motive and $s_k = \dim S_k$. With this formula and our counting we can calculate the trace of $T(p)$ on the spaces $S_{j,k}$ for all j and k. The results it gives agree with everything we know about $g=2$ modular forms. Inspired by our results Harder formulated a conjecture about congruences between $g=1$ and $g=2$ modular forms and we obtained a lot of numerical evidence for this, see [8, 6]. All of these things have been generalized to $g=2$ and level 2 in [1].

What about $g=3$? There we have a degree 2 map of stacks $M_3 \to A_3$. We now have local systems $V_{a,b,c}$ parametrized by triples (a,b,c) with $a \geq b \geq c \geq 0$. We are interested in vector-valued Siegel modular cusp forms of weight $(a-b, b-c, c+4)$, i.e. holomorphic functions $f : \mathcal{H}_3 \to W$ on the Siegel upper half space $\mathcal{H}_3$ to a finite-dimensional complex vector space W satisfying

$$f((a\tau+b)(c\tau+d)^{-1}) = \rho(c\tau+d)f(\tau)$$

where ρ is the irreducible representation of $\mathrm{GL}(3,\mathbb{C})$ on W of highest weight $a-b, b-c, c+4$.

We now have the following *conjectural formula* for the trace of the Hecke operator $T(p)$ on the space of cusp forms $S_{a-b,b-c,c+4}$:

$$\text{trace of Frobenius on } e_c(A_3 \otimes \mathbb{F}_p), V_{a,b,c}) - e_{3,\text{extra}}(a,b,c),$$

with $e_{3,\text{extra}}(a,b,c)$ given by

$$\begin{aligned} -e_c(A_2, V_{a+1,b+1}) - e_{2,\text{extra}}(a+1,b+1) \otimes S[c+2] \\ +e_c(A_2, V_{a+1,c}) + e_{2,\text{extra}}(a+1,c) \otimes S[b+3] \\ -e_c(A_2, V_{b,c}) - e_{2,\text{extra}}(b,c) \otimes S[a+4] \end{aligned}$$

The evidence we have is overwhelming and includes the following. It fits all the calculations we did over finite fields. The numerical Euler characteristic

$$\sum (-1)^i \dim H^i_c(A_3, V_{a,b,c})$$

is known by [2, 3] and this fits the results. We find that for $a+b+c \leq 60$ the space $S_{a-b,b-c,c+4}$ contributes a rank that is always divisible by 8. For $a=b=c$ it fits with the dimension formula for $\dim S_{0,0,c+4}$ for scalar-valued modular forms due to Tsuyumine. Moreover, we observed Harder-type congruences between $g=1$ and $g=3$ modular forms. We also have a precise conjectural formula for all the lifts from $g=1$ to $g=3$.

To illustrate this, assuming the conjecture we find for the eigenvalues of $T(p)$ with $p = 2, 3, 5$ and 7 on $S_{3,3,7}$ the values $2^3 \cdot 3^3 \cdot 5$, $2^6 \cdot 3^4 \cdot 5 \cdot 7$, $2^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 9749$ and $2^8 \cdot 5^3 \cdot 7^2 \cdot 8887$.

One can also look at the cohomology of M_3 instead of A_3. The degree 2 covering $M_3 \to A_3$ is ramified along the hyperelliptic locus. Unlike A_3 the moduli space M_3 can have cohomology for $a+b+c$ odd. This is related to Teichmüller modular forms that do not come from Siegel modular forms. An example is the modular form $\chi_9 = \sqrt{\chi_{18}}$ on M_3 that vanishes on the hyperelliptic locus and was studied by Ichikawa; we see it occurring in the cohomology of the local system $V_{5,5,5}$ on M_3.

References

[1] J. Bergström, C. Faber, G. van der Geer: Siegel modular forms of genus 2 and level 2: conjectures and cohomological computations. arXiv:0803.0917. *Int. Math. Res. Not.* 2008, Article ID rnn100, 20 p.

[2] J. Bergström, G. van der Geer: *The Euler characteristic of local systems on the moduli of curves and abelian varieties of genus three.* arXiv:0705.0293. *J. Topol.* **1**, 651-662 (2008).

[3] G. Bini, G. van der Geer: *The Euler characteristic of local systems on the moduli of genus 3 hyperelliptic curves*, Math. Ann. **332** (2005), 367–379.

[4] C. Faber, G. van der Geer: *Sur la cohomologie des systèmes locaux sur les espaces des modules des courbes de genre 2 et des surfaces abéliennes*, I, II. C.R. Acad. Sci. Paris, Sér. I, **338** (2004), 381–384, 467–470.

[5] G. van der Geer: *Siegel modular forms and their applications.* In: J. Bruinier, G. van der Geer, G. Harder, D. Zagier: The 1-2-3 of modular forms. Springer Verlag, 2008.

[6] G. van der Geer: *Rank one Eisenstein cohomology of local systems on the moduli space of abelian varieties.* arXiv:0802.2921

[7] E. Getzler: Resolving mixed Hodge modules on configuration spaces. *Duke Math. Journal* **96** (1999), 175–203.

