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Combinatorics

Organised by
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Abstract. Combinatorics is a fundamental mathematical discipline which focuses on the study of discrete objects and their properties. The current workshop brought together researchers from diverse fields such as Extremal and Probabilistic Combinatorics, Discrete Geometry, Graph theory, Combiantorial Optimization and Algebraic Combinatorics for a fruitful interaction. New results, methods and developments and future challenges were discussed. This is a report on the meeting containing abstracts of the presentations and a summary of the problem session.

Mathematics Subject Classification (2000): 05Cxx, 05Dxx, 52Bxx, 68Rxx.

Introduction by the Organisers

The workshop Combinatorics organised by Jeff Kahn (Piscataway), Angelika Steger (Zürich), and Benny Sudakov (Los Angeles) was held January 2nd - January 8th, 2011. Despite the early point in the year the meeting was extremely well attended with 52 participants from the US, Canada, Brazil, UK, Israel, and various European countries. The program consisted of 13 plenary lectures, accompanied by 19 shorter contributions and a vivid problem session led by Vera T. Sós. The plenary lectures were intended to provide overviews of the state of the art in various combinatorial areas and/or in-depth treatments of major new results. The short talks ranged over a wide variety of topics including in graph theory, coding theory, discrete geometry, extremal combinatorics, Ramsey theory, theoretical computer science, and probabilistic combinatorics. Special attention was paid throughout to providing a platform for younger researchers to present themselves and their results.

This report contains extended abstracts of the talks and the statements of the problems that were posed during the problem session. This was a particularly successful edition of the meeting Combinatorics, in large part because of the exceptional strength and range of the results discussed. Here we mention just a few of these, each of which involved spectacular progress on some well-known, longstanding conjecture. These few snapshots also provide a nice, if small, sample of the large variety of topics and methodologies that were presented during a fascinating week in Oberwolfach.

A family of graphs $\mathcal{F}$ is triangle-intersecting if for every $G, H \in \mathcal{F}$, the intersection $G \cap H$ contains a triangle. A celebrated conjecture of Simonovits and Sós from 1976 states that the largest triangle-intersecting families of graphs on a fixed set of n vertices are those obtained by fixing a specific triangle and taking all graphs containing it. This conjecture was recently proved by Ellis, Filmus, and Friedgut using spectral methods and discrete Fourier analysis; see the abstract of Ehud Friedgut's talk.

For a graph G, write $\Phi(G)$ for the number of perfect matchings of G. The Lovász-Plummer Conjecture (proposed in the 1970's) says that for bridgeless, cubic graphs G, $\Phi(G)$ grows exponentially in the number (say n) of vertices of G — this despite the fact that until two years ago it was not even known that the number must be more than *linear* in n. Not long before the Oberwolfach meeting, the Lovász-Plummer Conjecture was proved in full by Esperet, Kardoš, King, Král, and Norine, using a combination of ideas from graph theory and linear programming; see the Daniel Král's abstract.

A tournament is an orientation of a complete graph. Sumner's Universal Tournament Conjecture of 1971 says that any tournament on $2n - 2$ vertices contains every directed n-vertex tree. Following a long history of partial results, Sumner's conjecture has now been proved in full, assuming only that n is large, by Kühn, Mycroft and Osthus, using, *inter alia*, probabilistic ideas and a variant of Szemerédi's Regularity Lemma; see the abstract of Deryk Osthus.

A subject that's received a great deal of attention over the last decade or two concerns questions of the type: given a (large) finite set Γ and family $\mathcal{F}$ of subsets of Γ that can be shown to exhibit some structural property of interest, about how large a random subset of Γ will (probably) continue to exhibit said property? The last few years have seen breakthrough progress on such questions by Mathias Schacht (see his abstract) and (independently and using very different methods) Conlon and Gowers. These results provide very general information on questions of the above type, including, to give just two examples, the threshold for Szemerédi's (arithmetic progressions) Theorem to hold in random subsets of the integers, and verification of a much-studied conjecture of Kohayakawa, Łuczak, and Rödl (1997) on Turán-type problems in random graphs.

On behalf of all participants, the organisers would like to thank the staff and the director of the Mathematisches Forschungsinstitut Oberwolfach for providing such a stimulating and inspiring atmosphere.

Workshop: Combinatorics

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Abstracts

General questions about extremal graphs

László Lovász

Many questions in extremal graph theory can be phrased like this: what is the maximum of a certain linear combination of densities of given graphs $F_1, \dots, F_k$ in any graph G? Perhaps we have contraints on G, also in the form of fixing certain linear combinations of densities of $F_1, \dots, F_k$ in G. Over 60-70 years, a lot of questions of this type have been posed and many have been answered. The answers are often quite difficult. For example, the minimum density of triangles, subject to fixing the edge density, was only rather recently determined by Razborov [13], and even more recently extended from triangles to K_4-s by Nikiforov [12] and to all complete graphs by Reiher.

It is now possible to pose and in some cases answer some general questions about extremal graphs. In a previous talk in Oberwolfach, I posed some of these questions. I will add a new one, and report about substantial progress concerning them.

Besides the wealth of previous results, these general studies in extremal graph theory were made possible by the theory of graph limits. For two simple graphs F and G, let $t(F, G)$ denote the density of F is G, defined as the probability that a random map $V(F) \to V(G)$ preserves edges. A sequence $G_1, G_2, \dots$ of simple graphs is called *convergent*, if $|V(G_n)| \to \infty$ and $t(F, G_n)$ has a limit for every fixed F as $n \to \infty$ (see [1, 2]).

It was proved in [7] that every convergent sequence has a limit object in the form of a symmetric measurable function $W: [0,1]^2 \to [0,1]$. This represents the limit in the sense that

$$t(F, G_n) \to t(F, W) := \int_{[0,1]^V} \prod_{ij \in E} W(x_i, x_j) \prod_{i \in V} dx_i \qquad (n \to \infty).$$

These limit functions are called *graphons.*

Here are some general questions about extremal graphs.

