Mathematisches Forschungsinstitut Oberwolfach

Report No. 57/2010

DOI: 10.4171/OWR/2010/57

Mini-Workshop: Algebraic and Analytic Techniques for Polynomial Vector Fields

Organised by
Armengol Gasull, Bellaterra
Julia Hartmann, Aachen
Jaume Llibre, Bellaterra
Sebastian Walcher, Aachen

December 12th – December 18th, 2010

Abstract. Polynomial vector fields are in the focus of research in various areas of mathematics and its applications. As a consequence, researchers from rather different disciplines work with polynomial vector fields. The main goal of this mini workshop was to create new and consolidate existing interdisciplinary exchange on the subject.

Mathematics Subject Classification (2000): 34, 13A50, 13P, 13N, 37C.

Introduction by the Organisers

The mini workshop was attended by 17 participants, including a post doc from the USA funded by the "US Junior Oberwolfach Fellows" program of the US National Science Foundation.

The mini workshop brought together researchers from various areas of algebra and analysis who work on topics related to polynomial vector fields, and was interdisciplinary in that sense. Many of the participants did not personally know each other prior to the workshop and were also not familiar with all the research areas. Every participant was asked to give a survey talk about their specialty and its relation to the workshop subject. The participants took great care in preparing their talks and made an effort to keep them at a level accessible to a rather general audience while at the same time mentioning interesting recent developments. During each talk and afterwards, there was ample time for questions and a discussion. Additionally, there were several informal problem sessions throughout the week.

Even during the workshop one could see successful interaction as participants were suggesting new approaches or applications for the methods and results presented in the talks. As an example, the existence criterion for algebraically independent first integrals presented in Derksen's talk suggests a way to strengthen a result (due to Pereira) on the extactic of a polynomial vector field as well as additional results in that direction. The relationship of differential Galois theory to the center focus problem was also very interesting. The organisers are confident that the discussions started during the workshop will be continued and lead to fruitful collaborations. Many of the abstracts will be extended to survey articles in a special issue of the journal "Qualitative Theory of Dynamical Systems" and thus become available to a wider audience. To summarize, the workshop has broadened the view and the range of available tools for all participants.

Mini-Workshop: Algebraic and Analytic Techniques for Polynomial Vector Fields

Table of Contents

Abstracts

Computations with d-dimensional fast-slow Hopf bifurcations; d=2,3

Jean-Pierre Francoise

The center-focus problem for polynomial vector-fields relates closely to the extensions of the Hopf bifurcation (also called the Hopf-Takens bifurcation). It yielded several algebraic developements around the symbolic computation of the center set equations (Groebner basis, invariant theory,...). In the existing literature on degenerate Hopf bifurcation, we can identify seven different methods for finding periodic solutions. The method of Poincaré-Dulac normal form; the method of Lyapunov constants; the method of averaging; the method of intrinsic balancing; the Lyapunov-Schmidt method; the method of the succession function and the method of relative cohomology of forms derived by the author in 1996 and since then named the algorithm of the successive derivatives. It has been used in several other contexts (local Hilbert's 16th problem, zeros of Abelian integrals, perturbation of planar Hamiltonian systems, perturbation of elliptic sectors, see [M. Gentes, 2009]) and more recently was also used in Hamiltonian Dynamics ([Angoshtari, Jalali, 2007], [Francoise-Garrido-Gallavotti, 2010]).

The local Hilbert's problem asks for finding a bound to the number of limit cycles. It is related to the center-focus problem but it requires more analytic methods. The first notable contribution to the problem was due to N. N. Bautin in 1944. Since then, a complex-analytic version of Bautin's theory was developed by Y.Yomdin and the author [Francoise-Yomdin,1997] based on the computation of the so-called Bautin's ideal, computation of a Groebner basis, and Hironaka's division by an ideal.

The comparison between the averaging method and the Lyapunov-Schmidt method was discussed recently by [Buica-Francoise-Llibre, 2007].

It is natural in this subject to begin studying the degeneracies of 3-dimensional Hopf bifurcations. This seems to be a rather complicated task, taking into account the emergence of complicated dynamical systems phenomena like period-doubling bifurcation, tori bifurcations and their degeneracies as explained for instance at the end of the classical book of Guckenheimer-Holmes. One attempt is to start to investigate fast-slow systems. Even there some new phenomena occur like bursting and mixed-mode periodic solutions.

1. The fast-slow 2-dimensional case

We first give a short summary of the phenomena which can occur with one fast and one slow variable. We only consider one special cases which is representative of the general situation.

$$\begin{aligned} \epsilon\tfrac{dx}{dt} &= -y + f(x), \\ \tfrac{dy}{dt} &= x - c, \end{aligned} \tag{1}$$

We assume that this system (for $c = 0$) displays a stationary point at the origin and hence $f(0) = 0$ and that the eigenvalues of the Jacobian of the vector fields are purely imaginary at the point 0. The eigenvalues at point x are

$$\lambda_\pm = \frac{f'(x)}{2\epsilon} \pm \frac{1}{2}\sqrt{\frac{f'(x)^2}{\epsilon^2} - \frac{4}{\epsilon}}. \tag{2}$$

So that the eigenvalues become purely immaginary at 0 if and only if is the origin if a critical point of f. If this critical point is isolated, then the first condition for having a Hopf bifurcation is fullfilled. There exists then a normal form. The condition of regular Hopf bifurcation is that the first coefficient of this normal form is non zero. After some rescaling of time and variable: $t \mapsto \sqrt{\epsilon}t, x \mapsto x/\sqrt{\epsilon}$, the equation reads

$$\begin{aligned} \tfrac{dx}{dt} &= -y + f(x/\sqrt{\epsilon}), \\ \tfrac{dy}{dt} &= x \end{aligned} \tag{3}$$

Using the relative cohomology method, the first coefficient of the normal form is easily seen, in complex coordinates $x = \frac{1}{2}(z + \overline{z})$ $y = \frac{1}{2\mathrm{i}}(z - \overline{z})$, as the coefficient in $z\overline{z}$ of the 2-form

$$\omega = \frac{1}{\mathrm{i}} f'(\frac{\frac{1}{2}(z + \overline{z})}{2\sqrt{\epsilon}}) dz \wedge d\overline{z}. \tag{4}$$

It is then readily seen that if $f(x) = ax^2$, it will not contribute to the normal form. The first case is the cubic case $f(x) = ax^3$ and it will give a regular Hopf bifurcation. If instead we choose $f(x) = ax^5$ it will produce a degenerated Hopf bifurcation of order 2, also named Bautin bifurcation.

On top of it, the presence of the small term ϵ yields special effects. The scaling we made shows that the time characteristic of the Hopf bifurcation is $t/\sqrt{\epsilon}$ and the domain of existence is also reduced by a factor of $\sqrt{\epsilon}$. This justifies the terminology of singular Hopf bifurcation. We have indeed to distinguish between the "regular singular"' Hopf bifurcation and the degenerated singular Hopf bifurcation. A further study shows that this local situation is often mixed with a "global explosion of canards" where the local small limit cycle born near the Hopf bifurcation becomes very large for an exponentially small difference of the parameter c.

2. The Three-dimensional case

With the three-dimensional case of two fast variables and one slow, another remarkable effect can be observed in a slow crossing of a regular Hopf bifurcation. The following example is paradigmatic:

$$\begin{aligned} \tfrac{dx}{dt} &= x(z - x^2 - y^2) + y, \\ \tfrac{dy}{dt} &= y(z - x^2 - y^2) - x, \\ \tfrac{dz}{dt} &= \epsilon. \end{aligned} \tag{5}$$

In this system, the Hopf bifurcation is delayed and does not appear where "expected".

In a recent article [Guckenheimer, 2008], J. Guckenheimer discussed the case of 3-dimensional Hopf bifurcation with one fast variable and two slow. This is also called the singular Hopf bifurcation with two slow variables. The Hopf bifurcation appears close to a pseudo-stationary point of saddle-node type of type II. There is possibility in that case of mixed-mode oscillations where the small oscillations are related to the canards (generic and robust) discovered by E. Benoit.

Recently, we have focussed our study on the case of a 3-dimensional Hopf bifurcation with 3 time scales. There is the possibility of a slow crossing of a singular Hopf bifurcation where the delayed bifurcation is followed by an explosion of canards which relates to a model introduced in [Clement-Francoise, 2007].

References

[1] Angoshtari A. and Jalali M.A., *On the Existence of Chaotic Circumferential Waves in Spinning Disks* Chaos, **17**,(2007).

[2] A. Buica, JP Francoise and J. Llibre, *Periodic solutions of nonlinear periodic differential systems with a small parameter.* Com. Pure and Applied Analysis **6**, (2007) 103-111.

[3] F. Clément-JP. Francoise, *Mathematical modeling of the GnRH-pulse and surge generator*, SIAM Journal of Applied Dynamical Systems **6** (2007) 441–456.

[4] JP Francoise, *Successive derivatives of a first return map, application to the study of quadratic vector fields* Ergod. Th & Dyn. Syst. **16** (1996), 87–96.

[5] JP Francoise, P.L. Garrido and G. Gallavotti, *Pendulum, elliptic functions and relative cohomology classes* J. Math. Phys. **51** (2010), 032901, 12p.

[6] JP Francoise, Y. Yomdin, *Bernstein inequalities and applications to analytic geometry and differential equations.* Journal of Functional Analysis, **146**, (1997) 185–205.

[7] M. Gentes, *Center conditions and limit cycles for the perturbation of an elliptic sector*, Bull. Sci. Math.,**6** (2009) 597–643.

[8] J. Guckenheimer, *Singular Hopf bifurcation in systems with two slow variables* SIAM Journal of Applied Dynamical Systems **7** (2008) 1355–1377.

[9] M. Krupa, N. Popovic and N. Kopell, *Mixed-mode oscillations in three-time scales systems: a prototypical example* SIAM Journal of Applied Dynamical Systems **7** (2008) 361–420.

Center problem for Abel equations

Anna Cima

1. Motivation of the paper: Center Problem for planar systems

Consider the planar system of differential equations

$$\begin{aligned} \dot{x} &= -y + P(x,y), \\ \dot{y} &= x + Q(x,y), \end{aligned} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are polynomials starting with terms of degree 2 and maximum degree n. The classical problem of determine necessary and sufficient conditions on $P(x,y)$ and $Q(x,y)$ for system (1) to have a center at the origin is known as the Center-Focus Problem. It is known that equation (1) has a center at the origin when $P(x,y)$ and $Q(x,y)$ satisfy an infinite sequence of recursive conditions.

We focuse our attention to the case that $P(x,y) := P_n(x,y)$ and $Q(x,y) := Q_n(x,y)$ are homogeneous polynomials of degree n. When $n=2$ or $n=3$ this problem is absolutely solved from the works of Bautin, Kaptein and Sibirskii. In polar coordinates (ρ,θ), system (1) writes as

$$\begin{aligned}\dot\rho &= \rho^n f(\theta),\\ \dot\theta &= 1+g(\theta),\end{aligned} \tag{2}$$

with

$$f(\theta) = \cos(\theta)\,P_n(r\cos(\theta), r\sin(\theta)) + \sin(\theta)\,Q_n(r\cos(\theta), r\sin(\theta)),$$

and

$$g(\theta) = \cos(\theta)\,Q_n(r\cos(\theta), r\sin(\theta)) - \sin(\theta)\,P_n(r\cos(\theta), r\sin(\theta)).$$

Applying now the change introduced by Cherkas (see [14])

$$r = \frac{\rho^{n-1}}{1+g(\theta)\,\rho^{n-1}}$$

we get

$$\dot r = A(\theta)r^3 + B(\theta)r^2, \tag{3}$$

where $\dot r$ is the derivative in respect to θ and

$$A(\theta) = -(n-1)f(\theta)g(\theta)\,,\; B(\theta = g'(\theta) - (n-1)f(\theta).$$

We notice that $A(\theta)$ and $B(\theta)$ are homogeneous trigonometric polynomials of degree $2(n+1)$ and $n+1$ respectively.

Equation (3) is known as Abel equation. Now the Center-Focus problem of equation (1) has a translation in equation (3), that is, given r_0 small enough we look for necessary and sufficient conditions on $A(\theta)$ and $B(\theta)$ in order to assure that the solution of equation (3) with the initial condition $r(0) = r_0$ has the property that $r(0) = r(2\pi)$. We observe that this condition implies the periodicity of this solution.

Abel equation with polynomial coefficients is called the *Polynomial Abel equation.* The Center-Focus problem can be stated as in the trigonometric case. That is, to give necessary and sufficient conditions on $p(z)$ and $q(z)$ in order to have that the solutions of equation $\dot y = q(z)y^2 + p(z)y^2$ satisfy $y(a) = y(b)$ for a certains a and b for any solution with $y(0) = y_0$ small enough. Notice that now such condition does not imply the periodicity of $y(z)$. Nevertheless, in the last years the Center-Focus problem for the Polynomial Abel equation has done a big progress (see [4],[5], [6], [8], [15], [16], [19], for instance).

