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Mini-Workshop: 1-Motives

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Abstract. One-motives were introduced by Deligne in 1974 [10], as a generalization of the theory of semiabelian varieties. Viewed today, after Voevodsky's theory of mixed motives [31], it can be understood as motives of level ≤ 1. While Voevosdky's more general theory of mixed motives contains deep conjectures which at present seem to be out of reach, one-motives are much more accessible. In this mini-workshop, recent progresses were discussed: various aspects of one-motives and their realizations were explained, some applications in arithmetic algebraic geometry were given.

Mathematics Subject Classification (2000): 11: Number theory, 11G: Arithmetic algebraic geometry 14: Algebraic geometry, 14F: (Co)homology theory, 14K: Abelian varieties and schemes, 14L: Algebraic groups.

Introduction by the Organisers

The mini-workshop *1-motives* was well attended by participants with broad research areas. The talks covered the topics we describe here below. We first recall the general framework.

The existence of a category of pure motives over a given field k has been originally conjectured by Grothendieck [14]: its existence would result from the so called Standard Conjectures. According to Beilinson [6] and Deligne [9], the category of pure motives should be regarded as a subcategory of the abelian category of mixed motives. One does not know how to construct this category but if it exists, then also its derived category exists and one can try to understand the properties of this latter category. Beilinson in particular strongly developed this view point and formulated conjectures on motivic complexes. See also [7], [21] and [12]. See [24] for the aspects related to motivic homotopy.

A first concrete step towards the construction of mixed motives was provided by P. Deligne via level ≤ 1 mixed Hodge structures in [10] introducing the category of 1-motives over a field k. Deligne 1-motives have been introduced in the first talk by F. Andreatta. Voevodsky [31] constructed a triangulated category of motives DM that has been explained in a talk by J. Riou. Regarded as a two terms complex, a 1-motive can be understood as an object in Voevodsky's triangulated category DM and weight one Suslin-Voevodsky [30] cohomology is naturally 1-motivic. See [30] and [22]. In fact, Deligne's 1-motives generate the part of Voevodsky's triangulated category of motives coming from motives of varieties of dimension at most one as explained in a the second talk by J. Riou. This was indicated by Voevodsky, rationally, worked out by Morel and Orgogozo [25] and integrally, over a perfect field, up to inverting the exponential characteristic, is now proven in [4].

Therefore, 1-motives are compatible with the hypothetical motivic t-structure on DM and, with respect to the homotopy t-structure, they provide the so called 1-motivic sheaves (see [4]). Triangulated 1-motives are also compatible with biextensions as explained by C. Mazza in his talk. One can get categories on n-motivic sheaves that have been treated in the first talk by J. Ayoub. Actually, in a second talk, J. Ayoub has provided the construction of an abelian category explaining it as a candidate for mixed 2-motives in DM (see [1]).

Furthermore, Deligne constructed the 1-motive of a curve, realisation functors and settled down a list of conjectures asserting the algebraicity of certain level 1 mixed Hodge structures associated to the singular cohomology of complex algebraic varieties. These conjectures have been proven, rationally. Contributions have been given by several authors, among them J. Carlson, B. Kahn, A. Rosenschon, M. Saito, V. Srinivas and N. Ramachandran. In particular, in [4] it is shown that the full embedding of triangulated 1-motives in DM has (rationally) a left adjoint denoted LAlb. Applied to the motive of a variety X, it yields a bounded complex of 1-motives, denoted $\mathrm{LAlb}(X)$ that can be fully computed for smooth varieties and partly for singular varieties. As an application, this gives a full proof of Deligne's conjectures, rationally. Integrally, a proof is still missing.

In arithmetics, there are several tools that are deeply linked to 1-motives as it has been explained in the lectures by T. Szamuely. A first tool is that 1-motives provide a natural way to unify and generalize duality theorems for the Galois cohomology of commutative group schemes over local and global fields. For example, several duality theorems for the Galois and étale cohomology of 1-motives and the existence of a 12-term Poitou-Tate type exact sequence are proven in [15] and [13] for the function field case. The results give a common generalisation and a sharpening of well-known theorems by Tate on abelian varieties as well as results by Tate-Nakayama and Kottwitz on algebraic tori. These results are also used in [16] to study the Manin obstruction to the Hasse principle for rational points on torsors under semiabelian varieties over a number field. Another arithmetic application explained in T. Szamuely's talk is the function field case of the Brumer-Stark conjecture (which is related to the class group of a global field): indeed its proof by Deligne and a recent refinement due to Greither/Popescu both rely crucially on the properties of a certain 1-motive. Secondly, 1-motives are a natural tool for studying Kummer theory on semiabelian varieties over a field. Kummer-Chern-Eisenstein motives were constructed by A. Caspar in his talk. Moreover, the Tate conjectures for abelian varieties over number fields (which have been proven by Faltings) do have a similar formulation for 1-motives. The proof of this result is sketched by Jannsen [17], and a complete proof is now available in P. Jossen's thesis [18]. In P. Jossen's talk the Mumford-Tate conjecture for 1-motives has been discussed. Further, Raynaud introduced geometric monodromy for 1-motives [26] showing that the geometric monodromy is zero if and only if the 1-motive has potentially good reduction. Duality theorems for 1-motives with bad reduction was the subject of D. Park's talk. Motivic Integrals of $K3$ Surfaces over non-archimedean fields have been discussed in A. Stewart's talk.

It is possible to provide crystalline realizations of 1-motives [2] as indicated by F. Andreatta in his first talk. A crystalline version of Deligne's conjectures is not proven (for H^1 see [2]). Andreatta and Bertapelle [3] recently used the crystalline nature of the universal extension of a 1-motive to define a canonical Gauß-Manin connection on its de Rham realization as indicated by F. Andreatta in his second talk. As an application, they provide a construction of the so called Manin map from a motivic point of view. The study of the kernel of such a map plays a crucial rôle in Manin's proof of the geometric Mordell conjecture. Note that Kato and Trihan [19] used p-adic 1-motives to prove the conjectures of Birch and Swinnerton-Dyer in characteristic $p > 0$.

Finally, 1-motives with additive factors were defined by Laumon [20] in characteristic zero. H. Russell, in his first talk, provided how to deal with 1-motives with unipotent part in positive characteristics, allowing torsion and admitting Cartier duality [29]. This way, the 1-motive is no longer $\mathbb{A}^1$-invariant and a precise way to link this to the full motivic picture is to be understood. In [11] the universal regular quotient of the Chow group of points on projective varieties was constructed. H. Russell [28] reconstructed this by means of categories of rational maps to algebraic groups and using Cartier duality for Laumon 1-motives as explained and generalised in his second talk. Also additive higher Chow groups, as defined in [8] would fit into the picture. Rülling [27] showed that they compute the big de Rham-Witt complex. Over the complex numbers, formal (mixed) Hodge structures (of level 1) are introduced in [5] (see also [23]) in such a way that the Hodge realization of Deligne's 1-motives extends to a sharp realization for Laumon 1-motives, providing an equivalence of categories. This further extends providing sharp de Rham realization of Laumon 1-motives as indicated in the talk by A. Bertapelle.

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Mini-Workshop: 1-Motives

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Abstracts

1-Motives: origins and (p-adic) realizations

Fabrizio Andreatta
(joint work with Alessandra Bertapelle, Luca Barbieri-Viale)

Following [1] we explain how 1-motives originate via the simplicial Picard functor of varieties defined over a perfect field with special interest for positive characteristics. We then consider the crystalline realization of Deligne's 1-motives in positive characteristics and prove a comparison theorem with the De Rham realization of (formal) liftings to zero characteristic. This is proven using the crystalline nature of the universal extension of a 1-motive. Using [2] we prove that this result holds also for general bases. This allows to define a canonical Gauss-Manin connection on the de Rham realization of 1-motives. As an application we provide a construction of the so called Manin's map from a motivic point of view.

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2-Motives

Joseph Ayoub

The goal of this talk is to give a reasonable candidate for a category of mixed 2-motives over a field k. By "reasonable" we mean a category $\mathbf{M}_2(k)$ that shares some of the mirific properties that the conjectural category of mixed 2-motives is expected to enjoy. The plan is as follows. First, we give the definition of $\mathbf{M}_2(k)$. Then we explain the ideas behind the verification that $\mathbf{M}_2(k)$ is an abelian category.

Definition 1. *Let k be a perfect field. An object $M \in \mathbf{DM}_{\text{eff}}(k)$ is called a* mixed 2-motive*, or simply a* 2-motive*, if it satisfies the following conditions:*

(a) $H_i(M) = 0$ *for* $i \notin \{0, -1, -2\}$*;*
(b) $H_0(M)$ *is a* 0*-motivic sheaf;*
(c) $H_{-1}(M)$ *is a* 1*-motivic sheaf;*
(d) $H_{-2}(M)$ *is a* 2*-motivic sheaf which is* 1*-connected;*
(e) *if L is a non-zero* 0*-motivic sheaf, then $L[-1]$ is not a direct factor of M and* $Ext^1(H_{-2}(M), L) = 0$.