1. *Which inequalities between subgraph densities are valid?* To formalize this question, we consider an inequality

$$\sum_{i=1}^{n} a_i t(F_i, G) \geq 0$$

between subgraph densities, where $F_1, \dots, F_n$ are fixed graphs, and the a_i are arbitrary real (or, in an algorithmic setting, rational) coefficients, and ask: is it valid for all graphs G? Hatami and Norine [4] very recently proved that this question is undecidable. A key ingredient in their proof is the complexity of the triangle density vs. edge density problem mentioned above. On the other hand, it follows from the results of Lovász and Szegedy [10] that if we allow an arbitrarily small “slack”, then it becomes decidable.

2. *Can all valid inequalities be proved using just Cauchy-Schwarz?* Many proofs in extremal graph theory uses the Cauchy–Schwarz inequality (often repeatedly and in nontrivial ways). How general a tool is the Cauchy–Schwarz inequality in this context? Using the notions of graphons and graph algebras one can give an exact formulation of this question, which turns out to be analogous to Hilbert's 17th Problem about representing nonnegative polynomials as sums of squares. It turns out that the answer is negative (Hatami and Norine [4]), but it becomes positive if we allow an arbitrarily small error (Lovász and Szegedy [10]).

3. *Is there always an extremal graph?* Let us consider an optimization problem of the type:

$$\begin{aligned} \text{maximize } & \sum_{i=1}^{n} c_i t(F_i, G) \\ \text{subject to} & \sum_{i=1}^{n} a_i t(F_i, G) \geq 0. \end{aligned}$$

If we maximize over graphs G with a fixed number of nodes, then of course there is an extremal graph. But if we don't fix the number of nodes, then this does not follow. For example, there is no graph G minimizing $t(C_4, G)$ subject to $t(K_2, G) = 1/2$.

However, one can prove that there is always an extremal graphon, which then gives a "template" for asymptotically extremal graphs. This follows from the fact that the space of graphons is compact in an appropriate metric (the "cut metric"), which in turn is closely related to the Regularity Lemma, a fundamental tool in extremal graph theory [8, 11].

The existence of extremal graphons suggest new approaches to some old problems. For example, the Simonovits–Sidorenko Conjecture says in this language that for every bipartite graph F, the functional $t(F, W) - t(K_2, W)^{|E(F)|}$ is minimized by constant functions W. (In finite terms, this means that asymptotic minimizers are random graphs.) Now this suggests to prove that constant functions are at least local minimizers, and this in fact can be proved [5] by methods from analysis (studying a series expansion around the conjectured minimizer).

4. *Which graphs are extremal?* In other words, what is the pohssible structure of extremal graphs? Balázs Szegedy and I conjecture that every extremal problem has an extremal graphon that is *finite forcible* in the sense that they are determined by a finite number of prescribed subgraph densities; this corresponds to a unique asymptotic structure in finite graphs forced by finitely many subgraph densities.

Constant graphons (which are limits of random graphs) can be forced by the densities of edges and 4-cycles, by a result of Graham, Chung and Wilson [3]. This can be generalized to stepfunctions (limits of generalized random graphs) [6]. There are further nontrivial and quite interesting families, for example threshold graphons. A complete charaterization is an exciting but difficult open problem.

References

[1] C. Borgs, J. Chayes, L. Lovász, V.T. Sós, K. Vesztergombi: Counting graph homomorphisms, in: *Topics in Discrete Mathematics* (ed. M. Klazar, J. Kratochvil, M. Loebl, J. Matoušek, R. Thomas, P. Valtr), Springer (2006), 315–371.

[2] C. Borgs, J. Chayes, L. Lovász, V.T. Sós, K. Vesztergombi: Convergent Sequences of Dense Graphs I: Subgraph Frequencies, Metric Properties and Testing, *Advances in Math.* **219** (2008), 1801–1851.

[3] F. Chung, R.L. Graham and R.M. Wilson: Quasi-random graphs, *Combinatorica* **9** (1989), 345–362.

[4] H. Hatami and S. Norine: A solution to Lovász's seventeenth problem `http://arxiv.org/abs/1005.2382`

[5] L. Lovász: Subgraph densities in signed graphons and the local Sidorenko conjecture, `http://arxiv.org/abs/1004.3026`

[6] L. Lovász, V.T. Sós: Generalized quasirandom graphs, *J. Comb. Th. B* **98** (2008), 146–163.

[7] L. Lovász, B. Szegedy: Limits of dense graph sequences, *J. Comb. Theory B* **96** (2006), 933–957.

[8] L. Lovász, B. Szegedy: Szemerédi's Lemma for the analyst, *Geom. Func. Anal.* **17** (2007), 252–270.

[9] L. Lovász and B. Szegedy: Finitely forcible graphons (submitted) `http://arxiv.org/abs/0901.0929`

[10] L. Lovász and B. Szegedy: Random Graphons and a Weak Positivstellensatz for Graphs (submitted) `http://arxiv.org/abs/0902.1327`

[11] L. Lovász, B. Szegedy: Regularity partitions and the topology of graphons, in: *An Irregular Mind, Szemerédi is 70*, J. Bolyai Math. Soc and Springer-Verlag (2010) 415–446.

[12] V. Nikiforov: The number of cliques in graphs of given order and size `http://arxiv.org/abs/0710.2305`

[13] A.A. Razborov: Flag Algebras, *Journal of Symbolic Logic*, **72** (2007), 1239–1282.

Maximum union-free subfamilies

Jacob Fox

(joint work with Choongbum Lee and Benny Sudakov)

A set A of integers is *sum-free* if there are no $x, y, z \in A$ such that $x+y=z$. Erdős [8] in 1965 proved that every set of n nonzero integers contains a sum-free subset of size at least $n/3$. The proof is an influential application of the probabilistic method in extremal number theory. This result was rediscovered by Alon and Kleitman [3], who showed how to find a sum-free subset of size at least $(n+1)/3$. Finally, Bourgain [5] using harmonic analysis improved the lower bound to $(n+2)/3$. This result is the current state of the art for this problem. It is not even known if the constant factor $1/3$ is best possible.