In this talk we consider the more general equation

$$\dot r = A(\theta)r^n + B(\theta)r^m, \text{ with } n > m \tag{4}$$

assuming that $A(\theta)$ and $B(\theta)$ are trigonometric polynomials. The easiest case is when $m=1$ and $n=2$, that is the Riccati equation. Riccati equation can be solved explicitely and it turns out that (4) has a center if and only if two definite integrals depending on $A(\theta)$ and $B(\theta)$ vanishes. On the other hand when $m=1$ and $n>2$ the equation is reducible to one with $m=1$ and $n=2$.

2. Persistent Centers, Moment Conditions and the Composition Condition

In several papers the authors give the following sufficient condition for a center, named the *Composition Condition* (CC).

Let us denote by $\tilde{A}(\theta) = \int_0^\theta A(t)dt$ and $\tilde{B}(\theta)) = \int_0^\theta B(t)dt$.

Definition 1. The functions $A(\theta)$ and $B(\theta)$ satisfy the *Composition Condition* (CC) if there exist a periodic function $u(\theta)$ such that

$$\tilde{A}(\theta) = A_1(u(\theta)) \textit{ and } \tilde{B}(\theta) = B_1(u(\theta)) \tag{5}$$

for a certains functions A_1 and B_1.

The Composition Conjecture for the Abel differential equation states that (CC) is also a necessary condition to have a center. This Conjecture is false, see [2].

The polynomial version of the conjecture gave rise to several articles, trying to prove the conjecture or a weaker version of it (see....). Concerning this Conjecture, Briskin, Françoise and Yomdin proposed a simplified version of it. They asked for conditions that the center lies in a one parametric family of centers. Precisely, the question is to give conditions on $A(\theta)$ and $B(\theta)$ so that

$$\dot{r} = \epsilon\, A(\theta)r^n + B(\theta)r^m, \text{ with } n > m, \tag{6}$$

has a centre for all ϵ small enough. For the Abel equation it is known that such a condition implies that

$$\int_0^{2\pi} A(\theta)\tilde{B}^k(\theta)d\theta = 0\,,\, k \geq 0. \tag{7}$$

For each $k \in \mathbb{N}$, the expression $\int_0^{2\pi} A(\theta)\, \tilde{B}(\theta)^k\, d\theta$ is known as the *moment of* A *in respect to* B *of order* k. The question which arises is the converse, i. e., if we have a center with the vanishing of all the moments of A in respect to B, is it true that equation (6) has a centre for all ϵ small enough? It is known that this question has a negative answer, see [1], also [17] for the polynomial case. In the polynomial case, also in [17], a characterization of the pairs of polynomials p, q which satisfy the vanishing of all the moments of q in respect to p, is given.

What we prove is that equation (6) has a center for all ϵ if and only if equation

$$\dot{r} = \lambda\, A(\theta)r^n + \mu\, B(\theta)r^m, \text{ with } n > m \text{ and } \lambda, \mu \in \mathbb{R} \tag{8}$$

has a center for all $\lambda, \mu \in \mathbb{R}$. This motivates the following definition.

Definition 2. We say that equation (4) has a **persistent center** if

$$\dot{r} = \lambda\, A(\theta)r^n + \mu\, B(\theta)r^m,$$

has a center for all $\lambda, \mu \in \mathbb{R}$.

We also prove that if the center is persistent, then not only the moments of A in respect to B are zero, but the moments of B in respect to A must be also zero.

From these results it is natural to formulate two questions. Assume that a center of equation (4) satisfies the vanishing of all the moments, the moments of A in respect to B and the moments of B in respect to A.

Question 1. Is the Center persistent?
Question 2. Does it satisfy the Composition Condition?

In the talk I'll give examples which answer negatively both questions. We also will see that the vanishing of all the moments does not imply, en general, the existence of a centre.

We notice that the counterexample given in [17] satisfies that the moments of q in respect to p, vanishes, but the moments of p in respect to q, do not do it.

We introduce now the following definition.

Definition 3. We say that equation (4) has a **multipersistent center** if

$$\dot{r} = (\alpha\, A(\theta) + \beta\, B(\theta) r^n + (\gamma\, A(\theta) + \delta\, B(\theta) r^m,$$

has a center for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. In a similar way to the case of persistent centers, we see that multipersistent centers satisfy "multimoments" equal to zero, precisely:

$$\int_0^{2\pi} \tilde{A}^p(\theta)\tilde{B}^q(\theta)A(\theta)d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} \tilde{A}^p(\theta)\tilde{B}^q(\theta)B(\theta)d\theta = 0 \tag{9}$$

for all $p, q \in \mathbb{N}$. Results of Brudnyi let us to say that if the above equalities hold then the center satisfies the Composition Condition which in fact implies that the center is persistent.

The precise statements and the proofs of the mentioned results can be found in a preprint done jointly with Armengol Gasull and Francesc Mañosas.

References

[1] Alwash, M.A.M. *On a condition for a centre of cubic non-autonomous equations*, Proc. Roy. Soc. Edinburgh, 113A (1989), 289–291.

[2] Alwash, M.A.M. *The composition conjecture for Abel equation*, Expo. Math. 27, (2009), 241–250.

[3] Bautin, N. N., *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type*, Mat. Sbornik 30, (1952), 181–196, Amer. Math. Soc. Transl. Vol. 100 (1954), 1–19.

[4] Briskin, M., Françoise, J.P., Yomdin, Y. *Center condition II: Parametric and model center problem*, Isr. J. Math. 118, (2000), 61–82.

[5] Briskin, M., Françoise, J.P., Yomdin, Y. *Center condition III: Parametric and model center problem*, Isr. J. Math. 118, (2000), 83–108.

[6] Brudnyi, A. *An algebraic model for the center problem*, Bull. Sci. math. 128, (2004), 839–857.

[7] Brudnyi, A. *An explicit expression for the first return map in the center problem*, J. Diff. Eq. 206, (2004), 306–314.

[8] Brudnyi, A. *On center sets of ODEs determined by moments of their coefficients*, Bull. Sci. math. 130, (2006), 33–48.

[9] Brudnyi, A. *On the center problem for ordinary differential equations*, Amer. J. of Math. 128, no. 2, (2006), 419–451.

[10] Brudnyi, A. *Vanishing of higher-order moments on Lipschitz curves*, Bull. Sci. math. 132, (2008), 165–181.

[11] Chavarriga. J. *Integrable systems in the plane with center type linear part*, Appl. Math. (Warsaw) 22 ,(1994), 285–309.

[12] Chavarriga. J. and Ginè, J. *Integrability of a linear center perturbed by a fourth degree homogeneous polynomial*, Publ. Mat. 40 ,(1996), 21–39.

[13] Chavarriga. J. and Gin è, J. *Integrability of a linear center perturbed by a fifth degree homogeneous polynomial*, Publ. Mat. 41, no. 2,(1997), 335–356.
[14] Cherkas, L. A., *Number of limit cycles of an autonomous second-order system*, Differ. Eq. 5, (1976), 666–668.
[15] Cristopher, C. *Abel equations: composition conjectures and the model problem*, Bull. London Math. Soc. 32,(2000), 332–338.
[16] Briskin, M, Roytvarf, N. and Yomdin, Y. *Center conditions at infinity for Abel differential equations*, Annals of Math. 172,(2010), 437–483.
[17] Muzychuk, M. and Pakovich, F. *Solution of the polynomial moment problem*, Proc. London Math. Soc. 3, no. 99 (2009), 633–657.
[18] Pakovich, F. *A counterexample to the "Composition Conjecture"*, Proc. Amer. Math. Soc. 130,(2002), 3747–3749.
[19] Yomdin, Y. *The center problem for the Abel equation, composition of functions, and moment conditions*, Moscow Math. J. 3, no. 3, (2003), 1167–1195.

Linear and nonlinear control theory: a survey

Eva Zerz

The first contributions to mathematical systems and control theory date back as far as J. C. Maxwell's 1868 paper "On Governors". Still, it took another 100 years for systems theory to be recognized as a mathematical discipline in its own right. This is especially due to the pioneering work of R. E. Kalman, who introduced state space ideas into control theory. These models superseded the frequency domain approach that had been used by engineers up to then.

In the state space setting, a control system (for the purpose of this talk, we restrict to the time-invariant and input-affine case) takes the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t),$$

where x is the state function, u is the input function, and f and g are given functions which are assumed to be smooth, in this talk. One says that an input function u controls the system from state x_0 to state x_f in time $t_f \geq 0$ if the solution of the initial value problem with $x(0) = x_0$ satisfies $x(t_f) = x_f$. A system is called globally controllable if there exists, for any x_0, x_f in the state set $X \subseteq \mathbb{R}^n$, an input function $u \in \mathcal{U}$ that controls the system from x_0 to x_f in some finite time. Here, $\mathcal{U}$ denotes a set of admissible input functions. Sufficient conditions for several local versions of this concept can be formulated in terms of the smallest f-invariant Lie algebra that contains the columns of g [1, 2].

One is particularly interested in achieving such control goals by feedback laws

$$u(t) = \alpha(x(t)) + \beta(x(t))v(t),$$

which are used, for instance, to stabilize the system, or to make certain subsets of X invariant under $f + g\alpha$ [7]. This transformation of the input (combined with a transformation of the state) leads to the concept of feedback equivalence. For polynomial systems, feedback equivalence to a linear system (also known as exact linearization) can be decided by computing syzygies and checking integrability conditions [2, 3]. However, it turns out that this depends crucially on the choice of state and input variables of the system.

An alternative approach to modeling systems was proposed by J. C. Willems [4] in the 1980s. The system variables are treated on an equal footing. In this context, it clearly makes no sense to restrict to explicit first order differential equations. In the linear case, these systems admit a nice algebraic structure theory. Several specific problems in this area have been addressed:

• Structural properties of linear PDE and their algebraic characterization [5, 6]: Let D be a left Noetherian ring, F a left D-module, and q a positive integer. To any subset $M \subseteq D^{1\times q}$ one associates $\mathfrak{B}(M) = \{w \in F^q \mid mw = 0 \text{ for all } m \in M\}$, and to any subset $B \subseteq F^q$ one associates $\mathfrak{M}(B) = \{m \in D^{1\times q} \mid mw = 0 \text{ for all } w \in B\}$. This yields a Galois correspondence between the submodules of $D^{1\times q}$ and the Abelian subgroups of F^q. The image of $\mathfrak{B}$ consists of all linear systems, that is, $\mathcal{B} = \{w \in F^q \mid Rw = 0\}$ for some $R \in D^{g\times q}$. We have $\mathcal{B} \cong \mathrm{Hom}_D(\mathcal{M}, F)$, where $\mathcal{M} = D^{1\times q}/D^{1\times g}R$ is the so-called system module. If $\mathrm{Hom}_D(\cdot, F)$ is faithful, then we obtain order-reversing bijections between the submodules of $D^{1\times q}$ and linear systems in F^q. If this functor is additionally exact, a linear system $\mathcal{B}$ is parametrizable (that is, $\mathcal{B} = \{Lv \mid v \in F^r\}$ for some $L \in D^{q\times r}$) if and only if the system module $\mathcal{M}$ is torsionless. For several relevant choices of D and F, parametrizability amounts to controllability. In particular, this holds for $D = \mathbb{R}[\partial_1, \ldots, \partial_n]$ and $F = C^\infty(\mathbb{R}^n)$, i.e., for smooth solutions of linear partial differential equations with constant coefficients [5].

• Modeling from data – given observed trajectories, find a system of linear differential equations that describes them best [4]: Given a subset $B \subseteq F^q$, the goal is to construct $\mathfrak{M}(B)$. For polynomial trajectories, this has been solved for the cases where D is the ring of linear partial differential or difference equations with constant coefficients in a field [8, 9]. The case where D is the Weyl algebra has recently been addressed in [10], and it was shown that $\mathfrak{B}\mathfrak{M}(B)$ coincides with the linear span of B over the base field.

References

[1] H. Nijmeijer, A. J. van der Schaft. Nonlinear Dynamical Control Systems. Springer, 1990.

[2] A. Isidori. Nonlinear Control Systems. 3rd ed. Springer, 1995.

[3] G. Conte, C. Moog, A. M. Perdon. Algebraic Methods for Nonlinear Control Systems. Springer, 2007.

[4] A. C. Antoulas, J. C. Willems. A behavioral approach to linear exact modeling. *IEEE Transactions on Automatic Control* 38, 1776–1802 (1993).

[5] U. Oberst. Multidimensional constant linear systems. *Acta Applicandae Mathematicae* 20, 1–175 (1990).

[6] J.-F. Pommaret, A. Quadrat. Algebraic analysis of linear multidimensional control systems. *IMA Journal of Mathematical Control and Information* 16, 275–297 (1999).