The category of mixed 2*-motives is denoted by* $\mathbf{M}_2(k)$.

Some explanations are needed. Here, $\mathbf{DM}_{\text{eff}}(k)$ is Voevodsky's category of effective motives with rational coefficients. It can be defined as a full subcategory of $\mathbf{D}(Shv_{tr}^{Nis}(Sm/k, \mathbb{Q}))$, the derived category of Nisnevich sheaves with transfers on the category of smooth k-varieties. A complex K is in $\mathbf{DM}_{\text{eff}}(k)$ if its homology sheaves $H_i(K)$ are homotopy invariant for all $i \in \mathbb{Z}$. By a non-trivial theorem of Voevodsky, this is equivalent to the condition that the obvious maps $\mathbb{H}^n_{Nis}(X, K) \to \mathbb{H}^n_{Nis}(\mathbb{A} \times X, K)$ are isomorphisms for all $X \in Sm/k$ and $n \in \mathbb{Z}$. In particular, one sees that $\mathbf{DM}_{\text{eff}}(k)$ is a triangulated subcategory. The usual t-structure on the derived category of sheaves with transfers induces a t-structure on $\mathbf{DM}_{\text{eff}}(k)$ which is known as the *homotopy t-structure*. The heart of the homotopy t-structure is equivalent to the category $\mathbf{HI}(k)$ of homotopy invariant sheaves with transfers.

In [2], the notion of a n-motivic sheaf was introduced. Given a smooth k-variety X, we denote $h_0(X)$ the largest homotopy invariant quotient of the sheaf with transfers represented by X. Explicitly, $h_0(X)$ is the cokernel of

$$i_1^* - i_0^* : \underline{\text{hom}}(\mathbb{A}^1, \mathbb{Q}_{tr}(X)) \to \mathbb{Q}_{tr}(X).$$

Then a homotopy invariant sheaf with transfers $\mathcal{F}$ is n-motivic if it admits a presentation

$$\bigoplus_{\beta} h_0(Y_\beta) \to \bigoplus_{\alpha} h_0(X_\alpha) \to \mathcal{F} \to 0$$

where X_α and Y_β are smooth varieties of dimension $\leq n$. We denote $\mathbf{HI}_{\leq n}(k)$ the full subcategory of $\mathbf{HI}(k)$ whose objets are the n-motivic sheaves. We recall the following fact form [2].

Proposition 2. *For $n \in \{0, 1\}$, $\mathbf{HI}_{\leq n}(k) \subset \mathbf{HI}(k)$ is a thick abelian subcategory, i.e., stable by subobjects, quotients and extensions. Moreover, the obvious inclusion admits a left adjoint. These are denoted by:*

$$\pi_0 : \mathbf{HI}(k) \to \mathbf{HI}_{\leq 0}(k) \quad \textit{and} \quad Alb : \mathbf{HI}(k) \to \mathbf{HI}_{\leq 1}(k).$$

We say that $\mathcal{F} \in \mathbf{HI}(k)$ is 1-*connected* if $Alb(\mathcal{F}) = 0$. It is 0-*connected* if $\pi_0(\mathcal{F}) = 0$. Now, that all the terms of Definition 1 are explained, we can state the main theorem of [1].

Theorem 3. *The category $\mathbf{M}_2(k)$ is abelian.*

In the rest of the talk, we will explain the strategy of the proof of Theorem 3. The proof goes by first showing that some larger category ${}^2\mathcal{H}^{\mathcal{M}}(k)$ is abelian. The latter is the full subcategory of $\mathbf{DM}_{\text{eff}}(k)$ whose objects are called $(2, \mathcal{H})$-sheaves. An object $M \in \mathbf{DM}_{\text{eff}}(k)$ is a $(2, \mathcal{H})$-sheaf if it satisfies all the properties of Definition 1 except the one stating that $H_{-2}(M)$ is 2-motivic. In other words, instead of (d), we only ask that $H_{-2}(M)$ is 1-connected. Then we show that ${}^2\mathcal{H}^{\mathcal{M}}(k)$ is abelian by constructing a t-structure on $\mathbf{DM}_{\text{eff}}(k)$ whose heart is exactly the category of $(2, \mathcal{H})$-sheaves. The t-structure doing this job is the 2-*motivic t-structure*.

The 2-motivic t-structure is obtained from the homotopy t-structure by applying twice an abstract construction which we now explain. Let $\mathcal{T}$ be a triangulated category endowed with a t-structure $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$. Let $\mathcal{H}$ denotes the heart of $\mathcal{T}$. Assume that we are given a thick abelian subcategory $\mathcal{A} \subset \mathcal{H}$ and a left adjoint to the inclusion $\mathrm{F} : \mathcal{H} \to \mathcal{A}$. Assume also that for every exact sequence in $\mathcal{H}$:

$$0 \to A' \to A \to A'' \to 0$$

with $A'' \in \mathcal{A}$, the morphism $\mathrm{F}(A') \to \mathrm{F}(A)$ is a monomorphism. Then we have the following fact (cf. [1]).

Lemma 4. *We define a t-structure $({}^{\text{`}}\mathcal{T}_{\geq 0}, {}^{\text{`}}\mathcal{T}_{\leq 0})$ on $\mathcal{T}$ by the following conditions.*

- *An objet $P \in \mathcal{T}$ is in ${}^{\text{`}}\mathcal{T}_{\geq 0}$ iff $P \in \mathcal{T}_{\geq -1}$ and $H_{-1}(P)$ is* F*-connected (i.e., it is sent to* 0 *by* F*).*
- *An object $N \in \mathcal{T}$ is in ${}^{\text{`}}\mathcal{T}_{\leq 0}$ iff $N \in \mathcal{T}_{\leq 0}$ and $H_0(N)$ is in $\mathcal{A}$.*

Remark 5. The new t-structure $({}^{\text{`}}\mathcal{T}_{\geq 0}, {}^{\text{`}}\mathcal{T}_{\leq 0})$ is called a *perverted* t-structure. An objet A is in the heart of the perverted t-structure if it satisfies the following three conditions.

(a) $H_i(A) = 0$ for $i \notin \{0, -1\}$;
(b) $H_0(A)$ is on $\mathcal{A}$;
(c) $H_{-1}(A)$ is F-connected.

The construction of the n-motivic t-structures $({}^{n}\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^{n}\mathcal{T}^{\mathcal{M}}_{\leq 0}(k))$, for $n \in \{0, 1, 2\}$, goes by induction on n. For $n = 0$, it is simply the 0-motivic t-structure. For $n \in \{1, 2\}$, it is obtained by perverting the $(n-1)$-motivic t-structure with respect to the subcategory of $(n-1)$-motives. More precisely, we set.

Definition 6. *The* 1-motivic *t-structure $({}^{1}\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^{1}\mathcal{T}^{\mathcal{M}}_{\leq 0}(k))$ is obtained by perverting the homotopy t-structure using the subcategory $\mathbf{HI}_{\leq 0}(k) \subset \mathbf{HI}(k)$. The heart of the* 1*-motivic t-structure is denoted by ${}^{1}\mathcal{H}^{\mathcal{M}}(k)$ and its objects are called* $(1, \mathcal{H})$-sheaves*.*

These are objects $M \in \mathbf{DM}_{\mathrm{eff}}(k)$ such that $H_i(M) = 0$ for $i \notin \{0, -1\}$, $H_0(M)$ is 0-motivic, and $H_{-1}(M)$ is 0-connected. The homology functors with respect to the 1-motivic t-structure is denoted by ${}^{1}H_i$. In ${}^{1}\mathcal{H}^{\mathcal{M}}(k)$ we have special objects called 1-*motives.* They are defined as follows.

Definition 7. *An object $M \in \mathbf{DM}_{\mathrm{eff}}(k)$ is a* 1-motive *if $H_i(M) = 0$ for $i \notin \{0, -1\}$, $H_0(M)$ is a* 0*-motivic sheaf and $H_{-1}(M)$ is a* 0*-connected* 1*-motivic sheaf.*

It is easy to see the link between our definition and Deligne's classical definition of 1-motives. Moreover, it can be shown that $\mathbf{M}_1(k) \subset {}^{1}\mathcal{H}^{\mathcal{M}}(k)$ is a thick abelian subcategory and that the inclusion has a left adjoint $Alb : {}^{1}\mathcal{H}^{\mathcal{M}}(k) \to \mathbf{M}_1(k)$. Thus, the following definition makes sense.

Definition 8. *The* 2-motivic *t-structure $({}^{2}\mathcal{T}^{\mathcal{M}}_{\geq 0}(k), {}^{2}\mathcal{T}^{\mathcal{M}}_{\leq 0}(k))$ is obtained by perverting the* 1*-motivic t-structure using the subcategory $\mathbf{M}_1(k) \subset {}^{1}\mathcal{H}^{\mathcal{M}}(k)$. The heart of the* 2*-motivic t-structure is denoted by ${}^{2}\mathcal{H}^{\mathcal{M}}(k)$ and its objects are called* $(2, \mathcal{H})$-sheaves*.*

It is now a matter of unrolling the definitions to see that a $(2,\mathcal{H})$-sheaf is an object $M \in \mathbf{DM}_{\mathrm{eff}}(k)$ satisfying all the conditions of Definition 1 with the exception of $H_{-2}(M)$ being a 2-motivic sheaf. Theorem 3 follows then quite easily from this.