The analogous problem in extremal set theory has also been studied for a long time. A family of sets is called *union-free* if there are no three distinct sets X, Y, Z in the family such that $X \cup Y = Z$. An old problem of Moser asks: how large of a union-free subfamily does every family of m sets have? Denote this number by $f(m)$. The study of $f(m)$ has attracted considerable interest. Riddell observed that $f(m) \geq \sqrt{m}$ (this follows immediately from Dilworth's theorem, see below). Erdős and Komlós [9] determined the correct order of magnitude of $f(m)$ by proving that $f(m) \leq 2\sqrt{2m}+4$. They conjectured that $f(m) = (c - o(1))\sqrt{m}$ for some constant c, without specifying the right value of c. In 1972, Erdős and Shelah [10] improved both the upper and lower bound by showing that $\sqrt{2m}-1 < f(m) < 2\sqrt{m}+1$ (the lower bound was also obtained independently by Kleitman). Erdős and Shelah conjectured that their upper bound is asymptotically tight.

Conjecture 1. *$f(m) = (2 - o(1))\sqrt{m}$.*

We verify this conjecture and solve Moser's problem.

Theorem 2. *For all m, we have*

$$f(m) = \lfloor \sqrt{4m+1} \rfloor - 1.$$

Let $a \geq 2$ be an integer. A family of sets is called *a-union-free* if there are no $a+1$ distinct sets $X_1, \cdots, X_{a+1}$ such that $X_1 \cup \cdots \cup X_a = X_{a+1}$. Let $g(m,a)$ be the minimum over all families of m sets of the size of the largest a-union-free subfamily. In particular, $g(m,2) = f(m)$. The same proof which shows $f(m) > \sqrt{2m}-1$ also shows that $g(m,a) > \sqrt{2m}-1$. Recently, Barat, Füredi, Kantor, Kim and Patkos [4] proved that $g(m,a) \leq c(a + a^{1/4}\sqrt{m})$ for some absolute constant c and made the following conjecture on the growth of $g(m,a)$.

Conjecture 3. $\lim_{a\to\infty} \liminf_{m\to\infty} \frac{g(m,a)}{\sqrt{m}} = \infty$.

We prove this conjecture in the following strong form, which further gives the correct order of magnitude for $g(m,a)$.

Theorem 4. *For all $m \geq a \geq 2$, we have $g(m,a) \geq \max\{a, a^{1/4}\sqrt{m}/3\}$.*

The lower bound in Theorem 4 is tight apart from an absolute multiplicative constant factor by the above mentioned upper bound from [4]. Of course, if $m \leq a$, we have trivially $g(m,a) = m$.

For the proofs of these theorems, it is helpful to study the structure of the partial order on sets given by inclusion. Recall that a chain (antichain) in a poset is a collection of pairwise comparable (incomparable) elements. Dilworth's theorem [6] implies that any poset with m elements contains a chain or antichain of size at least $\sqrt{m}$. Notice that a chain or antichain of sets is a-union-free for all a, but the lower bound $g(m,a) \geq \sqrt{m}$ we get from this simple argument is not strong enough. For the proof of Theorem 4, we find considerably larger structures in posets which imply that the subfamily is a-union-free. The existence of such large structures in posets may be of independent interest.

References

[1] H. L. Abbott and D. Hanson, A problem of Schur and its generalizations, *Acta Arith.* **20** (1972), 172–185.

[2] M. Aigner, D. Duffus, and D. Kleitman, Partitioning a power set into union-free classes, *Discrete Math.* 88 (1991), 113–119.

[3] N. Alon and D. Kleitman, Sum-free subsets, in: *A tribute to Paul Erdős*, Cambridge Univ. Press, Cambridge, 1990, 13–26.

[4] J. Barat, Z. Füredi, I. Kantor, Y. Kim, and B.Patkos, Large B_d-free subfamilies, in preparation.

[5] J. Bourgain, Estimates related to sumfree subsets of sets of integers, *Israel J. Math.* **97** (1997), 71–92.

[6] R. P. Dilworth, A decomposition theorem for partially ordered sets, *Annals of Math.* **51** (1950), 161–166.

[7] P. Erdős, Some unsolved problems, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **6** (1961), 221–254.

[8] P. Erdős, Extremal problems in number theory, *Proceedings of Symposia in Pure Mathematics* **3** (1965), 181–189.

[9] P. Erdős and J. Komlós, On a problem of Moser, Combinatorial theory and its applications I, (Edited by P. Erdős, A. Renyi, and V. Sos), North Holland Publishing Company, Amsterdam (1969).

[10] P. Erdős and S. Shelah, On a problem of Moser and Hanson, Graph theory and applications, Lecture Notes in Math., Vol. 303, Springer, Berlin (1972), 75–79.

[11] D. Gunderson, V. Rödl, and A. Sidorenko, Extremal problems for sets forming boolean algebras and complete partite hypergraphs, *J. Combin. Theory Ser. A* 88 (1999), 342–367.

[12] D. Kleitman, On a combinatorial problem of Erdős, *Proc. Amer. Math. Soc.* **17** (1966), 139–141.

[13] D. Kleitman, Collections of subsets containing no two sets and their union, Combinatorics, Amer. Math. Soc., Providence, R.I. (1971), 153–155.

The Local Lemma is tight for SAT

Tibor Szabó

(joint work with Heidi Gebauer and Gábor Tardos)

We define a k-CNF formula as the conjunction of clauses that are the disjunction of exactly k distinct literals. A k-CNF formula is called a (k, s)-CNF formula if every variable appears in at most s clauses. The problem of satisfiability of (k, s)-CNF formulas is denoted by (k, s)-SAT.