[8] G. Harder: *A congruence between a Siegel and an elliptic modular form.* In: J. Bruinier, G. van der Geer, G. Harder, D. Zagier: The 1-2-3 of modular forms. Springer Verlag, 2008.

[9] R. Weissauer: The traces of the Hecke operators on the space of classical holomorphic Siegel modular forms. arXiv:0909.1744

Stability conditions for the local projective plane

Arend Bayer

(joint work with Emanuele Macrì)

I discussed our results [BM09] on the space of stability conditions on the derived category of the local $\mathbb{P}^2$, its group of autoequivalences, and its relation to mirror symmetry.

Motivation

Consider a projective Calabi-Yau threefold Y containing a projective plane Ideally, one would like to study the space of Bridgeland stability conditions on its derived category $\mathrm{D^b}(Y)$. Understanding the geometry of this space would give insights on the group of autoequivalences of $\mathrm{D^b}(Y)$ and give a global mirror symmetry picture. Understanding wall-crossing for counting invariants of semi-stable objects would have many implication for Donaldson-Thomas type invariants on Y.

However, no single example of stability condition on a projective Calabi-Yau threefold has been constructed. Instead, we study the full subcategory $\mathrm{D}^{\mathrm{b}}_{\mathbb{P}^2}(Y)$ of complexes concentrated on $\mathbb{P}^2 \subset Y$. Equivalently, we study the "local $\mathbb{P}^2$": the total space $X = \mathrm{Tot}\mathcal{O}_{\mathbb{P}^2}(-3)$ of the canonical bundle of $\mathbb{P}^2$, and its derived category $\mathcal{D}_0 := \mathrm{D}^{\mathrm{b}}_0(X)$ of coherent sheaves on X supported on the zero-section.

The space $\mathrm{Stab}(\mathcal{D}_0)$ of stability conditions on $\mathcal{D}_0$ is a three-dimensional complex manifold, coming with a local homeomorphism $\mathcal{Z}\colon \mathrm{Stab}(\mathcal{D}_0) \to \mathrm{Hom}(K(\mathcal{D}_0), \mathbb{C}) \cong \mathbb{C}^3$. The goal of this article is to study the space $\mathrm{Stab}(\mathcal{D}_0)$ as a test case for the properties we would expect in the case of Y; similar local example have been studied by Toda in [Tod08, Tod09].

This space $\mathrm{Stab}(\mathcal{D}_0)$ was first studied in [Bri06], where the author described an open subset, and conjectured a close relation to the Frobenius manifold of the quantum cohomology of $\mathbb{P}^2$. While this conjecture (and questions related to wall-crossing on $\mathrm{Stab}(\mathcal{D}_0)$) remains open, our results give a good description of a connected component of $\mathrm{Stab}(\mathcal{D}_0)$, explain its relation to autoequivalences of $\mathcal{D}_0$, and do give a global mirror symmetry picture.

Results

Our starting point is an explicit description of the "geometric chamber" U, which consists of stability conditions where all skyscraper sheaves $\mathcal{O}_x$ of points $x \in \mathbb{P}^2$ are stable.

For complex numbers $a, b \in \mathbb{C}$ with $\Im a > 0$ we define a map $Z\colon \mathrm{Coh_0X} \to \mathbb{C}$ given by

$$Z_{a,b}(E) = -\mathrm{ch}_2(E) + a \cdot \deg(E) + b \cdot \mathrm{rank}(E).$$

For $a \approx +i \cdot \infty$ and $b \approx +\infty$ one should think of this as a deformation of the map $Z(E) = i \cdot \deg(E) + \mathrm{rank}(E)$ that can be used to define slope-stability for sheaves on X. Let $B = -\frac{\Im b}{\Im a}$. Then a sheaf of slope $\mu = \frac{\deg(E)}{\mathrm{rank}(E)}$ will have $Z_{a,b}(E)$ in the upper half-plane if and only if $\mu > B$. This motivates the use of the "tilted" subcategory $\mathcal{A}^{\sharp(B)} \subset \mathcal{D}_0$ given by the following definitions:

$$\mathrm{Coh}^{>B} = \{F \in \mathrm{Coh}_0 X\colon \text{Any quotient of } F \text{ has slope } \mu > B\}$$
$$\mathrm{Coh}^{\leq B} = \{F \in \mathrm{Coh}_0 X\colon \text{Any subsheaf } F \text{ has slope } \mu \leq B\}$$
$$\mathcal{A}^{\sharp(B)} = \left\{E \cong \left(E^{-1} \xrightarrow{d} E^0\right) \in \mathcal{D}_0\colon \ker d \in \mathrm{Coh}^{\leq B}, \quad \mathrm{cok} d \in \mathrm{Coh}^{>B}\right\}$$

It is a standard fact that $\mathcal{A}^{\sharp(B)}$ is again an abelian category.