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Sharp de Rham realization

Alessandra Bertapelle

(joint work with Luca Barbieri-Viale)

Let $M = [u\colon L \to G]$ be a Deligne 1-motive over $\mathbb{C}$. The Hodge realization functor $M \mapsto (T_{\mathbb{Z}}(M), W_{\cdot}, F^{\cdot})$ provides an equivalence between the category of Deligne 1-motives $\mathcal{M}_1^{\mathrm{fr}}$ and the category $\mathrm{MHS}_1^{\mathrm{fr}}$ of free mixed Hodge structures of level ≤ 1 with polarizable Gr_{-1}^W (cf. [5]). In [3] this result has been extended to an equivalence between the abelian category of 1-motives with torsion and the whole category MHS_1.

By allowing vector groups and formal connected groups, i.e., by considering morphisms of fppf sheaves over $\mathbb{C}$, $u\colon L \to G$ with L a formal $\mathbb{C}$-group (with torsion free étale part) and G a connected smooth algebraic $\mathbb{C}$-group, one defines the category of Laumon 1-motives $\mathrm{M}_1^{a,\mathrm{fr}}$ which contains $\mathrm{M}_1^{\mathrm{fr}}$ as a full subcategory. In [1] the author introduces the category FHS_1 of formal Hodge structures of level ≤ 1 whose objects are pairs (H, V) with H a formal $\mathbb{C}$-group whose étale part is in MHS_1, V a $\mathbb{C}$-vector space with a fixed subspace V_0 together with several conditions on them. In [1], [2], it is shown how to extend the Hodge realization functor to an equivalence $T_{\oint}$ between the category of Laumon 1-motives with torsion and the category FHS_1. The construction of this functor, called the formal Hodge realization, is similar to that of the classical Hodge realization functor, except that it preserves memory of the Lie algebra of G.

In [5] Deligne also defines the de Rham realization $T_{\mathrm{dR}}(M)$ of a 1-motive M as the Lie algebra of the algebraic group $G^\natural$ where $M^\natural = [X \to G^\natural]$ is the universal extension of M and proves that there is an isomorphism $T_{\mathrm{dR}}(M) = T_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \mathbb{C}$ of filtered $\mathbb{C}$-vector spaces.

In this talk, after having recalled all the above constructions, we have shown how to define a sharp universal extension $M^\sharp = [L \to G^\sharp]$ of a Laumon 1-motive with torsion M as well as a sharp envelope functor $()^\sharp$ on FHS_1 so that

(a) $M^\sharp = M^\natural$ if M is a Deligne 1-motive,

(b) $()^\sharp$ generalizes the functor $\mathrm{MHS}_1 \to \mathrm{FHS}_1$, $H_{\mathbb{Z}} \mapsto (H_{\mathbb{Z}}, H_{\mathbb{Z}} \otimes \mathbb{C})$,

(c) $T_{\oint}(M)^\sharp = T_{\oint}(M^\sharp) = (L, \mathrm{Lie}(G^\sharp))$.
When X is a proper, but possibly singular variety, and M is the associated 1-motive (e.g. via generalized Pic) the sharp de Rham realization $T_\sharp(M) = \mathrm{Lie}(G^\sharp)$ is expected to correspond to the first cohomology group of a sharp cohomology of X (still to be constructed) that should carry more information about X than Betti cohomology.

Finally we have shown that the category of enriched Hodge structures of level ≤ 1 in [4] is equivalent to a subcategory of FHS_1.

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Kummer-Chern-Eisenstein motives

Alexander Caspar

Let $\mathbb{Q}(\sqrt{D})$ be a real quadratic field with $D \equiv 1 (\mathrm{mod}\, 4)$ and denote by $\mathcal{O}_F$ its ring of integers, by χ_D the primitive Dirichlet character modulo D and by $\epsilon \in \mathcal{O}_F^*$ a fixed generator of the totally positive units. We assume that the class number in the narrow sense is 1.

Let $S/\mathbb{Q}$ be the coarse moduli space over $\mathbb{Q}$ of abelian surfaces with real multiplication by $\mathcal{O}_F$ (a Hilbert modular surface) and the smooth toroidal compactification $j\colon S \hookrightarrow \widetilde{S}$.

In the first part we discussed a geometric construction of a motive, called the Kummer-Chern-Eisenstein motive, which is an extension of the Tate motive $\mathbb{Q}(-1)$ and the Dirichlet motive $\mathbb{Q}(0)\chi_D$:

$$H^2_{\mathrm{CHE}}(S, \mathbb{Q}(1)) \in \mathrm{Ext}^1_{\mathcal{MM}_{\mathbb{Q}}}(\mathbb{Q}(-1), \mathbb{Q}(0)\chi_D).$$

This motive is defined inside the motive $H^2_c(S, \mathbb{Q}(1))$ given by an extension from a long exact sequence of motives obtained by $j\colon S \hookrightarrow \widetilde{S}$. The name is due to the fact that the construction uses a suitable Eisenstein section and the first Chern class of line bundles $c_1(\mathcal{L}_1)$ and $c_1(\mathcal{L}_2)$, where $\mathcal{L}_1 \otimes \mathcal{L}_2 \simeq \Omega^2_{S/\mathbb{Q}}$ is the line bundle of modular forms.

In the second part we computed the l-adic and the Hodge-de Rham realizations of $H^2_{\mathrm{CHE}}(S, \mathbb{Q}(1))$:

$$\left(H^2_{\mathrm{CHE},l}(S, \mathbb{Q}(1)), H^2_{\mathrm{CHE},\infty}(S, \mathbb{Q}(1))\right) = (\epsilon, \log \epsilon).$$

Eventually we proved that they are isomorphic to the realizations of a Kummer motive and show how this is related to the Hodge 1-motive associated to S and its compactification.

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On the Mumford-Tate conjecture for 1-motives

Peter Jossen

Let k be a number field, let $\sigma : k \hookrightarrow \mathbb{C}$ be a complex embedding and let ℓ be a prime number. Denote by $\overline{k}$ the algebraic closure of k in $\mathbb{C}$. We fix a 1–motive

$$M = \left[\begin{tikzcd} & & Y \arrow[d, "u"'] & & \\ 0 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 0 \end{tikzcd}\right]$$

over k. Deligne [3] associates with M an ℓ–adic representation $\mathrm{V}_\ell M$, and with the pull–back M_σ of M to $\mathbb{C}$ via σ a rational mixed Hodge structure $\mathrm{V}_0 M_\sigma$. These objects depend naturally on M, hence come equipped with a *weight filtration*, induced by the weight filtration on M. The ℓ–adic representation $\mathrm{V}_\ell M$ is given by a continuous homomorphism

$$\rho_\ell : \Gamma_k \to \mathrm{GL}(\mathrm{V}_\ell M)$$

where $\Gamma_k := \mathrm{Gal}(\overline{k}|k)$ denotes the absolute Galois group of k, and we regard the Hodge structure $\mathrm{V}_0 M_\sigma$ as a $\mathbb{Q}$–linear algebraic representation

$$\rho_0 : \Gamma_{\mathrm{Hodge}} \to \mathrm{GL}_{\mathrm{V}_0 M_\sigma}$$

where Γ_{Hodge} denotes the absolute Hodge group, that is, the Tannakian fundamental group of the category of rational mixed Hodge structures. The image of ρ_ℓ is an ℓ–adic Lie subgroup of the ℓ–adic Lie group $\mathrm{GL}(\mathrm{V}_\ell M)$, and the image of ρ_0 is an algebraic subgroup of the linear algebraic group $\mathrm{GL}_{\mathrm{V}_0 M_\sigma}$. Set

$$\mathfrak{l}^M := \mathrm{Lie}(\mathrm{im}\rho_\ell) \subset \mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M) \quad \text{and} \quad \mathfrak{h}^M := \mathrm{Lie}(\mathrm{im}\rho_0) \subset \mathrm{End}_{\mathbb{Q}}(\mathrm{V}_0 M_\sigma)$$

According to [3], there is a canonical and natural isomorphism of filtered $\mathbb{Q}_\ell$–vector spaces $\mathrm{V}_0 M_\sigma \otimes \mathbb{Q}_\ell \cong \mathrm{V}_\ell M$, called the *comparison isomorphism*. Via this isomorphism, we consider the Lie algebra $\mathfrak{h}^M \otimes \mathbb{Q}_\ell$ as a subalgebra of $\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$. In my talk, I present the following result:

Main Theorem. *Inside* $\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M)$,

(i) *the inclusion* $\mathfrak{l}^M \subseteq \mathfrak{h}^M \otimes \mathbb{Q}_\ell$ *holds; and*
(ii) *the nilpotent radicals of* $\mathfrak{l}^M$ *and* $\mathfrak{h}^M \otimes \mathbb{Q}_\ell$ *are equal.*

