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Arbeitsgemeinschaft: Mathematical Billiards

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April 4th – April 10th, 2010

ABSTRACT. The workshop *Mathematical Billiards*, organised by Serge Tabachnikov (Penn State) and Serge Troubetzkoy (Marseille) was held April 4th–April 10th, 2010. This meeting was well attended by over 40 participants including a number of master and PhD students, with broad geographic representation. This workshop was a nice blend of researchers with various backgrounds who brought in their various point of views to cover the classics as well as recent advances in mathematical billiards and flat surfaces.

The report consists in the abstracts for the 18 lectures, followed by the abstracts for the 4 short talks that took place in the evenings. During the workshop, there was also a demo of the mathematical software Sage.

Mathematics Subject Classification (2000): 30F30, 37E35, 37A40.

Introduction by the Organisers

The Billiard system. The billiard dynamical system describes the motion of a free particle in a domain with a perfectly reflecting boundary.

More technically, a billiard table Q is a subset of a Riemannian manifold (usually \mathbb{R}^2) with a piece-wise smooth boundary. We define the billiard flow as follows: the billiard ball is a point particle, it moves along geodesic lines in Q with elastic collisions with ∂Q . The latter means that, at the impact point, the velocity vector of the particle is decomposed into two components, tangential and normal to ∂Q ; then the normal component instantaneously changes signs, whereas the tangential component remains the same, after which the free motion continues. In dimension two, this is the famous law of geometrical optics: the angle of incidence equals the angle of reflection.

Many mechanical systems with elastic collisions, that is, collisions preserving the total momentum and energy of the system, reduce to billiards. Perhaps the most famous example is an idealized gas made of massive elastically colliding balls. Here is an interesting lesser known example: the system of three elastically colliding point masses on a circle reduces, after fixing the center of mass, to the billiard inside an acute triangle whose angles depend on the ratios of masses. There are many physically motivated variations on billiards, such as magnetic billiards, in which free particles are subject to the action of a magnetic field.

The dynamical behavior of billiards is strongly influenced by the shape of the boundary. Billiards naturally fall into three classes: depending on whether the pieces of the boundary curve out, curve in, or are flat. In each of the cases the mathematical machinery used in the study is quite different. The presentation of talks below is organized accordingly.

The final group of talks will study outer (also known as dual) billiards, which are played outside a convex table Q in the Euclidean plane. Dual billiards are defined as follows. Fix an orientation of Q . Given a point x outside Q , draw the segment xy , with $y \in Q$, of the tangent line to Q such that its orientation agrees with that of Q . Extend this segment through y to the point $T(x)$ such that $\text{dist}(Tx, y) = \text{dist}(x, y)$. The map T of the exterior of Q to itself is the dual billiard transformation. This map is area-preserving; its definition extends to the spherical and hyperbolic geometry (in the former, outer and inner billiards are equivalent via the spherical duality). Outer billiards can be also defined in even-dimensional Euclidean spaces.

The following books are devoted to billiards: [1, 3, 8, 9].

Hyperbolic billiards. If the boundary of Q curves out, then parallel incoming orbits scatter, or disperse, producing hyperbolic behavior. A second mechanism of hyperbolicity exists: if two smooth curving in components are placed sufficiently far apart, then parallel orbits first focus, but then have time to diverge before the next collision. One of the main motivations of the study of hyperbolic billiards is Boltzmann's ergodic hypothesis, see [7].

The mathematical tools used to study hyperbolic billiards are the same as the ones used to study hyperbolic dynamical systems (Anosov systems, Axiom A systems, expanding maps, etc). There are serious additional difficulties, the presence of singularities (tangent orbits and orbits hitting non-smooth points of the boundary).

Elliptic billiards. The billiard in an ellipse is completely integrable: a subset of full measure in its phase space is foliated by invariant curves, corresponding to the billiard trajectories tangent to confocal ellipses and hyperbolas, the caustics of the billiard system (note however that one leaf of this foliation is singular: this is the invariant curve consisting of the trajectories that pass through the foci of the ellipse). Similar complete integrability holds for billiards inside ellipsoids in multi-dimensional space.

It turns out that part of this structure is shared by arbitrary convex tables. Lazutkin showed that one can apply the celebrated KAM theorem to show that

a set of positive measure of caustics exist for sufficiently smooth tables. Birkhoff showed that periodic orbits always exist in plane billiards with sufficiently smooth boundary with positive curvature. On the other hand, Mather proved that if the curvature vanishes at a point then the billiard possesses no caustics.

Polygonal billiards. Billiards in polygons come in two classes: rational and irrational polygons. Rational polygons are those for which the angles between sides are rational multiples of π . A rational billiard table determines a flat surface, this construction allows one to use the tools of Teichmüller theory to study rational billiards, and many deep results have been obtained this way, see, e.g., [4, 10]. Most of the polygonal talks will be on rational polygons, since in the irrational case there is essentially no machinery available, other than elementary geometry and computer simulation. As a consequence, the available results are considerably more scarce.

Dual or outer billiards. In the first volume of the *Mathematical Intelligencer*, Jürgen Moser wrote an article proposing the outer billiard as a toy model to study the question of the stability or not of the solar system [5]. The recent progress in the study of polygonal outer billiards is the subject of the two talks. See [2] for a survey of outer billiards and [6] for a monograph devoted to a special class of quadrilaterals, the irrational kites, for which some outer billiard orbits escape to infinity.

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Arbeitsgemeinschaft: Mathematical Billiards**Table of Contents**

Heiko Gimperlein	
<i>Introduction to Hyperbolic Billiards</i>	961
Marcello Seri	
<i>The Boltzmann-Sinai Conjecture, a sketch of the proof</i>	964
Nándor Simányi	
<i>Introduction to Hyperbolic Billiards</i>	965
Sylvie Oliffson Kamphorst	
<i>Billiards with external force</i>	967
Rafael Ramírez-Ros	
<i>Existence and non-existence of (convex) caustics</i>	969
Yuliy Baryshnikov	
<i>Periodic orbits in convex billiards</i>	972
Maxim Arnold	
<i>Birkhoff's conjecture</i>	975
Peter Spaeth	
<i>Interpolating Hamiltonians and length spectrum</i>	976
Karsten Kremer	
<i>Introduction to rational polygons</i>	977
Jon Chaika	
<i>Periodic Orbits for Billiards</i>	980
Frank Herrlich	
<i>The complexity of billiard coding</i>	982
Gabriela Schmithüsen	
<i>Coding billiards in regular $2k$-gons</i>	986
Thierry Monteil	
<i>Introduction to the theorem of Kerkhoff, Masur and Smillie</i>	989
Vincent Delecroix	
<i>Veech surfaces of genus 2 surfaces and L-shape rectangular billiard</i>	995
David Ralston	
<i>Billiards on Infinite Tables</i>	997
Glenn Merlet	
<i>Security, or finite blocking property</i>	1000

Stefan Müller	
<i>Dual (or outer) polygonal billiards</i>	1001
Nicolas Bedaride	
<i>Outer billiard on kites</i>	1004
Robert G. Niemeyer (joint with Michel L. Lapidus)	
<i>A Family of Periodic Orbits of the Koch Snowflake Fractal</i>	1005
Samuel Lelièvre (joint with Pascal Hubert, Serge Troubetzkoy)	
<i>Recurrence in periodic Ehrenfest wind-tree models</i>	1007
Martin Schmoll	
<i>Veech groups for holonomy free covers</i>	1009
Marcello Seri (joint with Giampaolo Cristadoro, Marco Lenci)	
<i>Recurrence of quenched random Lorentz tubes and related questions</i>	1010

Abstracts

Introduction to Hyperbolic Billiards

HEIKO GIMPERLEIN

A billiard is the dynamical system generated by the free motion of a point particle in a domain $Q \subset \mathbb{R}^2$ or \mathbb{T}^2 with piecewise smooth boundary $\partial Q = \Gamma_1 \cup \dots \cup \Gamma_k$. When the particle hits a boundary point at which the normal vector is well-defined, it is elastically reflected so that the angle of incidence equals the angle of reflection. The motion corresponds to a Hamiltonian system with a constant potential in Q that becomes infinite at ∂Q , and we are going to assume that the reflections do not have an accumulation point. The particle trajectories then give rise to a Hamiltonian flow $\{\Phi_t\}_{t \in \mathbb{R}}$ on (a dense subset of) the phase space $\Omega = S^*Q / \sim$, where \sim identifies the incoming and outgoing velocity vectors over ∂Q . Φ_t extends continuously to a multiple-valued map on Ω and preserves the Liouville measure μ . Its lack of differentiability at glancing points and corners is responsible for many of the particular difficulties of billiard systems, even though they correspond to a set of measure 0.

Question: Give an explicit example of a billiard with a cusp on the boundary, such that reflections accumulate in the tip in finite time.

The billiard dynamics is conveniently described in terms of wave fronts, i.e. families of particles parameterized by smooth curve segments with a continuous choice of a normal “velocity” vector field. Away from ∂Q , the curvature κ of the wave front evolves according to

$$\kappa_t = \frac{\kappa_0}{1 + t\kappa_0} .$$

Upon reflection, κ jumps according to the mirror equation

$$\kappa_{t+0} = \kappa_{t-0} + \frac{2\mathcal{K}(q)}{\cos(\varphi)}$$

depending on the angle of incidence φ and the curvature of the boundary $\mathcal{K}(q)$ at the point of reflection. The two equations completely determine the evolution, though practically they are most useful for the hyperbolic billiards defined below.

If all trajectories are eventually reflected, the dynamics can be reduced to ∂Q : A particle hitting the boundary corresponds to a point in $\mathcal{M} = \bigcup_{i=1}^k \Gamma_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, and the billiard map $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{M}$ maps a reflection to the next one. As for the flow, it is first defined on regular trajectories and then extends by continuity to a multiple-valued map on all of \mathcal{M} . Conversely, we may recover Φ_t from \mathcal{F} and the return-time $T : \mathcal{M} \rightarrow (0, \infty)$ of the particle to \mathcal{M} . When applied to the Liouville measure on Ω , this correspondence naturally yields the \mathcal{F} -invariant measure $\nu = \cos(\alpha) dr \wedge d\alpha$ on \mathcal{M} .

The billiard map is particularly useful to analyze the stability of the system. An important computation shows that if $(r_1, \alpha_1) = \mathcal{F}(r, \alpha)$ and the curvatures of the Γ_i are bounded, the derivative behaves like $\|D\mathcal{F}(r, \alpha)\| \sim \frac{1}{\cos(\alpha_1)}$, which is singular at glancing points. The singularities of \mathcal{F} are hence contained in $\partial\mathcal{M} \cup \mathcal{F}^{-1}(\partial\mathcal{M})$, and the explicit formula for ν shows

$$\int_{\mathcal{M}} \log^+ \|D\mathcal{F}^{\pm 1}\| \, d\nu < \infty .$$

To discuss linearized stability, we may appeal to a general theorem by Oseledets:

Theorem: Let \mathcal{M} be a Riemannian manifold (with corners), $\mathcal{N} \subset \mathcal{M}$ an open dense subset and $\mathcal{F} \in C^2(\mathcal{N}, \mathcal{M})$ a diffeomorphism onto its image. Suppose that \mathcal{F} preserves a probability measure ν and $\bigcap_{n \in \mathbb{Z}} \mathcal{F}^n(\mathcal{N})$ is of full measure. If $\int_{\mathcal{M}} \log^+ \|D\mathcal{F}^{\pm 1}\| \, d\nu < \infty$, then for each x in a full-measure $D\mathcal{F}$ -invariant subset $T_x\mathcal{M}$ admits a $D\mathcal{F}$ invariant decomposition $E_x^1 \oplus \cdots \oplus E_x^{m(x)}$ such that

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|D\mathcal{F}^n v\| = \lambda_x^i \quad (v \in E_x^i) .$$

If \mathcal{F} is ergodic, the Lyapunov exponents λ_x^i are a.e. constant. Loosely speaking, $\|D\mathcal{F}^n|_{E_x^i}\| \sim e^{n\lambda_x^i}$, so that for $\lambda_x^i > 0$ small perturbations in the direction E_x^i at time 0 are expected to grow exponentially in the future, making the long-term evolution “chaotic” and practically inaccessible.

If the Lyapunov exponents exist and are nonzero a.e., the map \mathcal{F} is said to be hyperbolic. For two-dimensional billiards, the preservation of the Liouville measure assures $\lambda_x^1 + \lambda_x^2 = 0$, and the billiard map is hyperbolic unless both exponents vanish.

Because of the above correspondence between the continuous and the discrete system, hyperbolicity turns out to be a property of the actual billiard flow on phase space: The tangent space of Ω splits into orthogonal subspaces $T^0\Omega$ and $T^\perp\Omega$ tangent resp. perpendicular to the flow, and the exponents associated to Φ_t on $T^\perp\Omega$ agree with those of \mathcal{F} . $T^0\Omega$ merely contributes an additional exponent 0.

A powerful method to determine the stable and unstable subspaces associated to the positive resp. negative Lyapunov exponents is the method of cone fields [1]. Here, a cone \mathcal{C}_x is determined by a line $\mathcal{L}_x \subset T_x\mathcal{M}$ and an angle $\alpha_x \in (0, \frac{\pi}{2})$, such that $\mathcal{C}_x = \{v \in T_x\mathcal{M} : \angle(v, \mathcal{L}_x) \leq \alpha_x\}$. Intuitively, application of $D\mathcal{F}^n$ to a cone should stretch out the unstable directions and “converge” to the unstable subspace E_x^u for $n \rightarrow \infty$. A convenient criterion goes back to Wojtkowski [4, 5]: If for almost all $x \in \mathcal{M}$ one finds cones \mathcal{C}_x^u satisfying the strict invariance condition $D\mathcal{F}(\mathcal{C}_x^u) \subset \text{int } \mathcal{C}_{\mathcal{F}(x)}^u \cup \{0\}$, then \mathcal{F} is hyperbolic and $E_x^u = \bigcap_{n=0}^{\infty} D\mathcal{F}^n(\mathcal{C}_{\mathcal{F}^{-n}(x)}^u)$. Replacing \mathcal{F} by \mathcal{F}^{-1} leads to the corresponding assertions for the stable subspace.

Once the infinitesimal case has been clarified, one would, of course, want to use it to investigate the long-time behavior of the flow. The appropriate notion is that of an unstable manifold, which is defined as a smooth curve $\mathcal{W}^u \subset \mathcal{M}$ such

that \mathcal{F}^{-n} is smooth on \mathcal{W}^u for all $n \geq 1$ and $\lim_{n \rightarrow \infty} |\mathcal{F}^{-n}(\mathcal{W}^u)| = 0$. Similarly, a stable manifold is an unstable manifold for \mathcal{F}^{-1} .

Dispersing billiards provide a prototypical example of a hyperbolic system. We consider the simple case where the Γ_i are nonintersecting, concave outward, smooth closed curves in \mathbb{T}^2 and all trajectories eventually get reflected. The two basic equations for the propagation of wave fronts assure that a dispersing wave front ($\kappa \geq 0$) will remain dispersive. A bit more work establishes hyperbolicity by applying Wojtkowski's criterion to the cone field

$$\mathcal{C}_x^u = \{(\text{linearized}) \text{ wave fronts at } x \text{ with } \kappa_{-0} \geq 0\}.$$

To construct stable and unstable manifolds through a point $x \in \mathcal{M}$, which lies on a regular trajectory, we recall that the singularities of \mathcal{F} are contained in $\partial\mathcal{M} \cup \mathcal{F}^{-1}(\partial\mathcal{M})$. Similarly, the singularities \mathcal{S}_n of \mathcal{F}^n , $n \in \mathbb{Z}$, are given by a union of compact smooth curves in \mathcal{M} , and we let $Q_n(x)$ the connected component of $\mathcal{M} \setminus \mathcal{S}_n$ containing x . $\overline{Q_n(x)}$ is a curvilinear polygon that shrinks to a line for $n \rightarrow -\infty$, and we obtain the maximal unstable manifold containing x quite explicitly as $W^u = \bigcap_{n \geq 1} \overline{Q_{-n}(x)} \setminus \{\text{endpoints}\}$. Replacing \mathcal{F} by \mathcal{F}^{-1} yields a maximal stable manifold through x .

We conclude with some remarks on the local ergodicity of dispersive billiards:

Theorem (Sinai [3]): Any $x \in \mathcal{M}$ which lies on at most one smooth singularity curve has an open neighborhood that belongs to a single ergodic component of \mathcal{F} .

Like most proofs of ergodicity for hyperbolic systems, Sinai's proof relies on Hopf's method and informally proceeds in two steps:

- 1) A.e. stable or unstable manifold belongs to a single ergodic component of \mathcal{F} .
- 2) Generic $x, y \in \mathcal{M}$ are connected by a finite sequence of stable and unstable manifolds $\mathcal{W}_1, \dots, \mathcal{W}_k$ such that $\mathcal{W}_i \cap \mathcal{W}_{i+1} \neq \emptyset$, a so-called Hopf chain.