Then for any $E \in \mathcal{A}^{\sharp(B)}$ the complex number $Z_{a,b}(E)$ will automatically lie in the closure of the upper half-plane. If we additionally require inequalities for a, b (coming from the Chern classes of stable vector bundles of slope $\mu = B$), then $Z_{a,b}(E)$ will be in the semi-closed upper half-plane $\{z\colon z \in \mathbb{R}_{>0} \cdot e^{i\pi\phi} \text{for } \phi \in (0, 1]\}$. This ensures that we get a well-behaved notion of stability in the category $\mathcal{A}^{\sharp(B)}$ by comparing the phase of $Z_{a,b}(E)$ with the phases $Z_{a,b}(E')$ of subobjects E' of E. Proving the existence of Harder-Narasimhan filtrations yields the first part of:

Theorem 1. *Whenever a, b satisfy above-mentioned inequalities, the above construction produces a stability conditions on $\mathcal{D}_0$. Further, any stability condition for which skyscraper shaves $\mathcal{O}_x$ of points are stable must be (up to rescaling) of this form.*

The exact form of the inequalities follows from the classical results by [DLP85] on Chern classes of stable vector bundles on $\mathbb{P}^2$.

Let $\overline{U}$ be the closure of U in $\mathrm{Stab}(\mathcal{D}_0)$. One can directly construct every *wall* of U, i.e. the components of the boundary $\partial U = \overline{U} \setminus U$, using exceptional vector bundles on $\mathbb{P}^2$. We use this to prove the following result:

Theorem 2. *The translates of $\overline{U}$ under spherical twists at exceptional vector bundles on $\mathbb{P}^2$ cover a connected component $\mathrm{Stab}^\dagger(\mathcal{D}_0)$ of $\mathrm{Stab}(\mathcal{D}_0)$.*

The translates of U are disjoint, and each translate is a chamber on which the moduli space of stable objects of class $[\mathcal{O}_x]$ is constant.

In [Bri06], Bridgeland described an open (but not dense) subset Stab_a of $\mathrm{Stab}^\dagger(\mathcal{D}_0)$ consisting of "algebraic" stability conditions that can be described in terms of quivers and exceptional collections on $\mathbb{P}^2$. By combining this description of Stab_a with the description given by Theorem 2, we establish:

Theorem 3. *The connected component $\mathrm{Stab}^\dagger(\mathcal{D}_0)$ is simply-connected.*

Using Theorem 2 we can classify all autoequivalences $\mathrm{Aut}^\dagger(\mathcal{D}_0)$ which preserve the connected component $\mathrm{Stab}^\dagger(\mathcal{D}_0)$:

Theorem 4. *The group $\mathrm{Aut}^\dagger(\mathcal{D}_0)$ is isomorphic to a product $\mathbb{Z} \times \Gamma_1(3) \times \mathrm{Aut}(X)$.*

Recall that $\Gamma_1(3)$ is isomorphic the group on two generators α and β subject to the relation $(\alpha\beta)^3 = 1$. As a subgroup of $\mathrm{Aut}^\dagger(\mathcal{D}_0)$ it is generated by the spherical twist at the structure sheaf $\mathcal{O}_{\mathbb{P}^2}$ of the zero-section $\mathbb{P}^2 \hookrightarrow X$, and by the tensor product with $\mathcal{O}_X(1)$; this was already observed in [Asp05].

The mirror partner of X is the universal family over the moduli space $\mathcal{M}_{\Gamma_1(3)}$ of elliptic curves with $\Gamma_1(3)$ level structures.[1] Let $\widetilde{\mathcal{M}}_{\Gamma_1(3)}$ be the universal cover, with the fundamental group $\Gamma_1(3)$ acting as the group of deck transformations.

Theorem 5. *There is an embedding $I \colon \widetilde{\mathcal{M}}_{\Gamma_1(3)} \hookrightarrow \mathrm{Stab}^\dagger(\mathcal{D}_0)$ which is equivariant with respect to the $\Gamma_1(3)$-action.*

Here the $\Gamma_1(3)$-action on $\mathrm{Stab}^\dagger(\mathcal{D}_0)$ is induced by the subgroup $\Gamma_1(3) \subset \mathrm{Aut}^\dagger(\mathcal{D}_0)$ identified in Theorem 4.

On the level of central charges, the embedding is given in terms of a Picard-Fuchs differential equation: for a fixed $E \in \mathcal{D}_0$, the function $(\mathcal{Z} \circ I)(z)(E) \colon \widetilde{\mathcal{M}}_{\Gamma_1(3)} \to \mathbb{C}$ is a solution of the Picard-Fuchs equation (i.e., a period in the mirror construction). In particular, while classical enumerative mirror symmetry gives an interpretation of their formal expansions at special points of M in terms of genus-zero

[1] This is a little too simplistic; more accurately, the mirror is a family an open subfamily of punctured quasi-projective elliptic curves.

Gromov-Witten invariants on Y, the space of stability conditions allows us to interpret solutions of Picard-Fuchs equations globally. The result is motivated by the conjectural picture for projective Calabi-Yau threefolds proposed in [Bri09, Section 7], and is based on computations of the monodromies of the Picard-Fuchs equation in the mathematical physics literature [AGM94, Asp05, ABK08].

References

[ABK08] Mina Aganagic, Vincent Bouchard, and Albrecht Klemm. Topological strings and (almost) modular forms. *Comm. Math. Phys.*, 277(3):771–819, 2008, arXiv:hep-th/0607100.

[AGM94] Paul S. Aspinwall, Brian R. Greene, and David R. Morrison. Measuring small distances in $N = 2$ sigma models. *Nuclear Phys. B*, 420(1-2):184–242, 1994, arXiv:hep-th/9311042.