Tovey [13] proved that while every $(3, 3)$-CNF formula is satisfiable (due to Hall's theorem), the problem of deciding whether a $(3, 4)$-CNF formula is satisfiable is already NP-hard. Dubois [2] showed that $(4, 6)$-SAT and $(5, 11)$-SAT are also NP-complete. Kratochvíl, Savický, and Tuza [9] defined the value $f(k)$ to be the largest integer s such that every (k, s)-CNF is satisfiable. They also generalized Tovey's result by showing that for every $k \geq 3$ $(k, f(k) + 1)$-SAT is already NP-complete. In other words, for every $k \geq 3$ the (k, s)-SAT problem goes through a kind of "complexity phase transition" at the value $s = f(k)$. On the one hand the $(k, f(k))$-SAT problem is trivial by definition in the sense that every instance of the problem is a "YES"-instance. On the other hand the $(k, f(k) + 1)$-SAT problem is already NP-hard, so the problem becomes hard from being trivial just by allowing one more occurrence of each variable. In fact, the complexity hardness jump is even greater: the problem of (k, s)-SAT is also MAX-SNP-complete for every $s > f(k)$ as was shown by Berman, Karpinski, and Scott [1] (generalizing a result of Feige [4] who showed that $(3, 5)$-SAT is hard to approximate within a certain constant factor).

A straightforward consequence of the Local Lemma is the following: if every clause of a k-CNF formula has a common variable with at most $2^k/e - 1$ other clauses, then the formula is satisfiable [9]. This immediately implies the lower bound

$$f(k) \geq \left\lfloor \frac{2^k}{ek} \right\rfloor .$$

From the other side Savický and Sgall [11] showed that $f(k) = O\left(k^{0.74} \cdot \frac{2^k}{k}\right)$. This was improved by Hoory and Szeider [8] who came within a logarithmic factor: $f(k) = O\left(\log k \cdot \frac{2^k}{k}\right)$. Recently, Gebauer [6] showed that the order of magnitude of the lower bound is correct and $f(k) = \Theta(\frac{2^k}{k})$.

More precisely, the construction of [6] gave $f(k) \leq \frac{63}{64} \cdot \frac{2^k}{k}$ for infinitely many k. The constant factor $\frac{63}{64}$ was clearly an artefact of the proof and there was no clear consensus (see [5]) where the asymptotic value of $f(k)$ should fall between the constants $1/e$ of [9] and $63/64$ of [6]. The goal of our investigation was to understand the limitations of the method of [6] and as it luckily turned out this was sufficient to determine the correct asymptotics.

Theorem 1.

$$f(k) = \left(\frac{2}{e} + O\left(\frac{1}{\sqrt{k}}\right)\right) \frac{2^k}{k}.$$

For the upper bound we use the fundamental binary tree approach of [6]. We define a suitable continuous setting for the construction of the appropriate binary trees, which allows us to study the problem via a differential equation. The solution of this differential equation corresponds to our construction of the binary trees, which then can be given completely discretely.
The lower bound is achieved via the lopsided version of the Lovász Local Lemma. The key of the proof is to assign the random values of the variables counter-intuitively: each variable is *more* probable to satisfy those clauses where it appears as a literal with its *less* frequent sign. The lower bound can also be derived from a theorem of Berman, Karpinski and Scott [1] tailored to give good lower bounds on $f(k)$ for small values of k.

Since the lopsided Lovász Local Lemma was fully algorithmized by Moser and Tardos [10] we now have that not only every (k, s)-CNF formula for $s = \lfloor 2^{k+1}/(e(k+1)) \rfloor$ has a satisfying assignment but there is also an algorithm that *finds* such an assignment in probabilistic polynomial time. Moreover, for just a little bit larger value of the parameter s one cannot find a satisfying assignment efficiently simply because already the decision problem is NP-hard.

Our upper bound construction also settles a couple of other related open questions from the survey paper of of Gebauer, Moser, Scheder, and Welzl [5]. The integer $l(k)$ is defined to be the largest integer number satisfying that whenever all clauses of a k-CNF formula intersect at most $l(k)$ other clauses the formula is satisfiable. Our construction shows that the lower bound on $l(k)$ given by the Local Lemma is asymptotically tight.

Theorem 2.

$$l(k) = \left(\frac{1}{e} + O\left(\frac{1}{\sqrt{k}}\right)\right)2^k.$$

Theorem 1 and Theorem 2 are another instances which show the tightness of the Lovász Local Lemma. The first such example was given by Shearer [12].

Finally, our construction also improves the constant factor in the best known upper bound in the Neighborhood Conjecture of Beck in the theory of positional games. However, the importance of this improvement is far less then the importance of Theorems 1 and 2 as in the Neighborhood Conjecture there is still an exponential gap between the known upper and lower bounds.

References

[1] P. Berman, M. Karpinski, and A. D. Scott, Approximation hardness and satisfiability of bounded occurrence instances of SAT. *Electronic Colloquium on Computational Complexity (ECCC)*, **10** (022), 2003.

[2] O. Dubois, On the r, s-SAT satisfiability problem and a conjecture of Tovey, *Discrete Appl. Math.* **26** (1990), 51-60.

[3] P. Erdős and J.Spencer, Lopsided Lovász local lemma and Latin transversals *Discrete Appl. Math.* **30**, (1991), 151–154.

[4] U. Feige, A threshold of $\ln n$ for approximating set cover, *J. ACM* **45**(4), (1998), 634–652.

[5] H. Gebauer, R. A. Moser, D. Scheder and E. Welzl, The Lovász Local Lemma and Satisfiability *Efficient Algorithms*, (2009), 30–54.

[6] H. Gebauer, Disproof of the Neighborhood Conjecture with Implications to SAT, *Proc. 17th Annual European Symposium on Algorithms (ESA)* (2009), LNCS 5757, 764–775.

[7] S. Hoory and S. Szeider. Computing unsatisfiable k-SAT instances with few occurrences per variable, *Theoretical Computer Science* **337**(1–3) (2005), 347–359,

[8] S. Hoory and S. Szeider, A note on unsatisfiable k-CNF formulas with few occurrences per variable, *SIAM J. Discrete Math* **20** (2), (2006), 523–528.

[9] J. Kratochvíl, P. Savický and Z. Tuza, One more occurrence of variables makes satisfiability jump from trivial to NP-complete, *SIAM J. Comput.* **22** (1), (1993), 203–210.

[10] R.A. Moser, G. Tardos, A constructive proof of the general lovász local lemma, *J. ACM* **57**(2), (2010)

[11] P. Savický and J. Sgall, DNF tautologies with a limited number of occurrences of every variable, *Theoret. Comput. Sci.* **238** (1–2), (2000), 495–498.