While 2) is based on the construction of stable and unstable manifolds, the heuristic idea behind 1) is as follows: If $M_{1/2} \subset \mathcal{M}$ are distinct ergodic components, then \mathcal{F} preserves the conditional ergodic measures $\mu_{1/2}$. The trajectory of μ_1 -a.e. $x \in M_1$ is distributed according to μ_1 , and similarly for $y \in M_2$. Therefore, given e.g. a stable manifold \mathcal{W}^s intersecting M_1 and M_2 , the trajectories of typical points $x \in \mathcal{W}^s \cap M_1$, $y \in \mathcal{W}^s \cap M_2$ are distributed according to μ_1 resp. μ_2 . But lying on \mathcal{W}^s , the future trajectories of x and y converge to each other and hence must lead to identical distributions. Contradiction to $\mu_1 \neq \mu_2$.

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The Boltzmann-Sinai Conjecture, a sketch of the proof

MARCELLO SERI

When Boltzmann was laying down the foundations of statistical physics, in the 1880s, he made the hypothesis that gases of hard balls are ergodic. Only in 1963 did Sinai give a formal mathematical foundation of the problem stating that the gas of $N \geq 2$ identical hard balls (of “not too big” radius) on a torus \mathbb{T}^ν , $\nu \geq 2$ is ergodic, of course on the submanifold of the phase space specified by the obvious conservation laws and provided that the Chernov-Sinai Ansatz holds true.

There is a standard way to transform a hard ball gas into an equivalent billiard and the difficulty in proving the conjecture is mainly due to the fact that it is semi-dispersing: in fact while in dispersing billiards strong hyperbolicity ensures a nice behavior, in the semi-dispersing situation it is more and more difficult to control what happens to the singularity manifolds and the flow.

What one would like to show is that almost every singular orbit is geometrically hyperbolic and it lies in the unique full measure ergodic component of the system. To achieve this goal one mainly has to get rid of the exceptional separating manifolds, i.e. codimension-one submanifolds of the phase space that separate distinct, open ergodic components of the billiard flow.

The Boltzmann-Sinai Conjecture was proved by Simányi in 1999 (see [2]). His proof can be divided in three steps.

- Step I Prove that every finite non-singular (i.e. smooth) trajectory segment $S^{[a,b]}x_0$ with a combinatorially rich symbolic collision sequence (in a well defined sense) is automatically “geometrically hyperbolic” (or *sufficient*), provided that the phase point x_0 does not belong to a countable union J of smooth sub-manifolds with codimension at least two (a “slim” set containing the exceptional phase points).
- Step II Assume as inductive step that all hard ball systems with N' balls ($2 \leq N' < N$) are (hyperbolic and) ergodic. Prove that there exists a slim set $E \subset \mathbf{M}$ such that for every phase point $x_0 \in \mathbf{M} \setminus E$ the entire trajectory $S^{\mathbb{R}}x_0$ contains at most one singularity and its symbolic collision sequence is combinatorially rich.
- Step III By using again the induction hypothesis, prove that almost every singular trajectory is sufficient in the time interval $(t_0, +\infty)$, where t_0 is the time moment of the singular reflection.

In this proof, the key point is given by the Main Lemma that states that *every separating manifold $J \subset \mathbf{M}$ would contain at least one sufficient phase point*. In fact the three steps above ensure that the Local Ergodic Theorem can be applied and the Main Lemma says that if the exceptional separating manifold J exists, it must contain some regular points that are in the same ergodic component as the points in a neighborhood, thus J cannot exist. In conclusion there must be

only one full measure connected ergodic component for the flow and the theorem is proved.

In the talk a brief historical introduction on the conjecture and the main ideas of this proof and of the proof of the Main Lemma were given.

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Introduction to Hyperbolic Billiards

NÁNDOR SIMÁNYI

Semi-dispersing billiards are defined as follows: We take a C^2 -smooth, connected, Riemannian manifold M without boundary and with a positive injectivity radius ρ , and we remove from M finitely many compact, geodesically convex subsets $B_i \subset M$ (the so-called scatterers, $i = 1, \dots, n$) that have C^1 -smooth boundaries ∂B_i . The billiard table (configuration space) is the set

$$B = M \setminus \bigcup_{i=1}^n \text{Int}(B_i).$$

We assume that B is compact. The billiard flow $(M, \{\Phi^t\} \mu)$ describes the uniform motion (with unit speed) of a point particle in B along geodesic lines, enduring elastic reflections when hitting a boundary component ∂B_i of B . If a trajectory ever hits an intersection $B_i \cap B_j$ ($i \neq j$), then such a trajectory is simply undefined. We always assume that all sectional curvatures κ of M are bounded above by a real number K .

It has been very well known since the early studies of mathematical billiards by Ya. G. Sinai in the 1960s [S63], [S70], that obtaining upper bounds (in particular, finiteness) for the number of collisions in terms of the length of trajectory segments plays a pivotal role in studying the fine ergodic and statistical properties of such systems. Such bounds are especially useful in effectively estimating the topological entropy of hard ball systems, as Burago, Ferleger and Kononenko showed in 1998, [BFK98].

Our goal is to review the main results in this area of research by also giving the audience a glimpse into the intricate geometric tools developed to tackle such problems. We will be discussing the geometric aspects (of the proofs) of the results below.

One of the early results is due to L. N. Vaserstein [V79] and G. Galperin [G81].

Theorem. If a natural *non-degeneracy condition* (see below) holds true for the semi-dispersing billiard flow $(M, \{\Phi^t\}, \mu)$, then in any trajectory the number of collisions during any finite time interval is finite.

Definition. The billiard table B (of a semi-dispersing billiard) is *non-degenerate* in an open subset U of M with the constant $C > 0$ if for every non-empty subset $I \subset \{1, 2, \dots, n\}$ and for every $y \in (U \cap B) \setminus \bigcap_{j \in I} B_j$

$$(1) \quad \max \left\{ \frac{\text{dist}(y, B_k)}{\text{dist}(y, \bigcap_{j \in I} B_j)} \mid k \in I \right\} \geq C,$$

whenever $\bigcap_{j \in I} B_j \neq \emptyset$. (We note that this is a local geometric property.)

Definition. B is *non-degenerate* in an open domain U of M if there exist constants $\delta > 0$, $C > 0$ such that B is non-degenerate with the constant C in any δ -ball of U .

In 1998 Burago, Ferleger, and Kononenko [BFK98] proved the following crucial result.

Theorem. Assume B is non-degenerate in an open neighborhood $U \subset M$ of a point $x \in \partial B$. Then there exists a neighborhood U_x of x (in M) and a number $P_x > 0$ such that every billiard trajectory entering U_x leaves it after making at most P_x collisions.

As an immediate corollary, we get that

Corollary. For every nondegenerate semi-dispersing billiard there exists a constant $P > 0$ such that every trajectory of the billiard flow makes no more than $P \cdot (t + 1)$ collisions during any time interval of length t .

For open ball systems in the euclidean space \mathbb{R}^k , the same authors also proved in [BFK98] the following theorem.

Theorem. The number of collisions of N elastic balls in \mathbb{R}^k is not larger than

$$\left(32 \cdot \sqrt{\frac{m_{\max}}{m_{\min}}} \cdot \frac{r_{\max}}{r_{\min}} \cdot N^{3/2} \right)^{N^2}.$$

Here m_{\max} (m_{\min}) denotes the maximum (minimum) mass of the particles, whereas r_{\max} (r_{\min}) is the maximum (minimum) value of the radii.

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Billiards with external force

SYLVIE OLIFFSON KAMPHORST

The billiard problem is the study of the movement of a point particle inside a region (the billiard table), undergoing collisions with its boundary. In the absence of external forces the movement between impacts is free and the particle moves along a polygonal line at constant speed. Moreover, if the collisions are elastic, the reflexion law applies and the kinetic energy is conserved.

When external forces act inside the region, the trajectory between impacts may no longer be a straight segment and the energy may also change. One can also consider external forces acting only at impacts, giving rise to inelastic collisions, gain or loss of energy, or change in the reflexion law. In a very general setup, the external forces may depend on the position, velocity and time. For a very nice review of the subject, including a list of recent results and open problems, see [10]. We mainly report the recent results of N. Chernov and D. Dolgopyat about the Lorentz gas under gravity [5, 6].

The laws of Statistical Mechanics and Thermodynamics rely on the Boltzmann-Sinai Ergodic Hypothesis, which states that molecules in a container will evolve in such a way that time average is equivalent to state average. The Lorentz Gas is a simplified version of a gas of molecules: in dimension 2, one particle (with mass equal 1) moves freely in a periodic array of circular scatters in the plane. The model reduced to its fundamental domain is equivalent to the dispersing Sinai Billiard, where the billiard table consists of a square with a disk removed from its center. The proof of ergodicity and the probability description [2, 3] of this billiard model is an important result to Statistical Mechanics. The position $q(t)$ of the Lorentz particle at time t evolves as a 2D Brownian motion. More precisely

$$\frac{q(t)}{\sqrt{t}} \Rightarrow \mathcal{N}(0, D)$$

where \mathcal{N} denotes the normal distribution and the diffusion matrix D is determined by the scatters. Observe that the result is independent of the constant kinetic energy $K = v^2/2$.

The Lorentz gas with a constant external force is introduced to study the motion of the particles of a gas under gravity or of charged atoms in a constant electric field. The motion between impacts is given by $\dot{q} = v, \dot{v} = g$. If the external force is small (compared to the initial kinetic energy of the particle), one expects the particle to describe a Brownian motion driven by the external force (under the effect of the collisions). The first results were obtained for the so called thermostated Lorentz gas [7, 8], in which the kinetic energy of the particle is fixed by means of a fictitious friction force (Gaussian thermostat) $\dot{q} = v, \dot{v} = g - \text{proj}_v g$. In this case (with the speed fixed to the value 1) we have

$$\frac{q(t) - at}{\sqrt{t}} \Rightarrow \mathcal{N}(0, D) \text{ with } \frac{a}{g} \rightarrow D$$

where the limit of the drift coefficient a expresses Ohm's law.

In [5] the Lorentz gas under constant external field and without thermostat is studied. If the initial kinetic energy is large and the external force is small, it is shown that

$$\frac{x(t)}{t^{2/3}} \Rightarrow \mathcal{B}$$

where x is the vertical position down from an initial level and \mathcal{B} is a distribution with probability density proportional to $\exp(-z^{3/2})/g$. This result applies to the closed board situation, where the particle hitting a top lid reflects down. In the open top board, where the particle bouncing to the top escapes, it is shown that the particle will escape with probability one.

To prove these results, a key observation is that the dynamics is described by the evolution of slow-fast variables. The fast variables are the position and the direction of motion, and the slow variable is the kinetic energy. The fast dynamics (collision dynamics) is well approximated by the thermostated Lorentz gas and thus the fast variables behave like normal random variables. This implies that the evolution of the kinetic energy is approximately described by a stochastic diffusion process \mathcal{K} satisfying the stochastic differential equation

$$d\mathcal{K} = \frac{\langle g, Dg \rangle}{2\sqrt{\mathcal{K}}} d\tau + (2\mathcal{K})^{1/4} \langle g, Dg \rangle^{1/2} dW_\tau, \quad \mathcal{K}(0) = K$$

where W_τ is the standard Brownian motion. The solution of this equation is related to square Bessel processes and this leads to the result stated above.

This was a brief summary of the contents of references below which present the Lorentz gas under gravity and included a very short list of other models: billiards with moving boundaries and the question of Fermi acceleration [9, 10, 12, 14, 15], non elastic billiards [16] and billiards in different metrics [1, 11, 13, 17].

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Existence and non-existence of (convex) caustics

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We consider the billiard dynamics inside a planar domain —a billiard table— whose border is a smooth closed convex curve: a particle follows straight lines inside the billiard table and it is reflected at the border following the rule “the angle of incidence equals the angle of reflection”. From now on, the term *convex* means that the border of the table has curvature everywhere non-negative, the term *strict* means that it has no *flat* points —points at which the curvature vanishes—, and the term *smooth* means that it admits a sufficiently high number of continuous derivatives, the number being different for each result.

We recall that a smooth curve inside the table is a *caustic* if a billiard trajectory, once tangent to it, stays tangent after every reflection. We refer to the books [9, 10] for a background on billiards and caustics.

There exist several negative and positive results about convex caustics. First, we shall describe some qualitative and quantitative non-existence theorems, which go back to Mather, Gutkin and Katok. Next, we shall state the classical existence result of Lazutkin, whose regularity was later improved by R. Douady. Finally, we shall present a negative result for higher dimensional tables found by Berger.

Theorem 1 (Mather [6]). *If the border of the table is C^2 and has some flat point, then there are no smooth convex caustics inside the table.*

This result follows from a formula known in geometrical optics as the mirror equation, see [10]. Mather used another method of proof based on the Lagrangian formulation of billiard dynamics. Both proofs are elementary.

Gutkin and Katok obtained the following quantitative versions of Mather’s theorem. Let d , w , and r be the the diameter, the width, and the inradius of the billiard table. Let $\underline{\kappa}$ and $\bar{\kappa}$ be the minimal and maximal values of the curvature of the border of the billiard table, and let L be its length.

Theorem 2 (Gutkin & Katok [4]). *If some of the following quantitative geometric conditions holds, the billiard table Ω contains a region Ω' free of convex caustics.*

<i>Condition</i>	<i>Description of Ω'</i>
$\sqrt{2\kappa}d^2 \leq r$	A disc of radius r' such that $r' > r - \sqrt{2\kappa}d^2$
$\sqrt{2\kappa}d^2 \leq w/3$	A disc of radius r' such that $r' > w/3 - \sqrt{2\kappa}d^2$
$\sqrt{2\kappa\bar{\kappa}}d^2 \leq 1$	A disc of radius r' such that $\bar{\kappa}r' > 1 - \sqrt{2\kappa\bar{\kappa}}d^2$
$\sqrt{2\kappa\bar{\kappa}}d^2 \leq 1$	A convex set such that $\text{Area}(\Omega \setminus \Omega') \leq \sqrt{2\kappa\bar{\kappa}}d^2 L$

We note that if the border of the table has a flat point, then $\underline{\kappa} = 0$ and $\Omega' = \Omega$, so we recover Mather's theorem. In particular, if we have a one-parameter family of strictly convex billiard tables Ω_t whose minimal curvature approaches zero at some critical parameter $t = t_*$ of the family, while the global shape of the table remains essentially unchanged, then the convex caustics are gradually pushed out to the boundary in the limit $t \rightarrow t_*$. An example of this situation is given by the strictly convex tables

$$\Omega_t = \{(x, y) \in \mathbb{R}^2 : x^2 + ty^2 + y^4 \leq 1\}, \quad t > t_* = 0.$$

A key step in the proof of this quantitative theorem is to establish a suitable set of upper and lower bounds on the Lazutkin parameter that arises in the string construction. This construction is a geometric method—similar to the gardener's method to draw ellipses with given foci—to draw all billiard tables with a given smooth convex closed caustic. These billiard tables are parameterized by the Lazutkin parameter, which quantifies the distance between the caustic and the border of the table. Small Lazutkin parameters correspond to caustics close to the border. Gutkin and Katok showed that too big Lazutkin parameters are incompatible with the geometric hypotheses of their theorem.

The non-existence of convex caustics implies the existence of billiard trajectories whose past and future behaviours differ dramatically. To be more precise, we say that a billiard trajectory is positively (respectively, negatively) ϵ -glancing if, for some bounce, the angle of reflection with the positive (respectively, negative) tangent vector is smaller than ϵ . Mather established, under the non-existence of smooth convex caustics, the existence of infinitely many billiard trajectories that are both positively and negatively ϵ -glancing for any $\epsilon > 0$. To bound the number of impacts $n = n(\epsilon)$ of such glancing billiard trajectories between its positive and negative ϵ -bounces as $\epsilon \rightarrow 0$ is an open problem, similar to bound the speed of Arnold diffusion in Hamiltonian Systems.

The only positive result of this talk is the following one.

Theorem 3 (Lazutkin [5], Douady [2]). *If the border of the table is C^6 and strictly convex, then there exists a collection of smooth convex caustics close to the border of the table whose union has positive area.*

Originally Lazutkin asked for C^{553} regularity. Douady reduced it to C^6 , and conjectured that C^4 regularity may suffice. There exist C^1 examples— C^2 except for a finite set of points—without caustics.

This result is deduced from an Invariant Curve Theorem for area-preserving twist maps on the annulus that was one of the first results in KAM theory. The reader is referred to the book [8] for a proof of the Invariant Curve Theorem in

the analytic case; the differentiable case contained in [5, 2] is technically more involved, so it is not recommended as a first reading.

As a by-product of standard KAM-like results, all the caustics obtained in Lazutkin's theorem have two important properties. First, they persist under small enough C^6 perturbations of the table. Second, their rotation numbers are poorly approximated by rational numbers since they belong to a Cantor set of the form

$$\mathcal{C} = \mathcal{C}_{\lambda, \tau, y_*} := \{y \in (0, y_*) : |y - m/n| \geq \lambda n^{-\tau}, \quad \forall n \in \mathbb{N}, m \in \mathbb{Z}\}$$

for some constants $\lambda > 0$, $\tau > 2$, and $y_* > 0$. This set can be viewed as the open interval $(0, y_*)$ with a countable number of small gaps centered at rational values.