[7] E. Zerz, S. Walcher, F. Güçlü. Controlled invariant varieties of polynomial control systems. Proc. 19th Symp. Mathematical Theory of Networks and Systems (MTNS), Budapest, 2010.

[8] E. Zerz. Characteristic frequencies, polynomial-exponential trajectories, and linear exact modeling. *SIAM Journal on Control and Optimization* 44, 1148–1163 (2005).

[9] E. Zerz. The discrete multidimensional MPUM. *Multidimensional Systems and Signal Processing* 19, 307–321 (2008).

[10] E. Zerz, V. Levandovskyy, K. Schindelar. Exact linear modeling with polynomial coefficients. To appear in *Multidimensional Systems and Signal Processing*, 2011.

Algorithmic Invariant Theory

Gregor Kemper

This talk gives an overview of algorithmic invariant theory.

The following is the standard situation of invariant theory: G is a linear algebraic group over an algebraically closed field K, and V is a finite-dimensional K-vector space with a linear G-action, given by a morphism $G \times V \to V$. This means that the action can be described by polynomial functions. A natural extension is to consider an affine K-variety X instead of V, which is then called a **G-variety**. The **invariant ring**

$$K[V]^G := \{f \in K[V] \mid f \circ \sigma = f \text{ for all } \sigma \in G\}$$

is the set of all polynomial functions $f\colon V \to K$ that are constant on every G-orbit. More generally, $K[X]^G$ is the set of all G-invariant regular functions on X. The following questions are central in invariant theory:

- *Hilbert's 14th Problem*: Is $K[V]^G$ (or, more generally, $K[X]^G$) finitely generated as a K-algebra?
- If so, how can we find generators?
- Which G-orbits can be separated by invariants, i.e., for which $x, y \in V$ does there exist $f \in K[V]^G$ with $f(x) \neq f(y)$?

Invariant theory has gone a long way towards answering Hilbert's 14th Problem. In fact, it has been shown by Hilbert, Nagata, Haboush and Popov that $K[X]^G$ is finitely generated for all G-varieties X if and only if G is reductive. Notice that the class of reductive groups includes all finite groups and all classical groups. However, there are instances of nonreductive groups G and linear representations V such that the invariant ring $K[V]^G$ is finitely generated. So the problem to classify all pairs (G, V) or (G, X) such that the invariant ring is finitely generated is still open.

The algorithmic side of the problem has been trailing behind the theoretical progress, but has by now almost caught up. In fact, we have an algorithm for computing generators of $K[X]^G$ in the case that G is reductive and X is a G-variety. A major step towards this algorithm is Derksen's algorithm (see [1]), which solves the problem for linearly reductive groups. In Derksen's algorithm, the ideal $D \subseteq K[V \times V]$ corresponding to the set

$$\{(v, w) \in V \times V \mid \text{ there exists } \sigma \in G \text{ with } w = \sigma(v)\}$$

plays a central role. This ideal has come to be known as the **Derksen ideal**. It has an algebraic description, and it can be computed by Gröbner basis techniques.

Apart from Derksen's algorithm, the Derksen ideal can also be used for computing invariant *fields*, i.e., the set $K(X)^G$ of invariants in the function field $K(X) = \mathrm{Quot}(K[X])$ (provided that X is irreducible). In fact, we have a remarkably simple algorithm that computes the invariant field for any linear algebraic group (see [5]). By an observation of Tobias Kamke [4], we can modify this algorithm in such a way that it computes a localization $K[X]^G_f$ of the invariant ring with $f \in K[X]^G$ nonzero. (This works under the hypothesis that

$K(X)^G = \mathrm{Quot}(K[X]^G)$, which is always satisfied if the connected component of G is unipotent.)

Once a localization $K[X]_f^G$ of the invariant ring is known, we have a *pseudo-algorithm* for computing $K[X]^G$ from this (see [3]). This procedure terminates after finitely many steps if and only if $K[X]^G$ is finitely generated, but we have no a priory upper bound for the running time. A further direction one can follow is to try to write the invariant ring as the ring of regular functions on a quasi-affine variety. By a result of Nagata and Winkelmann, there always exists such a quasi-affine variety, even if the invariant ring is not finitely generated (see [7, 8]).

A fairly recent trend in invariant theory has been the study of *separating invariants*. By definition, a subset $S \subseteq K[X]^G$ is called **separating** if for all pairs of points $x, y \in X$ the existence of $f \in K[X]^G$ with $f(x) \neq f(y)$ implies the existence of $f \in S$ with $f(x) \neq f(y)$. In other words, S has the same capabilities of separating orbits as $K[X]^G$. Since every generating subset is automatically separating, the concept of a separating subset is a *weakening* of the concept of a generating subset. But separating subsets have better properties than generating ones. For example,

- there always exists a finite separating subset (even if $K[X]^G$ is not finitely generated), and
- for G finite, there exists a separating set of homogeneous invariants in $K[V]^G$ of degree $\leq |G|$, even in the modular case. So Noether's bound always holds for separating invariants.

Separating invariants are also useful for computational purposes: In fact, there is an algorithms for computing separating invariants in the case that G is reductive (see [6]). Again, this algorithm uses the Derksen ideal. Since we also have algorithms for extending a separating subset into a generating subset, we obtain the above-mentioned algorithm for computing $K[X]^G$ for G reductive.

The following problems are still open:

- Find an algorithm for computing invariant rings of reductive groups acting on nonreduced algebras.
- Find an algorithm for computing a separating subset for G nonreductive.
- Find a test for finite generation of $K[X]^G$.
- Compute $K[X]^G$ as the ring of regular functions of a quasi-affine variety.
- Implement the known algorithms (i.e., beyond Derksen's algorithm and the finite groups case).

More information on invariant theory and its computational aspects can be found in [2].

References

[1] Harm Derksen, *Computation of Invariants for Reductive Groups*, Adv. Math. **141** (1999), 366–384.

[2] Harm Derksen, Gregor Kemper, *Computational Invariant Theory*, Encyclopaedia of Mathematical Sciences **130**, Springer-Verlag, Berlin, Heidelberg, New York 2002.

[3] Harm Derksen, Gregor Kemper, *Computing invariants of algebraic group actions in arbitrary characteristic*, Adv. Math. **217** (2008), 2089–2129.
[4] Tobias Kamke, *Algorithms for the Computation of Invariant Rings*, Dissertation, Technische Universität München 2009.
[5] Gregor Kemper, *The Computation of Invariant Fields and a Constructive Version of a Theorem by Rosenlicht*, Transformation Groups **12** (2007), 657–670.
[6] Gregor Kemper, *Computing Invariants of Reductive Groups in Positive Characteristic*, Transformation Groups **8** (2003), 159–176.
[7] Masayoshi Nagata, *Lectures on the Fourteenth Problem of Hilbert*, Tata Institute of Fundamental Research, Bombay 1965.
[8] Jörg Winkelmann, *Invariant Rings and Quasiaffine Quotients*, Math. Zeitschrift **244** (2003), 163–174.

Algorithms for Polynomial Differential Equations

Harm Derksen

Let $R = \mathbb{C}[X_1, \ldots, X_n]$ be the polynomial ring in n variables, and $K = Q(R) = \mathbb{C}(X_1, \ldots, X_n)$ be its quotient field. To a derivation $D = \sum_{i=1}^n P_i \frac{\partial}{\partial X_i} \in \mathrm{Der}_{\mathbb{C}}(R)$ we can associate a polynomial autonomous system of differential equations:

$$\dot{x(t)} = P(x(t)) \tag{1}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \text{ and } P = \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix}.$$

Given an intitial condition $x(0) = a$, the system (1) has a unique formal power series solution $x_a(t) \in \mathbb{C}[[t]]^n$. These power series converges for an open neighborhood of 0.

A function $f \in K$ is a rational first integral if $f(x_a(t))$ is constant for all $a \in \mathbb{C}^n$ for which $f(a)$ is defined. One can show that f is a first integral if and only if $Df = 0$. Let

$$K^D = \{f \in K \mid Df = 0\}$$

be the subfield of rational first integrals.

For $f \in K$ and a positive integer r we define

$$D^{(r)} f = \begin{pmatrix} f \\ Df \\ \vdots \\ D^{r-1} f \end{pmatrix}$$

If $f_1, \ldots, f_r \in K$ then we define the Wronski matrix by

$$W(f_1, \ldots, f_r) = \begin{pmatrix} D^{(r)} f_1 & D^{(r)} f_2 & \cdots & D^{(r)} f_r \end{pmatrix}.$$

Suppose that $\det W(f_1, \ldots, f_r) \neq 0$ and $\det W(f_1, \ldots, f_{r+1}) = 0$. Then there exist unique elements $g_1, \ldots, g_k \in K$ such that

$$D^{(r)} f_{r+1} = \sum_{i=1}^{r} g_i D^{(r)} f_i.$$

Lemma 1. *We have $g_1, \ldots, g_r \in K^D$.*

Lemma can be used to find elements in K^D by looking at Wronskians $\det W(m_1, m_2, \ldots, m_r)$ where $m_1, m_2, \ldots, m_r$ are monomials. Done systematically one can obtain a set of generators for K^D. If the transcendence degree $\mathrm{trdeg}(K^D : \mathbb{C})$ is known a priori, then this knowledge can be used as a termination criterion for an algorithm to compute generators of the field K^D.

For $a \in \mathbb{C}^n$, let Z_a be the Zariski closure of $\{x_a(t) \mid -\varepsilon < t < \varepsilon\}$. The definition of Z_a is independent of the choice of ε. Let us call $a \in \mathbb{C}^n$ generic if

$$\det W(m_1, \ldots, m_r) \neq 0 \Leftrightarrow \det W(m_1, \ldots, m_r)(a) \neq 0$$

for all positive integers r and all monomials $m_1, \ldots, m_r$. Such generic vectors exist, because the set of non-generic vectors is a countable union of hypersurfaces, and hence a set of measure 0.

Theorem 2. *For every $a \in \mathbb{C}^n$ we have*

$$\dim Z_a \leq \mathrm{trdeg}(K : K^D) = n - \mathrm{trdeg}(K^D : \mathbb{C}).$$

If a is generic, we have equality.

Corollary 3. *A system (1) has a Zariski dense trajectory if and only if $K^D = \mathbb{C}$.*

Suppose that we have a system

$$\dot{x}(t) = P(x(t), \beta) \tag{2}$$

where $\beta \in \mathbb{C}$ is a parameter, and $P \in \mathbb{C}[X_1, \ldots, X_n, Y]^n$.

Corollary 4. *If (2) has r algebraically independent first integrals for uncountably many values of β, then (2) has r algebraically independent first integrals for all values of β.*

Corollary (4) follows from (3).

Example 5. *Consider the system*

$$\left\{ \begin{array}{rcl} \dot{x} & = & x^2 + yQ(x, y) \\ \dot{y} & = & y + y^2 R(x, y) \end{array} \right. \tag{3}$$

where Q and R are polynomials. Suppose that (3) has a rational first integral. If we substitute $y = \varepsilon y$, then we have

$$\left\{ \begin{array}{rcl} \dot{x} & = & x^2 + \varepsilon yQ(x, y) \\ \dot{y} & = & y + \varepsilon y^2 R(x, y) \end{array} \right. \tag{4}$$

For $\varepsilon \neq 0$, (3) and (4) are equivalent, so (4) has a rational invariant. For $\varepsilon = 0$ we get

$$\left\{ \begin{array}{lcl} \dot{x} & = & x^2 \\ \dot{y} & = & y \end{array} \right. \tag{5}$$

By Corollary 4, (5) should have a rational first integral, but one can verify that it does not. This shows that (3) does not have a rational first integral.

A subset $\mathfrak{a} \subseteq \mathrm{Der}_{\mathbb{C}}(K)$ is said to be algebraically integrable if

$$\mathfrak{a} = \mathrm{Der}_{K^{\mathfrak{a}}}(K)$$

where

$$K^{\mathfrak{a}} = \{f \in K \mid \forall D \in \mathfrak{a}\ Df = 0\}.$$

There is a Galois correspondence between integrable subalgebras of $\mathrm{Der}_{\mathbb{C}}(K)$ and subfields of K (containing $\mathbb{C}$) that are algebraically closed within K.