Part (i) is an immediate consequence of a result of Deligne and Brylinski [2], stating that *every Hodge cycle of M is an absolute Hodge cycle.* We prove part (ii) by comparing the nilpotent radicals of $\mathfrak{l}^M$ and $\mathfrak{h}^M$ with the nilpotent radical of the Lie algebra of the motivic fundamental group of M. This object is in fact just a semiabelian variety $P(M)$ which we construct explicitly as follows. First, define a semiabelian variety $U(M)$ by requiring the following short exact sequence of k–group schemes

$$0 \to \mathcal{H}om(Y,T) \xrightarrow{(+,-)} \mathcal{H}om(Y,G) \times \mathcal{E}xt^1(M_A,T) \longrightarrow U(M) \to 0$$

to be exact. Here, M_A is the 1–motive $[Y \to A]$ deduced from M, viewed as a complex in degrees -1 and 0. The first arrow is given on points by sending t to the pair $(\iota_1(t), -\iota_2(t))$, where ι_1 is obtained by applying $\mathcal{H}om(Y,-)$ to the morphism $T \to G$ and where ι_2 is obtained by applying $\mathcal{E}xt^1(-,T)$ to the map $M_A \to Y[1]$. It follows rather directly from the construction that there are canonical isomorphisms of ℓ–adic representations and of mixed Hodge structures respecetively

$$\mathrm{V}_\ell U(M) \cong \mathrm{End}^W_{\mathbb{Q}_\ell}(\mathrm{V}_\ell M) \qquad \text{and} \qquad \mathrm{V}_0 U(M) \cong \mathrm{End}^W_{\mathbb{Q}}(\mathrm{V}_0 M)$$

where End^W means endomorphisms respecting the weight filtration. This already indicates that $P(M)$ should be a subgroup of $U(M)$. The map u corresponds to a rational point u of $\mathcal{H}om(Y,G)$, and viewing M as an extension of M_A by T we also get a rational point η on $\mathcal{E}xt^1(M_A,T)$.

Definition. We write $P(M)$ for the connected component of the algebraic subgroup of $U(M)$ generated by the image of (u,η), and name it *Lie algebra of the unipotent motivic fundamental group of M.*

A different construction of the semiabelian variety $P(M)$ was given by Bertolin in [1]. In order to prove part (ii) of our Main Theorem, we establish canonical and natural isomorphisms

$$\mathrm{V}_\ell P(M) \cong \mathrm{Nil}(\mathfrak{l}^M) \qquad \text{and} \qquad \mathrm{V}_0 P(M) \cong \mathrm{Nil}(\mathfrak{h}^M)$$

of ℓ–adic representations and of mixed Hodge structures respecetively. In the case where $T = 0$, that is, where the semiabelian variety G is an abelian variety, the isomorphism of Galois representations $\mathrm{V}_\ell P(M) \cong \mathrm{Nil}(\mathfrak{l}^M)$ was already known by work of Ribet [4].

The reductive quotients of $\mathfrak{l}^M$ and $\mathfrak{h}^M$ are canonically isomorphic to $\mathfrak{l}^A$, the Lie algebra of the image of Γ_k in $\mathrm{GL}(\mathrm{V}_\ell A)$, and to the Lie algebra of the Mumford–Tate group of A, provided $A \neq 0$ (if $A = 0$, then they are either both one dimensional in the case $T \neq 0$, or both trivial in the case $T = 0$). The classical Mumford–Tate conjecture predicts that inside $\mathrm{End}_{\mathbb{Q}_\ell}(\mathrm{V}_\ell A)$, the equality

$$\mathfrak{h}^A \otimes \mathbb{Q}_\ell = \mathfrak{l}^A$$

holds. Our Main theorem shows thus that if the classical Mumford–Tate conjecture holds for the abelian part A of M, then the corresponding statement for

1–motives, i.e. the equality $\mathfrak{h}^M \otimes \mathbb{Q}_\ell = \mathfrak{l}^M$ holds as well. The classical Mumford–Tate conjecture is known in a variety of cases, for instance if A is an elliptic curve (Serre), of real multiplication type (follows from Faltings), or if $\mathrm{End}_{\overline{k}} A = \mathbb{Z}$ and A is of dimension $2, 4, 6$ or odd (again Serre).

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Biextensions and 1-motives

Carlo Mazza

(joint work with Cristiana Bertolin)

The notion of biextension of two abelian groups by another abelian group is a classical one coming from [6, VII], later extended by Deligne in [4, 10.2] to the case where K_0 and K_1 are two complexes of abelian sheaves concentrated in degrees -1 and 0, and H is another abelian sheaf. His result $Biext^1(K_1, K_2; H) \cong Ext^1(K_1 \otimes^L K_2, H)$ suggests that the biextension of two 1-motives is related to their tensor product, which is outside the category of 1-motives for trivial reasons.

Cristiana Bertolin futher extended Deligne's work by defining the biextensions of two 1-motives (M_1 and M_2) by another 1-motive (M_3) and proved in [1] that

$$Biext^1(M_1, M_2; M_3) \cong Ext^1(M_1 \otimes^L M_2, M_3).$$

It was stated by Voevodsky, later proved by Orgogozo in [5], and generalized by Barbieri Viale and Kahn in [3], that there is a fully faithfull embedding Tot of the category of 1-motives into the category DM. Cristiana Bertolin and I proved in [2] that, after tensoring with $\mathbb{Q}$, the embedding realizes the connection between biextensions bilinear morphisms between 1-motives, i.e.,

$$Biext^1(M_1, M_2; M_3) \otimes \mathbb{Q} \cong Hom_{DM(\mathbb{Q})}(Tot(M_1) \otimes Tot(M_2), M_3).$$

This also answers a question in [3] because applying $LAlb$ on the right gives

$$Biext^1(M_1, M_2; M_3) \otimes \mathbb{Q} \cong Hom_{\text{1-isoMot}}(M_1 \otimes^1 M_2, M_3).$$

Here is a sketch of our proof: taking into account the different degree conventions by Deligne and Voevodsky, by [1] we get

$$Biext^1(M_1, M_2; M_3) \otimes \mathbb{Q} \cong Ext^1(M_1[1] \otimes^L M_2[1], M_3[1]) \otimes \mathbb{Q},$$

and by simple homological algebra we have

$$Ext^1(M_1[1] \otimes^L M_2[1], M_3[1]) \otimes \mathbb{Q} \cong Hom_{D(Sh)\otimes\mathbb{Q}}(M_1 \otimes^L M_2, M_3),$$

where $D(Sh)$ is the derived category of abelian sheaves. By [3, 4.4.1]:

$$Hom_{D(Sh)\otimes\mathbb{Q}}(M_1 \otimes^L M_2, M_3) \cong Hom_{DM(\mathbb{Q})}(Tot(M_1) \otimes Tot(M_2), M_3).$$

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Duality theorems for 1-motives with bad reduction

Donghoon Park

A smooth 1-motive $M = [u : X \to G]$ over S has another description (v, w, ψ):

(1) A homomorphism $v : X \to A$ given by $X \xrightarrow{u} G \xrightarrow{p} A$;

(2) A homomorphism $w : T^D \to A'$ coming from the extension $0 \to T \to G \xrightarrow{p} A \to 0$;

(3) A biextension morphism $\psi : X \times T^D \to \mathcal{P}_{A\times A'}$,

where T^D is the group of characters of T, A' is the dual abelain scheme of A, and $\mathcal{P}_{A\times A'}$ is a Poincaré biextension of A. In these terms, its dual 1-motive M^* is given by $(w, v, \widehat{\psi})$ where $\widehat{\psi}$ is derived from ψ by switching $X \leftrightarrow T^D$ and $A \leftrightarrow A'$. Thus the Cartier biduality theorem $(M^*)^* \cong M$ is an easy consequence of this definition. This equivalence is extended to the category of smooth 1-motives with torsion over S as follows. (Note that when S is the spectrum of any field, this category is not the same as the category $\mathcal{M}_k^{\mathrm{ab}}$ of 1-motives with torsion in [1], since an object of $\mathcal{M}_k^{\mathrm{ab}}$ is a quasi-isomorphism class of smooth 1-motives with torsion whose group scheme G is a semiabelian variety.)

Theorem 1. *(Cartier duality for 1-motives with torsion)* [5]
For a locally noetherian base scheme S, there is an additive category $\mathcal{M}_S^{\mathrm{tor}}$ containing both 1-motives and finite group schemes. The category $\mathcal{M}_S^{\mathrm{tor}}$ has a contravariant additive functor $^*: \mathcal{M}_S^{\mathrm{tor}} \to \mathcal{M}_S^{\mathrm{tor}}$ which gives rise to dual 1-motives and Cartier dual finite group schemes. In particular, the double dual ** of any object is the same as itself.