[Asp05] Paul S. Aspinwall. D-branes on Calabi-Yau manifolds. In *Progress in string theory*, pages 1–152. World Sci. Publ., Hackensack, NJ, 2005, arXiv:hep-th/0403166.

[BM09] Arend Bayer and Emanuele Macri. The space of stability conditions on the local projective plane. December 2009, 0912.0043.

[Bri06] Tom Bridgeland. Stability conditions on a non-compact Calabi-Yau threefold. *Comm. Math. Phys.*, 266(3):715–733, 2006, arXiv:math/0509048.

[Bri09] Tom Bridgeland. Spaces of stability conditions. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 1–21. Amer. Math. Soc., Providence, RI, 2009, arXiv:math/0611510.

[DLP85] J.-M. Drezet and J. Le Potier. Fibrés stables et fibrés exceptionnels sur $\mathbf{P}_2$. *Ann. Sci. École Norm. Sup. (4)*, 18(2):193–243, 1985.

[Tod08] Yukinobu Toda. Stability conditions and crepant small resolutions. *Trans. Amer. Math. Soc.*, 360(11):6149–6178, 2008, arXiv:math/0512648.

[Tod09] Yukinobu Toda. Stability conditions and Calabi-Yau fibrations. *J. Algebraic Geom.*, 18(1):101–133, 2009, arXiv:math/0608495.

Connectivity properties pertaining to $\mathcal{A}_g$ and $\mathcal{M}_g$

Eduard Looijenga
(joint work with Wilberd van der Kallen)

We work over $\mathbb{C}$ throughout. Let us first recall that if A is a principal polarized abelian variety of dimension $g > 0$, then the collection of its nonzero abelian subvarieties on which the polarization is principal is finite and that if $p(A)$ is the collection of its minimal elements, then the natural map $\prod_{P\in p(A)} P \to A$ is an isomorphism. We say that A is *decomposable* if $p(A) \neq \{A\}$. Such A define a closed subvariety $\mathcal{A}_{g,\mathrm{dec}}$ of the coarse moduli space $\mathcal{A}_g$.

Theorem 1. *We have* $H_k(\mathcal{A}_g, \mathcal{A}_{g,\mathrm{dec}}; \mathbb{Q}) = 0$ *for* $k \leq g - 2$.

Here is a stronger statement.

Theorem 2. *Regard* $\mathcal{A}_g$ *as the orbit space of the Siegel upper half space* $\mathcal{H}_g$ *by the group* $\mathrm{Sp}(2g, \mathbb{Z})$. *Then a complex-analytic cover* $(\tilde{\mathcal{A}}_g, \tilde{\mathcal{A}}_{g,\mathrm{dec}}) \to (\mathcal{A}_g, \mathcal{A}_{g,\mathrm{dec}})$ *defined by a torsion free subgroup* $\Gamma \subset \mathrm{Sp}(2g, \mathbb{Z})$ *is* $(g-2)$*-connected.*

This indeed implies Theorem 1: take Γ torsion free of finite index in $\mathrm{Sp}(2g,\mathbb{Z})$ and observe that $H_k(\mathcal{A}_g,\mathcal{A}_{g,\mathrm{dec}};\mathbb{Q})$ is a quotient of $H_k(\tilde{\mathcal{A}}_g,\tilde{\mathcal{A}}_{g,\mathrm{dec}};\mathbb{Q})$.

Let us write $\mathcal{H}_{g,\mathrm{dec}}$ for the locus of decomposables in $\mathcal{H}_g$. Theorem 2 in turn follows from

Theorem 3. *The subset $\mathcal{H}_{g,\mathrm{dec}}$ of $\mathcal{H}_g$ is a closed analytic subvariety that has the homotopy type of a bouquet of $(g-2)$-spheres.*

For this means that $\mathcal{H}_{g,\mathrm{dec}}$ may be regarded as the $(g-2)$-skeleton of a Γ-equivariant cellular decomposition of the contractible $\mathcal{H}_g$. More precisely, $\mathcal{H}_g$ is obtained from $\mathcal{H}_{g,\mathrm{dec}}$ by successively attaching cells of dimension $\geq g-1$ to $\mathcal{H}_{g,\mathrm{dec}}$ in a Γ-equivariant manner in such a way that no nontrivial element of Γ fixes a cell. This implies that $\tilde{\mathcal{A}}_g$ is obtainable from $\tilde{\mathcal{A}}_{g,\mathrm{dec}}$ by attaching cells of dimension $\geq g-1$ and so $(\tilde{\mathcal{A}}_g,\tilde{\mathcal{A}}_{g,\mathrm{dec}})$ is $(g-2)$-connected.