Theorem 6. *If $\mathfrak{a}_1, \mathfrak{a}_2 \subseteq \mathrm{Der}_{\mathbb{C}}(K)$ are algebraically integrable, then the Lie algebra $\mathfrak{a}$ generated by $\mathfrak{a}_1$ and $\mathfrak{a}_2$ is algebraically integrable, and $K^{\mathfrak{a}} = K^{\mathfrak{a}_1} \cap K^{\mathfrak{a}_2}$.*

This theorem gives us an algorithm for computing the intersection of two fields. Suppose that L_1 and L_2 are subfields of K which are algebraically closed within K. The subspaces of $\mathfrak{a}_1 = \mathrm{Der}_{L_1}(K)$ and $\mathfrak{a}_2 = \mathrm{Der}_{L_2}(K)$ are easily computed using linear algebra over K. With some more linear algebra, one finds the Lie algebra $\mathfrak{a}$ generated by $\mathfrak{a}_1$ and $\mathfrak{a}_2$. We have

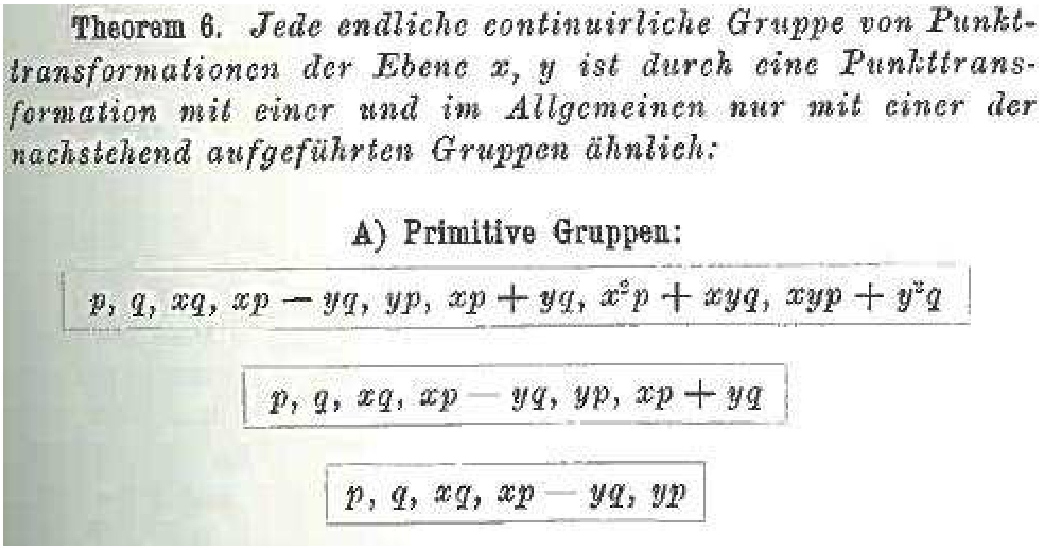
$$\dim_K \mathfrak{a} = \mathrm{trdeg}(K : K^D)$$

so the transcendence degree $\mathrm{trdeg}(K^D : \mathbb{C})$ is known. With a generalization of the algorithm mentioned earlier, one can compute generators of $K^{\mathfrak{a}} = K^{\mathfrak{a}_1} \cap K^{\mathfrak{a}_2} = L_1 \cap L_2$.

Lie algebras of vector fields, a survey

Jan Draisma

This survey talk concerns finite-dimensional Lie subalgebras of the Lie algebra $\mathrm{Der}\,\mathbb{C}[[x_1, \ldots, x_n]]$ of derivations of the formal power series algebra $\mathbb{C}[[x_1, \ldots, x_n]]$. The study of such Lie algebras was initiated by Sophus Lie at the end of the 19th century. Motivated by his insight that symmetries of differential equations can help in finding their solutions, he set out to classify such Lie algebras up to the group $\mathrm{Aut}(\mathbb{C}[[x_1, \ldots, x_n]])$ of formal coordinate changes (though of course his terminology was different). He spent special attention to *transitive* subalgebras, which are those that contain an element of the form $\frac{\partial}{\partial x_i}$ + higher-order terms for every $i = 1, \ldots, n$. For example, in $n = 1$ variable, there are exactly three classes of such Lie algebras, namely, $\langle p \rangle_{\mathbb{C}}$, $\langle p, xp \rangle_{\mathbb{C}}$, and $\langle p, xp, x^2p \rangle_{\mathbb{C}}$, where p stands for $\frac{\partial}{\partial x}$. Lie classified (transitive) Lie subalgebras up to three variables. Fragment 1 shows part of his classification in two variables. He did not publish the entire classification in three variables, because it involved too many straightforward calculations; he refers to this in the first sentence of Fragment 2.



Theorem 6. *Jede endliche continuirliche Gruppe von Punkttransformationen der Ebene x, y ist durch eine Punkttransformation mit einer und im Allgemeinen nur mit einer der nachstehend aufgeführten Gruppen ähnlich:*

A) Primitive Gruppen:

$p, q, xq, xp - yq, yp, xp + yq, x^2p + xyq, xyp + y^2q$

$p, q, xq, xp - yq, yp, xp + yq$

$p, q, xq, xp - yq, yp$

Figure 1. Fragment from [6, Page 71] (there the statement of the theorem continues).

A transitive Lie algebra $\mathfrak{g}$ has a natural subalgebra $\mathfrak{h}$, consisting of all derivations that stabilise the maximal ideal. The pair $(\mathfrak{g}, \mathfrak{h})$ serves as a (very) local algebraic model for the action of a finite-dimensional Lie group on a manifold. For example, the first Lie algebra listed under A in Fragment 1 corresponds, via the translation $p \to \frac{\partial}{\partial x}, q \to \frac{\partial}{\partial y}$, to the action of the projective general linear group on the projective plane. In higher dimensions, classifying all transitive Lie algebras is a daunting task, so that one has to redefine one's goals. A beautiful result says that roughly any pair $(\mathfrak{g} \subseteq \mathfrak{h})$ of abstract Lie algebras with $\mathfrak{h}$ of codimension n in $\mathfrak{g}$ can be realised as a transitive Lie algebra in n variables [5, 1]. I sketch Blattner's coordinate-free proof of this fact: he defines a commutative, associative multiplication on $A := \mathrm{Hom}_{U(\mathfrak{h})}(U(\mathfrak{g}), \mathbb{C})$ and shows that $\mathfrak{g}$ acts naturally on this algebra by derivations. Then he shows that A is isomorphic to the algebra of formal power series, which gives the desired realisation.

In another direction, pairs $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h}$ is *maximal* among all subalgebras of $\mathfrak{g}$ are called *primitive*. These have been classified: First, when $\mathfrak{g}$ is not simple, Morozov classifies the possibilities in [7]. Second, maximal-dimensional subalgebras of *simple* Lie algebras are classified by Dynkin in [3, 4]. I review both classifications.

At the end of the talk, I describe a beautiful question of Lie's as to whether one can realise transitive pairs by vector fields with more modest functions as coefficients, such as polynomials and exponentials; see Fragment 2. I explain how Blattner's proof gives an answer to Lie's question in some cases (see [2]), and speculate about the general case.

References

[1] Robert J. Blattner. Induced and produced representations of Lie algebras. *Trans. Am. Math. Soc.*, 144:457–474, 1969.

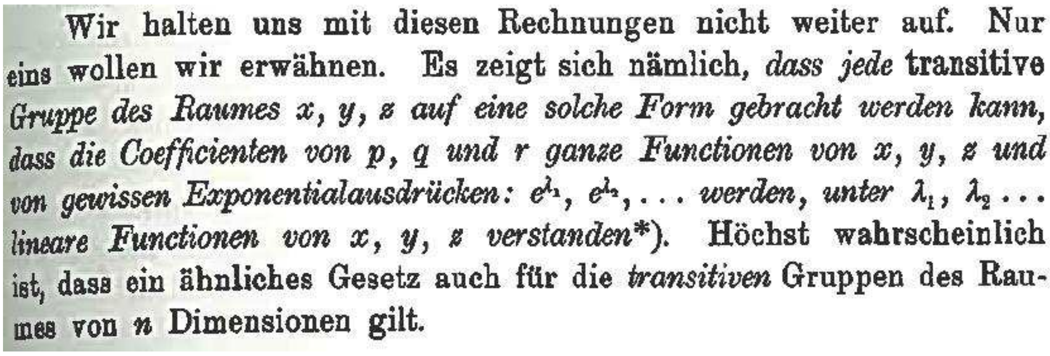

Wir halten uns mit diesen Rechnungen nicht weiter auf. Nur eins wollen wir erwähnen. Es zeigt sich nämlich, *dass jede* transitive *Gruppe des Raumes x, y, z auf eine solche Form gebracht werden kann, dass die Coefficienten von p, q und r ganze Functionen von x, y, z und von gewissen Exponentialausdrücken: e^{λ_1}, e^{λ_2}, ... werden, unter λ_1, λ_2 ... lineare Functionen von x, y, z verstanden**). Höchst wahrscheinlich ist, dass ein ähnliches Gesetz auch für die *transitiven* Gruppen des Raumes von n Dimensionen gilt.

FIGURE 2. Fragment from [6, Page 177].

[2] Jan Draisma. On a conjecture of Sophus Lie. In *Differential Equations and the Stokes Phenomenon*, Singapore, 2002. World Scientific Publishing Company, Incorporated. Proceedings of a workshop held at Groningen University from May 28–30, 2001.
[3] Eugene B. Dynkin. Maximal subgroups of the classical groups. *Am. Math. Soc. Transl.*, II. Ser. 6:245–378, 1957.
[4] Eugene B. Dynkin. Semisimple subalgebras of semisimple Lie algebras. *Am. Math. Soc. Transl.*, II. Ser. 6:111–244, 1957.
[5] Victor W. Guillemin and Shlomo Sternberg. An algebraic model of transitive differential geometry. *Bull. Am. Math. Soc.*, 70:16–47, 1964.
[6] Sophus Lie and Friedrich Engel. *Transformationsgruppen.* B.G. Teubner, Leipzig, 1893.
[7] V.V. Morozov. Sur les groupes primitifs. *Rec. Math. Moscou n. Ser.*, 5:355–390, 1939.
The author is supported by an NWO Vidi grant.

Jet groupoids and the invariance of geometric structures

MOHAMED BARAKAT

One of the original motivations behind SOPHUS LIE's work was the desire to develop a symmetry and solvability theory for ordinary and partial differential equations analogous to GALOIS' theory for univariate polynomial equations. In this analogy the solvability of a univariate polynomial equation by a tower of radical extensions should correspond to the solvability of ODEs by a cascade of integrations.

0.1. **Galois' setup.** Let K be a field, $f \in K[x]$ a monic irreducible separable polynomial over K, and $V(f)$ the vanishing locus of f, i.e. the set of roots of f in the separable closure $\bar{K}$.

To describe the GALOIS group of f as the symmetry group of $V(f)$ we first consider the set-stabilizer subgroup

$$G_{V(f)} := \{\sigma \in \mathrm{Aut}(\bar{K}) \mid \sigma(V(f)) = V(f)\}$$

of the full automorphism group $G := \mathrm{Aut}(\bar{K})$. In words, $G_{V(f)}$ is the biggest subgroup of $\mathrm{Aut}(\bar{K})$ which acts on $V(f)$. This action[1] is described by a homomorphism $\alpha : G_{V(f)} \to S_n$ where $n = |V(f)|$.

[1] Since f is irreducible the action is transitive on the n roots and since f is separable $n = \deg f$.

The GALOIS group of f can now be defined[2] as the coimage of α

$$\mathrm{Gal}(f) := G_{V(f)}/\ker\alpha.$$

By definition, the GALOIS group $\mathrm{Gal}(f)$ acts effectively and transitively on $V(f)$ and is isomorphic to the image of α in S_n.

0.2. **Lie's setup.** Let us now describe LIE's setup starting with a system

$$\Delta = \Delta(x, u_\mu) = 0$$

of (ordinary or parital) differential equations. A solution $u = f(x)$ is a locally defined analytic function $f : O \subset \mathbb{R}^n \to \mathbb{R}^m$ such that $\Delta(x, \frac{\partial^{|\mu|} f}{\partial x^\mu}(x)) = 0$.

LIE made several attempts to mimic GALOIS' setup. The first step was to find a substitute for the group $G = \mathrm{Aut}(\bar{K})$ mapping the class of locally defined analytic functions into itself. It was natural for him to consider *geometric* local actions on the space of such functions, i.e. those induced by local transformations on the total space $E = \mathbb{R}^n \times \mathbb{R}^m$. Each such local transformation maps the graph of one function into the graph of another. This led him to consider what we now call **Lie pseudo-groups** of point and contact transformations and their associated **Lie algebras of vector fields** [Lie60]. The subgroup $G_{V(f)}$ then corresponds to the sub-pseudo-group $G_\Delta = G_{\mathrm{Sol}(\Delta)}$ mapping the set of solutions of Δ onto itself. So the second step was to describe G_Δ without referring to the unknown solutions of Δ for which he introduced the notion of **prolongation** of diffeomorphisms (and of vector fields) as another incarnation of the chain rule. The partial differential equations defining such sub-pseudo-groups are nowadays called **Lie equations** [KS72]. TRESSE showed in [Tre94] that LIE equations can be transformed into a so-called **Lie form**

$$\Phi(y_\mu) = \omega(x),$$

where $\Phi(y_\mu)$ are differential invariants of an action of the higher order general linear group $\mathrm{GL}_q(\mathbb{R}^\ell)$ (cf. [Kob95]). VESSIOT proved in [Ves03] that the integrability conditions of LIE equations of transitive pseudo-groups are given by a set of equations

$$I_k[\omega(y)] = c_k\,\omega(x)$$

with some constants c_k and showed how to use these constants in the classification of transitive pseudo-groups.