Let R be a discrete valuation ring and let K be the field of fractions of R. In this case, a semiabelian variety G_K over K admits a unique Néron model G. The Néron model of $\mathbb{G}_{m,K}$ will be denoted by $\mathcal{G}_{m,R}$. Using the equivalence between the category of $\mathbb{G}_{m,K}$-biextensions of A_K and A'_K and the category of

$\mathcal{G}_{m,R}$-biextensions of A and A', where A_K and A'_K are abelian varieties and where A and A' are their Néron models respectively, we define the Néron model of a 1-motive (v_K, w_K, ψ_K) by (v_R, w_R, ψ_R), where (v_R, w_R, ψ_R) are unique smooth liftings over R of (v_K, w_K, ψ_K). We also see the equivalence between (v_K, w_K, ψ_K) and the 2-term complex $[u_R : X_R \to G]$ given by a homomorphism of Néron models. The category $\mathcal{M}^{\text{Nér}}$ of Néron models of 1-motives has a duality functor * and is closed under this functor ([6]) and the category $\mathcal{M}^{\log}$ of log 1-motives in [3] has a natural embedding to the category $\mathcal{M}^{\text{Nér}}$ and this embedding preserves their duality functors.

We also consider a 1-motive with bad reduction over R, $[0 \to \mathbb{G}_{m,R}]$ for instance. To get the $\mathbb{G}_{m,R}$-prolongation of a Poincaré biextension $\mathcal{P}_{A_K \times A'_K}$, we need a vanishing condition of some bilinear pairing of component groups of special fibers of their integral models. When R is complete with the finite residue class field, for an abelian variety A_K over K and its dual A'_K if we consider their Néron models, this pairing is perfect ([4]). Let Γ and let Γ' be subgroups of component groups of these Néron models such that they exactly annihilate each other under this pairing. Their corresponding smooth integral models are denoted by $\mathcal{A}^{\Gamma}$ and $\mathcal{A}'^{\Gamma'}$ and we can get some duality between $\mathcal{A}^{\Gamma}$ and $\mathcal{A}'^{\Gamma'}$. Now we define a *1-motive with bad reduction* M over R by a group scheme homomorphism $u : X \to G$, where X is locally $\mathbb{Z}^r$ for the étale topology and G is given by an extension of $\mathcal{A}^{\Gamma}$ by a torus T, for some Γ, and then this is equivalent to the lifted triple (v_R, w_R, ψ_R) of (v_K, w_K, ψ_K) ([6]). When X and T^D are constant group schemes, we get an integral version of the arithmetic duality theorem in [2].

Theorem 2. *(Arithmetic duality for 1-motives with bad reduction)* [6]

Let R be as above then the pairing $\mathbb{H}^i_c(R, M) \times \mathbb{H}^{2-i}(R, M^*) \to \mathbb{H}_c(R, \mathbb{G}_{m,R}) = \mathbb{Q}/\mathbb{Z}$ is perfect after appropriate completion, where M has constant X and T^D, where M^* is dual to M and where $\mathbb{H}^i$ (respectively $\mathbb{H}^i_c$) means étale hypercohomology (respectively hypercohomology with compact support).

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Triangulated 1-motives

Joël Riou

In these two talks, I introduced the triangulated category of motives $DM_-^{\mathrm{eff}}(k)$ defined by Voevodsky for any perfect field k and explained how the derived category of the abelian category of 1-motives up to isogenies could be embedded in this category $DM_-^{\mathrm{eff}}(k)$. This was an exposition based on the article [1], which contains proofs of results previously announced in [2, §3.4].

We let $\mathbf{Sch}_k$ be the category of quasi-projective schemes over k and $\mathbf{Sm}_k$ be the full subcategory of $\mathbf{Sch}_k$ consisting of smooth schemes.

Definition 1. *Let $X \in \mathbf{Sch}_k$. The nth symmetric product $\mathbf{Sym}^n X$ of X is defined as the quotient scheme of the product X^n by the action of the symmetric group. We let $\mathbf{Sym}^\infty X$ be the scheme $\sqcup_{n\geq 0}\mathbf{Sym}^n X$.*

The obvious pairings $\mathbf{Sym}^m X \times \mathbf{Sym}^n X \to \mathbf{Sym}^{m+n} X$ for all m, n lead to a structure of a monoid scheme on $\mathbf{Sym}^\infty X$.

Definition 2. *Let Y and X be objects in $\mathbf{Sm}_k$. A naïve finite correspondence $Y \rightsquigarrow X$ is an element in the group completion $\mathbf{Hom}_k(Y, \mathbf{Sym}^\infty X)^+$ of the monoid $\mathbf{Hom}_k(Y, \mathbf{Sym}^\infty X)$.*

It is easy to define the composition of naïve finite correspondences. A more technical notion of finite correspondences is defined in [2]. Both notions coincides when we invert the characteristic exponent. *A fortiori*, there is no difference when we tensor morphisms with $\mathbf{Q}$.

Definition 3. *The category of smooth correspondences $\mathbf{SmCor}_k$ has the same objects as $\mathbf{Sm}_k$: they are denoted $[X]$ for $X \in \mathbf{Sm}_k$. The group of morphisms $[Y] \to [X]$ is the group of finite correspondences $Y \rightsquigarrow X$.*

We have an obvious functor $\mathbf{Sm}_k \to \mathbf{SmCor}_k$.

Definition 4. *A presheaf with transfers $\mathcal{F}$ is an additive functor $\mathbf{SmCor}_k^{opp} \to \mathbf{Ab}$ where $\mathbf{Ab}$ is the category of abelian groups. A presheaf with transfers $\mathcal{F}$ is a sheaf with transfers for the étale topology (resp. the Nisnevich topology) if the induced presheaf on the category $\mathbf{Sm}_k$ is a sheaf for the corresponding topology. The category of Nisnevich sheaves with transfers is denoted $\mathbf{AbShv}_{Nis}^{tr}$.*

Definition 5. *Let $X \in \mathbf{Sm}_k$. We define the presheaf with transfers $\mathbf{Z}_{tr}(X) = ([U] \longmapsto \mathbf{Hom}_{\mathbf{SmCor}_k}([U], [X]))$.*

These presheafs $\mathbf{Z}_{\mathrm{tr}}(X)$ are étale sheaves with transfers. Other examples are given by the following construction:

Proposition 6. *Let G be a commutative k-group scheme. Then the sheaf (denoted by G) represented by G (i.e. $U \longmapsto G(U) = \mathbf{Hom}_k(U, G)$) is equipped with transfers.*

One has to attach a map $\gamma^\star\colon G(V) \to G(U)$ to the action of a finite correspondence $\gamma = [U] \to [V]$. For simplicity, assume that it is a naïve finite correspondence and that U is connected. Then, γ corresponds to a morphism $U \to \mathbf{Sym}^n(V)$. Let $s\colon V \to G$ an element in $G(V)$. We define $\gamma^\star s \in G(U)$ as the composition:

$$U \xrightarrow{\gamma} \mathbf{Sym}^n V \xrightarrow{\mathbf{Sym}^n s} \mathbf{Sym}^n G \to G$$

where the last morphism $\mathbf{Sym}^n G \to G$ is the morphism induced by the morphism $G^n \to G$ given by the sum in the commutative group scheme G.

Definition 7. *An object K in the derived category of $D^-(\mathbf{AbShv}^{tr}_{Nis})$ is $\mathbf{A}^1$-local if for any $X \in \mathbf{Sm}_k$ and $i \in \mathbf{Z}$, the obvious map is an isomorphism:*

$$\mathbf{H}^i_{Nis}(X, K) \xrightarrow{\sim} \mathbf{H}^i_{Nis}(\mathbf{A}^1 \times X, K)$$

where $\mathbf{H}^i_{Nis}$ denotes hypercohomology groups for the Nisnevich topology.

An important result by Voevodsky is that if K is a sheaf lying in degree 0, it suffices to check this condition when $i = 0$.

Definition 8. *The category $DM^{\mathit{eff}}_-(k)$ is defined as the full triangulated subcategory of $D^-(\mathbf{AbShv}^{tr}_{Nis})$ consisting of $\mathbf{A}^1$-local objects. The inclusion functor $DM^{\mathit{eff}}_-(k) \to D^-(\mathbf{AbShv}^{tr}_{Nis})$ has a left adjoint which is denoted $L_{\mathbf{A}^1}$. For any $X \in \mathbf{Sm}_k$, we define the motive of X: $M(X) = L_{\mathbf{A}^1}\mathbf{Z}_{tr}(X)$. The category of effective geometric motives $DM^{\mathit{eff}}_{gm}(k)$ over k can be defined as the pseudoabelian envelope of the full triangulated subcategory of $DM^{\mathit{eff}}_-(k)$ generated by the objects $M(X)$ for all $X \in \mathbf{Sm}_k$.*

Definition 9. *The motive $\mathbf{Z}$ is defined as $M(Spec\, k)$. The motive $\mathbf{Z}(1)$ is*

$$\ker(M(\mathbf{P}^1) \to M(Spec\, k))[-2],$$

i.e., we have a canonical decomposition $M(\mathbf{P}^1) = \mathbf{Z} \oplus \mathbf{Z}(1)[2]$.