An irreducible component of $\mathcal{H}_{g,\mathrm{dec}}$ defines a decomposition of $(\mathbb{Z}^{2g},\langle\,,\,\rangle)$ into two perpendicular unimodular summands; if the genera of these summands are g' and g'' (so that $g=g'+g''$), then this component is isomorphic to $\mathcal{H}_{g'}\times\mathcal{H}_{g''}$. Conversely, any such decomposition of $(\mathbb{Z}^{2g},\langle\,,\,\rangle)$ determines an irreducible component of $\mathcal{H}_{g,\mathrm{dec}}$. An intersection of such irreducible components is given by a (finite) decomposition of $\mathbb{Z}^{2g}$ into pairwise perpendicular proper unimodular sublattices (to which we shall refer as a *proper unimodular decomposition* of $(\mathbb{Z}^{2g},\langle\,,\,\rangle)$ and is isomorphic to the corresponding product of Siegel upper half spaces. In particular it is contractible. We conclude that the covering of $\mathcal{H}_{g,\mathrm{dec}}$ by its irreducible components is a closed covering that satisfies the Leray property. The nerve of this covering is given by the poset of unimodular decompositions of $(\mathbb{Z}^{2g},\langle\,,\,\rangle)$, with $\leq$ standing for "is refined by". So by Weil's nerve theorem, 3 follows from

Theorem 4. *The poset of proper unimodular decompositions of $(\mathbb{Z}^{2g},\langle\,,\,\rangle)$ is spherical of dimension $g-2$.*

This theorem is derived with the help of a standard argument from.

Theorem 5. *The poset of proper unimodular sublattices of $(\mathbb{Z}^{2g},\langle\,,\,\rangle)$ (with $\leq$ being '$\subseteq$') is spherical of dimension $g-2$.*

This is what we regard in this context as our main result. It is proved with the help of a nerve theorem for posets that we obtain using techniques introduced by Quillen and Maazen. Both statements and proofs of Theorems 4 and 5 remain valid if we replace in $(\mathbb{Z}^{2g},\langle\,,\,\rangle)$, the base ring $\mathbb{Z}$ by any Euclidean ring R. This enables us to improve somewhat on earlier work of Charney:

Theorem 6. *If R is an Euclidean ring, then the natural map $H_i(\mathrm{Sp}(2g,R),\mathbb{Z})\to H_i(\mathrm{Sp}(2g+2,R),\mathbb{Z})$ (induced by the obvious inclusion) is an isomorphism for $g\geq 2i+3$ and surjective for $g=2i+2$.*

These results have a counterpart for $\mathcal{M}_g$. Let $\mathcal{M}_g^c\supset\mathcal{M}_g$ parameterize the stable genus g curves with compact Jacobian so that $\Delta_g^c:=\mathcal{M}_g^c-\mathcal{M}_g$ parameterizes the singular ones among them.

Theorem 7. *We have* $H_k(\mathcal{M}_g^c, \Delta_g^c; \mathbb{Q}) = 0$ *for* $k \leq g-2$.

This is derived in a similar fashion as we did for the pair $(\mathcal{A}_g, \mathcal{A}_{g,\mathrm{dec}})$ via a chain of intermediate results that starts with:

Theorem 8. *The separating curve complex in genus* g *is* $(g-3)$*-connected.*

We recall the definition of this complex. Fix a closed connected, orientable surface S_g of genus g. An embedded circle $\alpha \subset S_g$ is called a *separating curve* if $S_g - \alpha$ has two connected components, none of which is an open disk. It is clear that such an α defines a unimodular decomposition of $H_1(S_g; \mathbb{Z})$ with two (nonzero) summands. The separating curve complex $\mathcal{C}_{\mathrm{sep}}(S_g)$ has as its vertices the isotopy classes of separating curves and we stipulate that a finite nonempty set of these spans a simplex of $\mathcal{C}_{\mathrm{sep}}(S_g)$ if its elements can be simultaneously be represented by curves that are pairwise disjoint. Special cases of Theorem 8 were previously obtained by Farb and Ivanov (0-connectivity for $g \geq 3$), Putman (1-connectivity for $g \geq 4$) and Hatcher-Vogtmann ($\lfloor \frac{g-3}{2} \rfloor$-connectivity). The proof of Theorem 8 is inductive in nature and this forces us to prove such a statement also about the case of pointed surfaces (a statement, whose formulation is not quite obvious, see [1, Theorem 5]). Other major input is a theorem of Harer.

Observe that the complex $\mathcal{C}_{\mathrm{sep}}(S_g)$ is acted on by the mapping class group Γ_g. Now let us recall that the natural map $\Gamma_g \to \mathrm{Sp}(H_1(S_g; \mathbb{Z})) \cong \mathrm{Sp}(2g, \mathbb{Z})$ is surjective. Its kernel, here denoted T_g, is called the *Torelli group*. We have a natural poset map from the barycentric subdivision of $\mathcal{C}_{\mathrm{sep}}(S_g)$ to the poset of proper unimodular decompositions of $H_1(S_g; \mathbb{Z})$. This map is easily seen to factor through $T_g \backslash \mathcal{C}_{\mathrm{sep}}(S_g)$ and it can be shown that the resulting map from $T_g \backslash \mathcal{C}_{\mathrm{sep}}(S_g)$ to the poset of proper unimodular decompositions of $H_1(S_g; \mathbb{Z})$ (or rather, its geometric realization) is a $\mathrm{Sp}(2g, \mathbb{Z})$-equivariant homotopy equivalence.

References

[1] Eduard Looijenga: *Connectivity of complexes of separating curves,* available at arXiv:1001.0823.