On the contrary, *resonant* caustics—the ones whose tangent trajectories are closed polygons—have rational rotation numbers and can be destroyed under arbitrarily small perturbations of the billiard table. See the example in [7].

Finally, let us consider the higher dimensional case. That is, we deal with hypersurfaces of the Euclidean n -dimensional space instead of smooth curves of the plane, for any $n \geq 3$. We assume that we have three open hypersurfaces V_- , U , and V_+ of class C^2 with non-degenerate second fundamental form at their respective points y_- , x , and y_+ . We also suppose that any line tangent to V_- at a point enough close to y_- intersects transversely U at a point close to x and its reflection is tangent to V_+ at a point close to y_+ . All these hypotheses are local.

Theorem 4 (Berger [1]). *Under these hypotheses of regularity, non-degeneracy, and tangent reflection, the billiard hypersurface U is part of a quadric Q , and both caustic hypersurfaces V_- and V_+ are part of the same quadric Q' confocal to Q .*

The proof follows from a duality argument once a higher dimensional version of the planar mirror equation is established. In particular, the principal curvatures of the hypersurface U at points close to x play a role similar to the one played by the (planar) curvature in the planar case.

Gruber proved a similar theorem under weaker regularity hypotheses in [3].

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Periodic orbits in convex billiards

YULIY BARYSHNIKOV

Let $\mathcal{D} \subset \mathcal{R}^d$ be a convex billiard domain (with smooth enough boundary ∂D). We are concerned here with *n-periodic trajectories* of the billiard map. They are important from the ergodic theory viewpoint, and also appear at the quasiclassical interface between billiard dynamics and spectral theory for Laplace operator on \mathcal{D} .

The overarching principle in understanding periodic trajectories in billiards is to *dispense with the dynamics, and consider instead the space of cyclically ordered n-tuples on ∂D , potential candidates for periodic orbits*. Topology and differential geometry of this space can be used to understand the structure of *n*-periodic orbits of the billiard map.

In this talk two manifestations of this principle have been discussed: one dealing with estimating from below the number of distinct periodic orbits, and the other dealing with the billiard domains with many periodic orbits.

CYCLIC CONFIGURATION SPACES AND PERIODIC ORBITS

Consider the space $\text{Cyc}_n = (\partial D)^n - \cup_i \Delta_{ii+1}$, of *n*-tuples of points on ∂D with $x_i \neq x_{i+1}$, *i* running cyclically through $1, \dots, n$. As is well known, the periodic orbits of the billiard mapping correspond to the critical points of *length functional* $l : \text{Cyc}_n \rightarrow \mathcal{R}_+$ given by $l(x) = \sum_i |x_i - x_{i+1}|$. This opens the door for the Morse theoretic estimates $m_l \geq h_l$ for the number of non-degenerate critical points m_l of a smooth function on a manifold via its Betti numbers, $h_l = \text{rank} H_l(M)$ (singular homologies with coefficients in \mathcal{R}): these estimates work for generic billiard boundaries ∂D . For general boundaries, one would need to use some version of Lusternik-Schnirelman category, a far harder to compute invariant of a topological space.

The archetypal result in the area belongs to Birkhoff and deals with the planar case, $d = 2$. Before stating it, recall that for in this dimension (where $\partial D = S^1$), one can define the *rotation number* r of a trajectory $x \in \text{Cyc}_n$ as the index of the mapping of $S^1 \rightarrow \partial D$ constructed by gluing together oriented subsegments $[x_i, x_{i+1}]$ of ∂D . Rotation number r takes values from 1 through $(n-1)$; reversing the ordering of the points takes $r \mapsto n-r$, making only the $r = 1, \dots, \lceil (n-1)/2 \rceil$ values interesting geometrically.

Theorem 5. *For any n , there exist at least 2 geometrically distinct¹ (that is, not obtained from each other by cyclically changing or reflecting the indices of the vertices) n -periodic orbits of the billiard map for each rotation number $r = 1, \dots, \lceil (n-1)/2 \rceil$.*

The key observation here is that the space Cyc_n is a disjoint union of manifolds with corners corresponding to trajectories with different rotation numbers; each

¹One should note here that for r not coprime with n , the periodic trajectories include those traversing themselves several times.

component is homeomorphic to the product of the circle with a ball. One can easily verify that there exists a gradient flow of the length function l transversal to the boundary, yielding two critical points for each of the components.

Generalizations of Birkhoff's result to higher dimensions follow the same route (using Morse, or Lusternik-Schnirelman estimates), with the critical difference of far more involved topology of the space cyc_n of cyclic trajectories. The first steps towards these generalizations were done by Babenko [4]. The main body of results on the higher-dimensional versions of Birkhoff's theorems were obtained by Farber, Tabachnikov [5, 6] and, recently, Karasev [8]. Some of the key ingredients of these results were: the idea [4] to study the topology of Cyc_n (clearly independent of the specific boundary shape) using the round spheres, which leads to nice symmetries of l ; the work [12] on the topology of configuration spaces on general manifolds, which bear quite a few similarities with Cyc_n ; the usage of equivariant (with respect to the natural action of the dihedral group) cohomologies [5].

For generic boundaries, Farber and Tabachnikov [5] found the lower bound on the total number of n -periodic trajectories to be $(n-1)(d-1)$.

For general boundaries, the situation is far more complicated; I will just summarize some of the strongest results obtained thus far:

- The number of geometrically distinct n -periodic trajectories, n an odd integer, $d \geq 3$ is at least $\lfloor \log_2(n-1) \rfloor + d - 1$ [5];
- n if $d \geq 4$ is even, and $\lceil n/2 \rceil + 1$ if $d \geq 3$ is odd [6];
- If in addition n is prime, then the lower bound improves to $(n-1)(d-2) + 2$ [8];

It is not clear how much more information can be squeezed out of the topology of Cyc_n ; perhaps some work towards construction of billiard domains with few periodic trajectories might be useful.

NON-INTEGRABLE DISTRIBUTIONS AND BILLIARD DOMAINS WITH MANY PERIODIC ORBITS

What about the billiard domains with *many* periodic trajectories, specifically with k -parametric families of those? Again, to approach this question, it is useful to consider k -parametric families of cyclic tuples of points in \mathcal{R}^d , and ask, when they form a family of trajectories in some (undefined yet) billiard domain.

We consider a slightly different version of the space of cyclic configurations $Cycl_n$ with distinct consecutive points, dropping the condition that $x_i \in \partial D$. On this manifold, one can construct $\mathcal{B} \subset TCycl$, a codimension n distribution (i.e. a subbundle of the tangent bundle, see [10] for definitions and survey) given as the annihilator of the forms $\omega_i = p_i^*(d_i(l))$, where d_i is the differential of the length function l considered as a function of x_i ; p_i is the projection $p_i : x \mapsto x_i$. In words, the distribution \mathcal{B} is generated by the infinitesimal moves of x_i 's orthogonal to the bisectors of the angle $x_{i-1}x_ix_{i+1}$. One can easily prove the following result:

Theorem 6. *A germ of a k -dimensional manifold U in Cycl_n is a germ of k -parametric family of n -periodic trajectories in a (smooth) billiard domain if and only if U is an integral manifold for the distribution \mathcal{B} (that is $TU \subset \mathcal{B}|_U$).²*

The following family of examples is easy to visualize: for $d = 2$ and $n = 2$, the billiard domains with 1-parametric complete families of 2-periodic trajectories (that implies that each point on ∂D belongs to a two-periodic trajectory) are exactly the planar bodies of constant width. One can think of a mechanical system consisting of an axis (of length w) with a couple of independent wheels attached at its ends and moving without sliding; a trajectory which rotates the axis by 180° while keeping the wheels moving with positive speed each sweeps a curve of constant width w ; at the same time its motion is described by a non-holonomic distribution locally diffeomorphic to the standard contact structure in \mathcal{R}^{33}

The distribution \mathcal{B} is completely non-integrable on the level sets of l , meaning that there are no further global integrals for \mathcal{B} , or, equivalently, that the Lie brackets of the vector fields tangent to \mathcal{B} generate the tangent spaces to $\{l = \text{const}\}$.

This construction seems to be useful to address the following (surprisingly stubborn) conjecture by Ivrii:

There are no convex planar billiard domains with 2-dimensional families of n -periodic orbits.

In particular, the proof of (known, due to Rychlik – and others, [11]) the this result for 3-[periodic orbits becomes a rather transparent exercise in non-integrable distributions. More generally, there exists an algorithm (Cartan’s prolongation method) which implies that if a germ of k -dimensional integral manifold for a real-analytic distribution (such as \mathcal{B}) does not exist, this will be detected after a finite number of steps, involving differentiations and algebraic operations [9]. Unfortunately for computers, even the $n = 4$ case seems to be too involved for a purely machine-generated solution.

On the positive side, the distributional approach led to the following results:

- There are no 2-dimensional families of 3- or 4-periodic trajectories in the dual planar billiards ([7, 13], resp);
- The spherical billiards with 2-dimensional families of 3-periodic trajectories are all equivalent to the domain bounded by right equilateral triangle [1].

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²A warning: we consider here a somewhat relaxed notion of a billiard trajectory, paying attention only to the “equal angles of reflection” law, ignoring potential conflicts caused by convexity, collinearity etc.

³One might wonder why the 4-dimensional configuration space Cycl_2 in \mathcal{R}^2 is modeled on a 3-dimensional contact manifold: in fact, the length is the global integral for \mathcal{B} , and one can reduce the space to the level hypersurfaces of l .

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Birkhoff's conjecture

MAXIM ARNOLD

Elliptic billiards represent the most ancient part of the mathematical billiards theory. Arising from geometrical optics, elliptic billiards are now considered as a toy model for the general theory of integrable Hamiltonian systems. That is the main reason why elliptic billiards attract so much attention. Besides their extremely simple structure, billiards in elliptic tables develop almost every phenomenon any low-dimensional Hamiltonian system has.

An ellipse can be considered somehow as an internal perturbation of a disc. Thus many qualitative properties of the billiard transform may survive under such perturbation. For example it is well-known that elliptic billiards are also completely integrable. Any trajectory, except those which pass through the foci, is tangent to some confocal conic. As the interior of a disk is foliated by caustics having the form of concentric circles, the neighbourhood of the boundary of an elliptic billiard table is foliated by caustics having the form of confocal ellipses.

Birkhoff conjectured that the only billiards in strictly convex domains with this property are elliptic billiards. It is hard to believe that such a natural question still remains unsolved.

The aim of the present talk is to provide the proof of Bialy's theorem (see [1],[2]) which is a partial answer to the conjecture. If the whole phase space of the billiard

transform is foliated by continuous closed invariant curves then the billiard table is a disk.

In this purely educational talk, the concepts of convex billiard phase space and area form will be introduced. I shall provide the connection between the area form on the phase space of a billiard transform and the area form for the space of oriented lines in order to derive isoperimetric inequality and mirror equation. These two expressions lead to the cornerstone contradiction of the miraculous looking proof of Bialy's result.

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Interpolating Hamiltonians and length spectrum

PETER SPAETH

Let Ω be a bounded strictly convex domain in \mathbb{R}^2 with smooth boundary. The collection $\Gamma(m, n)$ of periodic billiard trajectories within Ω is non-empty for any winding number $0 < m/n < 1/2$ by the Poincaré–Birkhoff Theorem. So a natural question to then ask is what can be determined about the lengths of periodic billiard trajectories within Ω . In the early 1980s Marvisi and Melrose [1] studied the lengths of periodic trajectories close to the boundary of Ω .

Let $L(\Gamma(m, n))$ be the collection of all lengths of elements of $\Gamma(m, n)$ and denote

$$\mathcal{L}(\Omega) = \bigcup_{m,n} L(\Gamma(m, n)) \cup \mathbb{N}L_\infty,$$

where L_∞ is the length of the boundary $\partial\Omega$ of Ω . $\mathcal{L}(\Omega)$ is closed and every number mL_∞ is a limit point from below of $\cup_n L(\Gamma(m, n))$ whenever $m \geq 1$. Set

$$T_{m,n} = \sup L(\Gamma(m, n)), \quad t_{m,n} = \inf L(\Gamma(m, n)).$$

To study periodic billiard trajectories in a neighborhood of $\partial\Omega$, up to derivatives of all order, the billiard map agrees with a so-called interpolating Hamiltonian [1]. Hence the billiard map can be reinterpreted as the Hamiltonian flow of the interpolating Hamiltonian. The following result is deduced from this point of view.

Theorem 7. [1] *For every $m \geq 1$ and every $k \in \mathbb{N}$*

$$(1) \quad n^k (T_{m,n} - t_{m,n}) \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Furthermore there exist constants $c_{k,m} = c_{k,m}(\Omega)$ such that

$$(2) \quad T_{m,n} \sim mL_\infty + \sum_{k=1}^{\infty} c_{k,m} n^{-2k} \quad (\text{as } n \rightarrow \infty).$$

□

The constants $c_{k,m}$ depend on the geometry of the domain Ω and, remarkably, are related to the spectrum of the Dirichlet problem

$$\begin{cases} \Delta u = \lambda^2 u & \text{in } \Omega \subset \mathbb{R}^2, \\ u|_{\partial\Omega} = 0. \end{cases}$$

We would also like to remark that if $\partial\Omega$ is analytic one can improve the polynomial convergence in (1) with exponential convergence [2].

Now for outer billiards, one studies not the length spectrum but rather the area spectrum, in other words the areas enclosed by periodic outer billiard trajectories. Tabachnikov [4] obtains a similar expansion to that of equation (2).

Tabachnikov also asks if the area spectrum can be related to the spectrum of some differential operator as is the case for the inner billiard above. Because the outer billiard map is invariant under affine transformations so too should such an operator be.

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Introduction to rational polygons

KARSTEN KREMER

Let Q be a polygonal billiard table. If Q is simply connected and its angles are rational multiples of π , then the billiard flow on Q corresponds to the geodesic flow F^t on a translation surface M . After explaining the construction of M and introducing some basic facts about translation surfaces we show that the billiard flow on Q is minimal in almost every direction. The main reference for this talk is [1].

1. UNFOLDING

The main idea for investigating billiards on a polygonal table is to reflect the table instead of reflecting the ray. This process of unfolding the billiard table is called the *Katok-Zemliakov-construction*: Each time the billiard ray hits a side s of Q we reflect the table Q using the reflection σ_s on s . Then the ray continues travelling on a straight line on $\sigma_s(Q)$. When the ray again hits a side $\sigma_s(t)$ (for some side t of Q) the table is again reflected. Note that $\sigma_{\sigma_s(t)} = \sigma_s \sigma_t \sigma_s$, hence all possible reflections are contained in the group $A(Q) := \langle \sigma_s : s \text{ side of } Q \rangle$ which is a subgroup of the group $\mathbb{R}^2 \rtimes \text{O}_2(\mathbb{R})$ of motions in the plane. A ray in direction v is mapped by σ_s to a ray in direction $\bar{\sigma}_s(v)$, where $\bar{\sigma}_s$ is the projection of σ_s to the second component. Let $G(Q)$ be the subgroup of $\text{O}_2(\mathbb{R})$ generated by the $\bar{\sigma}_s$.

Definition 1. Q is rational if $G(Q)$ is finite.

Being a rational polygon thus means that once started a billiard ray can only move in finitely many directions. If Q is simply connected this is equivalent to the angles of Q being $\frac{k_i}{n_i}\pi$ for $k_i, n_i \in \mathbb{N}$ coprime ($i = 1, \dots, \ell$). Then $G(Q)$ is isomorphic to the dihedral group D_N , where $N = \text{lcm}(n_i)$.

The phase space of the billiard was $Q \times S^1 / \sim$ with $(x, v) \sim (x, \bar{\sigma}_s(v))$ for $x \in s$. We now replace this by $M := Q \times G(Q) / \sim$ with $(x, g) \sim (x, \bar{\sigma}_s \circ g)$ for $x \in s$. As $G(Q)$ is finite this is an oriented compact surface with a *flat metric*, i. e. constant curvature 0 except in the vertices of Q (called *saddle points*). The billiard flow on Q with starting direction θ now becomes the geodesic flow F^θ on M .

Remark 2. $g(M) = 1 + \frac{N}{2} \left(\ell - 2 - \sum_{i=1}^{\ell} \frac{1}{n_i} \right)$

Proof. We use the tiling of M by copies of Q to compute the Euler characteristic $\chi(M)$: The number of faces is $2N$, the number of edges ℓN , and the number of vertices $\sum \frac{2N}{2n_i} = N \sum \frac{1}{n_i}$. Thus we have $2 - 2g(M) = \chi(M) = N(2 - \ell + \sum \frac{1}{n_i})$. \square

2. TRANSLATION SURFACES

The surface M is defined as a set of (possibly transformed) copies of Q glued together by translations. We identify the copies of Q with subsets of \mathbb{C} , thus M becomes a Riemann surface, where (almost) all coordinate change maps are translations:

Definition 3. A Riemann surface X together with an atlas $z_\alpha : U_\alpha \rightarrow \mathbb{C}$ is called a translation surface with singularities $\Sigma = \{P_1, \dots, P_n\} \subset X$ if

- $\varphi = z_\alpha \circ z_\beta^{-1} : z \mapsto z + c$ for $(U_\alpha \cup U_\beta) \cap \Sigma = \emptyset$
- $P_i \in U_\beta \Rightarrow z_\beta(P_i) = 0$ and $\varphi : z \mapsto z^k$ for $k \in \mathbb{N}_{>1}$.