In the talk I will try to relate all these things to the modern language of natural bundles, LIE derivates, jet groupoids, and jet algebroids. I will also give examples showing how the above mentioned integrability conditions cover all classical integrability conditions in differential geometry [KN63, KN69, Pom78].

[2] It is more common to start with the subgroup $G_K := \mathrm{Aut}_K(\bar{K}) = \{\sigma \in \mathrm{Aut}(\bar{K}) \mid \sigma_{|K} = \mathrm{id}_K\} \leq G_{V(f)}$, the so-called **absolute Galois group of** K, which automatically acts on $V(f)$ and then to define the GALOIS group as $\mathrm{Gal}(f, K) := G_K/\ker\alpha_{|G_K}$. This is equivalent to the above definition since f is irreducible over K. The drawback of this latter definition is that it has no obvious counterpart in LIE's setup.

References

[KN63] Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry. Vol I*, Interscience Publishers, a division of John Wiley & Sons, New York-Lond on, 1963. MR MR0152974 (27 #2945)

[KN69] ———, *Foundations of differential geometry. Vol. II*, Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969. MR MR0238225 (38 #6501)

[Kob95] Shoshichi Kobayashi, *Transformation groups in differential geometry*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1972 edition. MR 1336823 (96c:53040)

[KS72] Antonio Kumpera and Donald Spencer, *Lie equations. Vol. I: General theory*, Princeton University Press, Princeton, N.J., 1972, Annals of Mathematics Studies, No. 73. MR 52 #1805

[Lie60] Sophus Lie, *Gesammelte Abhandlungen. Siebenter Bd*, Herausgegeben von der Akademie der Wissenschaften zu Leipzig und dem Norwegischen Mathematischen Verein durch Friedrich Engel und Poul Heegard, B. G. Teubner, Leipzig, H. Aschehoug & Co., Oslo, 1960. MR 0122662 (23 #A3)

[Pom78] J.-F. Pommaret, *Systems of partial differential equations and Lie pseudogroups*, Mathematics and its Applications, vol. 14, Gordon & Breach Science Publishers, New York, 1978, With a preface by André Lichnerowicz. MR MR517402 (81f:58046)

[Tre94] Ar. Tresse, *Sur les invariants différentiels des groupes continus de transformations*, Acta Math. **18** (1894), no. 1, 1–3. MR 1554846

[Ves03] Ernest Vessiot, *Sur la théorie des groupes continus*, Ann. Sci. École Norm. Sup. (3) **20** (1903), 411–451. MR 1509031

Linear ODEs with reductive Galois group

Camilo Sanabria Malagón

Set $n \in \mathbb{N}$, with $n > 1$. Let Γ be a compact Riemann surface, $k := \mathbb{C}(\Gamma)$ the field of meromorphic functions over Γ and $v \in \mathrm{Der}_{\mathbb{C}}(k) \setminus \{0\}$ a nontrivial derivation of k. Let

$$L(y) := a_n v^n(y) + a_{n-1} v^{n-1}(y) + \ldots + a_1 v(y) + a_0 y$$

be a differential operator with $a_i \in k$, $i \in \{0, 1, \ldots, n\}$. In this case, we say that the linear ODE $L(y) = 0$ is defined over Γ.

1. Background

1.1. **Projective equivalence and pullbacks.** Let $\pi : \Gamma \to \Gamma_0$ be a finite ramified covering of Riemann surfaces and $L_0(y_0) = 0$ a linear ODE defined over Γ_0, i.e.

$$L_0(y_0) := b_n v_0^n(y_0) + b_{n-1} v_0^{n-1}(y_0) + \ldots + b_1 v_0(y_0) + b_0 y_0$$

where $b_i \in k_0 := \mathbb{C}(\Gamma_0)$, $i \in \{0, 1, \ldots, n\}$, and $v_0 \in \mathrm{Der}_{\mathbb{C}}(k_0) \setminus \{0\}$. We say that $L(y) = 0$ is a *pullback* of $L_0(y_0) = 0$ if $y = y_0 \circ \pi$ is a solution of $L(y) = 0$ whenever y_0 is a solution of $L_0(y_0) = 0$.

Let $L_1(y_1) = 0$ be another linear ODE defined over Γ. We say that $L(y) = 0$ is *projectively equivalent* to $L_1(y_1) = 0$ if, for some analytic function f over $U \subseteq \Gamma$ with $v(f)/f \in k$, $y = f \cdot y_1$ is a solution of $L(y) = 0$ whenever y_1 is a solution of $L_1(y_1) = 0$.

1.2. **A theorem by F. Klein and Standard equations.** A classical theorem due to F.Klein states that, up to pullback and up to projective equivalence, any irreducible second order linear ODE defined over $\mathbb{P}^1(\mathbb{C})$ with algebraic solutions is a hypergeometric equation.

Theorem 1 (Klein, 1877 [7, 8]). *If $L(y) = y'' + a_1(z)y' + a_0(z)y \ \ (a_1, a_0 \in \mathbb{C}(z))$ is irreducible and $L(y) = 0$ has algebraic solutions then the solutions are of the form*

$$y = f \cdot {}_2F_1\,{}_{(a,b,c)}\,(P(z))$$

where $P(z), f'/f \in \mathbb{C}(z)$ and ${}_2F_1\,{}_{(a,b,c)}$ is a hypergeometric function.

This result was extended to arbitrary compact Riemann surfaces by B. Dwork and F. Baldassarri [1, 2]. An algorithmic implementation for Klein's result has been crafted in a joint work of M. Berkenbosch, M. van Hoeij and J-A. Weil [3]. The algorithm relies on the fact that the hypergeometric functions appearing in the theorem correspond to Galois coverings of $\mathbb{P}^1(\mathbb{C})$ by $\mathbb{P}^1(\mathbb{C})$, therefore these can be listed using Schwarz triples.

By introducing the concept of *standard equation* M. Berkenbosch managed to extend Klein's Theorem to order three [3]: up to pullback and up to projective equivalence, all third order irreducible linear ODE defined over $\mathbb{P}^1(\mathbb{C})$ with algebraic solutions is a standard equations.

Standard equations can be defined as follow. Let $y_1, \dots, y_n$ be a full-system of solutions of $L(y) = 0$, we say that $L(y) = 0$ is *standard* if

$$k \subseteq \mathbb{C}\left(v^{i-1}(y_j)\right)_{i,j \in \{1,\dots,n\}} \text{ and } w := \det\left(v^{i-1}(y_j)\right) \in \mathbb{C}.$$

An algorithmic implementation for an extension of Klein's theorem would require a solution to the problem of listing or classifying standard equations. To solve this problem we can use ruled surfaces. Before explaining the solution, we need two more concepts.

1.3. **Galois correspondence and a theorem of E. Compoint.** Let $S \subset \Gamma$ be the collection of singular points of $L(y) = 0$. Given a $p \in \Gamma \setminus S$, there is an open neighborhood $U \subseteq \Gamma$ of p over which there is a full system of analytic solutions $y_i : U \to \mathbb{C}$, $i \in \{1, \dots, n\}$, for $L(y) = 0$.

Denote by $\mathcal{H}_\Gamma(U)$ the collection of analytic functions over U and by Φ the evaluative k-morphism

$$\begin{aligned} \Phi : k[X^i_j, \frac{1}{det}] &\longrightarrow \mathcal{H}_\Gamma(U) \\ X^i_j &\longmapsto v^{i-1}(y_j) \end{aligned}$$

where $k[X^i_j, \frac{1}{det}]$ is the ring of polynomials in $n \times n$ variables with coefficients in k and det, the determinant polynomial $\det(X^i_j)$, is inverted. Denote by I the kernel of Φ.

We define a right-action of $GL_n(\mathbb{C})$ on $k[X^i_j, \frac{1}{det}]$ as follow. To $g = (g^i_j) \in GL_n(\mathbb{C})$ we assign the k-morphisms

$$g : X^i_j \longmapsto \sum_l X^i_l g^l_j.$$

The Galois group G of $L(y) = 0$ is

$$G := \{g \in GL_n(\mathbb{C}) \mid g(I) \subseteq I\}.$$

In particular, the restriction to G of the right $GL_n(\mathbb{C})$-action induces a right G-action by k-morphisms on

$$k[v^{i-1}(y_j), \frac{1}{w}] \simeq k[X^i_j, \frac{1}{det}]/I.$$

Explicitly, to $g = (g^i_j) \in G$ we assign the k-morphism $g : v^{i-1}(y_j) \mapsto \sum_l v^{i-1}(y_l)g^l_j$.

The Galois correspondence [9, Theorem 1.27] implies that if $P(X^i_j) \in k[X^i_j, \frac{1}{det}]$ is G-invariant then $P(v^{i-1}(y_j)) \in k$.

Theorem 2 (Compoint, 1998 [5, 4])**.** *Assume G is reductive and unimodular. Let $P_1, \dots, P_r \in \mathbb{C}[X^i_j]$ be homogeneous generators of the $\mathbb{C}$-algebra of G-invariants and $f_l \in k$ such that $P_l(v^{i-1}(y_j)) = f_l$, $l \in \{1, \dots, r\}$. Then the ideal I is generated by the polynomials $P_1 - f_1, \dots, P_r - f_r$.*

2. Ruled Surfaces

From now on we assume G is reductive, unimodular and non-connected (e.g. $G \subset SL_n(\mathbb{C})$ is finite and non-trivial). The right G-action on $\mathbb{C}[X^i_j]$ defines a left G-action on $\mathbb{C}^{n\times n}$. Let $\Pi : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}/G$ be the canonical projection assigning to each point its G-orbit. We fix the coordinate systems (X^i_j) for $\mathbb{C}^{n\times n}$ and $(P_1, \dots, P_r)$ for $\mathbb{C}^{n\times n}/G$.

The solution functions $y_1, \dots, y_n$ induce an analytic map

$$\begin{array}{rcl} \Psi : U & \longrightarrow & \mathbb{C}^{n\times n} \\ q & \longmapsto & \left(v^{i-1}(y_j)(q)\right). \end{array}$$

The composition with Π, $\Psi^G := \Pi \circ \Psi$, induces an algebraic map

$$\begin{array}{rcl} \Psi^G : U & \longrightarrow & \mathbb{C}^{n\times n}/G \\ q & \longmapsto & (f_1(q), \dots, f_r(q)). \end{array}$$

Since the map Ψ^G is algebraic, we can extended it to a map over $\Gamma \setminus S$. Abusing notation, we will also denote the extension by Ψ^G. The image

$$V := \Psi^G(\Gamma \setminus S) \subset \mathbb{C}^{n\times n}/G$$

is an algebraic curve. The curve V gives a cone $C(V) \subset \mathbb{C}^{n\times n}/G$ with vertex at the origin.

Finally, blowing-up $C(V)$ at the vertex we obtain a ruled surface S [6, Example V.2.11.4]. This ruled surfaces S, obtained starting from $L(y) = 0$, is isomorphic to the ruled surface that we would obtain starting from $L_0(y_0) = 0$ and from

$L_1(y_1) = 0$ if $L(y) = 0$ is the pullback of $L_0(y_0) = 0$ or $L(y) = 0$ is projectively equivalent to $L_1(y_1) = 0$.

We conclude that S is invariant up to pullback and up to projective equivalence and so ruled surfaces can be used to list standard equations.

References

[1] F. Baldassarri and B. Dwork, *On second order linear differential equations with algebraic solutions*, American Journal of Mathematics **101** (1979):42–76.

[2] F. Baldassarri, *On second order linear differential equations with algebraic solutions on algebraic curves*, American Journal of Mathematics **102** no. 3 (1980): 517–535.

[3] M. Berkenbosch, *Algorithms and moduli spaces for differential equations*, Séminaires & Congrès **13** (2006): 1–38.

[4] F. Beukers, *The maximal differential ideal is generated by its invariants*, Indagationes Mathematicae N.S. **11** no. 1 (2000): 13–18.

[5] E. Compoint, *Differential equations and algebraic relations*, Journal of Symbolic Computation **25** (1998): 705–725.

[6] R. Hartshorne, *Algebraic Geometry*, Graduate Text in Mathematics **52**, Springer-Verlag (1977).

[7] F. Klein, *Über lineare Differentialgleichungen I*, Mathematische Annalen **11** (1877):115–118.

[8] F. Klein, *Über lineare Differentialgleichungen II*, Mathematische Annalen **12** (1878):167–179.

[9] M. van der Put, M.F. Singer, *Galois Theory of linear Differential Equations*, Lecture notes in Mathematics **1444**, Springer-Verlag (1997).