Theorem 10 (Suslin-Voevodsky)**.** *The object $\mathbf{Z}(1)[1]$ identifies with the sheaf with transfers represented by the multiplicative group $\mathbf{G}_m$ (see proposition 6).*

The definition $M(X) \otimes M(Y) := M(X \times Y)$ for $X, Y \in \mathbf{Sm}_k$ extends to a $\otimes$-product on the triangulated category $DM^{\mathrm{eff}}_-(k)$. Then, $\mathbf{Z}(q)$ for $q \geq 0$ can be defined as the qth $\otimes$-power of $\mathbf{Z}(1)$ (for $q < 0$, we set $\mathbf{Z}(q) = 0$). This allows us to define motivic cohomology:

Definition 11. *For any $X \in \mathbf{Sm}_k$, $p, q \in \mathbf{Z}$, we set*

$$H^p(X, \mathbf{Z}(q)) = \mathbf{Hom}_{DM^{\mathit{eff}}_-(k)}(M(X), \mathbf{Z}(q)[p]).$$

If we let $1 - \mathbf{isoMot}_k$ be the category of 1-motives over k up to isogenies, one may define a functor ι from $1 - \mathbf{isoMot}_k$ to the category of complexes in $\mathbf{AbShv}^{\mathrm{tr}}_{\mathrm{Nis}}$ that sends $[X \to G]$ to $\cdots \to 0 \to X \otimes \mathbf{Q} \to G \otimes \mathbf{Q} \to 0 \ldots$ (with X in degree 0 and G in degree 1) where the sheaves represented by X and G are equipped with transfers as in proposition 6. Using standard results on abelian varities, one may easily show that the image of this functor consists of $\mathbf{A}^1$-local objects, and one may prove that this construction induces a functor $\iota\colon D^{\mathrm{b}}(1 - \mathbf{isoMot}_k) \to DM^{\mathrm{eff}}_-(k)$.

Theorem 12 ([1]). *The functor $\iota\colon D^b(1-\mathbf{isoMot}_k) \to DM_-^{eff}(k)$ is a fully faithful triangulated functor and its image is the pseudoabelian triangulated subcategory of $DM_{gm}^{eff}(k;\mathbf{Q})$ generated by motives of (smooth) curves over k.*

The strategy of the proof is the following:

- The additive category $1-\mathbf{isoMot}_k$ is abelian. All objects M are equipped with a functorial weight filtration $0 = W_{\leq -3}M \subset W_{\leq -2}M \subset W_{\leq -1}M \subset W_{\leq 0}M = M$ such that the functors $W_{\leq w}$ are exact, and the subcategories of $1-\mathbf{isoMot}_k$ consisting of *pure* 1-motives of a given weight are semisimple.
- This abelian category $1-\mathbf{isoMot}_k$ has cohomological dimension ≤ 1 (i.e., in this category, the $\mathbf{Ext}^2$ are trivial).
- If K is a bounded acyclic complex in $1-\mathbf{isoMot}_k$, then $\iota K \simeq 0$. This shows that ι induces a functor $D^{\mathrm{b}}(1-\mathbf{isoMot}_k) \to DM_-^{\mathrm{eff}}(k)$.
- By *dévissage*, one has to show that for any tuple (M, M') of objects in $1-\mathbf{isoMot}_k$ that are pure of weights w and w', the canonical map
$$\mathbf{Ext}^q_{1-\mathbf{isoMot}_k}(M, M') \to \mathbf{Hom}_{D^-(\mathbf{AbShv}^{\mathrm{tr}}_{\mathrm{Nis}})}(\iota M, \iota M'[q])$$
is a bijection.

The last step is done using a case-by-case study. The source is known to be trivial when $q \neq 0, 1$ and in some other cases depending on the weights. Most of the work that has to be done is to show the homologous vanishing for the target.

To do this, one also needs to understand how curves and 1-motives are related. This involves a duality result. More precisely, the $\otimes$-product admits a partial right adjoint $\mathbf{RHom}^{\mathrm{eff}}\colon DM_{\mathrm{gm}}^{\mathrm{eff}}(k)^{\mathrm{opp}} \times DM_-^{\mathrm{eff}}(k) \to DM_-^{\mathrm{eff}}(k)$. In the one hand, a consequence of the identification (see [3]):
$$H^{2n}(X, \mathbf{Z}(n)) \simeq CH^n(X)$$
for any $X \in \mathbf{Sm}_k$ is a Poincaré duality result: if $X \in \mathbf{Sm}_k$ is projective and smooth of pure dimension d, then the class of the diagonal in $X \times X$ induces an isomorphism $M(X) \xrightarrow{\sim} \mathbf{RHom}^{\mathrm{eff}}(M(X), \mathbf{Z}(d)[2d])$. In the other hand, if X is a smooth and projective curve, the theory of the Picard scheme and theorem 10 allows a computation of the homology sheaves (at least for the étale topology) of $\mathbf{RHom}^{\mathrm{eff}}(M(X), \mathbf{Z}(1)[2])$ which shows that $M(X) \otimes \mathbf{Q}$ is in the image of the functor ι. Then, some $\mathbf{Hom}$ groups can be computed as sheaf cohomology groups of X for the Zariski topology, in which case we will get some vanishing as the cohomological dimension shall be ≤ 1.

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1-motives with unipotent part

Henrik Russell

In this talk, the definitions of 1-motives by Deligne resp. Laumon were extended as follows: A 1-motive with unipotent part over an algebraically closed field is a tuple $(\mathcal{F}, L, A, G, \mu)$, where $\mathcal{F}$ is a formal group whose Cartier dual is algebraic, L an affine algebraic group, A an abelian variety, G an extension of A by L and $\mu : \mathcal{F} \to G$ a homomorphism in the category of sheaves of abelian groups (for the fppf-topology). The category of these 1-motives contains torsion and admits duality. In particular, every smooth connected commutative algebraic group over k has a dual in this category.

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Albanese varieties with modulus and Hodge theory

Henrik Russell
(joint work with Kazuya Kato)

Let X be a proper smooth variety over a field k of characteristic zero, Y an effective divisor on X (with multiplicity). The generalized Albanese variety $\mathrm{Alb}(X, Y)$ of X of modulus Y is a higher dimensional analog of the generalized Jacobian with modulus of Rosenlicht-Serre; i.e., $\mathrm{Alb}(X, Y)$ is defined by a universal mapping property w.r.t. rational maps from X to connected commutative algebraic groups of modulus $\leq Y$.

In this talk, $\mathrm{Alb}(X, Y)$ was constructed in an algebraic way: $\mathrm{Alb}(X, Y)$ is the Cartier dual of the Laumon 1-motive $[\mathcal{F}_{X,Y} \to \mathrm{Pic}^0_X]$. We explain this step by step: Let $\underline{\mathrm{Div}}_X$ denote the sheaf of relative Cartier divisors on X. This sheaf admits a class map to the Picard functor; let $\underline{\mathrm{Div}}^0_X := \underline{\mathrm{Div}}_X \times_{\mathrm{Pic}_X} \mathrm{Pic}^0_X$. Then $\mathcal{F}_{X,Y} = (\mathcal{F}_{X,Y})_{\mathrm{et}} \times (\mathcal{F}_{X,Y})_{\mathrm{inf}}$ is the formal subgroup of $\underline{\mathrm{Div}}_X$ given by

$$\begin{aligned}(\mathcal{F}_{X,Y})_{\mathrm{et}} &= \left\{ D \in \underline{\mathrm{Div}}^0_X(k) \;\middle|\; \mathrm{Supp}(B) \subset \mathrm{Supp}(Y) \right\} \\ (\mathcal{F}_{X,Y})_{\mathrm{inf}} &= \exp\left(\widehat{\mathbb{G}}_a \otimes_k \Gamma\big(X, \mathcal{O}_X\left(Y - Y_{\mathrm{red}}\right)/\mathcal{O}_X\big)\right)\end{aligned}$$

where Y_{red} is the underlying reduced divisor of Y. The map $\mathcal{F}_{X,Y} \to \mathrm{Pic}^0_X$ is the composition of the inclusion $\mathcal{F}_{X,Y} \subset \underline{\mathrm{Div}}^0_X$ and the class map $\underline{\mathrm{Div}}^0_X \to \mathrm{Pic}^0_X$.