On a translation surface we have $\frac{dz_\alpha}{dz_\beta} = \varphi' = 1$, therefore $dz_\alpha = dz_\beta$, thus we can define globally the (so called *Abelian*) differential

$$dz := \begin{cases} dz_\alpha & \text{on } U_\alpha \\ 0 & \text{at } P_i \end{cases}$$

on X . On the other hand given an Abelian differential ω on X , we can define charts $U \rightarrow \mathbb{C}$ by $x \mapsto \int_{x_0}^x \omega$ for some arbitrary $x_0 \in U$. Then two different charts differ only by a constant, hence these charts define a translation structure on X .

For a *half-translation surface* we also allow coordinate change maps φ of the form $z \mapsto -z + c$. In this case we get $dz_\alpha = \pm dz_\beta$, thus we cannot define a differential dz globally, but for the quadratic differential dz^2 this is possible. We can also get back the half-translation structure from a quadratic differential ω as above, we only have to replace ω by a branch of its square root when integrating.

3. $\text{SL}_2(\mathbb{R})$ -ACTION ON TRANSLATION SURFACES

For a translation surface $S = (X, \{z_\alpha\})$ and a matrix $A \in \text{SL}_2(\mathbb{R})$ we define

$$A \cdot X := (X, \{c_A \circ z_\alpha\}) \text{ where } c_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2, z \mapsto Az.$$

This is again a translation surface, thus we have an action of $\mathrm{SL}_2(\mathbb{R})$ on the set of translation surfaces. The stabiliser $\Gamma(S)$ of S under this action is called the *Veech group* of S .

Example 4. Let S be the regular octagon of side length 1 with opposite edges glued together. Obviously the rotation $\frac{1}{2}\sqrt{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ by 45° is an element of $\Gamma(S)$. We also can divide S into two horizontal cylinders: one with height 1 and width $1 + \sqrt{2}$, the other with height $\frac{1}{2}\sqrt{2}$ and width $2 + \sqrt{2}$. Then the matrix $\begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix}$ maps both cylinders onto themselves and induces the identity on the boundary of the cylinders. Hence it is also contained in $\Gamma(S)$.

4. MINIMALITY OF F^θ

Let F^θ be the geodesic flow in direction θ on the surface M with at least one saddle point. We denote the orbit of x by $F_*(x)$ and its half orbits by $F_+(x) := \{F_t(x) : t > 0\}$ and $F_-(x) := \{F_t(x) : t < 0\}$.

Definition 5. F^θ is minimal if every trajectory $F_*^\theta(P)$ is dense.

Lemma 6. If a periodic trajectory exists in direction θ , then there is also a cylinder and a saddle-connection in this direction.

Proof. Parallel trajectories stay parallel if there is no saddle point between them. Thus by moving the starting point of a periodic trajectory a bit perpendicular to θ we get a cylinder. The only obstruction to enlarging this cylinder further is a saddle-connection. \square

Lemma 7. If $F_+^\theta(P)$ is infinite and $\beta := (P, Q]$ is an interval perpendicular to θ , then $F_+^\theta(P)$ hits β again.

Proof. The set $S := \{\text{first intersection of } F_-^\theta(X) \text{ with } \beta : x \in \Sigma \cup \{P\}\}$ is finite. Choose Q' on β such that $(P, Q']$ contains no point of S . Now the strip $F_+^\theta((P, Q'])$ hits β again (because its area is bounded) without hitting P or Σ before (by definition of S). \square

Theorem 8. If there is no saddle-connection in direction θ , then F^θ is minimal.

Proof. Suppose there is $X \in M$ such that the closure of its orbit $A := \overline{F_*^\theta(X)}$ is not equal to M . Then A is an F^θ -invariant set. Choose P in its boundary ∂A , and let α be an interval containing P perpendicular to θ . As $P \in \partial A$ there is $Q \in \alpha \setminus A$.

The complement A^c of A is open, thus Q is contained in an open interval $\beta \subset \alpha \cap A^c$. Enlarge β as far as possible to find $P' \in A$ such that $(P, Q] \subset A^c$. There is no saddle-connection and by Lemma 6 also no periodic trajectory in direction θ , hence $F^\theta(P')$ is infinite in (at least) one direction. By Lemma 7 therefore $F^\theta(P')$ hits $(P', Q] \subset A^c$. This contradicts $P' \in A$. \square

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Periodic Orbits for Billiards

JON CHAIKA

Recall that if P is a polygon the billiard flow F on $P \times S^1$ is defined by the rule that (x, θ) travels at unit speed in direction θ in the interior of P and at the boundary θ changes according to elastic collision. Given a polygon P we say that the trajectory of (x, θ) is periodic if there exists L such that $F^L(x, \theta) = (x, \theta)$. Notice that $F^L(x, \theta) = (x, \theta)$ does not necessarily imply that $F^L(y, \theta) = (y, \theta)$.

It is often convenient to invoke symbolic dynamics for billiards. Given a polygon P with n sides label the sides by $\{1, 2, \dots, n\}$. One can identify a point (x, θ) with the sides it hits. That is let $\tau : P \times S^1 \rightarrow \{1, 2, \dots, n\}^{\mathbb{Z}}$ by $\tau(x, \theta) = \dots c_{-1}, c_0, c_1, \dots$ where c_i is the label of the i^{th} side hit by the billiard flow of (x, θ) . This map is defined for the full measure set of (x, θ) whose orbits avoid the vertices of P . If the orbit of x in direction θ is periodic then the coding is as well. The combinatorial class of a periodic trajectory is this coding and all of its shifts (which form a finite set).

A seminal result in the study of periodic trajectories deals with the asymptotics of periodic trajectories with a given length. Let $N(P, L)$ denote the number of combinatorially different orbits of length less than or equal to L .

Theorem 1. (Masur) *For every rational polygon P there exist constants c_P, C_P such that $\liminf_{L \rightarrow \infty} \frac{N_P(L)}{L^2} \geq c_P$ and $\limsup_{L \rightarrow \infty} \frac{N_P(L)}{L^2} \leq C_P$.*

Question 1. For every rational polygon P does $\lim_{L \rightarrow \infty} \frac{N_P(L)}{L^2}$ exist?

Vorobets has computed estimates on c_P, C_P which still leave an enormous gap. Masur also showed that the directions of periodic trajectories were dense. Using Masur's result this was improved.

Theorem 2. (Boshernitzan, Galperin, Krüger, Troubetzkoy) *Let P be a rational polygon, for a residual set of $x \in P$ the set of periodic directions is dense in S^1 .*

The previous two results provide asymptotics, but one can ask for explicit periodic orbits.

Theorem 3. (Boshernitzan and Galperin, Stepin, Vorobets) *Let P be a rational polygon and S be a side. If $x \in S$ and θ is perpendicular to S then the trajectory of (x, θ) is periodic (if it avoids vertices).*

The proof is based on the Poincaré recurrence theorem and the fact that a trajectory in a rational polygon only travels in finitely many angles.

This result has been applied to general right triangles.

Theorem 4. (*Troubetzkoy*) *Every right triangle has a periodic trajectory in the direction perpendicular from its base.*

In fact he showed that most points on the base belong to a periodic orbit in this direction.

Definition 1. *A periodic trajectory in a polygon P is called stable if there is a neighborhood of P such that all the polygons in the neighborhood have periodic trajectories with the same combinatorics.*

Fix a direction. For each side k let α_k denote the angle this side makes with the fixed direction.

Theorem 5. (*Galperin, Stepin, Vorobets*) *A periodic trajectory with combinatorics (a_1, a_2, \dots, a_r) is stable iff each side appears equally often with odd subscripts and even subscripts.*

Theorem 6. (*Galperin, Stepin, Vorobets*) *There are no stable periodic trajectories in rectangles nor in countably many right triangles.*

Theorem 7. (*Hooper*) *No right triangle has a stable periodic orbit.*

Question 2. Does every triangle have a periodic orbit?

It is classical that any triangle with angles less than 90° has a periodic trajectory, the Fagnano orbit. This orbit is given by marking the intersection of altitude with the opposite side. There is a periodic orbit connecting these three points. This construction fails whenever a triangle has an angle greater than or equal to 90° because one of the altitudes no longer lies in the interior of the triangle.

Theorem 8. (*Schwartz*) *Every triangle with angles less than 100° has a periodic trajectory.*

The proof of this result is involved. It requires finding infinite families of periodic orbits for triangles. In particular the $(30^\circ, 60^\circ, 90^\circ)$ right triangle presents complications.

Theorem 9. (*Schwartz*) *For every $\epsilon > 0$ there exists a triangle within ϵ of the $(30^\circ, 60^\circ, 90^\circ)$ triangle whose smallest periodic orbit has length greater than $\frac{1}{\epsilon}$.*

This is much stronger than saying that $(30^\circ, 60^\circ, 90^\circ)$ has no stable periodic orbits. It proves that an infinite number of different combinatorics of stable periodic orbits are found in open sets close to $(30^\circ, 60^\circ, 90^\circ)$.

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The complexity of billiard coding

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Let $Q \subset \mathbb{R}^2$ be a compact polygon with sides C_1, \dots, C_k . As usual, a *billiard ray* in Q is a straight line that is reflected at the sides of Q according to the rule “incoming angle = outgoing angle”; if it hits a vertex, the billiard ray stops. For any billiard ray L in Q , the sides that it hits define a *cutting sequence* $c(L) = (c_0, c_1, \dots)$ where the c_i are elements of $\{C_1, \dots, C_k\}$.

We consider two types of coding associated with Q :

A) Directional coding

Fix a billiard ray L which does not hit a vertex of Q . We define \mathcal{L}_L to be the language of subwords of $c(L)$. In particular

$$\mathcal{L}_L(n) = \{\text{subwords of } c(L) \text{ of length } n\}.$$

B) Q -coding

Here we consider subwords of the cutting sequences of all billiard rays in Q . Any such subword is also the beginning of the cutting sequence of a billiard ray which starts on the boundary $\Gamma = \partial Q$ of Q . We call this language \mathcal{L}_Q , explicitly

$$\mathcal{L}_Q(n) = \{\text{initial segment of length } n \text{ of some ray } L \text{ emanating at } \Gamma\}.$$

For a language \mathcal{L} we denote by $p_{\mathcal{L}}(n)$ the number of different words of length n in \mathcal{L} ; the function $p_{\mathcal{L}}$ is called the *complexity* of \mathcal{L} .

The main results on the complexity of the two types of coding associated with billiard rays in Q are:

Theorem 8 (Hubert [3]). *Let Q be convex and rational with angles $\frac{k_i}{r_i}\pi$, $i = 1, \dots, k$. Assume k_i and r_i relatively prime and let $r = \text{lcm}(r_1, \dots, r_k)$. Let L be a billiard ray in Q that does not hit a vertex and is not parallel to a saddle connection. Then for all sufficiently large n we have*

$$p_L(n) = n(k - 2)r + 2r,$$

where we write p_L instead of $p_{\mathcal{L}_L}$.

Theorem 9 (Cassaigne, Hubert and Troubetzkoy [1]). *For any rational polygon Q we have*

$$p_Q(n) \sim n^3,$$

where again we write p_Q instead of $p_{\mathcal{L}_Q}$.

Here \sim means that the ratio $\frac{p_Q(n)}{n^3}$ is bounded by constants from above and from below. The statement in [1] requires Q to be convex, but N. Bedaride showed that this assumption is not necessary.

Theorem 10 (Galperin, Krüger and Troubetzkoy [2]). *For an arbitrary polygon Q , the complexity of \mathcal{L}_Q grows subexponentially, i. e.*

$$\lim_{n \rightarrow \infty} \frac{\log p_Q(n)}{n} = 0.$$

The classical example for Q is the square. Here any line L with an irrational slope α satisfies the hypothesis of Theorem 1. Moreover any such line is dense in Q . It follows that the language \mathcal{L}_L depends only on α , not on the initial point. By elementary reasoning one finds $p_L(n) = 4n + 4$ for any $n \geq 1$. Traditionally, the two vertical sides of the square are labelled by the same symbol, say 0, and the two horizontal sides by 1. Then the complexity of the cutting sequence is reduced to $p(n) = n + 1$; such sequences are known as *Sturmian sequences*.

For the proof of the first two theorems, the polygon Q is unfolded by reflections at the sides to a compact surface X . Note that with the notation of Theorem 1, X is triangulated by $2r$ copies of Q . The total number of sides in this triangulation is kr , and the number of vertices is $\sum_{i=1}^k \frac{r}{r_i}$. The copies of Q endow $X - V(X)$ with a translation structure, where $V(X)$ is the set of vertices. Each vertex v is a cone singularity for this structure of order k_i if v is a copy of the i th vertex of Q . The sum of the orders of the vertices is $\sum_{i=1}^k \frac{r}{r_i} \cdot k_i$. Since on the other hand, the sum of the angles in Q is $\sum_{i=1}^k \frac{k_i}{r_i} \cdot \pi = (k - 2)\pi$, we find the useful formula

$$\sum_{v \in V(X)} \text{ord}(v) = (k - 2) \cdot r.$$

The billiard ray L on Q transforms into a straight line L' on X . This line intersects the edges of the triangulation of X in a new cutting sequence $c(L')$. We label the edges of X as $E(X) = \{C_{ij} : i = 1, \dots, k; j = 1, \dots, r\}$. Then $c(L')$ has entries in $E(X)$ and defines a new language $\mathcal{L}_{L'}$ with a complexity $p' = p_{\mathcal{L}_{L'}}$. The proof of Theorem 1 is achieved by the following two propositions:

Proposition 1. $p'(n) = n(k - 2)r + 2r$ for all n .

Proposition 2. $p'(n) = p_L(n)$ for all sufficiently large n .

For the proof of Proposition 1, we consider the difference $p'(n + 1) - p'(n)$: Any contribution to this difference corresponds to a ray which is parallel to L and intersects n edges of X properly and then ends in a vertex. Conversely, in any

vertex $v \in V(X)$ there are $\text{ord}(v)$ different directions parallel to L' . Therefore we find with the help of the above formula

$$p'(n+1) - p'(n) = \sum_{v \in V(X)} \text{ord}(v) = (k-2) \cdot r.$$

Since $p'(1) = k \cdot r$, the proposition follows by induction.

For the proof of Proposition 2, we consider the closure W of the set of all $\sigma^m(c(L'))$ of the cutting sequence under iterates of the shift map σ . The closure is taken in the space $E(X)^{\mathbb{N}}$ of all sequences with entries in the set of edges of X , endowed with the product topology for the discrete topology on X . It turns out that W consists of the cutting sequences $c(\tilde{L})$ for lines \tilde{L} parallel to L' , and of sequences which, after an arbitrary finite initial segment, are of that type. Now if $p_L(n)$ would be strictly smaller than $p'(n)$ for infinitely many n , we could find sequences z_n and y_n of distinct elements in $\mathcal{L}_{L'}(n)$ that have the same image in \mathcal{L}_L . These sequences could be assumed to converge in W . The limits would have to be equal which would contradict our assumption.

The main ingredient in the proof of Theorem 2 is counting *generalized diagonals* in the polygon Q , i. e. billiard rays that begin and end in a vertex. On the unfolded surface X , they correspond to *saddle connections*, i. e. geodesics connecting two vertices of X .

Recall that there are countably many generalized diagonals even in the simplest case where Q is the square: Any line with a rational slope gives a periodic orbit thus its parallel through a vertex is a generalized diagonal.

For a saddle connection s on X which is not an edge, denote by $l(s)$ the number of its *links*, i. e. connected pieces cut out by the edges of X . Thus s starts in a vertex $v_0(s)$, then crosses $l(s) - 1$ edges, and finally ends in a vertex $v_1(s)$.

For $n \geq 0$ let $N(n)$ be the number of generalized diagonals in Q whose corresponding saddle connections s on X satisfy $l(s) \leq n$. Note that $N(0) = k$, the number of vertices of Q , and that the edges of Q are not counted as generalized diagonals.

The main result of [1] relates the number of generalized diagonals of Q to the complexity $p = p_{\mathcal{L}_Q}$ of the billiard coding on Q :

Theorem 11 ([1] Thm. 1.1).

$$p(n) = \sum_{i=0}^{n-1} N(i).$$

From this result, Theorem 2 follows using Masur's well known estimates of the number of saddle connections:

Theorem 12 (Masur [4], [5]). *There exist constants C_1, C_2 such that for all n ,*

$$C_1 \cdot n^2 \leq N(n) \leq C_2 \cdot n^2.$$

It should be noted that Theorem 4 holds for arbitrary polygons, while Theorem 5 makes essential use of the fact that Q is rational. As already mentioned, the proof of Theorem 4 in [1] requires Q to be convex, but N. Bedaride found a proof without this assumption.