Hilbert's sixteenth problem for Liénard equations

Magdalena Caubergh

Polynomial Liénard equations are planar differential equations associated to the second order scalar differential equations

$$x'' + f(x)\,x' + g(x) = 0, \tag{1}$$

where the functions f and g are polynomials of degree n and m respectively. They occur as models or at least as simplifications of models in many domains of science. In this survey talk attention goes to isolated periodic orbits or so-called limit cycles of (1) as a contribution to Hilbert's 16th Problem.

Recall that Hilbert's 16th Problem essentially asks for a uniform bound $\mathcal{H}(n)$ for the maximum number of limit cycles of a planar polynomial vector field

$$x' = \sum_{i,j=0}^{n} a_{ij} x^i y^{j-i} \text{ and } y' = \sum_{i,j=0}^{n} b_{ij} x^i y^{j-i} \text{ where } a_{ij}, b_{ij} \in \mathbb{R}, 0 \le i, j \le n,$$

uniformly in terms of the degree n. This problem is more than 100 years old and its investigation has produced many papers contributing to the wide development of the theory of Dynamical Systems. It is not known whether a uniform upper bound only depending on the degree of the vector field might exist, not even when the degree is two. Even Dulac's theorem to prove that for individual vector fields the number of limit cycles is finite was far from trivial, see e.g. [10].

Solution programmes for Hilbert's 16th Problem mostly consist in its reduction to several subproblems, based on either considering local cyclicity problems [13] or restricting the class of vector fields to a particular simpler class, see e.g. [10] for an overview.

In the following we denote by $\mathcal{N}(m,n)$ the maximum number of limit cycles of (1). Part of the 13th Problem that S. Smale put on his list of problems for the 21st century deals with Hilbert's 16th Problem restricted to the classical (polynomial) Liénard equations, i.e. the case $g(x) = x$ in the differential equations (1) (see [15]). Moreover Smale suggests that the maximal number of limit cycles $\mathcal{N}(1,n)$ for classical Liénard equations grows at most by an algebraic law of type n^d where d is a universal constant.

The problem for classical Liénard equations when the degree of f is equal to 2 is solved; the result in [12] shows that $\mathcal{N}(1,2) = 1$. Besides there is the so-called Lins, de Melo and Pugh Conjecture, stating that the maximal number of limit cycles is equal to l if $g(x) = x$ and the degree of f is $2l$ or $2l+1$.

Of course there is the recent counter-example to that conjecture, for limit cycles in singular perturbations, due to Dumortier, Panazzolo and Roussarie (see [7]), but it does not contradict the possibility for the growth of the number of limit cycles to be linear. In [7] classical Liénard equations are presented with degree of f equal to $2l$ and having at least $l+1$ limit cycles; hence one limit cycle more than conjectured by Lins, de Melo and Pugh.

In fact, in [12], they prove that, under these assumptions, there are at most l small amplitude limit cycles. Lloyd and Lynch considered the similar problem for generalized Liénard equations [11]. In most cases, they prove an upper bound for the number of small amplitude limit cycles, that can bifurcate out of a single non-degenerate singularity.

Later Coppel proved in [4] that $\mathcal{N}(2,1) = 1$. In [5] and [9] it is shown that $\mathcal{N}(3,1) = 1$ and in [6] it is shown that $\mathcal{N}(2,2) = 1$. Up to now, as far as we know, only these four cases have been completely investigated. Recently progress has been made towards proving the finiteness part of Hilbert's 16th Problem for classical Liénard equations. In [2] the study of the finiteness part of Smale's 13th Problem is reduced to singular perturbation problems:

Theorem 1. *Let $K > 0$. Then there exists a finite number $N(n,K)$ such that for $\|a\| \leq K$ the classical Liénard equation $\mathcal{L}_a^n$ of degree n, i.e. (1) with*

$$f(x,a) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-2} + x^{n-1} \text{ and } g(x) = x,$$

has at most $N(n,K)$ limit cycles.

In this talk we recall a basic argument to prove the exact upper bounds $\mathcal{N}(1,2) = \mathcal{N}(2,1) = \mathcal{N}(2,2) = 1$. Next we provide some insight in the techniques underlying Theorem 1 and finally discuss a programme for the general case, i.e., when $g(x) \neq x$ (see [3]).

To prove Theorem 1 we apply the localization method of Roussarie [14], reducing the global Smale problem to the study of (local) cyclicity of limit periodic sets. Limit periodic sets are subsets consisting of singularities and regular orbits, that can produce limit cycles by perturbation, and the cyclicity is the maximal number of limit cycles that they can generate in a perturbation. This localization method requires an appropriate compactification of the phase plane, as well as the chosen space of Liénard equations itself. Besides the compactification process, due to the fact that the 'central system' is too degenerate to permit a study of its unfolding without a blow up, the method includes a desingularization.

In this way the boundary of the space of Liénard equations is made by Hamiltonian and singular perturbation problems. These boundary problems both exhibit different phenomena. The study of the cyclicity problem for classical Liénard equations of odd degree (i.e. n is even) that do not belong to the boundary is easy and well-known among specialists. In this case limit cycles stay at a uniform distance from infinity (see [14]). For classical Liénard equations of even degree (i.e. n is odd) this is no longer true and then the main problem consists in studying limit cycles that come close to infinity. These limit cycles are so-called large amplitude limit cycles of which it is shown in [2] that there are at most l.

References

[1] M. Caubergh and J.P. Françoise, *Generalized Liénard equations, cyclicity and Hopf-Takens bifurcations*, Qualitative Theory of Dynamical Systems **5(2)**(2004), 195–222.

[2] M. Caubergh and F. Dumortier, *Hilbert's 16th problem for classical Liénard equations of even degree*, Journal of Differential Equations **244** (2008), 1359–1395.

[3] M. Caubergh, F. Dumortier and S. Luca, *Cyclicity of unbounded semi-hyperbolic 2-saddle cycles in polynomial Liénard systems*, Discrete and Continuous Dynamical Systems **27(3)** (2010), 963-980.

[4] W. A. Coppel, *Some Quadratic Systems with at Most One Limit Cycle*, Dynamics Reported **2** (1988), Wiley, New York, 61–68.

[5] F. Dumortier and C. Li, *On the uniqueness of limit cycles surrounding one or more singularities for Liénard equations*, Nonlinearity **9** (1996), 1489–1500.

[6] F. Dumortier and C. Li, *Quadratic Liénard Equations with Quadratic Damping*, Journal of Differential Equations **139 (1)** (1997), 41–59.

[7] F. Dumortier, D. Panazzolo and R. Roussarie, *More limit cycles than expected in Liénard equations*, Proceedings of the American Mathematical Society **135** (2007), 1895–1904.

[8] F. Dumortier and R. Roussarie, *Birth of canard cycles*, Discrete and Continuous Dynamical Systems **A 2(4)** (2009) 723–781.

[9] F. Dumortier and C. Rousseau, *Cubic Liénard equations with linear damping*, Nonlinearity **3** (1990), 1015–1039.

[10] Yu. S. Ilyashenko, *Centennial history of Hilbert's 16th Problem*, Bulletin of the American Mathematical Society (N.S.) **39** (2002), 301–354.

[11] N.G. Lloyd and S. Lynch, *Small-amplitude limit cycles of certain Liénard systems*, Proceedings of the Royal Society London **418** (1988), 199–208.

[12] A. Lins, W. de Melo and C.C. Pugh, *On Liénard's equation*, Lecture Notes in Mathematics **597** (1997), 335-357, Springer, Berlin.

[13] R. Roussarie, *Bifurcations of Planar Vector Fields and Hilbert's Sixteenth Problem*, Progress in Mathematics **164** (1998), Birkhauser-Verlag, Basel.

[14] R. Roussarie, *Putting a boundary to the space of Liénard Equations*, Discrete and Continuous Dynamical Systems **A 17 (2)** (2007), 441-448.

[15] S. Smale, *Mathematical Problems for the Next Century*, Mathematical Intelligencer **20 (2)** (1998), 7–15.

Centers and Limit Cycles

Colin Christopher

In this talk I will give a general background to the center-focus problem, and then to show why the problem is interesting: both in what it tells us about the distinctive algebraic features of polynomial vector fields, and also in the simple concrete estimates it gives of the number of limit cycles which can exist in these vector fields.

In more detail, we consider systems of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are polynomials with a critical point at the origin which is either a center or a focus. Such systems can be brought to the form

$$\dot{x} = -y + \lambda x + p(x, y), \qquad \dot{y} = x + \lambda y + q(x, y). \tag{2}$$

In general, although the linear terms of (2) give a center when $\lambda = 0$, the system will not have a center at the origin, and a number of limit cycles will bifurcate from the origin as λ is perturbed away from zero in a (multiple) Andronov-Hopf bifurcation.

To see this, we choose a one-sided analytic transversal at the origin with a local analytic parameter c, and represent the return map by an expansion

$$c \mapsto h(c) = c + \sum_{i=1}^{\infty} \alpha_i \, c^i, \tag{3}$$

which turns out to be analytic in c and also in the parameters of the system.

The stability of the origin is clearly given by the sign of the first non-zero α_i, and we have a center if and only if all the α_i are zero. Because the transversal is crossed twice by every trajectory, the α_{2k} must vanish as soon as the previous α_i, $i < 2k$ vanish.

If α_{2k+1} is the first non-zero term, then at most k limit cycles can bifurcate from the origin. Provided we have sufficient choice in the coefficients α_i, we can also obtain that many limit cycles in a simultaneous bifurcation from the critical point. We call the α_{2i+1} the *Lyapunov quantities* of the critical point, and denote them $L(i)$.

When $\lambda = 0$, the $L(i)$ turn out to be polynomials in the parameters of the system. By the Hilbert basis theorem, the vanishing of all the $L(i)$ must be equivalent to the vanishing of the first N of them, for some integer N. Thus the set of points where we have a center must be an algebraic set, which we call the *center variety*.

Clearly, the closer that the origin is to being a center, the more limit cycles we can obtain, and the more information we get about the general configurations of limit cycles in polynomial systems (Hilbert's 16th Problem).

Unfortunately, although computations of the $L(i)$ are not difficult, analyzing their common roots is computationally intractable except for all but the simplest families of systems. Worse, there is no a-priori bound on the value of N we need to take, so that for each potential set of conditions for a center, we need to check that these conditions do indeed give rise to a center.

The *center-focus problem* asks for the criteria which determine whether a critical point whose linear parts give a center, really is a center.

In contrast to the situation above, the mechanisms which give centers seem to fit into just two nice patterns. One is the existence of a first integral or an integrating factor of the form

$$e^{g/h} \prod f_i^{l_i},$$

where f_i, f and g are polynomials, and $f_i = 0$ and $h = 0$ define invariant algebraic curves of the system (2). We call such a function a *Darboux* function.

The second is the existence of an algebraic symmetry, that is a map $(x, y) \mapsto (X(x, y), Y(x, y))$, where X and Y are algebraic functions of x and y, which keeps the system fixed but "reverses" time.

Both these mechanisms clearly have important global algebraic consequences for the systems which exhibit them, and demonstrates the fascination of the center-focus problem as an extreme local to global principle.

Since the center types seem to be better known, it would make sense, therefore, to work directly from families of centers.

In this talk I will give examples of how perturbing systems from centers can indeed push the bounds of our knowledge of limit cycle configurations, and also will give some indications about our current state of knowledge on the center-focus problem.

Further details can be found in [1] and the first part of the book [2]

References

[1] Colin Christopher *Estimating limit cycle bifurcations from centers*, in Differential Equations with Symbolic Computation (ed. D. Wang and Z. Zheng), Springer (2005), 23–36.

[2] Colin Christopher and Chengzhi Li, *Limit cycles of differential equations.* Advanced Courses in Mathematics. CRM Barcelona. Birkhuser (2007).

Abelian Integrals and Limit Cycles

Chengzhi Li

Consider the planar differential systems

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \tag{1}$$

where P_n and Q_n are real polynomials of degree at most $n \geq 2$. The second half of the famous Hilbert's 16th problem is asking for the maximum number of limit cycles of system (1), denoted by $H(n)$, for all P_n and Q_n, and asking for possible relative positions of the limit cycles. A *limit cycle* of system (1) is an isolated closed orbit. For *a given system* (1) the number of limit cycles is finite (Dulac-Ilyashenko-Écalle), but the original Hilbert's 16th problem is still open even for the case $n = 2$, and there is no answer if $H(2)$ is finite or not. About the relative positions of limit cycles, J. Llibre and G. Rodríguez proved that any configuration of limit cycles is realizable by a polynomial system of certain degree.

Now we consider a perturbation system $X_{H,\varepsilon}$ of a Hamiltonian system X_H:

$$\frac{dx}{dt} = -\frac{\partial H(x,y)}{\partial y} + \varepsilon f(x,y), \quad \frac{dy}{dt} = \frac{\partial H(x,y)}{\partial x} + \varepsilon g(x,y), \tag{2}$$

where H, f and g are polynomials of degree m, n and n, and ε is a small parameter.