[12] J.B. Shearer, On a problem of Spencer, *Combinatorica* **5**, (1985), 241–245.

[13] C.A. Tovey, A simplified NP-complete satisfiability problem, *Discr. Appl. Math.* **8** (1), (1984), 85–89.

Packing tight Hamilton cycles

Po-Shen Loh
(joint work with Alan Frieze and Michael Krivelevich)

Introduction. Hamilton cycles occupy a position of central importance in graph theory, and are the subject of countless results, from Dirac's Theorem [4] to many more modern investigations in graphs, digraphs, hypergraphs, and random and pseudo-random instances of these objects. See, e.g., any of [1, 8, 9, 12, 13, 14].

There has also been a long history of research concerning conditions for the existence of multiple edge-disjoint Hamilton cycles. Indeed, Nash-Williams discovered that the Dirac condition already guarantees not just one, but at least $\lfloor \frac{5}{224}n \rfloor$ edge-disjoint Hamilton cycles. His questions in [15, 16, 17] started a line of investigation, leading to recent work by Christofides, Kühn, and Osthus [3], who answered one of his conjectures asymptotically by proving that minimum degree $\big(\frac{1}{2} + o(1)\big)n$ is already enough to guarantee $\frac{n}{8}$ edge-disjoint Hamilton cycles.

For random graphs, these "packings" with Hamilton cycles are even more complete. Bollobás and Frieze [2] showed that for every fixed r, one can typically find r edge-disjoint Hamilton cycles in the random graph process as soon as the minimum degree reaches $2r$. Kim and Wormald [10] established a similar result for random r-regular graphs, proving that such graphs typically contain $\lfloor r/2 \rfloor$ edge-disjoint Hamilton cycles. The previous statements are of course best possible, but invite the natural question of what happens when r is allowed to grow. Along these lines, Frieze and Krivelevich showed in [5] that one can pack $\lfloor \frac{\delta}{2} \rfloor$ Hamilton cycles in $G_{n,p}$, up to $p \le \frac{(1+o(1))\log n}{n}$, where δ is the minimum degree of the graph. For large p, they discovered in [6] that one can pack almost all edges into Hamilton cycles. This was later improved to essentially the full range of p by Knox, Kühn, and Osthus [11].

In the hypergraph setting, the study of this Hamilton cycle packing problem was initiated by Frieze and Krivelevich in [7]. Although the notion of a Hamilton cycle in an ordinary graph is clear, there are several ways to generalize the notion to hypergraphs. Indeed, for any $1 \le \ell \le k$, we may define a k-uniform hypergraph C to be a *Hamilton cycle of type* ℓ if there is a cyclic ordering of the vertices of C so that every edge consists of k consecutive vertices, and every pair of consecutive edges E_{i-1}, E_i in C (according to the natural ordering of the edges) has $|E_{i-1} \setminus E_i| = \ell$. The extreme cases $\ell = 1$ and $\ell = k-1$ are the most obvious generalizations of graph Hamiltonicity, and cycles of those types are often called *tight* and *loose*, respectively. In [7], the first two authors studied the problem of covering almost all the edges of a random k-uniform hypergraph with disjoint Hamilton cycles of a fixed type ℓ. They considered ℓ on the looser end of the spectrum, determining sufficient conditions for the cases $\ell \ge k/2$. However, their methods did not extend to the regime $\ell < k/2$, which seemed more difficult.

Our contribution. In this work, we introduce several new techniques which enable us to prove the first results for packing *tight* Hamilton cycles, i.e., with $\ell = 1$.

We show that under certain natural pseudo-random conditions, almost all edges of a 3-uniform hypergraph on n vertices can be covered by edge-disjoint tight Hamilton cycles, for n divisible by 4. This implies the following main result. Here, $H_{n,p;3}$ denotes the random 3-uniform hypergraph obtained by taking each triple independently with probability p.

Theorem 1. *Suppose that ϵ, n, p satisfy $\epsilon^{45} np^{16} \gg \log^{21} n$. Then whenever n is a multiple of four, $H_{n,p;3}$ can have all but at most $\epsilon^{1/15}$-fraction of its edges covered by a disjoint union of tight Hamilton cycles **whp**.*

In other words, for $p \gg \frac{\log^{\Theta(1)} n}{n^{1/16}}$, we can pack all but $o(1)$-fraction of the edges into tight Hamilton cycles. It appears very likely that our approach can be naturally extended to the general k-uniform case, but the analysis necessarily becomes more involved. Also, although both results are stated for n divisible by 4, we expect that they are true in general. Note, however, that a divisibility condition is unavoidable in the general case of packing Hamilton cycles of type ℓ in k-uniform hypergraphs, since ℓ must divide n.

Main techniques. The key insight in our proof is the following connection between tight Hamilton cycles in H and Hamilton cycles in an associated digraph. For a random permutation $v_1, v_2, \ldots, v_n$ of the vertices of H, define an $\frac{n}{2}$-vertex digraph D with vertex set $\{(v_1, v_2), (v_3, v_4), \ldots, (v_{n-1}, v_n)\}$. Note that each vertex of D corresponds to an ordered pair of vertices of H, so D will have an even number of vertices, since the number of vertices of H is a multiple of 4. Place a directed edge from (v_i, v_{i+1}) to (v_j, v_{j+1}) if and only if both hyperedges $\{v_i, v_{i+1}, v_j\}$ and $\{v_{i+1}, v_j, v_{j+1}\}$ are present in H. In this construction, Hamilton cycles in D give rise to tight Hamilton cycles in H.

To extract edge disjoint Hamilton cycles from a digraph D with an even number of vertices, we use an approach similar to that taken in [7]. Let $w_1, w_2, \ldots, w_{2m}$ be a random permutation of the vertices of D with $m = n/4$, and define $A = \{w_1, w_2, \ldots, w_m\}$ and $B = \{w_{m+1}, \ldots, w_{2m}\}$. Define a bipartite graph Γ with bipartition (A, B), and place an edge between $w_i \in A$ and $w_j \in B$ whenever $\overrightarrow{w_i w_j}$ and $\overrightarrow{w_j w_{i+1}}$ are both edges of D. Now perfect matchings in Γ give rise to Hamilton cycles in D, and previous approaches in [7] show how to pack perfect matchings in pseudo-random bipartite graphs.