A key role in the proof of Theorem 4 is played by the *bispecial words* in the language $\mathcal{L} = \mathcal{L}_Q$. They are defined as follows: For $n \geq 1$ and $c = (c_1, \dots, c_n) \in \mathcal{L}(n)$, let $m_l(c)$ be the number of edges e such that $ec = (e, c_1, \dots, c_n) \in \mathcal{L}(n+1)$, similarly let $m_r(c)$ be the number of e such that $ce \in \mathcal{L}(n+1)$ and $m_b(c)$ the number of pairs (e_1, e_2) of edges such that $e_1ce_2 \in \mathcal{L}(n+2)$. An element $c \in \mathcal{L}(n)$ is called bispecial if both $m_r(c) > 1$ and $m_l(c) > 1$. Denote by $B(n)$ the set of bispecial elements in $\mathcal{L}(n)$. With these notations, the following lemma can be proved by elementary reasoning:

Lemma 1 ([1] Thm. 2.1). *For $n \geq 0$ let $s(n) = p(n+1) - p(n)$. Then*

$$s(n+1) - s(n) = \sum_{c \in B(n)} (m_b(c) - m_l(c) - m_r(c) + 1).$$

For $c \in B(n)$ let $gd(c)$ be the number of generalized diagonals in Q whose corresponding saddle connections s on X satisfy $l(s) = n+1$ and whose cutting sequence is $c(s) = c$. Note that by definition

$$N(n) - N(0) = \sum_{i=0}^{n-1} \sum_{c \in B(i)} gd(c).$$

The crucial step in the proof of Theorem 4 now is

Lemma 2 ([1] Lemma 3.1). *For any $n \geq 1$ and any $c \in B(n)$*

$$m_b(c) = m_l(c) + m_r(c) + gd(c) - 1.$$

It follows by Lemma 1 that $s(n+1) - s(n) = \sum_{c \in B(n)} gd(c)$ for any $n \geq 0$. Now Theorem 4 easily follows using the above formula for $N(n)$.

For the proof of Lemma 2 consider the space

$$T\Gamma = \{(x, \varphi) : x \in \Gamma = \partial Q, 0 < \varphi < \pi\}$$

of tangent vectors at boundary points of Q that are directed to the interior of Q . $T\Gamma$ consists of k rectangles, each containing the billiard rays in Q that start at a specific side. These rectangles are considered as the *0-cells* of $T\Gamma$.

Let $f : T\Gamma \rightarrow T\Gamma$ be the billiard map, that sends a vector (x, φ) to the vector of the corresponding ray at the point of its first reflection. The map f is not defined in the points where the corresponding billiard ray runs directly into a vertex. The set of these points is a union of curves that subdivide $T\Gamma$ into subsets called *1-cells*. Inductively we obtain *n-cells* as the connected components of the complement of the set where f^n is not defined.

Note that for all billiard rays in a fixed n -cell, the first n entries of the cutting sequence are the same. More precisely, the n -cells correspond bijectively to the words of length n in \mathcal{L}_Q .

Now fix $c \in \mathcal{L}(n)$ and denote by C the corresponding n -cell. Then $m_b(c)$ is the number of cells obtained by subdividing C by the curves where f^n or f^{-1} is not defined. The number of these curves is $i = m_r(c) - 1$ and $j = m_l(c) - 1$, resp. Each intersection point of two such curves corresponds to a generalized diagonal; thus the total number of intersection points is $l = gd(c)$. It is proved in [1] Lemma 3.1 that all these intersection points are ordinary double points. Thus we obtain a cell decomposition of C with $m_b(c)$ faces, $l + l_0$ vertices (where l_0 denotes the number of vertices on ∂C) and $i + l + j + l + l_0$ edges. Euler's formula gives

$$1 = m_b(c) + l + l_0 - i - l - j - l - l_0,$$

which proves Lemma 2.

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Coding billiards in regular $2k$ -gons

GABRIELA SCHMITHÜSEN

In this talk we study cutting sequences of geodesics rays in translation surfaces obtained from regular $2k$ -gons ($k \geq 2$). The content is taken from [1], [2] and [3].

Suppose that we are given a translation surface obtained from glueing parallel edges of a polygon with $2k$ edges. Suppose further that the edges of the polygon are labelled with the letters a_1, \dots, a_k , where edges that are identified on the translation surface carry the same letter. Recall that the *cutting sequence* of a geodesic τ is the bi-infinite word in the letters a_1, \dots, a_k obtained by reading off the labels of the edges that τ crosses. The following two questions naturally arise:

- Which bi-infinite sequences over the alphabet $\mathcal{A} = \{a_1, \dots, a_k\}$ are cutting sequences?
- Can one reobtain the geodesic from its cutting sequence?

These questions can be satisfactorily answered, if the polygon is a regular $2k$ -gon. We will first describe the classical result for the square (i.e. $k = 2$) that states that the cutting sequences of non-periodic geodesics are precisely the Sturmian sequences. We then explain the recent result of Smillie and Ulcigrai which gives a nice explicit description of the cutting sequences in the same flavour for regular $2k$ -gons with $k \geq 3$.

1. The torus - case

Consider the torus obtained by glueing parallel edges of a unit square and let us label the vertical edges by 0 and the horizontal edges by 1. We may lift a geodesic τ on the torus to a line l on its universal covering, the Euclidean plane \mathbb{R}^2 . The cutting sequence of τ becomes the cutting sequence of the line l with the unit lattice $\Lambda = \langle e_1, e_2 \rangle$ where e_1 and e_2 are the two standard basis vectors of \mathbb{R}^2 . We assume that the line l does not meet any vertex of the lattice and that the slope of l is irrational. A first simple but crucial observation is that if the geodesic line $l : y = mx + b$ has slope $m \leq 1$, then it cannot cross two consecutive horizontal edges, i.e. its cutting sequence does not contain two consecutive 1's. Similarly, if $m \geq 1$ it does not contain two consecutive 0's. One calls a sequence *of type 0* if it does not contain the subword 11 and *of type 1* if it does not contain the subword 00. The fact that cutting sequences on the torus are of type 0 or of type 1 also follows from the result that the complexity of the cutting sequence c is $p_c(n) = n + 1$ (use $n = 2$), as it was shown in the previous talk. Recall that the *complexity* $p_c(n)$ is the number of subwords of c of length n . Since we assume that the slope of the line l is not rational, we have the following remark.

Remark 1. The cutting sequence of a geodesic on the torus is a *Sturmian sequence*, i.e. it has complexity $n + 1$ and is not eventually periodic.

It turns out that this actually is the only restriction and that both of our questions are answered by the following theorem.

Theorem 2. ([1, Theorem 6.4.22]) Every Sturmian sequence is the cutting sequence of some geodesic on the torus and uniquely determines it.

To give some idea of the proof, we explain how to reobtain the slope m of a line l from its cutting sequence. This involves two steps: in the first step one develops a deriving process, which leads to the so called *additive* and *multiplicative coding sequence*. In a second step, one reobtains m from the multiplicative cutting sequence using the *continued fraction algorithm*.

Definition 3. Suppose that c is a Sturmian sequence of type 0. Then c consists of blocks of 0's which are separated by single 1's. The derived sequence c' is obtained from c by removing one 0 from each block. If c is of type 1, one similarly obtains c' by removing one 1 from each block of 1's.

The derived sequence c' of a cutting sequence c can be geometrically understood as follows. Consider the sheared lattice $\Lambda' = f_A(\Lambda)$, where f_A is the affine map

$$f_A : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ with } A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ in type 0 and } A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ in type 1.}$$

Observe that looking at its image on the torus, we have in the first case replaced the vertical segment and in the second case the horizontal segment by the diagonal. From this one directly sees that the cutting sequence of the same geodesic with respect to Λ' instead of Λ is c' . Instead of taking the cutting sequence of l with respect to $\Lambda' = f_A(\Lambda)$, one may alternatively take the cutting sequence of $l' := f_A^{-1}(l)$ with respect to the original lattice Λ . It follows in particular that c' is again a cutting sequence. We denote by s_0 the type of c , by s_1 the type of c' and recursively by s_{n+1} the type of $c^{(n+1)} = (c^{(n)})'$. The sequence s_n is then the *additive coding sequence* of c . We obtain from this the *multiplicative coding sequence* a_n by counting consecutive equal letters in s_n , i.e. let $\psi(c) = c^{(n)}$, where $n \geq 1$ is minimal such that $c^{(n)}$ is of different type than c and define $a_n \geq 1$ recursively to be the smallest number such that $(\psi^n(c))^{(a_n)} = \psi^{n+1}(c)$.

A short calculation shows that the slope m' of the line l' is $\frac{m}{1-m}$ in type 0 and $m - 1$ in type 1. From this it is fast to see, that

$$m = \frac{1}{a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}}$$

is the continued fraction $[0; a_0, a_1, a_2, a_3, \dots]$ in type 0. A similar result holds in type 1.

2. The $2k$ - gon

In their recent work [3], Smillie and Ulcigrai mimic the approach for the torus and obtain a beautifully explicit answer for the translation surface obtained by glueing parallel edges of a regular $2k$ -gon for $k \geq 3$. We deal in this talk only with the case $k = 4$. This is not really a restriction, since the other cases work very much in the same way.

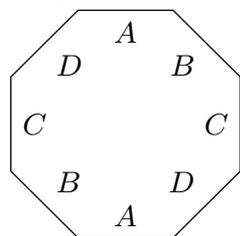


Figure 1: The case $k = 4$
– the regular octagon



Figure 2: The admissibility graph

A first observation is that instead of the two types of slopes in the case of the torus, one should distinguish eight different types for the direction of the geodesic τ , namely: $\theta \in [0, \frac{\pi}{8})$, $\theta \in [\frac{\pi}{8}, 2\frac{\pi}{8})$, \dots and $\theta \in [\frac{7}{8}\pi, \pi]$. Here θ denotes the angle that τ forms with the horizontal direction. One easily reads off from Figure 1 that

for $\theta \in [0, \frac{\pi}{8})$ the cutting sequence of τ defines a bi-infinite path in the admissibility graph in Figure 2. The other seven cases are obtained from the case $\theta \in [0, \frac{\pi}{8})$ by applying a symmetry of the surface. In order to transfer the deriving process, let us summarize what we did in the torus case. We applied an affine map depending on the type of the slope. Then c' was the cutting sequence of the preimage of the line. In the octagon case something similar turns out to work. Consider the affine map $\gamma = \sigma \circ \nu$, where σ is the reflection on the vertical axis and ν is the parabolic element fixing the horizontal direction, such that the derivatives of σ and ν are:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2(1 + \sqrt{2}) \\ 0 & 1 \end{pmatrix}, \text{ respectively.}$$

For the other seven types, one obtains the affine map γ by conjugating $\sigma \circ \nu$ by the corresponding symmetry of the surface. τ' is then by definition the preimage of the geodesic τ under γ . A main ingredient of [3] is to give a combinatorial description of the cutting sequence of τ' : Let c be a bi-infinite sequence that describes a path in the graph in Figure 2. Form the sequence c' by keeping only sandwiched letter, i.e. letters which are preceded and followed by the same letter. E.g. the word ... CACCCDBDCD... becomesACBC... . We call c' the *derived sequence* of c .

Theorem 4. If $c = c(\tau)$ is the cutting sequence of a geodesic τ , then for the derived sequence c' holds: $c' = c(\tau')$.

Similarly as in the torus case one obtains a multiplicative coding sequence in the letters $0, 1, \dots, 7$, by keeping track of the type when applying the deriving process. This is used to describe which cutting sequences may occur and to reobtain the slope of the geodesic from its coding by a continued fraction-like algorithm.

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Introduction to the theorem of Kerkhoff, Masur and Smillie

THIERRY MONTEIL

The theorem of Kerkhoff, Masur and Smillie [5] asserts that for any connected translation surface S (in particular for any rational polygonal billiard), and for almost every $\theta \in \mathbb{S}^1$, the flow in the direction θ is uniquely ergodic.

1. ROUGH IDEA

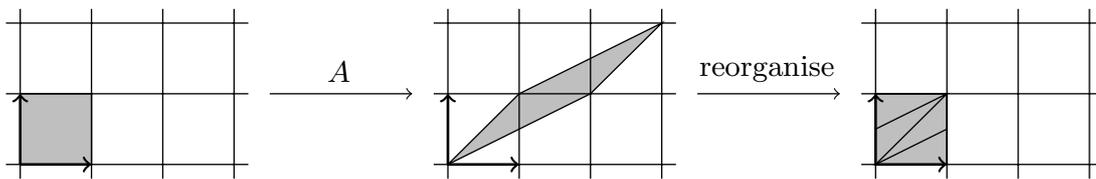
To define the flow on a polygonal billiard, we need the Euclidean notion of angle, whereas for translation surfaces we only need the affine notion of straight line.

In particular, we can apply the matrix $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on a translation surface S without changing the dynamical properties of the flow defined on it.

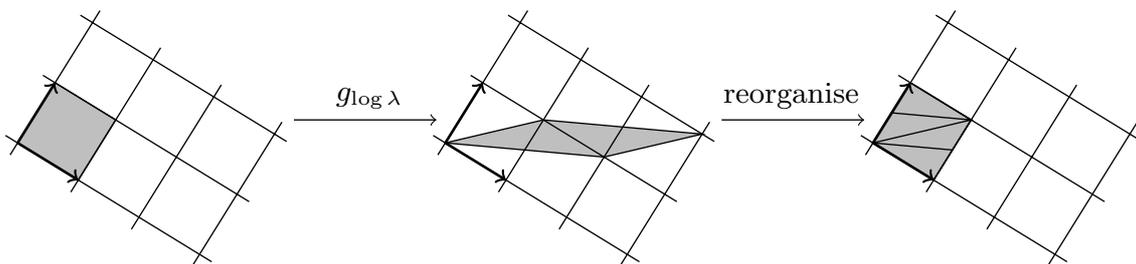
By contracting the vertical direction, g_t accelerates the time of the vertical flow, so that the asymptotic behaviour of the trajectory $\{g_t S\}$ on the space of translation surfaces will provide some informations about the dynamics of the vertical flow defined on S .

2. REMARK

Applying the flow g_t to the standard flat torus $\mathbb{R}^2/\mathbb{Z}^2$ will lead to a degenerate torus (its vertical meridians are shrunk), but it is not always the case, since it is sometimes possible to reorganise the translation surface while applying g_t . For example, let us consider the action of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on the standard flat torus, which is well defined since $A \in SL(2, \mathbb{Z})$.



The matrix A is diagonalisable, with two orthogonal eigenlines corresponding to the eigenvalues $\lambda = (3 + \sqrt{5})/2$ and λ^{-1} . If we rotate the torus so that the eigenlines become vertical and horizontal, we obtain a new torus S and the action of A on $\mathbb{R}^2/\mathbb{Z}^2$ corresponds to the action of $g_{\log \lambda}$ on S , hence $g_{\log \lambda} S = S$ and the trajectory $\{g_t S\}$ is periodic.



3. STRATEGY OF THE PROOF

The proof of the theorem is split into three parts:

- (1) In order to deal with the asymptotic behaviour of the trajectory $\{g_t S\}$, we will define a convenient topology on the set of translation surfaces and provide a criterion for compactness.

- (2) Masur's criterion [6]: if $\{g_t S\}$ does not converge to infinity (that is, if there exists a subsequence $t_n \rightarrow \infty$ such that $g_{t_n} S$ stays in a given compact), then the vertical flow is uniquely ergodic.
- (3) For any translation surface S and for almost every $\theta \in \mathbb{S}^1$, the flow in the direction θ does not converge to infinity (meaning that the previously discussed degeneration is a rare phenomenon).

We will focus on the first two parts.

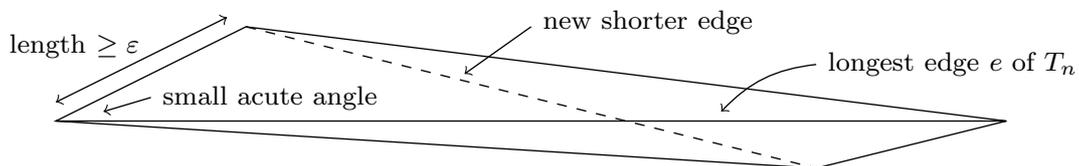
4. TOPOLOGY ON THE SET OF TRANSLATION SURFACES

Any translation surface can be triangulated so that the edges are saddle connections, and any collection of saddle connections having disjoint interiors can be extended to such a triangulation. Thanks to the Euler characteristic, the number of triangles in a triangulation depends only on the number of singularities and on the genus of the surface.

Let S be a translation surface and T be a triangulation of S . We define a small neighbourhood of S by letting the edges of T (viewed as a vectors of \mathbb{R}^2) move slightly around their initial position. In particular, two nearby translation surfaces admit triangulations that have the same combinatorics of glueing (and therefore have the same genus). Each saddle connection in S can be written as a sum of edges of T , so that the choice of the triangulation is not relevant.

Let $\text{systole}(S)$ denote the length of a shortest saddle connection in S . Let us prove that, given $g \geq 1$ and $\varepsilon > 0$, the set of translation surfaces of genus g (and of constant area 1) satisfying $\text{systole}(S) \geq \varepsilon$ is compact.