For $k = \mathbb{C}$, the following Hodge theoretic description was given:

Theorem 1. (1) *We have an exact sequence*

$$0 \to \mathrm{Alb}(X, Y) \to H^{2n}(X, \mathcal{D}_{X,Y}(n)) \xrightarrow{\deg} \mathbb{Z} \to 0,$$

where for $r \in \mathbb{Z}$*,* $\mathcal{D}_{X,Y}(r)$ *denotes the kernel of the surjective homomorphism of complexes* $\mathcal{D}_X(r) \to \mathcal{D}_Y(r)$ *with* $D_X(r)$ *the Deligne complex*

$$[\mathbb{Z}(r) \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^{r-1}_X]$$

and $\mathcal{D}_Y(r)$ *the similar complex*

$$[\mathbb{Z}(r)_Y \to \mathcal{O}_Y \xrightarrow{d} \Omega^1_Y \xrightarrow{d} \dots \xrightarrow{d} \Omega^{r-1}_Y].$$

(2) *We have an exact sequence*

$$H^{n-1}(X, \Omega^n_X) \to H^{2n-1}_c(X-Y, \mathbb{C}/\mathbb{Z}(n)) \oplus H^{n-1}(X, \Omega^n_X/J\Omega^n_X) \to \mathrm{Alb}(X,Y) \to 0.$$

For the proof of this theorem a category $\mathcal{H}_1$ of "mixed Hodge structures with additive parts" is used. This category yields a Hodge theoretic description of the category of Laumon 1-motives and is closely related to the formal Hodge structures from [1] and the enriched Hodge structures from [2]. The definition of $\mathcal{H}_1$ aims to stick close to the classical language of Hodge structures and to express duality in a simplest possible way.

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Motivic Integrals of K3 Surfaces over Non-Archimedean Fields

Allen J. Stewart

(joint work with Vadim Vologodsky)

1. Motivic Integral.

Let R be a complete discrete valuation ring with perfect residue field k and with fraction field K. The motivic integral of a smooth proper Calabi-Yau variety, X, over K is element of the ring $K_0(Var_k)_{loc}$, obtained from the Grothendieck ring $K_0(Var_k)$ of algebraic varieties over k by inverting the element $\mathbb{Z}(-1) := [\mathbb{A}^1]$, defined by the formula

$$\int_X := \sum_i [V_i^\circ](r_i - \min_i r_i). \tag{1}$$

Here $\mathcal{V}$ is a weak Néron model of X *i.e.*, a smooth scheme over R whose generic fiber is X and such that every point of X with values in an unramified extension $K' \supset K$ extends to a R'-point of $\mathcal{V}$, V_i° are the connected components of the special fiber of $\mathcal{V}$, and the numbers $r_i \in \mathbb{Z}$ are defined from the equation $div\, \omega = \sum_i r_i[V_i^\circ]$, for a nonzero top degree differential form $\omega \in \Gamma(X, \omega_X)$. A weak Néron model always exists but is almost never unique. The key result proven by Loeser and Sebag ([2]) is that the right-hand side of equation (1) is independent of the choice of $\mathcal{V}$. If $k = \mathbb{F}_q$, the image of the motivic integral under the homomorphism

$$K_0(Var_{\mathbb{F}_q})_{loc} \to \mathbb{Z}_{(q)} \quad [Z] \rightsquigarrow |Z(\mathbb{F}_q)| \tag{2}$$

is equal to the volume $\int_{X(K)} |\omega|$, for an appropriately normalized $\omega \in \Gamma(X, \omega_X)$.

In this talk we express the motivic integral of K3 surfaces over $\mathbb{C}((t))$ with strictly semi-stable reduction in terms of the associated limit Hodge structures.

2. Limit Hodge structure.

Schmid and Steenbrink associated with every smooth projective variety over the field K_{mer} of meromorphic functions on an open neighborhood of zero in the complex plane a mixed Hodge structure, called the limit Hodge structure ([3],[4]).

Building upon the Schmid-Steenbrink construction, with every smooth projective variety X over $\mathbb{C}((t))$ we associate a mixed Hodge structure $H^m(\lim X)$ equipped with the monodromy action, called the limit Hodge structure. A rough idea: Steenbrink attached a mixed Hodge structure to every normal crossing log scheme over the log point ([5]). Applying his construction to the special fiber Y of a strictly semi-stable model $\overline{X}$ of X over $R = \mathbb{C}[[t]]$ we get our $H^m(\lim X)$. We prove the independence of the choice of a model and the functoriality.

3. Motivic integral of K3 surfaces over $\mathbb{C}((t))$.

In order to state our result we need to introduce a bit of notation. Let X be a smooth projective K3 surface over $K = \mathbb{C}((t))$ and let

$$H^2(\lim X) = (H^2(\lim X, \mathbb{Z}), W_i^{\mathbb{Q}} \subset H^2(\lim X, \mathbb{Q}), F^i \subset H^2(\lim X, \mathbb{C}))$$

be the corresponding limit Hodge structure. Assume that the monodromy acts on $H^2(\lim X, \mathbb{Z})$ by a unipotent operator. Then, its logarithm is known to be integral ([1]):

$$N : H^2(\lim X, \mathbb{Z}) \to H^2(\lim X, \mathbb{Z}). \tag{3}$$

Set $W_i^{\mathbb{Z}} = W_i^{\mathbb{Q}} \cap H^2(\lim X, \mathbb{Z})$. The morphisms

$$Gr\, N^i : W_{i+2}^{\mathbb{Z}}/W_{i+1}^{\mathbb{Z}} \to W_{2-i}^{\mathbb{Z}}/W_{1-i}^{\mathbb{Z}}, \quad i = 1, 2 \tag{4}$$

are injective and have finite cokernels. Let $r_i(X, K)$ be their orders.

Theorem 1. *Let X be a smooth projective K3 surfaces over K. Assume that X has a strictly semi-stable model over $R = \mathbb{C}[[t]]$ and that the operator N is not equal to 0. Let s be the smallest integer such that $N^s = 0$. Then s is either 2 or 3 and for every finite extension $K_e \supset K$ of degree e the motivic integral of the K3 surface $X_e = X \otimes_K K_e$ over K_e is given by the following formulas.*

(a) *If $s = 2$ then*

$$\int_{X_e} = 2\mathbb{Z}(0)-(e\sqrt{r_1(X,K)}+1)[E(X)]+20\mathbb{Z}(-1)+(e\sqrt{r_1(X,K)}-1)[E(X)](-1)+2\mathbb{Z}(-2),$$

where $E(X)$ is the elliptic curve defined by the rank 2 Hodge structure on $W_1^{\mathbb{Z}} = W_1^{\mathbb{Q}} \cap H^2(\lim X, \mathbb{Z})$.

(a) *If $s = 3$ then*

$$\int_{X_e} = \left(\frac{e^2 r_2(X,K)}{2} + 2\right)\mathbb{Z}(0) + (20 - e^2 r_2(X,K))\mathbb{Z}(-1) + \left(\frac{e^2 r_2(X,K)}{2} + 2\right)\mathbb{Z}(-2).$$

Note, that if $N = 0$ the K3 surface X has a smooth proper model over R whose special fiber Y (and, thus, the motivic integral) is determined by the polarized pure Hodge structure $H^2(\lim X, \mathbb{Z})$.

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The arithmetic of 1-motives I-II

Tamás Szamuely

In my lectures I tried to explain how certain questions in arithmetic lead to considerations about 1–motives. I presented in some detail my work with David Harari on arithmetic duality theorems for 1-motives and its application to local-global questions for rational points on homogeneous spaces of algebraic groups. I also surveyed work of Deligne on the function field case of the Brumer–Stark conjecture and its recent refinement by Greither and Popescu.

1. Rational points

In the study of the Hasse principle for rational points for varieties over number fields there is a by now classical method going back to the 1970 ICM lecture of Manin [10] that justifies the existence of counterexamples in many (though not all) cases. To explain it, denote by $X(\mathbf{A}_k)$ the set of adelic points of a smooth variety X defined over a nuber field k, and by $\mathrm{Br}\, X$ its cohomological Brauer group. Recall that for a completion k_v of k at a finite place we have an isomorphism $\mathrm{Br}\, k_v \cong \mathbf{Q}/\mathbf{Z}$ by local class field theory; for $k_v = \mathbf{R}$ we have $\mathrm{Br}\,\mathbf{R} \cong \mathbf{Z}/2\mathbf{Z}$ which we may view as a subgroup of $\mathbf{Q}/\mathbf{Z}$. Manin defined a pairing

$$X(\mathbf{A}_k) \times \mathrm{Br}\, X \to \mathbf{Q}/\mathbf{Z}, \quad [(P_v), \alpha] \mapsto \sum \alpha(P_v)$$

where the evaluation map $\alpha \mapsto \alpha(P_v)$ is induced by contravariant functoriality of $\mathrm{Br}\, X$ and the sum is taken inside $\mathbf{Q}/\mathbf{Z}$ (it is known to be finite). If the sequence (P_v) is the diagonal image of a k-rational point, then the pairing with any $\alpha \in \mathrm{Br}\, X$ gives zero by the global reciprocity law of class field theory. So denoting by $X(\mathbf{A}_k)^{\mathrm{Br}}$ the left kernel of the above pairing we have the implication $X(\mathbf{A}_k)^{\mathrm{Br}} = \emptyset \Rightarrow X(k) = \emptyset$. This is the Manin obstruction to the Hasse principle. It is said to be the only obstruction if the converse implication holds.