[2] Wilberd van der Kallen, Eduard Looijenga: *Spherical complexes attached to symplectic lattices,* available at arXiv:1001.0883.

The connected components of the moduli spaces containing the Burniat surfaces

Fabrizio Catanese
(joint work with Ingrid Bauer)

1. What is.. a Burniat surface?

The so called Burniat surfaces were constructed by Pol Burniat in 1966 ([3]), where the method of singular bidouble covers was introduced in order to solve the geography problem for surfaces of general type.

The special construction of surfaces with geometric genus $p_g(S) = 0$, done in [3], was brought to attention by Chris Peters, who explained Burniat's calculation of invariants in the modern language of algebraic geometry, and nowadays the name of Burniat surfaces is reserved for these surfaces with $p_g(S) = 0$.
The birational structure of Burniat surfaces is rather simple to explain:

let $P_1, P_2, P_3 \in \mathbb{P}^2$ be three non collinear points (which we assume to be the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$), and let $D_i = \{\Delta_i = 0\}$, for $i \in \mathbb{Z}/3\mathbb{Z}$, be the union of three distinct lines through P_i, including the line $D_{i,1}$ which is the side of the triangle joining the point P_i with P_{i+1}.

Assume that $D = D_1 \cup D_2 \cup D_3$ consists of nine different lines.

Definition 1. *A Burniat surface S is the minimal model for the function field*

$$\mathbb{C}(x,y)(\sqrt{\frac{\Delta_1}{\Delta_2}}, \sqrt{\frac{\Delta_1}{\Delta_3}}).$$

Proposition 2. *Let S be a Burniat surface, and denote by m the number of points, different from P_1, P_2, P_3, where the curve D has multiplicity at least three. Then $0 \leq m \leq 4$, and the invariants of the smooth projective surface S are:*

$$p_g(S) = q(S) = 0, K_S^2 = 6 - m.$$

The heart of the calculation, based on the theory of bidouble covers, as illustrated in [5], is that the singularities where the three curves have multiplicities $(3,1,0)$ lower K^2 and $p_g - q$ both by 1, while the singularities where the three curves have multiplicities $(1,1,1)$ lower K^2 by 1 and leave $p_g - q$ unchanged.

One understands the biregular structure of S through the blow up W of the plane at the points of D $P_1, P_2, P_3, \dots P_m$ of multiplicity at least three.

W is a weak Del Pezzo surface of degree $6 - m$ (i.e., a surface with nef and big anticanonical divisor).

Proposition 3. *The Burniat surface S is a finite bidouble cover (a finite Galois cover with group $(\mathbb{Z}/2\mathbb{Z})^2$) of the weak Del Pezzo surface W. Moreover the bicanonical divisor $2K_S$ is the pull back of the anticanonical divisor $-K_W$. The bicanonical map of S is the composition of the bidouble cover $S \to W$ with the anticanonical quasi-embedding of W, as a surface of degree $K_S^2 = K_W^2$ in a projective space of dimension $K_S^2 = K_W^2$.*

2. The main classification theorem

Fixing the number $K_S^2 = 6 - m$, one sees immediately that the Burniat surfaces are parametrized by a rational family of dimension $K_S^2 - 2$, and that this family is irreducible except in the case $K_S^2 = 4$.

Definition 4. *The family of Burniat surfaces with $K_S^2 = 4$ of nodal type is the family where the points P_4, P_5 are collinear with one of the other three points P_1, P_2, P_3, say P_1.*

The family of Burniat surfaces with $K_S^2 = 4$ of non-nodal type is the family where the points P_4, P_5 are never collinear with one of the other three points.

Our main classification result of Burniat surfaces is summarized in the following table giving information concerning the families of Burniat surfaces: more information will be given in the subsequent theorems.

K^2	dim	is conn. comp.?	is rational?	π_1
6	4	yes	yes	$1 \to \mathbb{Z}^6 \to \pi_1 \to (\mathbb{Z}/2\mathbb{Z})^3$
5	3	yes	yes	$\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^3$
4, non nodal	2	yes	yes	$\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^2$
4, nodal	2	yes	yes	$\mathbb{H} \oplus (\mathbb{Z}/2\mathbb{Z})^2$
3	1	no: $\subset$ 4-dim. irr. component	yes	$\mathbb{H} \oplus \mathbb{Z}/2\mathbb{Z}$
2	0	no: $\in$ conn. component of standard Campedelli	yes	$(\mathbb{Z}/2\mathbb{Z})^3$

Theorem 5 (Classification Theorem I)**.** *The three respective subsets of the moduli spaces of minimal surfaces of general type $\mathfrak{M}^{min}_{1,K^2}$ corresponding to Burniat surfaces with $K^2 = 6$, resp. with $K^2 = 5$, resp. Burniat surfaces with $K^2 = 4$ of non nodal type, are irreducible connected components, normal, rational of respective dimensions 4,3,2.*

Moreover, the base of the Kuranishi family of such surfaces S is smooth.