For this, let S_n be a sequence of translation surfaces of genus g whose systole is larger than ε . To get compactness, we have to ensure that it is possible to find a triangulation of each S_n whose edges have uniformly bounded length. We can achieve this by starting from any triangulation T_n of S_n , and assume that the longest edge e of T_n is very long. This edge bounds two triangles whose edges have length at least ε . Since the area of S_n is 1 and e is the longest edge, the angles that the triangles make with e are very small.



So we can reorganise the triangulation of S to get a better triangulation, by replacing (with a flip) e by a shorter saddle connection whose length is smaller by a constant of at least $\varepsilon/2$. So after finitely many such reorganisations, we get a triangulation of S_n whose edges have uniformly bounded length (the bound is of the order of $1/\varepsilon$).

Since the number of triangles is bounded, there are only finitely many combinatorics of glueings, so that we can assume that the type of the triangulation is fixed along a subsequence. Up to another subsequence, each edge of each triangle

converges, so that we can construct a limit translation surface S_∞ .

We also would like to say that two points in two close surfaces are close to each other if they are close in a common triangulation. Some tricky stuff can happen near a surface which admits two symmetric triangulations (different points of the surface will be identified). This problem can be solved by considering marked translation surfaces to break the symmetry, we will not take care about this later.

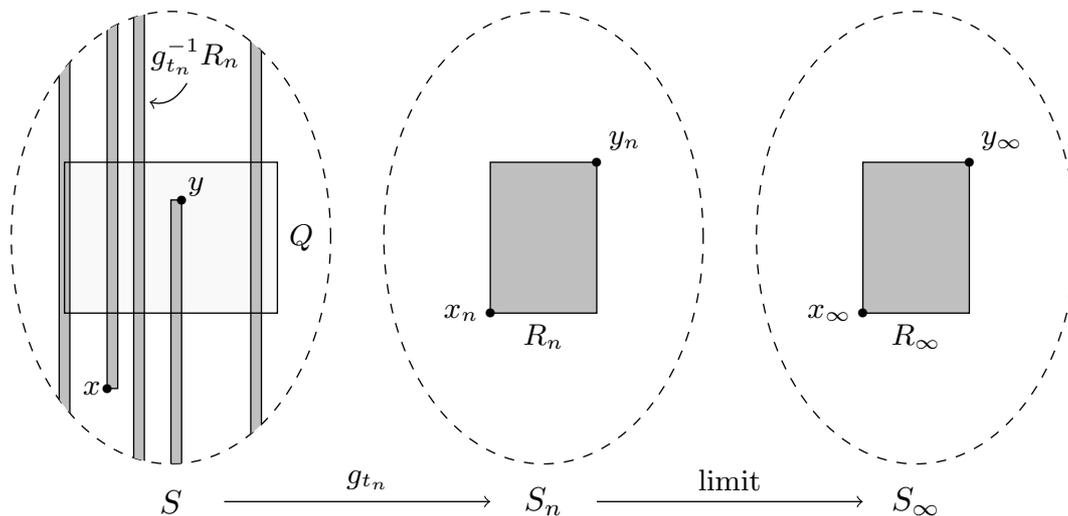
5. PROOF OF MASUR'S CRITERION

Let S be a translation surface whose trajectory $\{g_t S\}$ does not converge to infinity. Let $\{t_n\}$ be a subsequence such that $S_n := g_{t_n} S$ converges to a translation surface S_∞ .

Assume by contradiction that the vertical flow in S is not uniquely ergodic: there exist two distinct ergodic probability measures $\mu \neq \nu$ that are invariant under the vertical flow. Let Q be a horizontal rectangle in S such that $\mu(Q) \neq \nu(Q)$.

Let x be a generic point for μ . We can follow the trajectory $\{x_n\}$ of x on $\{S_n\}$ under g_{t_n} . Passing to a subsequence, we can assume that this trajectory converges to some $x_\infty \in S_\infty$. Do the same for y with ν .

Let us first assume that there exists an open set that does not meet any singularity which contains a rectangle R_∞ in S_∞ such that x_∞ (resp. y_∞) is the lower-left (resp. upper-right) corner of the rectangle. So, for n big enough, we can still embed a rectangle R_n in S_n , whose dimensions (w_n, h_n) are very close to the ones of R_∞ , and such that x_n (resp. y_n) is the lower-left (resp. upper-right) corner of it. Let us apply $g_{t_n}^{-1}$ to R_n : we get a very long rectangle in S (of height $e^{t_n} h_n$).

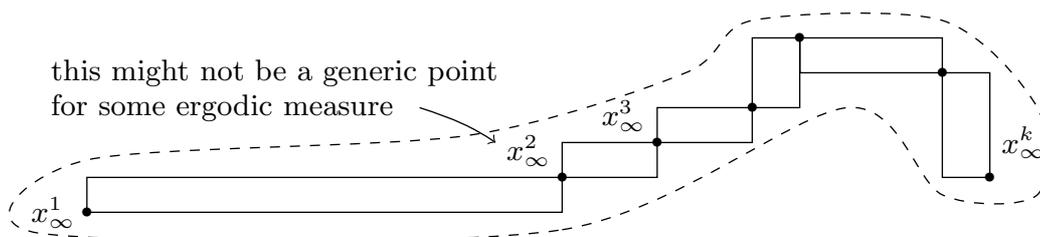


Its right side corresponds to the orbit of x under the vertical flow from time 0 to time $e^{t_n} h_n$, and its left side corresponds to the orbit of y under the vertical flow from time $-e^{t_n} h_n$ to time 0. If ϕ^t denotes the vertical flow on S , Birkhoff's ergodic theorem applied to the characteristic function of Q tells us that

$$\frac{1}{T} \left(\int_{t=0}^T \chi_Q(\phi^t(x)) dt - \int_{t=-T}^0 \chi_Q(\phi^t(y)) dt \right) \xrightarrow{T \rightarrow \infty} \mu(Q) - \nu(Q) \neq 0$$

For $T = e^{t_n} h_n$, the parenthesis on the left side is the difference between the length of the intersection of Q with the right side of the rectangle $g_{t_n}^{-1}(R_n)$ and the length of the intersection of Q with the left side of the rectangle $g_{t_n}^{-1}(R_n)$, which is bounded by two times the height of Q (a defect happens when $g_{t_n}^{-1}(R_n)$ is astride a vertical side of Q , which can happen at most twice). So, we get a contradiction when n goes to infinity.

We assumed the possibility to embed a nice rectangle R_∞ in S_∞ with x_∞ and y_∞ as opposite corners. If this is not the case, since x and y are not on the vertical of some singularity, we can ensure (up to shifting some elements of the subsequence $\{t_n\}$) that $\{x_n\}$ and $\{y_n\}$ stay uniformly far from the singularities, so that x_∞ and y_∞ are not singularities of S_∞ . Then, since S_∞ is connected, there exists a path in S_∞ between x_∞ and y_∞ , which can be surrounded by an open set not meeting any singularity (by compactness). So, there exists a finite sequence $x_\infty = x_\infty^1, x_\infty^2, \dots, x_\infty^k = y_\infty$ such that each rectangle with opposite corners x_∞^i and x_∞^{i+1} lies in the open set.



If, up to taking more subsequences, each x_∞^i is a limit point of the trajectory of some point x^i in S under g_{t_n} that is generic for some invariant ergodic measure μ^i (for ϕ_t), then we can apply the previous reasoning on each rectangle and prove that $\mu = \mu^1 = \mu^2 = \dots = \mu^k = \nu$, which concludes the proof.

To ensure this, we can notice that each x_∞^i ($1 \leq i \leq k - 1$) can be moved a bit, so, given a small open neighbourhood U_∞^i of x_∞^i in the open set, we have to find a good substitute for x_∞^i in U_∞^i . The open set U_∞^i can be backported to an open set U_n^i in S_n , for n big enough. This set and therefore its preimage $g_{t_n}^{-1}U_n^i$ have uniformly positive Lebesgue measure (in n). Since the Lebesgue measure is an average of ergodic measures, there exists an ergodic measure μ^i (for ϕ_t in S) that gives positive measure to the set of points that belong to infinitely many $g_{t_n}^{-1}U_n^i$, in particular, there exists a generic point x^i for μ^i in S such that the trajectory $\{g_{t_n} x^i\}$ has a limit point x_∞^i in U_∞^i .

6. RELATED RESULTS

6.1. Approximating irrational polygonal billiards by rational ones. Boshernitzan and Katok [5] proved that the set of n -gons on which the billiard flow is ergodic is a dense G_δ subset of the set of n -gons. The result also holds if we restrict ourselves to a subspace X of n -gons such that for any N , the set of rational tables P with $|G(P)| \geq N$ is dense in X (e.g. the set of right-angled triangles).

Vorobets [9] gave a quantitative version of this theorem: if P is a polygonal billiard table whose angles $\theta = (\theta_1, \dots, \theta_n)$ are such that there exist infinitely many rationals of the form $P/q = (p_1/q, \dots, p_n/q)$, with $\gcd(p_1, \dots, p_n, q) = 1$ and $\|\theta - P/q\|_\infty \leq 1/2^{2^{2^q}}$, then the billiard flow is ergodic on P .

6.2. Hausdorff dimension of the set of non-uniquely ergodic directions.

Masur [6] proved that for any translation surface S , the Hausdorff dimension of the set of non-uniquely ergodic directions is less than or equal to $1/2$. Cheung [2] proved that this bound is sharp: there exists translation surfaces whose set of non-uniquely ergodic directions has Hausdorff dimension equal to $1/2$. Masur and Smillie [7] proved that for any connected component \mathcal{C} of any stratum (in genus at least 2), there exists $\delta > 0$ such that for any generic translation surface S in the component \mathcal{C} , the Hausdorff dimension of the set of non-uniquely ergodic directions is δ .

6.3. Slow divergence still implies unique ergodicity.

Cheung and Eskin [3] proved that there exists $\varepsilon > 0$, depending only on the stratum of the translation surface S , such that the condition $\liminf_{t \rightarrow \infty} t^\varepsilon \text{ systole}(g_t S) > 0$ implies that the vertical flow is uniquely ergodic.

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Veech surfaces of genus 2 surfaces and L-shape rectangular billiard

VINCENT DELECROIX

Flat surfaces can be seen from two points of view

- a compact Riemann surface S together with a non null Abelian differential $\omega \in \Omega(S) - \{0\}$ up to scaling of the Abelian differential and isomorphisms of the surface,
- a set of polygons of \mathbb{R}^2 with identification on their boundary up to cut and paste equivalence.

The flat metric associated to the second definition corresponds to the symmetric tensor $|\omega|^2$. To pass from the first definition, to the second, one has to use a coordinate chart for which $\omega = dz$ or to consider the foliations induced by the vector fields X and Y of S satisfying $\omega(X) = 1$ and $\omega(Y) = i$. We refer to [5] for details.

Starting from a flat surface one can forget its flat structure and consider only its complex structure. The main argument of McMullen in the classification of genus 2 surfaces consists at looking at all flat structures that exist on a given Riemann surface. For a Riemann surface S of genus g , the set of Abelian differential $\Omega(S)$ is a complex vector space of dimension g . Hence one can think of the space of flat surfaces of genus 2 as a topological complex vector bundle of dimension 2 over the moduli space \mathcal{M}_g of Riemann surfaces. The main object used in McMullen's work to understand the set of flat structures on a given Riemann surface, is the Jacobian which is a complex tori endowed with a polarization constructed from the vector space of Abelian differential $\Omega(S)$ and the first homology group $H_1(S, \mathbb{Z})$.

1. GEOMETRY AND DYNAMIC OF $SL_2(\mathbb{R})$ ACTION

1.1. Symmetries of a flat surface. As shown in Thierry Monteil's talk, there is a well defined action of $SL_2(\mathbb{R})$ on the set of flat surfaces. Let us consider the following particular elements of $SL_2(\mathbb{R})$

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \quad r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad n_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Proposition 3. *Any element in $SL_2(\mathbb{R}) - \{id\}$ is conjugated to exactly one of the preceding. An element of $SL_2(\mathbb{R}) - \{id\}$ is called*

- *hyperbolic if it is conjugated to an g_t ,*
- *elliptic if it is conjugated to an r_θ ,*
- *parabolic if it is conjugated to an n_t .*

In THIERRY'S TALK, a link was made between the recurrence of the surface S in the stratum under the flow $G_\theta = \{r_\theta g_t r_{-\theta}\}_{t \in \mathbb{R}}$ and unique ergodicity of the directionnal flow in direction θ in the surface S . We refer to his talk for details about geometry of hyperbolic transformations.

The geometry of parabolic transformations is associated to periodic orbits (see [2])

Proposition 4. *Let (S, ω) be a flat surface. Then there exists $t > 0$ such that $n_t \cdot (S, \omega) = (S, \omega)$ if and only if the horizontal direction is completely periodic and the associated cylinders have commensurable moduli.*

1.2. $SL_2(\mathbb{R})$ **orbit.** There is a natural metric on \mathcal{M}_g that gives rise to nice embedding of the $SL_2(\mathbb{R})$ orbit.

Proposition 5. *Let (S, ω) be a flat surface, then there is a local isometry*

$$\mathbb{H} \simeq SL_2(\mathbb{R})/SO(2) \rightarrow \mathcal{M}_g$$

induced by the $SL_2(\mathbb{R})$ action on (S, ω) .

Moreover, McMullen for its classification uses the fact that this isometry extends to the moduli space of Jacobian of curves of genus g \mathcal{A}_g (this moduli space is the Siegel half-plane quotiented by a symplectic group).

Proposition 6 (Kra, [3]). *The (S, ω) be a flat surface then the composition*

$$\mathbb{H} \rightarrow \mathcal{M}_g \rightarrow \mathcal{A}_g$$

is an isometry.

1.3. **Dynamical properties of Veech surfaces.** The shape of the $SL_2(\mathbb{R})$ orbit of a flat surface gives information on the surface. Veech surfaces have “small” orbits because of their symmetries. Veech proves

Theorem 13 (Veech alternative). *Let (S, ω) be a Veech surface. The directional flow in the vertical direction is either parabolic or uniquely ergodic.*

This theorem allows us to have a precise understanding of periodic directions on a Veech surface. If (S, ω) is a Veech surface, each direction for which there exists a saddle connection is completely periodic and the geometry of the associated cylinder decomposition are associated to cups of the surface $\mathbb{H}/SL(S, \omega)$.

2. CALTA AND MCMULLEN CLASSIFICATION IN GENUS 2

In [1] it is proved using geometry of periodic directions that Veech surface in genus 2 can be completely described. She showed in particular that the periods of a surface must live in a quadratic field. In [3] using parametrization of Jacobian multiplications by Hilbert modular surfaces and some argument strongly related to the dimension 2 McMullen gives another point of view on Calta’s classification.

Theorem 14 ([1], [3]). *A flat surface $(S, \omega) \in \mathcal{H}(2)$ is a Veech surface if and only if the Jacobian of S admits real multiplication and ω is an eigenform for this action.*

In particular, all periods live in a quadratic field which is the trace group of the Veech group.

For the other stratum in genus 2 McMullen obtained the following theorem.

Theorem 15 (McMullen). *The only non primitive Veech surface in $\mathcal{H}(1, 1)$ is the double pentagon.*

Theorem 14 can be used to give explicit examples of Veech surface obtained from billiard.

Theorem 16 ([3]). *The L-shape rectangular billiard $L(a, b)$ gives a Veech surface under the unfolding procedure if and only if $a = x + z\sqrt{d}$ and $b = y + z\sqrt{d}$ for some $x, y, z \in \mathbb{Q}$ with $x + y = 1$ and $d \in \mathbb{N}$. In the latter case, the trace field of $P(a, b)$ is $\mathbb{Q}(\sqrt{d})$.*

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Billiards on Infinite Tables

DAVID RALSTON

1. THE INFINITE STAIRCASE

The *Infinite Staircase* is a flat surface which is non-compact. Formally, the surface is comprised of infinitely many 2 by 1 rectangles R_i , where each rectangle has its own vertical edges identified, and then

$$\begin{aligned} ([0, 1] \times \{0\})_i &\sim ([1, 2] \times \{1\})_{i-1}, \\ ([1, 2] \times \{0\})_i &\sim ([0, 1] \times \{1\})_{i+1}. \end{aligned}$$

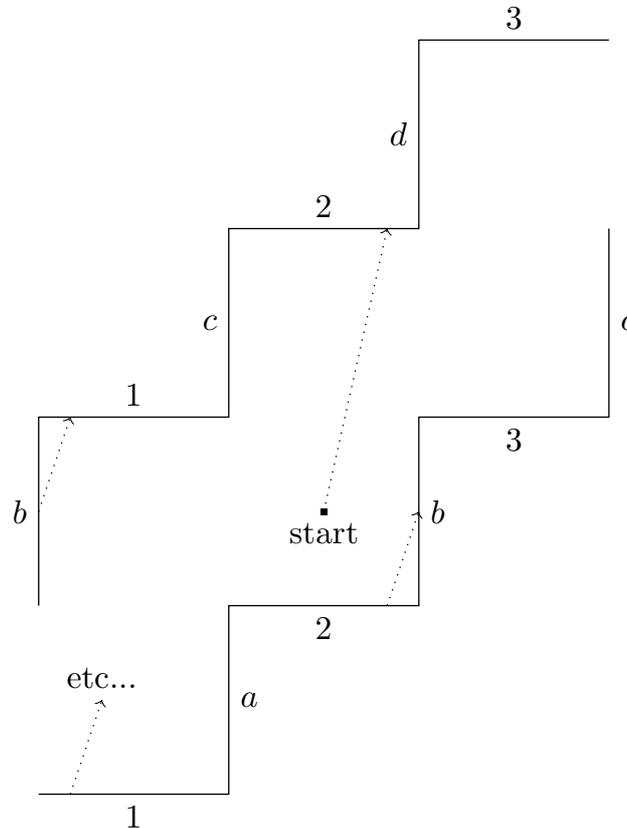
Because all identifications take place between parallel sides, the resulting structure is a flat surface, albeit of infinite area. The flow F_θ is simply the linear flow at angle θ on this flat surface. See Figure 1.