However, not all Hamilton cycles in D arise from perfect matchings in one particular Γ. Similarly, not all Hamilton cycles in H arise from Hamilton cycles in a single D. We overcome both obstacles with the same iterative approach, which we illustrate for the hypergraph packing. Roughly speaking, instead of stopping after generating a single D, we sequentially generate digraphs $D_1, D_2, \ldots, D_r$ in the above manner, extracting a large set of edge disjoint directed Hamilton cycles from each, and deleting the corresponding edge-disjoint Hamilton cycles from H. At each step, we verify that the pseudo-random properties are maintained. We repeat the process until we have packed the required number of cycles.

References

[1] B. Bollobás, **Random Graphs**, Academic Press, London (1985).

[2] B. Bollobás and A. Frieze, On matchings and Hamiltonian cycles in random graphs, in: **Random Graphs '83 (Poznan, 1983)**, North-Holland Math. Stud., 118, North-Holland, Amsterdam (1985), 23–46.

[3] D. Christofides, D. Kühn, and D. Osthus, Edge-disjoint Hamilton cycles in graphs, submitted.

[4] G. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69–81.

[5] A. Frieze and M. Krivelevich, On two Hamilton cycle problems in random graphs, *Israel Journal of Mathematics* **166**, 221–234.

[6] A. Frieze and M. Krivelevich, On packing Hamilton cycles in ϵ-regular graphs, *J. Combin. Theory Ser. B*, **94** (2005) 159–172.

[7] A. Frieze and M. Krivelevich, Packing Hamilton cycles in random and pseudo-random hypergraphs, submitted.

[8] H. Hán and M. Schacht, Dirac-type results for loose Hamilton cycles in uniform hypergraphs, *J. Combin. Theory Ser. B* **100** (2010), 332–346.

[9] P. Keevash, D. Kühn, R. Mycroft, and D. Osthus, Loose Hamilton cycles in hypergraphs, submitted.

[10] J.H. Kim and N. Wormald, Random matchings which induce Hamilton cycles, and Hamiltonian decompositions of random regular graphs, *J. Combin. Theory Ser. B* **81** (2001), 20–44.

[11] F. Knox, D. Kühn, and D. Osthus, Approximate Hamilton decompositions of random graphs, submitted.

[12] M. Krivelevich and B. Sudakov, Sparse pseudo-random graphs are Hamiltonian, *J. Graph Theory* **42** (2003), 17–33.

[13] D. Kühn, R. Mycroft, and D. Osthus, Hamilton ℓ-cycles in uniform hypergraphs, *J. Combin. Theory Ser. A*, to appear.

[14] D. Kühn and D. Osthus, A survey on Hamilton cycles in directed graphs, submitted.

[15] C. Nash-Williams, Hamiltonian lines in graphs whose vertices have sufficiently large valencies, in: **Combinatorial Theory and its Applications, III**, North-Holland (1970), 813–819.

[16] C. Nash-Williams, Edge-disjoint Hamiltonian circuits in graphs with vertices of large valency, in: **Studies in Pure Mathematics**, Academic Press (1971), 157–183.

[17] C. Nash-Williams, Hamiltonian arcs and circuits, in: **Recent Trends in Graph Theory**, Springer (1971), 197–210.

Cycle-saturated graphs with minimum number of edges

Zoltán Füredi

(joint work with Younjin Kim)

A graph G is said to be H-**saturated** if
— it does not contain H as a subgraph, but
— the addition of any new edge (from $E(\overline{G})$) creates a copy of H.
Let $\mathrm{sat}(n, H)$ denote the *minimum* size of an H-saturated graph on n vertices. Given H, it is difficult to determine $\mathrm{sat}(n, H)$ because this function is not necessarily monotone in n, neither in H. Recent surveys are by J. Faudree, Gould, and Schmitt [10] (2010+), and by Pikhurko [18] (2004). It is known [16] that

$$\mathrm{sat}(n, H) < c_H n.$$

However, the $\lim_{n\to\infty} \mathrm{sat}(n,H)/n$ does not necessarily exists, Pikhurko (2009). The classical theorem of Erdős, Hajnal, and Moon [8] (1964) states that

$$\mathrm{sat}(n, K_t) = \binom{t-2}{2} + (t-2)(n-t+2).$$

A similar statement holds for hypergraphs, see Bollobás [5] (1965) and its generalizations by Kalai [15], Frankl [13], Alon [1], using Lovász' algebraic method. Remarkable asymptotics were given by Alon, Erdős, Holzman, and Krivelevich [2] (1996) (saturation and degrees). For multiple copies of K_p Faudree, Ferrara, Gould, and Jacobson [11] (2009) determined $\mathrm{sat}(tK_p, n)$ for $n > n_0(p,t)$.

What is the saturation number for the cycle, C_k? Most cases are unsolved.

Theorem 1 (ZF and Y. Kim, 2010+). *For* $n \geq k \geq 5$,

$$\mathrm{sat}(n, C_k) = n + \frac{n}{k} + O(\frac{n}{k^2} + k^2).$$

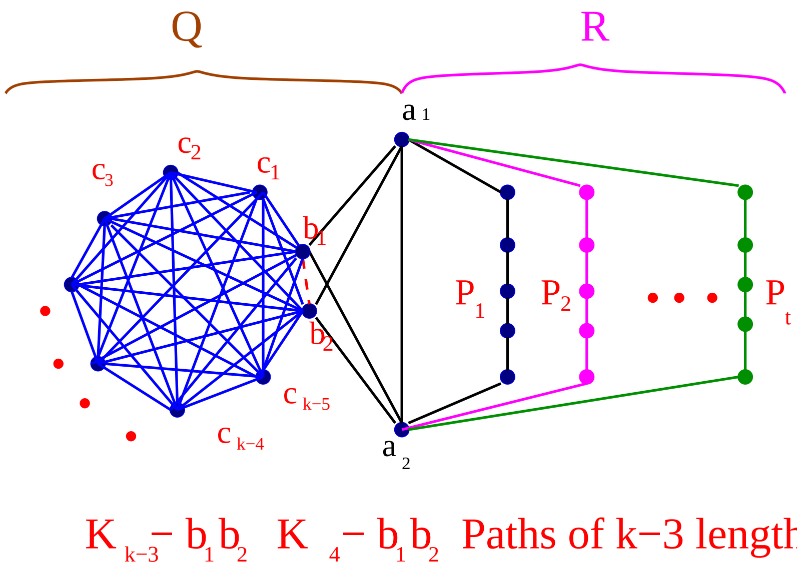


Figure. 1.