Observe that the above result for $K^2 = 6$ was first proven by Mendes Lopes and Pardini in [6]. We showed in [1] the stronger theorem

Theorem 6 (Primary Burniat's Theorem)**.** *Any surface homotopy equivalent to a Burniat surface with $K^2 = 6$ is a Burniat surface with $K^2 = 6$.*

For $K^2 = 2$ another realization of the Burniat surface is as a special element of the family of Campedelli surfaces, Galois covers of the plane with group $(\mathbb{Z}/2\mathbb{Z})^3$ branched on seven lines (one for each non trivial element of the group). For the Burniat surface we have the special configuration of a complete quadrilateral together with its three diagonals.

For $K^2 = 3$ work in progress of the authors shows that the general deformation of a Burniat surface is a Galois covering with group $(\mathbb{Z}/2\mathbb{Z})^2$ of a cubic surface with at least three singular points, and with branch locus equal to three plane sections. It is still an open question whether the closure of this set is again a connected component of the moduli space.

3. Nodal Burniat surfaces and Murphy's law[2].

A new phenomenon occurs for nodal surfaces, confirming Vakil's 'Murphy's law' philosophy ([7]). To explain it, recall that indeed there are two different structures for the moduli spaces of surfaces of general type.

One is the moduli space $\mathfrak{M}^{min}_{\chi,K^2}$ for minimal models S having $\chi(\mathcal{O}_S) = \chi$, $K_S^2 = K^2$, the other is the Gieseker moduli space $\mathfrak{M}^{can}_{\chi,K^2}$ for canonical models X having $\chi(\mathcal{O}_X) = \chi$, $K_X^2 = K^2$. Both are analytic spaces (the latter is actually known to be a quasiprojective) and there is a natural holomorphic bijection $\mathfrak{M}^{min}_{\chi,K^2} \to \mathfrak{M}^{can}_{\chi,K^2}$. Their local structure as complex analytic spaces is the quotient of the base of the Kuranishi family by the action of the finite group $Aut(S) = Aut(X)$.

In [4] series of examples were exhibited where $\mathfrak{M}^{can}_{\chi,K^2}$ was smooth, but $\mathfrak{M}^{min}_{\chi,K^2}$ was everywhere non reduced. For nodal Burniat surfaces with $K_S^2 = 4$ both spaces are everywhere non reduced, but the nilpotence order is higher for $\mathfrak{M}^{min}_{\chi,K^2}$; this is a further pathology, which adds to the ones presented in [4] and in [7].

Theorem 7 (Classification theorem II = Murphy's law[2])**.** *The subset of the Gieseker moduli space $\mathfrak{M}^{can}_{1,4}$ of canonical surfaces of general type X corresponding to Burniat surfaces S with $K_S^2 = 4$ and of nodal type is an irreducible connected component of dimension 2, rational and everywhere non reduced.*

More precisely, the base of the Kuranishi family of X is locally analytically isomorphic to $\mathbb{C}^2 \times Spec(\mathbb{C}[t]/(t^m))$, where m is a fixed integer, $m \geq 2$.

The corresponding subset of the moduli space $\mathfrak{M}^{min}_{1,4}$ of minimal surfaces S of general type is also everywhere non reduced.

More precisely, the base of the Kuranishi family of S is locally analytically isomorphic to $\mathbb{C}^2 \times Spec(\mathbb{C}[t]/(t^{2m}))$.

An interesting question is to determine the above integer m explicitly.

References

[1] I. Bauer, F. Catanese, *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces*, arxiv:0909.3699 (2009).

[2] I. Bauer, F. Catanese, *Burniat surfaces II: secondary Burniat surfaces form three connected components of the moduli space*, arxiv:0911.1466 (2009).

[3] P. Burniat, *Sur les surfaces de genre $P_{12} > 1$.* Ann. Mat. Pura Appl. (4) 71 (1966) 1–24.

[4] F. Catanese, *Everywhere nonreduced moduli spaces.* Invent. Math. 98 (1989), no. 2, 293–310.

[5] F. Catanese, *Singular bidouble covers and the construction of interesting algebraic surfaces.* Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 97–120, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, (1999).

[6] M. Mendes Lopes, R. Pardini, *A connected component of the moduli space of surfaces with $p_g = 0$.* Topology 40 (2001), no. 5, 977–991.

[7] R. Vakil, *Murphy's law in algebraic geometry: badly-behaved deformation spaces.* Invent. Math. 164 (2006), no. 3, 569–590.

Reporter: Daniel Greb

Participants

Prof. Dr. Dan Abramovich
Department of Mathematics
Brown University
Box 1917
Providence , RI 02912
USA

Prof. Dr. Ingrid Bauer-Catanese
Lehrstuhl für Mathematik VIII
Universität Bayreuth
NW - II
95440 Bayreuth

Prof. Dr. Arend Bayer
Department of Mathematics
University of Connecticut
196 Auditorium Road
Storrs , CT 06269-3009
USA

Dr. Kai A. Behrend
Dept. of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver , BC V6T 1Z2
CANADA

Dr. Jonas Bergstrom
Dept. of Mathematics
University of Amsterdam
Plantage Muidergracht 24
NL-1018 TV Amsterdam

Dr. Janko Böhm
FR 6.1 - Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken

Dr. Christian Böhning
Mathematisches Institut
Georg-August-Universität
Bunsenstr. 3-5
37073 Göttingen