It is often interesting to restrict the Manin pairing to subquotients of $\mathrm{Br}\, X$. We shall be interested in the subquotient Б(X) defined as follows. Consider the natural maps $\mathrm{Br}\, k \xrightarrow{\pi} \mathrm{Br}\, X \xrightarrow{\rho} \mathrm{Br}\,(X \times_k \overline{k})$ and set $\mathrm{Br}_{\,a} X := \ker(\rho)/\mathrm{Im}\,(\pi)$. Then take Б$(X) \subset \mathrm{Br}_{\,a}(X)$ to be the subgroup of locally trivial elements. As the image of $\mathrm{Br}\, k$ in $\mathrm{Br}\, X$ pairs trivially with adelic points (again by the global reciprocity law), the Manin pairing induces a pairing with $\mathrm{Br}_{\,a} X$ and finally with Б(X). We still have of course $X(\mathbf{A}_k)^{\text{Б}} = \emptyset \Rightarrow X(k) = \emptyset$, with $X(\mathbf{A}_k)^{\text{Б}}$ defined similarly as $X(\mathbf{A}_k)^{\mathrm{Br}}$. The group Б$(X)$ is often more interesting than $\mathrm{Br}\, X$ because if one assumes that the Tate–Shafarevich group of the Albanese variety of X is finite, it is also finite, and in some cases even explicitly computable.

The main theorem of our paper [7] with David Harari now states:

Theorem 1.1. *Given a torsor X under a semi-abelian variety G over a number field whose abelian quotient has finite Tate–Shafarevich group, we have*

$$X(\mathbf{A}_k)^{\text{Б}} \neq \emptyset \Rightarrow X(k) \neq \emptyset,$$

i.e. the Manin obstruction associated with Б(X) is the only obstruction to the Hasse principle.

This result was known for $G = A$ an abelian variety (Manin himself) or G a torus (Sansuc [11]) but the general case was a long-standing open question; see e.g. Skorobogatov's book ([12], p. 133).

The main idea of the proof is (as already in Manin's case) to relate the Manin pairing

$$\langle\ ,\ \rangle_M : \quad X(\mathbf{A}_k) \times \text{Б}(X) \to \mathbf{Q}/\mathbf{Z}$$

to a Cassels–Tate type pairing

$$\langle\ ,\ \rangle_{CT} : \quad \text{Ш}(M) \times \text{Ш}(M^*) \to \mathbf{Q}/\mathbf{Z}$$

for the 1–motive $M = [0 \to G]$.

In fact, in our previous paper [6] with Harari we have defined such a pairing for an arbitrary 1-motive defined over a number field and proved:

Theorem 1.2. *Let k be a number field and M a 1-motive over k with Cartier dual M^*. There exists a canonical pairing*

$$\text{Ш}^1(M) \times \text{Ш}^1(M^*) \to \mathbf{Q}/\mathbf{Z}$$

which is non-degenerate modulo divisible subgroups. If one assumes the finiteness of the Tate–Shafarevich group of the abelian variety quotient of M, it is a perfect pairing of finite groups.

Here $Ш^1(M)$ is defined as the group of locally trivial elements in the first hypercohomology group $\mathbf{H}^1(k, M)$. The theorem generalizes the classical duality results of Cassels [4] and Tate [14] on abelian varieties as well as duality theorems of Kottwitz ([8], [9]) on tori.

Taking up the proof sketch of Theorem 1.1, we return to the special case $M = [0 \to G]$. The method is to construct a map $\iota : Ш(M^*) \to Б(X)$ such that for all adelic points (P_v) of X and all $\alpha \in Б(X)$ the formula

$$\langle (P_v), \iota(\alpha)\rangle_M = \langle [X], \alpha\rangle_{CT} \tag{1.1}$$

holds. To understand the formula, note first that the torsor X is known to have a cohomology class $[X]$ in the group $H^1(k, G) = \mathbf{H}^1(k, M)$; it is a trivial class if and only if X has a k-point (see e.g. [12], pp. 18–19). Hence the assumption $X(\mathbf{A}_k) \neq \emptyset$ implies that $[X] \in Ш^1(M)$. The left hand side does not depend on the choice of (P_v) because elements of $Б(X)$ are 'locally constant' by definition. Now assume that the map ι exists and formula (1.1) holds. Then the assumption $X(\mathbf{A}_k)^{Б} \neq \emptyset$ together with (1.1) implies that $[X]$ is orthogonal to the whole of $Ш^1(M^*)$ under the pairing $\langle\ ,\ \rangle_{CT}$. Thus $[X] = 0$ by Theorem 1.2, i.e. $X(k) \neq \emptyset$.

Borovoi, Colliot-Thélène and Skorobogatov [3] have generalized Theorem 1.1 to homogeneous spaces under an arbitrary connected algebraic group. The precise statement is the same as in Theorem 1.1, except that G is a connected algebraic group, and X is a homogeneous space of G whose geometric points have connected stabilizers. There is, however, an additional restriction on the number field k: it must be totally imaginary. In fact, the same paper contains a quite surprising example ([3], Proposition 3.16) of a connected non-commutative and non-linear algebraic group over $\mathbf{Q}$ for which the statement fails. This shows that over arbitrary number fields general connected algebraic groups behave differently from commutative or linear ones.

The proof of their generalization uses techniques going back to Borovoi's papers [1] and [2] to reduce to the case of a torsor under a semi-abelian variety, where our Theorem 1.1 can be applied. Thus 1–motives play a key role in the proof of this general result about algebraic groups as well.

2. The Brumer–Stark conjecture over function fields

Let K be a finite abelian extension of $\mathbf{Q}$, and S a finite set of places of K containing infinite and ramified places. Define

$$\theta_S(s) := \prod_{P \notin S} (1 - \sigma_P^{-1}(NP)^{-s})^{-1},$$

where σ_P is the Frobenius of the prime ideal P in the commutative Galois group $G := \mathrm{Gal}(K|\mathbf{Q})$. This is a formal power series in s, with coefficients in the group ring $\mathbf{Z}[G]$. The Brumer–Stark conjecture, as formulated e.g. in [13], is:

Conjecture 2.1. The class group Cl_K is annihilated by $e \cdot \theta_S(0)$ for any S as above, where e is the order of the group μ_K of roots of unity in K.

A variant of the above conjecture can be formulated as follows. Consider another finite set T of places with $T \cap S = \emptyset$, and set

$$\theta_{S,T}(s) := \prod_{P \in T} (1 - \sigma_P^{-1}(NP)^{1-s})^{-1} \theta_S(s).$$

Then one has:

Conjecture 2.2. The T-class group $Cl(\mathcal{O}_T)$ is annihilated by $\theta_{S,T}(0)$ for any S, T as above (modulo a minor technical condition on T).

This implies the previous conjecture, because the elements $1 - \sigma_P^{-1}(NP)$ can be shown to generate the annihilator of the $\mathbf{Z}[G]$-module μ_K ([13], Chapter IV, Lemma 1.1).

The function field analogue of the latter conjecture can be formulated as follows. Let $X \to Y$ be a Galois cover of proper smooth curves defined over a finite field $\mathbf{F}_q$, with abelian Galois group G. Let S, T be disjoint finite sets of closed points of X, with S containing the ramified primes. Set

$$\theta_{S,T}(u) := \prod_{P \in T} (1 - \sigma_P^{-1}(qu)^{\deg(P)}) \prod_{P \notin S} (1 - \sigma_P^{-1}(u)^{\deg(P)})^{-1}.$$

The following analogue of the Brumer–Stark conjecture was proven in the 1980's by Deligne and, independently, by Hayes.

Theorem 2.3. The generalized Picard group $\mathrm{Pic}^0_T(X)$ is annihilated by $\theta_{S,T}(1)$ for any S, T as above.

Recently, Greither and Popescu [5] proved the following refinement.

Theorem 2.4. The element $\theta_{S,T}(1)$ lies in the Fitting ideal of the $\mathbf{Q}/\mathbf{Z}$-dual of the $\mathbf{Z}[G]$-module $\mathrm{Pic}^0_T(X)$.

This is indeed a refinement, because the Fitting ideal of a finitely generated module is contained in its annihilator, which is also the annihilator of its dual.

The above results are relevant to our topic because at the heart of both Deligne's proof of Theorem 2.3 and the refinement 2.4 by Greither–Popescu the 1–motive

$$M = [\mathbf{Z}^{\overline{S}} \to \mathrm{Pic}^0_{\overline{T}}(\overline{X})]$$

over $\overline{\mathbf{F}}_q$ plays a key role. Here $\overline{S}$, $\overline{T}$ are the sets of closed points above S, T, respectively. The key result of Deligne (whose proof is given in Chapter V of [13]) is the formula

$$\theta_{S,T}(u) = \det(\mathbf{Q}_\ell[G])(1 - F \cdot u \mid T_\ell(M) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell)$$

where $T_\ell(M)$ is the ℓ-adic realization of the 1–motive M and F is the absolute Frobenius. It is derived from the Grothendieck–Lefschetz trace formula in étale cohomology for the curve $\overline{X}$ by dévissage. From this Theorem 2.3 follows by a Cayley–Hamilton argument, but 2.4 requires a more refined analysis of the Fitting ideal.

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