Though the flat surface is of infinite area, we may still define its Veech group in the normal way, and the Veech group of this surface is seen to be generated by the matrices

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, following the argument of [5], the dynamics of any trajectory starting with rational slope p/q is essentially unchanged (in terms of periodic cylinders) under the action of an element of this Veech group. However, we see that both matrices leave invariant the property that exactly one of p, q is even or neither are even. It is a classical result (and easy to verify) that *any* rational slope may be achieved using these two actions (and their inverses) starting with only the two

FIGURE 1. The Infinite Staircase: a trajectory begins at a point and flows linearly, with opposite vertical edges identified (labelled here 1, 2, 3...) and opposite horizontal edges identified (labelled here $a, b, c, d...$).



slopes $1/1$ and $0/1$. As the flow in slope zero clearly consists of infinitely many periodic cylinders and the flow in slope one consists of two nonrecurrent cylinders, the same behavior is experienced by rational slopes with one or no even terms, respectively.

In irrational directions, by tracking the return to any linear flow to the “bottom of a rectangle,” which happens at constant intervals, we see that the flow can be represented by the following transformation:

$$T(x, n) = (x + \alpha, n - \chi_{[0,1/2)}(x) + \chi_{[1/2,1)}(x)),$$

after identifying the vertical edges of the staircase to form a cylinder and simply considering the rectangles to be of width one. When the flow hits the left half of the bottom of a rectangle, it is sent down a level, but when it hits the right half, it continues up to the next level. This cylinder transformation, a skew product over an irrational circle rotation, has been extensively studied, shown to be ergodic in [2].

The underlying irrational circle rotation is uniquely ergodic, which implies very strong uniform convergence for ergodic sums. In the infinite measure situation,

however, distributional properties are considerably more intricate, with certain anomalous behaviors depending on the continued fraction expansion of α studied in [6][1]; the infinite measure scenario admits stranger and less predictable behavior than the much simpler irrational rotation.

2. \mathbb{Z} -COVERS OF FLAT SURFACES

One natural way to generalize this situation is to consider what are termed \mathbb{Z} -covers of flat surfaces; considering a flat surface to be a compact Riemannian surface M with a set of punctures P and a flat metric ($P \neq \emptyset$ in all but the genus one setting), consider an element $\omega \in H_1(M, P; \mathbb{Z})$. Then we create an infinite measure surface by taking an infinite collection $(M \setminus P)_i$ for $i \in \mathbb{Z}$ with identification along ω . When a linear flow crosses ω from one side, the flow is transferred to a higher index copy, and when the flow crosses from the other side, it is transferred to a lower index. In this way it is natural to see that the crossing number is intrinsic in the formal definition; see [3].

It is shown further in [3] that recurrence of the resulting flow on a flat surface of infinite area is equivalent to ergodicity of the same flow on the original surface if the relative loop ω was chosen to have zero holonomy. Necessity is not difficult to see: nonzero holonomy of ω will manifest itself as the linear flow in a ‘typical’ direction θ crossing ω more from one side than the other, resulting in a drift in the linear flow on the infinite surface. The sufficiency, however, relies on a result of Schmidt [7] and is highly reliant on the fact that the crossing number, which was used to define the transition from one copy of $(M \setminus P)$ to another, has one-dimensional range. The questions of studying recurrence or even ergodicity of the flow in \mathbb{Z}^2 -covers of flat surfaces are therefore still very much unanswered. Recent progress in the study of the *Ehrenfest Wind-Tree Model*, a periodic distribution of rectangular obstacles in the plane (and a \mathbb{Z}^2 -cover of a flat surface), appears in [4] and was extensively mentioned in another talk at this workshop.

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Security, or finite blocking property

GLENN MERLET

A translation surface or a billiard, is said to have the *finite blocking property* (FBP) if, for any two points O and A , there is finite set of points, disjoint from $\{O, A\}$ that intersect every trajectory between O and A . Such a surface is also called secure.

This property is preserved by linear action, and finitely branched coverings, so that it can be studied for translation surfaces rather than billiards. The torus has the FBP because all the middles of segments between two points in an unfolding are mapped on only four points $M, M + (1/2, 0), M + (1/2, 0), M + (1/2, 1/2)$ in the torus, thus all finitely branched covering of the torus (FBCT) have the FBP.

In a series of four papers [3, 4, 5, 6], T. Monteil has shown that having the FBP is equivalent to being a FBCT for several classes of surfaces, including Veech surfaces, n -regular polygons, (which have the FBP if and only if n is 3, 4 or 6), convex translation surfaces or L-shaped translation surfaces (which have the FBP if and only if the lengths of their edges are commensurable). For Veech surfaces, the result has been proved independently by E. Gutkin in [2].

The strongest result is the following : a translation surface with FBP is *purely periodic*, that is its flow is periodic in every direction that admits a periodic orbit and in any purely periodic translation surface, the holonomy generated by the periodic orbits is a lattice. Since the finitely branched covering of the torus are the translation surface whose holonomy is a lattice, the equivalence between having the FBP and being a FBCT holds for the surfaces whose holonomy is generated by the periodic orbits. This includes convex translation surfaces and an open dense set of surfaces with full measure. However, the conjecture of equivalence between having the FBP and being a FBCT is still open for general translation surfaces. Actually, some surfaces whose holonomy is not generated by their periodic orbits are known, but they have the FBP.

T. Monteil also proved the equivalence for Veech surfaces and for surfaces of genus 2, without introducing holonomy. For surfaces of genus 2, it deeply relies on the classification of surfaces with genus 2 by K. Calta in [1].

In this talk, we concentrate on an important lemma that states that a translation surface that contains two adjacent cylinders with non-commensurable perimeters can not have the FBP and on the proof that FBP implies pure periodicity. The proofs are based on elegant elementary arguments of geometry, constructing infinitely many trajectories in unfoldings of the surfaces and using similar triangles two prove that FBP would lead some ratios to be both rational and irrational.

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Dual (or outer) polygonal billiards

STEFAN MÜLLER

Let Q be an oriented strictly convex closed curve in the plane \mathbb{R}^2 (the dual billiard table). For a point x in the exterior $E \subset \mathbb{R}^2$ of Q , draw the unique tangent line (if Q is not smooth, the supporting line) to Q through the point x whose orientation agrees with that of Q . If A denotes the point of tangency, the dual billiard map T is defined by $Tx = r_Ax$, where r_A is the reflection in the point A .

If Q is convex but not strictly convex, let U be the union of lines containing straight segments of Q . Then T is defined on $E \setminus U$ (cf. corners in inner billiards). For generic $x \in E$, all iterations $\{T^kx \mid -\infty < k < \infty\}$ (the orbit of x) are defined.

In this talk, let Q be a convex polygon. The dual billiard map is discontinuous (along U). Locally (in regions bounded by U), T is a reflection in the same vertex. In particular, it is area-preserving, and a neighborhood of an n -periodic point consists of periodic points of period n or $2n$ (the latter only occurs for n odd).

Question 1. Are the orbits of T bounded? Periodic?

In this talk, we prove the following results.

Theorem 2 ([2]). If Q is quasi-rational, all orbits are bounded.

Theorem 3 ([2]). If Q is rational, all orbits are periodic.

Theorem 4 ([4]). For any Q , T has a periodic orbit.

See also the surveys [1, 3]. The question whether for a general polygon, some orbits may escape to infinity, is discussed in the next talk.

Fix a generic point $o \in E$. There is a unique vertex A_1 of $Q = Q_0$ such that $To = r_{A_1}o$. Let $Q_1 = r_{A_1}Q_0$ be the reflection of Q in its vertex A_1 (cf. unfolding technique for inner billiards). Denote by T_1 the dual billiard map corresponding to the dual billiard table Q_1 . We obtain a vertex A_2 of Q_1 such that $T_1o = r_{A_2}o$. Proceeding inductively, we obtain dual billiard maps T_k (use the inverse map T^{-1} to define $W^k(Q) := Q_k$ for negative integers) and a *necklace of polygons* $\mathcal{N}(Q, o) = \{Q_k \mid -\infty < k < \infty\}$. The following proposition allows us to study necklace dynamics in place of the orbits of T regarding our questions above.

Proposition 5. For a dual billiard map T ,

- (i) $T = r_{A_1}T_1r_{A_1}$,
- (ii) $T^k o = r_{A_1} \cdots r_{A_k} o$, $-\infty < k < \infty$,

- (iii) $\mathcal{N}(Q, o)$ is bounded if and only if the T -orbit of o is, and
- (iv) $\mathcal{N}(Q, o)$ is periodic if and only if the T -orbit of o is.

The proof of (i) is a simple geometric argument, while (ii) is proved by induction on k using (i). Since reflections preserve distance, (iii) follows directly from (ii), i.e. $\text{dist}(T^k o, Q) = \text{dist}(o, Q_k)$, and (iv) is proved similarly.

Note that each Q_k belongs to the collection \mathcal{P} of polygons P not containing o that are either translations or reflections in some point of the plane of Q . Each such P has two distinct vertices A_+ and A_- , where $W(P)$ is the reflection of P in the vertex A_+ , and $W^{-1}(P)$ is the reflection of P in the vertex A_- .

We next introduce the important notions of necklace vector and necklace polygon. Consider straight lines through the point o (the “origin”) parallel to the sides of the n -gon Q (there are $m \leq n$ such lines). This partitions the plane into $2m$ cones C_i with right boundary rays R_i (counted clockwise mod $2m$). For $P \in \mathcal{P}$, P contained in a cone C belonging to the above partition of the plane, define the *necklace vector* $\vec{c} = A_- A_+$. It is straightforward to prove this is independent of the choice of P . Next choose a point A_1 on the ray R_1 , and follow the necklace vector \vec{c}_1 of the cone C_1 until intersection with R_2 at the point A_2 . Continue this process until return to R_1 to obtain a polygonal line $Q' = A_1 \dots A_{2m+1}$. $Q' = Q'(Q)$ is called the *necklace polygon* of Q . The name is justified by the following result.

Proposition 6. Q' has the following properties.

- (i) $A_{2m+1} = A_1$, i.e. Q' is a closed polygon,
- (ii) Q' is determined by Q uniquely up to translations and dilations (from the choices of o , R_1 , and A_1),
- (iii) Q' is convex, and
- (iv) Q' is a centrally symmetric $2m$ -gon.

For (i), if $A_{2m+1} \neq A_1$, Q' spirals into o (replace T by its inverse if necessary). Continue Q' to an infinite polygonal line in both directions. One can now show that $\{Q_k \mid k > 0\}$ remains a bounded distance from o , while $\{Q_{-k} \mid k > 0\}$ spirals away to infinity (it is unbounded). Suppose Q is rational, that is, its vertices lie in some lattice (the dual billiard map is invariant under affine linear maps of the plane, so one may assume the lattice is \mathbb{Z}^2). In particular, the set $\{Q_k \mid -\infty < k < \infty\}$ is discrete. Being discrete and bounded, the set $\{Q_k \mid k > 0\}$ is finite, thus periodic. But then $\{Q_{-k} \mid k > 0\}$ is periodic as well, a contradiction to unboundedness. This proves (i) for rational polygons. Any polygon Q can be approximated by rational ones, and Q' clearly depends continuously on Q (in the obvious sense). Therefore the general case of (i) follows from the case of rational polygons previously established. The proof of (ii) and (iii) is easy, and (iv) can be shown by recalling that the definition of the necklace vector implies $\vec{c}_{m+i} = -\vec{c}_i$.

With the notations from above, there exists real numbers t_1, \dots, t_m satisfying $A_i A_{i+1} = t_i \vec{c}_i$, which by part (ii) of the previous proposition are defined up to a common multiple, or $(t_1 : \dots : t_m) \in \mathbb{R}P^m$ up to cyclic permutation. Q is called

quasi-rational if the t_i are rational multiples of each other, or $(t_1 : \dots : t_m) \in \mathbb{Q}P^m$. For example, every rational or regular polygon is quasi-rational.

We are now in a position to prove Theorem 1. Each $P \in \mathcal{P}$ is uniquely determined by the vertex A_+ and the information whether it is obtained from Q by a translation or a reflection (denoted by \pm respectively). Let S_i^\pm be the set of points A such that the corresponding $P \in \mathcal{P}$ intersects the ray R_i , and also write $S_i = S_i^+ \sqcup S_i^-$ (thought of as subsets of two different copies of the plane). The necklace dynamics induce maps $F_i: S_i \rightarrow S_{i+1}$ as follows: add the necklace vector \vec{c}_i until it intersects the ray R_{i+1} . (The points A move under the necklace map W along the necklace vectors, and the polygons themselves are translated under W^2 by twice the necklace vector.) The first return map $F: S_1 \rightarrow S_1$ is defined as the composition of all F_i . It is easy to see that the necklace $\mathcal{N}(Q, o)$ is bounded if and only if the first return map is bounded.

Next consider translations $R_{i+1} - k\vec{c}_i$ of R_{i+1} . Together with the rays R_i and $R_i + \vec{c}_i$ they define parallelograms with sides \vec{a}_i and \vec{b}_i (pointing away from the origin) and one diagonal \vec{c}_i . By definition of the map F_i , we have $F_i(x + 2\vec{a}_i) = F_i(x) + 2\vec{b}_i$. If Q is quasi-rational, we may without loss of generality assume the t_i are integers (they are defined up to a common multiple only). Then the point A_i lies on $R_{i+1} - t_i\vec{c}_i$, and the triangles oA_iA_{i+1} and $\vec{a}_i\vec{b}_i\vec{c}_i$ are similar. As a consequence, $A_iA_{i+1} = t_i\vec{c}_i$, $oA_i = t_i\vec{a}_i$, and $oA_{i+1} = t_i\vec{b}_i = t_{i+1}\vec{a}_{i+1}$. We thus have $F_i(x + 2t_i\vec{a}_i) = F_i(x) + 2t_i\vec{b}_i = F_i(x) + 2t_{i+1}\vec{a}_{i+1}$. Iterating this equality, we obtain for the first return map $F(x + 2t_1\vec{a}_1) = F(x) + 2t_1\vec{a}_1$, i.e. F is periodic.

It therefore suffices to show the orbits of points in the first two parallelograms (viewed from o) are bounded. Call their union π . By definition, $F^k(x) = G_k(x) + i_k(x)2t_1\vec{a}_1$ for some map $G_k: \pi \rightarrow \pi$ and an integer $i_k(x) \geq -1$. The return map F and its iterates are of course invertible, and making the same argument for F^{-k} we see that $-i_k(x) \geq -1$, or $i_k(x) \in \{-1, 0, 1\}$. That proves that $F^k(x)$ remains a bounded distance from π . The proof of Theorem 1 is complete.

Theorem 2 follows at once: a rational polygon is quasi-rational, so its orbits are bounded by Theorem 1. As remarked above, its orbits are also discrete, therefore they must be finite, i.e. periodic.

To prove Theorem 3, consider a closed polygon P with sides $\pm 2p_i\vec{c}_i$ for positive integers p_1, \dots, p_m to be determined in the course of the proof. Fix a necklace polygon Q' (recall this was determined up to translations and dilations only), and consider dilations qQ' for real $q > 0$. A moment's thought convinces the reader that if $|qt_i - 2p_i|$ is sufficiently small (say less than some $\epsilon > 0$), the orbit of a point x in the cone C_1 near the point qA_1 under the necklace dynamics follows precisely the polygon P , in particular, its orbit is closed, i.e. x is a periodic point. In order to choose p_1, \dots, p_m, q appropriately, consider the linear flow in the direction (t_1, \dots, t_m) on the m -torus $\mathbb{R}^m / (2\mathbb{Z})^m$. Then $|qt_i - 2p_i| < \epsilon$ for some integers p_i if and only if the image of the origin under the time- q map is ϵ close to the origin. By Poincaré recurrence, this happens infinitely many times. Thus the proof in fact establishes the existence of infinitely many fixed points.

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Outer billiard on kites

NICOLAS BEDARIDE

We are interested in outer dual billiard on polygon. A kite is a polygon K_A with vertices $(0; -1), (-1; 0), (0; 1), (A; 0)$ where $A \in (0 : 1)$. Schwartz in [1] consider outer billiard on the kite and prove the following theorem where $E = \mathbb{R} * \{-1\}$.