Suppose that the level curves of X_H contain a family of ovals $\{\gamma_h\}$, filling up an annulus for $h \in (a,b)$. Then we may define the Abelian integral

$$I(h) = \oint_{\gamma_h} f(x,y)dy - g(x,y)dx. \tag{3}$$

A natural question is: How many periodic orbits of X_H keep being unbroken and become the periodic orbits of the perturbed system (2) for small ε ?

By Poincaré-Pontryagin Theorem, we can answer this question as follows (see Theorem 2.4 of part II in [2], for example). If $I(h)$ is not identically zero for $h \in (a,b)$, then (i) a necessary condition of a limit cycle bifurcating from γ_h is $I(h) = 0$; (ii) if h is a simple zero of $I(h)$, then a hyperbolic limit cycle can bifurcate from γ_h; (iii) if $h \in (a,b)$ is a root of $I(h)$ of multiplicity k, then at most k limit cycles can bifurcate from γ_h; and (iv) the total number of limit cycles bifurcating from the annulus $\{\gamma_h : h \in (a,b)\}$ is bounded by the maximum number of isolated zeros (taking into account their multiplicities) of $I(h)$ for $h \in (a,b)$.

In general, V. I. Arnold repeatedly proposed the following problem:

For fixed integers m and n find the maximum $Z(m,n)$ of the numbers of isolated zeros of the Abelian integrals (3).

If take $m = n+1$, then system (2) is a special form of system (1), close to a Hamiltonian system X_H. In this sense the above problem usually is called the *weak (or tangential, infinitesimal)* Hilbert's 16th problem, and the number $\tilde{Z}(n) = Z(n+1,n)$ can be chosen as a lower bound of the Hilbert number $H(n)$.

If the unperturbed system is integrable but non-Hamiltonian, one has to use a integrating factor, say $\mu(x,y) = 1/R(x,y)$, and the perturbed system can be written in the form

$$\dot{x} = -\frac{\partial F(x,y)}{\partial y}R(x,y) + \varepsilon f(x,y), \quad \dot{y} = \frac{\partial F(x,y)}{\partial x}R(x,y) + \varepsilon g(x,y), \tag{4}$$

and associated to it we define the (generalized, or pseudo) Abelian integral

$$I(h) = \oint_{\gamma_h} \mu(x,y)\,(f(x,y)dy - g(x,y))dx, \tag{5}$$

where $\{\gamma_h\}$ are the family of ovals contained in the level curves $\{F(x,y) = h\}$. Since $\mu(x,y)$, in general, is not a rational function, the study of the number of zeros of (5) is more difficult than the study for (3).

If $I(h) \equiv 0$, one has to consider higher order approximation, an algorithm to compute higher order Abelian integrals was given in 1996 by J.-P. Françoise.

A. N. Varchenko and A. G. Khovanskii proved in 1984 that for given m and n the number $Z(m,n)$ is uniformly bounded. But to find an explicit expression for

$Z(m,n)$, even to find an explicit bound to $Z(m,n)$, is very hard, see a recent paper by G. Binyamini, D. Novikov and S. Yakovenko [1]. There are many works dealing with restrictions on H or on the class of f and g. We list some of them below.

- It is natural to think about to find $\tilde{Z}(n) = Z(n+1,n)$ for lower n, and this was done by several authors, and succeeded only for $n = 2$ (1993 to 2002).
- For elliptic Hamiltonian $H = y^2 + P_k(x)$, where P_k is a polynomial of degree k, there is a series of works to study the number of zeros of $I(h)$ for different classes of perturbations.
- F. Takens and V. I. Arnold proposed the $1 : q$ resonance problem. Except for the case $q = 4$, the codimension 2 cases have been completed solved, and codimension ≥ 3 cases are partially solved. The study is related to Abelian integrals.
- The study of quadratic perturbation of quadratic integrable and non-Hamiltonian systems was done for some special classes.
- Many authors studied this problem for certain H under perturbations f and g, belonging to some function classes. H, f and g are not necessarily polynomials.

There are some methods to study the number of zeros of Abelian integrals: the method based on the Picard-Fuchs equation; the method based on the Argument Principle, the averaging method, the method of using Chebyshev property, and the method based on complexification of the Abelian integrals.

Similar to Hilbert's 16th problem, its weak form is still far from completely solved. Some new methods, new approaches, and new techniques need to be developed.

All above mentioned references can be found in the second part of [2], except the recent paper [1].

References

[1] G. Binyamini, D. Novikov and S. Yakovenko, *On the number of zeros of Abelian integrals*, Invent. Math. **181** (2010), 227–289.

[2] C. Christopher and C. Li, *Limit Cycles of Differential Equations*, Birkhäuser Verlag, 2007, Basel-Boston-Berlin.

Darboux theory of integrability

Xiang Zhang
(joint work with Jaume Llibre)

The algebraic theory of integrability is a classical one, which is related with the first part of the Hilbert's 16th problem. This kind of integrability is usually called Darboux integrability, which provides a link between the integrability of polynomial vector fields and the number of invariant algebraic hypersurfaces that they have. In this talk we shall study the existence of a first integral for polynomial vector fields in $\mathbb{R}^n$ or $\mathbb{C}^n$ with $n \geq 2$ via the Darboux theory of integrability.

Consider the *polynomial vector fields* in $\mathbb{C}^n$

$$\mathcal{X} = \sum_{i=1}^{n} P_i(x)\frac{\partial}{\partial x_i}, \qquad x = (x_1, \ldots, x_n) \in C^n,$$

where $P_i = P_i(x) \in \mathbb{C}[x]$ have no common factor for $i = 1, \ldots, n$, and $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ the ring of all complex polynomials in the variables $x_1, \ldots, x_n$. The integer $d = \max\{\deg P_1, \ldots, \deg P_n\}$ is the *degree* of the vector field $\mathcal{X}$.

Let $f = f(x) \in \mathbb{C}[x]$. We say that $\{f = 0\} \subset \mathbb{C}^n$ is an *invariant algebraic hypersurface* of $\mathcal{X}$ if there exists a polynomial $K_f \in \mathbb{C}[x]$ such that $\mathcal{X}(f) = fK_f$.

If $f, g \in \mathbb{C}[x]$ are coprime, we say that $\exp(g/f)$ is an *exponential factor* of $\mathcal{X}$ if there exists a polynomial $L_e \in \mathbb{C}[x]$ of degree at most $d - 1$ such that $\mathcal{X}(\exp(g/f)) = \exp(g/f)L_e$.

Let $\mathbb{C}_m[x]$ be the $\mathbb{C}$–vector space of polynomials in $\mathbb{C}[x]$ of degree at most m. Then it has dimension $R = \begin{pmatrix} n+m \\ n \end{pmatrix}$. Let $v_1, \ldots, v_R$ be a base of $\mathbb{C}_m[x]$. Denote by M_R the $R \times R$ matrix

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_R \\ \mathcal{X}(v_1) & \mathcal{X}(v_2) & \cdots & \mathcal{X}(v_R) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}^{R-1}(v_1) & \mathcal{X}^{R-1}(v_2) & \cdots & \mathcal{X}^{R-1}(v_R) \end{pmatrix}, \tag{1}$$

where $\mathcal{X}^{k+1}(v_i) = \mathcal{X}(\mathcal{X}^k(v_i))$. An irreducible invariant algebraic hypersurface $f = 0$ of degree m has *algebraic multiplicity* k if $\det M_R \not\equiv 0$ and k is the maximum positive integer such that f^k divides $\det M_R$; and it has *no defined algebraic multiplicity* if $\det M_R \equiv 0$.

A *Darboux first integral* is a first integral of the form

$$\left(\prod_{i=1}^{r} f_i^{l_i}\right)\exp(g/h),$$

where f_i, g and h are polynomials, and the l_i's are complex numbers.

The classical Darboux theory of integrability in $\mathbb{C}^n$ with $n \geq 2$ is the following.

Theorem A *Assume that the polynomial vector field $\mathcal{X}$ in $\mathbb{C}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces $f_i = 0$ for $i = 1, \ldots, p$. Then the following statements hold.*

(*a*) *If $p \geq N + 1$, then the vector field $\mathcal{X}$ has a Darboux first integral, where* $N = \begin{pmatrix} n+d-1 \\ n \end{pmatrix}$.

(*b*) *If $p \geq N + n$, then the vector field $\mathcal{X}$ has a rational first integral.*

Statement (*a*) of Theorem A is due to Darboux [3, 4]. Statement (*b*) of Theorem A was proved by Jouanolou [5] using tools of algebraic geometry, and also has an elementary proof for dimension 2 given by Christopher and Llibre [1] in 2000 and for arbitrary dimension given by Llibre and Zhang [8] in 2010.

The next results improve Theorem A taking into account the exponential factors and tell us how to construct first integrals using Darboux theory of integrability.

Theorem B *Suppose that a polynomial vector field $\mathcal{X}$ of degree d in $\mathbb{C}^n$ admits p irreducible invariant algebraic hypersurfaces $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$ and q exponential factors* $\exp(g_j/h_j)$ *with cofactors L_j for $j = 1, \ldots, q$.*

(i) *There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$, if and only if the (multi–valued) function*

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \ldots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} \tag{2}$$

is a first integral of $\mathcal{X}$.

(ii) *If $p + q \geq N + 1$, then the vector field $\mathcal{X}$ has a Darboux first integral.*

(iii) *If $p + q \geq N + n$, then the vector field $\mathcal{X}$ has a rational first integral.*

(iv) *For $n = 2$, there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\mathrm{div}(P, Q)$ if and only if function (2) is an integrating factor of $\mathcal{X}$.*

The following theorem improves Theorem A taking into account not only the invariant algebraic hypersurfaces but also their algebraic multiplicities.

Theorem C *Assume that the polynomial vector field $\mathcal{X}$ in $\mathbb{C}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces.*

(*i*) *If some of these irreducible invariant algebraic hypersurfaces has no defined algebraic multiplicity, then the vector field $\mathcal{X}$ has a rational first integral.*

(*ii*) *Suppose that all the irreducible invariant algebraic hypersurfaces $f_i = 0$ has defined algebraic multiplicity q_i for $i = 1, \ldots, p$. If $\mathcal{X}$ restricted to each hypersurface $f_i = 0$ having multiplicity larger than 1 has no rational first integral, then the following statements hold.*

  (*a*) *If $\sum_{i=1}^{p} q_i \geq N + 1$, then the vector field $\mathcal{X}$ has a Darboux first integral, where N is the number defined in Theorem A.*

  (*b*) *If $\sum_{i=1}^{p} q_i \geq N + n$, then the vector field $\mathcal{X}$ has a rational first integral.*

Statement (*i*) follows from Theorem 3 of Pereira [9] (see also Theorem 5.3 of [2] for dimension 2). Statement (*ii*) was proved by Llibre and Zhang in [6], where we showed by examples that the additional condition is necessary

The following result, due to Llibre and Zhang [7], improves Theorem A in $\mathbb{R}^n$ taking into account the algebraic multiplicity of the hyperplane at infinity.

Theorem D *Assume that the polynomial vector field $\mathcal{X}$ in $\mathbb{R}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces $f_i = 0$ for $i = 1, \ldots, p$ and the invariant hyperplane at infinity.*

(*i*) *If some of these irreducible invariant algebraic hypersurfaces or the invariant hyperplane at infinity has no defined algebraic multiplicity, then the vector field $\mathcal{X}$ has a rational first integral.*

(*ii*) *Suppose that all the irreducible invariant algebraic hypersurfaces $f_i = 0$ have defined algebraic multiplicity q_i for $i = 1, \dots, p$ and that the invariant hyperplane at infinity has algebraic multiplicity k. If the vector field restricted to each invariant hypersurface including the hyperplane at infinity having algebraic multiplicity larger than 1 has no rational first integral, then the following hold.*

(*a*) *If $\sum_{i=1}^{p} q_i + k \geq N + 2$, then the vector field $\mathcal{X}$ has a Darboux first integral, where N is defined in Theorem A.*

(*b*) *If $\sum_{i=1}^{p} q_i + k \geq N + n + 1$, then the vector field $\mathcal{X}$ has a rational first integral.*

The Darboux theory of integrability has been successfully applied to the study of some physical models, of the center–focus problem and so on.

References

[1] C. Christopher and J. Llibre, *Integrability via invariant algebraic curves for planar polynomial differential systems*, Annals of Differential Equations **16** (2000), 5–19.

[2] C. Christopher, J. Llibre and J.V. Pereira, *Multiplicity of invariant algebraic curves in polynomial vector fields*, Pacific J. Math. **229** (2007), 63–117.

[3] G. Darboux, *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré (Mélanges)*, Bull. Sci. Math. 2ème série **2** (1878), 60–96; 123–144; 151–200.