Our construction for a k-cycle saturated graph for $n = t(k-4) + k - 1$ can be read from the picture below. For other n, one can add $r < k - 4$ pendant edges.

The case of C_4 was established by Ollmann [17] (1972) and by Y. Chen [6] (2009) proved a conjecture of Fisher, Fraughnaugh, Langley [12].

$$\mathrm{sat}(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor \text{ for } n \geq 5. \tag{1}$$

$$\mathrm{sat}(n, C_5) = \lceil \frac{10(n-1)}{7} \rceil \text{ for } n \geq 21. \tag{2}$$

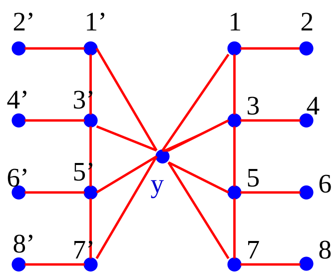
The best previous general bounds were due to Barefoot, Clark, Entringer, Porter, Székely, and Tuza [3] (1996) and Gould, Łuczak, and Schmitt [14]

$$\left(1+\frac{1}{2k+8}\right)n \le \mathrm{sat}(n,C_k) \le (1+\frac{2}{k-\epsilon(k)})n+O(k^2)$$

where $\epsilon(k)=2$ for k even ≥ 10, $\epsilon(k)=3$ for k odd ≥ 17.

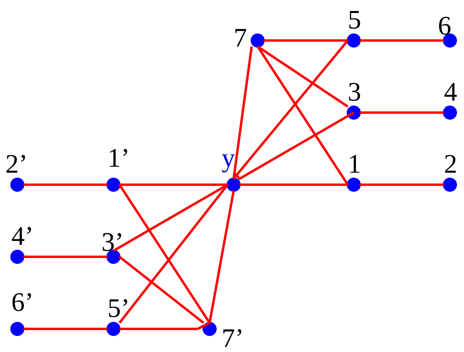
A graph G is H-**semisaturated** (formerly called *strongly* saturated) if $G+e$ contains more copies of H than G does for $\forall e \in E(\overline{G})$. Let $\mathrm{ssat}(n,H)$ be the minimum size of an H-semisaturated graph. Obviously, $\mathrm{ssat}(n,H) \leq \mathrm{sat}(n,H)$.

It is known that $\mathrm{ssat}(n,K_p) = \mathrm{sat}(n,K_p)$ (it follows from Frankl/Alon/Kalai generalizations of Bollobás theorem) and $\mathrm{ssat}(n,C_4) = \mathrm{sat}(n,C_4)$ (Tuza, 1989). Below we have a C_5-semisaturated graph (every vertex can be reached by a path of length 2 from y). The picture on the right is the extremal C_5-saturated graph.



n = 1+8t
e = 11t

Figure 1.



n = 1+ 7t
e = 10t

Figure 2.

Conjecture 2. $\mathrm{ssat}(n,C_5) = \frac{11}{8}n + O(1)$.

Theorem 3 (ZF and Y. Kim, 2010+)**.** *For* $n \geq k \geq 5$,

$$\mathrm{ssat}(n,C_k) = n + \frac{n}{2k} + O(\frac{n}{k^2}+k^2).$$

These cycle-semisaturated graphs have paths of length $k-4$ with spikes.

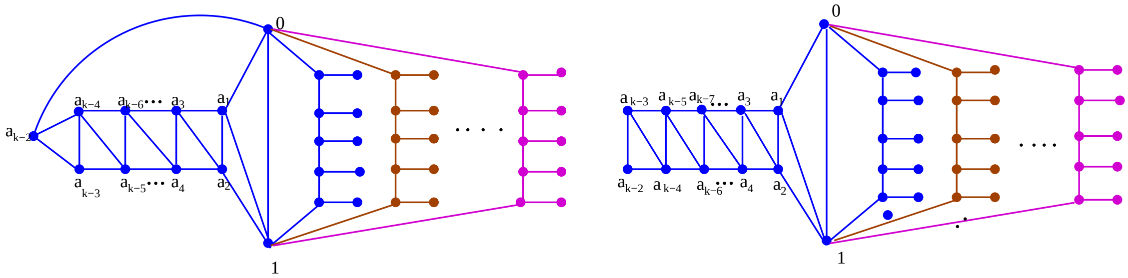
Conjecture 4. *Both of our constructions for* $\mathrm{sat}(n,C_k)$ *and* $\mathrm{ssat}(n,C_k)$ *are optimal whenever* $n > n_0(k)$.

References

[1] Noga Alon, *An extremal problem for sets with applications to graph theory*, J. Combin. Theory Ser. A, **40** (1985), 82–89.

[2] Noga Alon, Paul Erdős, Ron Holzman, Michael Krivelevich, *On k-saturated graphs with restrictions on the degrees*, J. Graph Theory, **23** (1996), 1–20.

[3] C. A. Barefoot, L. H. Clark, R. C. Entringer, T. D. Porter, L. A. Székely, Zs. Tuza, *Cycle-saturated graphs of minimum size*, Discrete Mathematics **150** (1996), 31–48.



(a) k is odd (b) k is even

Figure.2.