Theorem 17. *Let A be an irrational number, then there exists an uncountable set $C \subset E$ such that all defined orbits in C are unbounded. For all defined orbit of point in E , either the orbit is periodic or the point is in C . The set of periodic points in E is open and dense. The set C has zero lebesgue measure, the map $A \mapsto \text{Hdim}(C_A)$ is almost every where constant and the first return map to C is defined and conjugated to an odometer.*

In this talk, we will present the structure of the proof of this theorem, following [1]. The principal idea is to approximate A by a suitable sequence p_n/q_n of odd rationals, and to study in detail the outer billiard in K_{p_n/q_n} . Then the limit of periodic orbit will converge in Hausdorff topology to an unbounded orbit. A rational p/q is odd if pq is an odd number. First the first return map to E is well defined and denoted ψ . The main tool is **the arithmetic graph**. It is defined by the map

$$T : \mathbb{Z}^2 \rightarrow E \\ (m, n) \mapsto (2Am + 2n + 2\alpha, (-1)^{m+n+1})$$

This map is injective if A is irrational, and injective on ball of radius $1/q$ if $A = p/q$. Now the graph has vertices in \mathbb{Z}^2 , and there is a edge between (m, n) and (m', n') if and only if $\psi T(m, n) = T(m', n')$ and the two points are at distance at most $\sqrt{2}$. Denote this graph by Γ and $\hat{\Gamma}$ the component containing $(0, 0)$. The map T must be understood as the description of the first return map to E of a point $(\alpha, -1)$. Three theorems are proved on the structure of this graph: First $\hat{\Gamma}$ is the disjoint union of closed polygons and embedded infinite polygonal arc (**embedding theorem**). Secondly the graph is diriged by six families of lines (**Hexagrid theorem**). Finally $\hat{\Gamma}$ has a cantor structure (**Copy theorem**). The proofs of these three theorems involve another theorem: the **master picture theorem**. This theorem relates the structure of the arithmetic graph to a partition of a three dimensional torus in polyhedrons. The proof of this theorem uses two important lemmas, the **pinwheel lemma** and the **torus lemma**. The pinwheel lemma is a generalization of a well known result on outer billiard.

The second part of the proof consists in the approximation of A by a **strong sequence**. For an odd number $\hat{\Gamma}$ is an infinite periodic polygonal arc. A strong sequence is a sequence of odd rationals which fulfills some hypothesis related to Γ_{p_n/q_n} . Roughly speaking each period of the graph must contain a part of $\Gamma_{p_{n-1}/q_{n-1}}$. Then there exists a subsequence of graphs which converge in the Hausdorff topology to a graph Γ_∞ . We can prove that this graph is the arithmetic graph of a point $(\alpha, -1)$ in E . Then the orbit of this point will be unbounded since the graph rises infinitely far away from the base line. The construction can be generalized to obtain a cantor set of unbounded orbits. The existence of the strong sequence is due to a combination of arithmetic results and combinatorics structure of Γ for odd rational numbers.

Then a deep analysis of the structure of periodic point in a kite corresponding to a strong sequence allows to obtain some dynamical results. Once again, the structure of the arithmetic graph via the Hexagrid theorem is in the heart of the proof.

After this work, a lot of open questions appear. For a dynamical point of view we can ask:

- what is the dynamic for points outside E ?
- Is almost every orbit in a kite periodic ?
- What can be done for other polygons ?

In symbolic dynamics we can define a coding map on four letters by associating a letter to each cone in which the outer billiard map is continuous. This map is of zero entropy, thus the complexity function is not exponential.

- Can we describe the language in a given Kite ?
- What is the complexity of this language ?

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A Family of Periodic Orbits of the Koch Snowflake Fractal

ROBERT G. NIEMEYER

(joint work with Michel L. Lapidus)

The Koch snowflake KS is a fractal that encloses a region with finite area; see Figure 1. However, the nowhere differentiability of the closed non-rectifiable curve prevents one from considering $\Omega(KS)$ as a well-defined billiard (i.e., there is a priori no well-defined phase space “ $\Omega(KS) \times S^1 / \sim$ ” where \sim is the standard equivalence relation on the boundary that identifies outward pointing vectors with inward pointing vectors). In order to understand how one may begin to define the compact region $\Omega(KS)$ with boundary KS as a billiard, we investigate the behavior of orbits in $\Omega(KS_n)$, where KS_n is the n th prefractal approximation of KS ; see Figure 1.

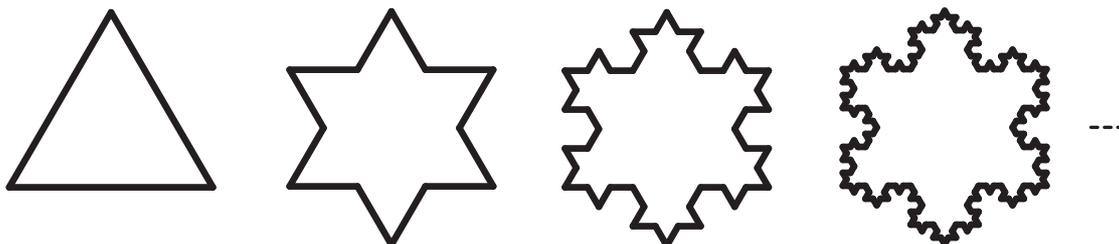


FIGURE 1. $\Omega(KS_i)$, $i = 0, 1, 2, 3$. The ellipses in the figure are meant to indicate that the process continues ad infinitum. $\Omega(KS)$ would then be the limiting object.

The compact region $\Omega(KS_n)$ is a rational billiard with interior angles measuring $4\pi/3$ and $\pi/3$. The associated flat surface $\mathcal{S}(KS_n)$ has (nonremovable) conic singularities at points corresponding to the obtuse angles of the prefractal approximations, and we may extend the geodesic flow on $\mathcal{S}(KS_n)$ continuously at the (removable) conic singularities corresponding to acute angles. Moreover, $\mathcal{S}(KS_n)$ is a branched cover of the hexagonal torus $\mathcal{S}(KS_0) := \mathcal{S}(\Delta)$ with two branch points.

A consequence of the fact that $\mathcal{S}(KS_n)$ is a branched cover over the surface $\mathcal{S}(KS_0)$ corresponding to the equilateral triangle billiard $\Omega(KS_0)$ is that directions which are periodic in $\Omega(KS_0)$ are periodic in $\Omega(KS_i)$ for all $i \geq 0$, and vice-versa; the same holds for uniquely ergodic directions. In addition, the Veech group $\Gamma(KS_i)$ is a subgroup of $\Gamma(KS_0)$ for all $i \geq 0$.

We are then in a position to define what we call a *compatible sequence of orbits*. Such a sequence is comprised of orbits with compatible initial conditions (x_i^0, θ_i^0) . These initial conditions are either all (eventually) the same or contain base-points x_i^0 on the boundaries KS_i such that they are all collinear in the direction θ_0^0 . Consequently, $(x_i^0, \theta_i^0) = (x_0^0, \theta_0^0)$ for all i .

In this talk, we show that it is possible to construct a family of periodic orbits of the Koch snowflake that is both dynamically relevant and geometrically relevant. We call such orbits piecewise Fagnano orbits for their obvious geometric relation to the Fagnano orbit of the equilateral triangle. Such orbits can be phrased as inverse limits of piecewise Fagnano orbits of the prefractal approximations; see Figure 2. We detail many of the claims made in this talk in [LaNi2].

We then make conjectures on the existence of an associated “fractal flat surface” and the existence of a well-defined phase space “ $\Omega(KS) \times \tilde{S}^1$ ”, where \tilde{S}^1 is presumably a collection of inward pointing directions. It is the case that the genus of the surfaces $\mathcal{S}(KS_i)$ increases with the number of sides k_i of KS_i . Since the Veech group $\Gamma(KS_n)$ of $\mathcal{S}(KS_n)$ is a subgroup of $\Omega(KS_0)$, we conjecture that the inverse limit of Veech groups will be dynamically relevant and offer insight into how to properly define a “fractal flat surface” $\mathcal{S}(KS)$ in such a way that 1) a geodesic flow is defined and 2) such a flow is dynamically equivalent to the yet to be determined billiard flow on the conjectured phase space “ $\Omega(KS) \times \tilde{S}^1$ ”. For



FIGURE 2. The primary piecewise Fagnano orbits $pp\mathcal{F}_i$ of $\Omega(KS_i)$, $i = 0, 1, 2$.

more detailed conjectures and statements of open problems, please see [LaNi1] on the Mathematics ArXiv (arXiv:0912.3948).

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Recurrence in periodic Ehrenfest wind-tree models

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(joint work with Pascal Hubert, Serge Troubetzkoy)

In the original wind-tree model proposed by Tatyana and Paul Ehrenfest in 1912 [EhEh] for a Lorentz gas, a point particle evolves horizontally and vertically bouncing elastically on randomly located identical square obstacles at a $\pi/4$ angle with the trajectory.

The periodic wind-tree model $T_{a,b}$ is a variation where obstacles are $a \times b$ rectangles aligned on the \mathbb{Z}^2 lattice. J. Hardy and J. Weber in 1980 [HaWe] study trajectories of slope ± 1 in tables $T_{a,b}$ with $a + b = 1$, using skew products over rotations and the Denjoy–Koksma inequality. When $a/b \notin \mathbb{Q}$ they show the

particle recurs and diffuses ‘abnormally’, i.e. staying up to time t in a domain whose diameter increases like $\log t \log \log t$.

We let (a, b) vary in the whole parameter space $(0, 1)^2$ and consider all possible starting directions for the billiard flow. The complete understanding achieved in recent years of the $\mathrm{SL}(2, \mathbb{R})$ orbits of L-shaped translation surfaces in genus two, and in particular of torus coverings (Kani [Ka1, Ka2], Hubert–Lelievre [HuLe], McMullen [Mc]), allows for a precise description of the billiard flow on $T_{a,b}$. For $(a, b) \in \mathbb{Q}^2$ it has a strong link to the linear flow on a square-tiled L-shaped translation surface.

Using this, we prove:

- for a class of rational parameters, the existence of completely periodic directions
- for another class of rational parameters, the escaping of certain trajectories, and a lower bound for the rate of escape in almost all directions.

These results extend to a dense G_δ of parameters:

Theorem 18. *There is a dense G_δ of parameters (a, b) for which*

- *the billiard flow in $T_{a,b}$ is recurrent*
- *the set of periodic points is dense in the phase space of $T_{a,b}$*
- *$\forall k \geq 1$, for a.e. slope α , for a.e. starting point x , the following estimate holds for the diffusion of the directional billiard flow ϕ_t^α :*

$$\limsup_{t \rightarrow \infty} \frac{\mathrm{dist}(\phi_t^\alpha x, x)}{\log t \log \log t \dots \log \log \dots \log t} = +\infty.$$

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Veech groups for holonomy free covers

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We calculate the lattice Veech groups for (branched) cyclic covers

$$p_{w,z} : \Sigma_{k,l}(w, z) \rightarrow \mathbb{T}$$

of the standard torus $\mathbb{T} := \mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$. The construction of the covers $p_{w,z}$ is explained below.

To begin, let $p : \mathbb{C} \rightarrow \mathbb{T}$ be the standard projection $z \mapsto [z] := z \bmod \mathbb{Z} \oplus i\mathbb{Z}$. Denote the oriented line segment connecting $0 \in \mathbb{C}$ with $z \in \mathbb{C}$ by $[0, z]$. The standard projection of the line segment $[0, z]$ to \mathbb{T} defines a relative cycle $\llbracket 0, z \rrbracket \in H_1(\mathbb{T}, \{[0], [z]\}; \mathbb{Z})$. In a similar fashion we associate to each pair $(w, z) \in \mathbb{C}^2 \setminus \{0\}$ and $(k, l) \in \mathbb{N}^2$ with $\gcd(k, l) = 1$ a relative homology class

$$k\llbracket 0, w \rrbracket + l\llbracket 0, z \rrbracket \in H_1(\mathbb{T}, \{[0], [w], [z]\}; \mathbb{Z}).$$

Relative homology classes of that shape are called *k-l-slit classes*.

By a standard topological construction we can associate each (nontrivial) k-l-slit class a cyclic \mathbb{T} cover unbranched outside $\{[0], [w], [z]\} \subset \mathbb{T}$. The cover $p_{w,z} : \Sigma_{k,l}(w, z) \rightarrow \mathbb{T}$, is up to minor differences for particular configurations of $(w, z) \in \mathbb{C}^2 \setminus \{0\}$ the cover associated to the cycle $k\llbracket 0, w \rrbracket + l\llbracket 0, z \rrbracket \in H_1(\mathbb{T}, \{[0], [w], [z]\}; \mathbb{Z})$ by means of the topological construction. Pulling back the standard 1-form $dz \in \Omega^1(\mathbb{T})$ gives a holomorphic 1-form $\omega_{w,z} \in \Omega(\Sigma_{k,l}(w, z))$ and defines a translation structure on $\Sigma_{k,l}(w, z)$. We call a cyclic translation cover of the shape $(\Sigma_{k,l}(w, z), \omega_{w,z})$ a k-l-slit surface. We are interested in the Veech group

$$\Gamma(\Sigma_{k,l}(w, z)) := \Gamma(\Sigma_{k,l}(w, z), \omega_{w,z}),$$

whenever that group is a lattice subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

k-l-slit surfaces which are Lattice surfaces. With the convention

$$\Sigma_{k,l}(n) := \Sigma_{k,l}\left(\frac{l}{n}, -\frac{k}{n}\right),$$

we have:

Theorem 19. *Any k-l-slit surface $\Sigma_{k,l}(w, z)$ ($\gcd(k, l) = 1$) lies on the $\mathrm{SL}_2(\mathbb{Z})$ orbit of one of the surface $\Sigma_{k,l}(n)$ for some $n \in \mathbb{N}$. The Veech-groups of the $\Sigma_{k,l}(n)$ are congruence groups of level 1, namely*

$$\Gamma(\Sigma_{k,l}(n)) = \Gamma_1(n).$$

The way we prove the Theorem is to study the *Hurwitz space* $\mathcal{H}_{k,l}^\infty$ (for translation structures also known as *Kernel foliation*) of k-l-slit covers. Because of the translation structure of the covers, that space $\mathcal{H}_{k,l}^\infty$ admits not only a translation structure, but is also a \mathbb{T} -cover. The $\mathrm{SL}_2(\mathbb{Z})$ action on k-l-slit covers equals the (natural) $\mathrm{SL}_2(\mathbb{Z})$ action on the kernel foliation $\mathcal{H}_{k,l}^\infty$. The k-l-slit surfaces with *trivial holonomy* define a translation torus inside $\mathcal{H}_{k,l}^\infty$ containing all isomorphy classes of k-l-slit surfaces which are lattice surfaces. The $\mathrm{SL}_2(\mathbb{Z})$ action on the

zero holonomy torus is (up to some coordinate changes) well studied and gives the above Theorem.

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Recurrence of quenched random Lorentz tubes and related questions

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(joint work with Giampaolo Cristadoro, Marco Lenci)

A Lorentz tube is a system of a particle (or, from a statistical viewpoint, many non-interacting particles) freely moving in a domain extended in one direction and performing elastic collisions with randomly placed obstacles. These kinds of “extended billiards” are, on the one hand, paradigms of systems where some transport properties can be studied in a rigorous mathematical way and, on the other hand, reliable models for real situations. The primary interest in their study lies on such properties as recurrence, diffusivity, and transmission rates. Unfortunately, few rigorous results are available and their proofs typically rely on some periodic structure.

In [1] a more realistic situation is taken into account: the so-called quenched disorder.

Consider the billiard dynamics in a cylinder-like set that is tessellated by countably many translated copies of the same d -dimensional polytope and place in each copy a random configuration of “good” semi-dispersing obstacles. The ensemble of dynamical systems thus defined, one for each global choice of scatterers, is called *quenched random Lorentz tube*.

For $d = 2$ is proved that, under general conditions, almost every system in the ensemble is recurrent.

Assumed that the scatterers are smooth enough, mutually disjoint, and not too big or small, and assumed that the distribution of the scatterers is ergodic under shift and the finite horizon condition is fulfilled, one can construct for almost all the configurations a suitable Poincaré section with an associated 1-dimensional commutative discrete cocycle such that the new system has finite measure and is ergodic.

The key point of the proof is that the cocycle can be proved to be recurrent and this recurrence is equivalent to the recurrence of the corresponding Lorentz tube. The ergodicity, then, follows in a quite standard way because it is possible to define the local stable and unstable manifolds and the connectivity of almost all the pairs of point in the Poincaré section is guaranteed by the assumptions.

Here a remark has to be done to say that for such systems, the very fundamentals of ordinary ergodic theory do not work, in particular the Poincaré Recurrence Theorem can be applied only on the reduced finite-measure system but this would give no information as whether the system is totally recurrent, totally transient, or mixed.

In this talk the main idea of the proof were discussed and some related problem and extensions were presented, with particular care to $d > 2$ systems and “exotic” $d = 2$ tubes.

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