[4] G. Darboux, *De l'emploi des solutions particulières algébriques dans l'intégration des systèmes d'équations différentielles algébriques*, C. R. Math. Acad. Sci. Paris **86** (1878), 1012–1014.

[5] J.P. Jouanolou, *Equations de Pfaff algébriques*, in Lectures Notes in Mathematics **708**, Springer–Verlag, New York/Berlin, 1979.

[6] J. Llibre and X. Zhang, *Darboux Theory of Integrability in $\mathbb{C}^n$ taking into account the multiplicity*, J. of Differential Equations **246** (2009), 541–551.

[7] J. Llibre and X. Zhang, *Darboux theory of integrability for polynomial vector fields in $\mathbb{R}^n$ taking into account the multiplicity at infinity*, Bull. Sci. Math. **133** (2009), 765–778.

[8] J. Llibre and X. Zhang, *Rational first integrals in the Darboux theory of integrability in $\mathbb{C}^n$*, Bull. Sci. Math. **134** (2010), 189–195.

[9] J.V. Pereira, *Vector fields, invariant varieties and linear systems*, Annales de l'institut Fourier **51** (2001), 1385-1405.

Inverse Problems of the Darboux Theory of Integrability

Chara Pantazi

(joint work with C. Christopher, J. Llibre, S. Walcher)

Introduction. We consider the planar (complex in general) polynomial differential system

$$\dot{x} = P(x, y), \ \dot{y} = Q(x, y) \tag{1}$$

and its associated vector field $X = P\partial/\partial x + Q\partial/\partial y$. We are interested in the following *Problems*: (1) Find all invariant algebraic curves of vector field X. (2) Decide whether Darboux integrating factor exists. These problems are very important because: (i) They are classical problems (Poincaré and later Darboux), (ii) They are related to a question due to Prelle and Singer (1983) [8, 9]: For planar ODE, does it exist an elementary (Liouville) first integral? (iii) There is a direct connection to qualitative properties of system (1). Note that these problems are directly connected to the famous Poincaré problem about the degree of invariant curves, [1, 2, 12].

Since these problems are difficult [10] we deal mainly with the following *Inverse Problems*: *(1) Inverse problem for curves:* We consider irreducible pairwise relatively prime polynomials

$$f_1, \ldots, f_r, \text{ and } f := f_1 \cdots f_r$$

Find all the polynomial vector fields that have all $f_i = 0$ invariant.
(2) Inverse problem for integrating factors: We furthermore consider nonzero complex constants $d_1, \cdots, d_r$. Find all the polynomial vector fields with a given Darboux integrating factor of the form $R = f_1^{-d_1} \cdots f_r^{-d_r}$. We are interested in investigating these inverse problems because: (i) They provide a better understanding of obstacles to elementary (Liouville) integrability, (ii) they are necessary for characterization (classification), in particular for integrating factor case, (iii) they allow the construction of vector fields with special properties.

The inverse problem for curves. We consider the irreducible pairwise relatively prime polynomials

$$f_1, \ldots, f_r, \text{ and we denote by } f = f_1 \cdots f_r.$$

Vector fields of type

$$X = \widetilde{X} \cdot f + a \cdot X_f, \qquad \mathcal{V}^0$$

where $\widetilde{X}$ is an arbitrary polynomial vector field and a is an arbitrary polynomial. Moreover, this type of vector fields (we call them the 'trivial' ones), forms a subspace of $\mathcal{V}$ which will be denoted by $\mathcal{V}^0$. Vector fields of type

$$X = \widetilde{X} \cdot f + \sum_i a_i \frac{f}{f_i} \cdot X_{f_i}, \qquad \mathcal{V}^1$$

where $\widetilde{X}$ is an arbitrary polynomial vector field and a_i are arbitrary polynomials, form a subspace of $\mathcal{V}$ which will be denoted by $\mathcal{V}^1$. In general, $\mathcal{V}^0 \subseteq \mathcal{V}^1 \subseteq \mathcal{V}$.
Note that for a given f the cofactors L of all vector fields admitting $f = 0$ form an ideal and a necessary and sufficient condition for the existence of $X = (P, Q)$ is

$$L \in \langle f_x, f_y \rangle : \langle f \rangle \,. \tag{2}$$

A polynomial vector field X satisfies equation (2) with $L \in \langle f_x, f_y \rangle$ if and only if $X \in \mathcal{V}^0$. Additionally, the map sending vector field to cofactor induces an isomorphism (see also [11, 4])

$$\mathcal{V}/\mathcal{V}^0 \cong (\langle f_x,\, f_y\rangle : \langle f\rangle) \,/\, \langle f_x,\, f_y\rangle \,.$$

of finite dimensional vector spaces.

We introduce two generic nondegeneracy conditions (see also [6]):
(ND1) Each $f_i = 0$ is nonsingular.
(ND2) All singular points of $f = 0$ have multiplicity one (thus when two irreducible components intersect, they intersect transversally, and no more than two irreducible components intersect at one point).
Note that if conditions (ND1)and (ND2) hold then $\mathcal{V} = \mathcal{V}^1$.

We also have the following result.
Theorem *The dimension of $\mathcal{V}_f/\mathcal{V}_f^0 \geq$ # of singular points of $f = 0$. The equality holds if every singular point of $f = 0$ has multiplicity one. In particular, if $f = 0$ has no singular points then $\mathcal{V}_f = \mathcal{V}_f^0$.*

An algorithmic approach Here we present an algorithm to find vector fields that are not trivial [4].

- Find the Gröbner basis G of $\langle f_x,\, f_y\rangle$ with respect to some fixed monomial ordering.
- Only finitely many monomials $m_1, \ldots, m_d$ are not multiples of some leading monomial in G.
- The classes $m_i + \langle f_x,\, f_y\rangle$ form a basis of $\mathbb{C}[x, y]/\langle f_x,\, f_y\rangle$.
- Obtain cofactors from kernel of the map

$$M_f\, :\, g + \langle f_x,\, f_y\rangle \mapsto f \cdot g + \langle f_x,\, f_y\rangle \,.$$

- Vector fields are obtained, in principle, from defining equation and Gröbner.

Inverse problem for integrating factors. Let $f_1, \cdots, f_r$ be irreducible and pairwise coprime polynomials. We denote by $f = f_1 \cdots f_r$, and we consider the nonzero complex constants $d_1, \cdots, d_r$.

The vector fields X having a Darboux integrating factor of the form

$$R = \left(f_1^{d_1} \cdots f_r^{d_r}\right)^{-1}$$

satisfy the relation $-d_1L_1 - \cdots - d_rL_r + div(X) = 0$. Additionally, such vector fields form a linear space which will be denoted by $\mathcal{F} = \mathcal{F}(d_1, \ldots, d_r)$ (and obviously $\mathcal{F} \subseteq \mathcal{V}$).

We note that for a given arbitrary polynomial g, we can consider the vector field (in general, not polynomial) $Z_g = Z_g^{(d_1,\ldots,d_r)}$, see also [3]. Note that Z_g is the Hamiltonian vector field of the function $g/\left(f_1^{d_1-1} \cdots f_r^{d_r-1}\right)$. Since we are interested in polynomial vector fields we consider the vector field

$$f_1^{d_1} \cdots f_r^{d_r} \cdot Z_g = f \cdot X_g - \sum_{i=1}^{r} (d_i - 1) g \frac{f}{f_i} \cdot X_{f_i}, \qquad \mathcal{F}^0$$

and note that such vector fields admit the integrating factor $(f_1^{d_1} \cdots f_r^{d_r})^{-1}$. Vector fields of this type form a subspace $\mathcal{F}^0 = \mathcal{F}^0(d_1, \cdots, d_r)$ of $\mathcal{F}$ and we also call them 'trivial'. Obviously, $\mathcal{F}^0 \subseteq \mathcal{F} \subseteq \mathcal{V}_f$.

Remark *In order to investigate the structure of the quotient space*

$$\mathcal{F}(d_1, \ldots, d_r)/\mathcal{F}_f^0(d_1, \ldots, d_r)$$

one may replace d_i by 1 if d_i is a positive integer, and by $d_i - k_i$ with any positive integer k_i otherwise. In particular we may assume that each d_i has real part ≤ 1.

Additionally, we may use the concept of morphisms in order to 'break' some singularities and to obtain the following result of finiteness dimension, [7].

Theorem *One always has*

$$\dim \big(\mathcal{F}(d_1, \ldots, d_r)/\mathcal{F}^0(d_1, \ldots, d_r)\big) < \infty.$$

References

[1] M. Carnicer, *The Poincaré problem in the nondicritical case.* Ann. Math. **140** (1994), 289–294.

[2] D. Cerveau, A. Lins Neto, *Holomorphic foliations in $CP(2)$ having an invariant algebraic curve.* Ann. Inst. Fourier **41** (1991), 883–903.

[3] C. Christopher, J. Llibre, C. Pantazi, S. Walcher, *Inverse problems for multiple invariant curves.* Proc. Roy. Soc. Edinburgh **137A**, (2007) 1197 - 1226.

[4] C. Christopher, J. Llibre, C. Pantazi and S. Walcher, *Inverse problems for invariant algebraic curves: Explicit computations.* Proc. Roy. Soc. Edinburgh **139A**, (2009), 1 - 16.

[5] C. Christopher, J. Llibre, C. Pantazi, S. Walcher, *Darboux integrating factors: Inverse problems.* J. Differential Eqs, **250** (1) (2011), 1–25.

[6] C. Christopher, J. Llibre, C. Pantazi, X. Zhang: *Darboux integrability and invariant algebraic curves for planar polynomial systems.* J. Phys. A **35**, (2002), 2457–2476.

[7] J. Llibre, C. Pantazi, S. Walcher, *Morphisms and inverse problems for Darboux integrating factors.* Preprint, 15 pp. (2008).

[8] M. J. Prelle and M. F. Singer, *Elementary first integrals of differential equations.* Trans. Amer. Math. Soc. **279** (1983), 613–636.

[9] M.F. Singer, *Liouvillian first integrals of differential equations.* Trans. Amer. Math. Soc. **333** (1992), 673–688.

[10] D. Schlomiuk, *Elementary first integrals and algebraic invariant curves of differential equations.* Expo. Math. **11** (1993), 433–454.

[11] S. Walcher, *Plane polynomial vector fields with prescribed invariant curves.* Proc. Royal Soc. Edinburgh **130A** (2000), 633 - 649.

[12] S. Walcher, *On the Poincaré problem.* J. Differential Eqs. **166** (2000), 51–78.

Reporter: Julia Hartmann

Participants

Dr. Mohamed Barakat
Fachbereich Mathematik
T.U. Kaiserslautern
Erwin-Schrödinger-Straße
67653 Kaiserslautern

Prof. Dr. Magdalena Caubergh
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus Universitari
E-08193 Bellaterra

Prof. Dr. Colin Jeffrey Christopher
School of Computing & Mathematics
University of Plymouth
GB-Plymouth, Devon PL4 8AA

Prof. Dr. Anna Cima
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus Universitari
E-08193 Bellaterra

Prof. Dr. Harm Derksen
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor , MI 48109-1043
USA

Prof. Dr. Jan Draisma
Department of Mathematics
Eindhoven University of Technology
P.O.Box 513
NL-5600 MB Eindhoven

Prof. Dr. Jean-Pierre Francoise
Laboratoire Jacques-Louis Lions
Universite Paris 6
4, Place Jussieu
F-75252 Paris Cedex 05

Prof. Dr. Armengol Gasull
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus Universitari
E-08193 Bellaterra

Dr. Julia Hartmann
Lehrstuhl A für Mathematik
RWTH Aachen
Templergraben 55
52062 Aachen

Prof. Dr. Gregor Kemper
Zentrum Mathematik
TU München
Boltzmannstr. 3
85748 Garching b. München

Prof. Dr. Chengzhi Li
Department of Mathematics
Peking University
100871 Beijing
P.R.CHINA

Prof. Dr. Jaume Llibre
Departament de Matematiques
Universitat Autonoma de Barcelona
Campus Universitari
E-08193 Bellaterra

Dr. Chara Pantazi
Departament de Matematica Aplicada I
EPSEB, Edifici P
Universita Politecnica de Catalunya
Av. del Doctor Maranon, 44-50
E-08028 Barcelona

Camilo Sanabria
Department of Mathematics & Computer Sc.
Bronx Community College
CP 315
2155 University Avenue
Bronx , NY 10453
USA

Prof. Dr. Sebastian Walcher
Lehrstuhl A für Mathematik
RWTH Aachen
Templergraben 55
52062 Aachen

Prof. Dr. Eva Zerz
Lehrstuhl D für Mathematik
RWTH Aachen
Templergraben 64
52062 Aachen

Prof. Dr. Xiang Zhang
Shanghai Jiaotong University
Department of Mathematics
200240 Shanghai
P.R.CHINA