Dr. Tom Bridgeland
Dept. of Pure Mathematics
University of Sheffield
Hicks Building
Hounsfield Road
GB-Sheffield S3 7RH

Prof. Dr. Fabrizio Catanese
Lehrstuhl für Mathematik VIII
Universität Bayreuth
NW - II
95440 Bayreuth

Dr. Alessandro Chiodo
Institut Fourier Grenoble
BP 53
F-38041 Grenoble

Prof. Dr. Ionut Ciocan-Fontanine
School of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis MN 55455-0436
USA

Prof. Dr. Carel Faber
Matematiska Institutionen
Kungl. Tekniska Högskolan
Lindstedtsvägen 25
S-10044 Stockholm

Prof. Dr. Barbara Fantechi
S.I.S.S.A.
Via Beirut 2 - 4
I-34151 Trieste

Prof. Dr. Gavril Farkas
Institut für Mathematik
Humboldt-Universität
10099 Berlin

Prof. Dr. Gerard van der Geer
Korteweg-de Vries Instituut
Universiteit van Amsterdam
Postbus 94248
NL-1090 GE Amsterdam

Prof. Dr. Angela Gibney
Department of Mathematics
University of Georgia
Athens , GA 30602
USA

Dr. William Danny Gillam
Department of Mathematics
Columbia University
2990 Broadway
New York , NY 10027
USA

Dr. Lothar Göttsche
International Centre for Theoretical
Physics (ICTP)
Strada Costiera 11
I-34100 Trieste

Patrick Graf
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Dr. Daniel Greb
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Prof. Dr. Samuel Grushevsky
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton , NJ 08544-1000
USA

Prof. Dr. Paul Hacking
Department of Mathematics
University of Massachusetts
Lederle Graduate Research Tower
710 North Pleasant Street
Amherst , MA 01003-9305
USA

Prof. Dr. Klaus Hulek
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover

Prof. Dr. Daniel Huybrechts
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Stefan Kebekus
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Prof. Dr. Sandor Kovacs
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle , WA 98195-4350
USA

Prof. Dr. Andrew Kresch
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Prof. Dr. Yuan-Pin Lee
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City , UT 84112-0090
USA

Prof. Dr. Manfred Lehn
Institut für Mathematik
Johannes-Gutenberg Universität Mainz
Staudingerweg 9
55099 Mainz

Margherita Lelli-Chiesa
Institut für Mathematik
Humboldt-Universität
10099 Berlin

Prof. Dr. Max Lieblich
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle , WA 98195-4350
USA

Prof. Dr. Jun Li
Department of Mathematics
Stanford University
Stanford , CA 94305-2125
USA

Daniel Lohmann
Mathematisches Institut
Universität Freiburg
Eckerstr. 1
79104 Freiburg

Prof. Dr. Eduard J.N. Looijenga
Mathematisch Instituut
Rijksuniversiteit Utrecht
P.O.Box 80.010
6 Budapestlaan
NL-3584 CD Utrecht

Dr. Cristina Manolache
Institut für Mathematik
Humboldt-Universität
10099 Berlin

Dr. Margarida Melo
Departamento de Matematica da
Universidade de Coimbra
Apartado 3008
Coimbra 3001-454
PORTUGAL

Prof. Dr. Martin Möller
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 6-10
60325 Frankfurt am Main

Prof. Dr. Ian Morrison
Department of Mathematics
Fordham University
441 East Fordham Road
Bronx , NY 10458
USA

Prof. Dr. Stefan Müller-Stach
Institut für Mathematik
Johannes-Gutenberg-Universität Mainz
Staudingerweg 9
55128 Mainz

Prof. Dr. Shigeru Mukai
RIMS
Kyoto University
Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Angela Ortega
Institut für Mathematik
Humboldt-Universität Berlin
Rudower Chaussee 25
10099 Berlin

Prof. Dr. Nicola Pagani
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
S-100 44 Stockholm

Prof. Dr. Alexander Polishchuk
Department of Mathematics
University of Oregon
Eugene , OR 97403-1222
USA

Dr. Sönke Rollenske
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Frank-Olaf Schreyer
FB Mathematik & Informatik
Campus E2.4
Universität des Saarlandes
66123 Saarbrücken

Prof. Dr. Georg Schumacher
Fachbereich Mathematik und
Informatik
Philipps-Universität
Lahnberge
35032 Marburg

Prof. Dr. Edoardo Sernesi
Dipartimento di Matematica
Universita degli Studi "Roma Tre"
Largo S. Leonardo Murialdo 1
I-00146 Roma

Nicola Tarasca
Institut für Mathematik
Humboldt-Universität
10099 Berlin

Dr. Orsola Tommasi
Institut für Algebraische Geometrie
Leibniz Universität Hannover
Welfengarten 1
30167 Hannover

Prof. Dr. Ravi Vakil
Department of Mathematics
Stanford University
Stanford , CA 94305-2125
USA

Prof. Dr. Alessandro Verra
Dipartimento di Matematica
Universita Roma 3
Largo S. Leonardo Murialdo 1
I-00146 Roma

Dr. Filippo Viviani
Dipartimento di Matematica
Universita degli Studi Roma Tre
Largo S. L. Murialdo, 1
I-00146 Roma