[4] Tom Bohman, Maria Fonoberova, Oleg Pikhurko. *The saturation function of complete partite graphs*, J. Comb., **1** (2010), 149–170.
[5] B. Bollobás, *On generalized graphs*, Acta Math. Acad. Sci. Hungar. **16** (1965), 447-452.
[6] Ya-Chen Chen, *Minimum C_5-saturated graphs*, J. Graph Theory **61** (2009), 111–126.
[7] Ya-Chen Chen, *All minimum C_5-saturated graphs*, J. Graph Theory.
[8] P. Erdős, A. Hajnal, J. W. Moon, *A problem in graph theory*, Amer. Math. Monthly **71** (1964), 1107-1110.
[9] P. Erdős and Ron Holzman, *On maximal triangle-free graphs*, J. Graph Theory, **18** (1994), 585–594.
[10] J. Faudree, R. Faudree, J. Schmitt, *A survey of minimum saturated graphs and hypergraphs*, manuscript, May 2010.
[11] Ralph Faudree, Michael Ferrara, Ronald Gould, and Michael Jacobson, *tK_p-saturated graphs of minimum size*, Discrete Math., **309** (2009), 5870–5876.
[12] D. C. Fisher, K. Fraughnaugh, L. Langley, *On C_5-saturated graphs with minimum size*, Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1995), *Congr. Numer.* **112** (1995), 45-48.
[13] Peter Frankl, *An extremal problem for two families of sets*, European J. Combin., **3** (1982), 125–127.
[14] Ronald Gould, Tomasz Łuczak, John Schmitt, *Constructive upper bounds for cycle-saturated graphs of minimum size*, Electronic J. Combinatorics **13** (2006), Research Paper 29, 19 pp.
[15] Gil Kalai, *Weakly saturated graphs are rigid*, In Convexity and graph theory (Jerusalem, 1981), North-Holland Math. Stud. **87** pp. 189–190. North-Holland, Amsterdam, 1984.
[16] L. Kászonyi, Zs. Tuza, *Saturated graphs with minimal numbers of edges*, J. Graph Theory **10** (1986), 203–210.
[17] L. Taylor Ollmann, *$K_{2,2}$ saturated graphs with a minimal number of edges*, In Proceedings of the Third Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1972), pages 367–392, Boca Raton, Fla., 1972. Florida Atlantic Univ.
[18] O. Pikhurko, *Results and open problems on minimum saturated hypergraphs*, Ars Combin. **72** (2004), 111-127.

The Lipschitz constant of the RSK correspondence

Nati Linial

(joint work with Nayantara Bhatnagar)

The Robinson-Schensted-Knuth (RSK) correspondence maps an arbitrary permutation $\pi \in S_n$ bijectively to an ordered pair of Young tableaux of the same shape $\lambda = \lambda(\pi)$. How much can $\lambda(\pi)$ change as we mildly vary π? Specifically, if we pre-multiply π by t transpositions, to what extent can λ change? We begin with the case when $t = 1$ and show that the resulting Young diagram can differ from λ on at most $\sqrt{n/2}$ cells. We show that this bound is tight by giving explicit constructions of permutations π for which this bound is attained where the diagrams differ in at least $(1 - o(1))\sqrt{n/2}$ cells. We then turn to consider the same question for larger t and show that the corresponding diagram changes in at most $O(\sqrt{nt \ln t})$ cells. The best constructions we know nearly match this bound and yield, e.g., $(1 - o(1))\sqrt{nt/2}$ changes for $t = o(n)$.

A *standard Young tableau* (SYT or tableau) of size n with entries from $[n]$ is a diagram whose cells are filled with the elements of $[n]$ in such a way that the entries are strictly increasing from left to right along a row as well as from top to bottom down a column. The *shape* of a tableau T, denoted $sh(T)$ is the partition corresponding to the diagram of T.

The RSK correspondence discovered by Robinson and further extended by Knuth is a bijection between the set of permutations S_n and pairs of tableaux of size n of the same shape. This bijection is intimately related to the representation theory of the symmetric group, the theory of symmetric functions, and the theory of partitions.

We ask to what extent λ changes as π changes slightly. In order to make our question concrete, we need to specify two measures of distance: One between permutations and the other between diagrams. For permutations we use pre-multiplication by transpositions. An *adjacent transposition* is a permutation of the form $(i, i+1)$. We denote the least number of adjacent transpositions that transform the permutation π to τ by $d(\pi, \tau)$. Recall that $d(\cdot, \cdot)$ is the graph metric in the Cayley graph of S_n w.r.t. the generating set of adjacent transpositions $(1, 2), (2, 3), \cdots, (n, n-1)$. We will say that two permutations π and τ are *at distance* t if $d(\pi, \tau) = t$. If λ and μ are two diagrams, define their distance to be $\Delta(\lambda, \mu) := \frac{1}{2}\sum_{i=1}^{n} |\lambda_i - \mu_i|$. Let π and τ be any two permutations. We are interested in the Lipschitz constant of this mapping, i.e., we wish to determine

$$\max \Delta(\lambda(\pi), \lambda(\tau))$$

over all $\pi, \tau \in S_n$ with $d(\pi, \tau) = t$.

Here are our main results:

- Let π and τ be permutations in $\mathcal{S}_n$ with respective Young diagrams λ and μ, and suppose that $d(\pi, \tau) = 1$. Then $\Delta(\lambda, \mu) \leq \sqrt{\frac{n}{2}}$. The bound is tight.
- More generally, if $d(\pi, \tau) = t$, then $\Delta(\lambda, \mu) \leq O(\sqrt{nt \ln t})$. For $t < n/2$ this bound is tight up to the logarithmic term.

Computing the partition function for perfect matchings in a hypergraph

Alexander Barvinok
(joint work with Alex Samorodnitsky)

Let us fix an integer $k > 1$ and let V be a finite ground set, $|V| = km$ for a positive integer m. Let $\binom{V}{k}$ denote the family of all k-subsets $S \subset V$. A family $W = \left\{w_S : \ S \in \binom{V}{k}\right\}$ of non-negative numbers w_S attached to the k-subsets S of V is called a *weight* on $\binom{V}{k}$. We say that weight W is *positive* if $w_S > 0$ for all $S \in \binom{V}{k}$.

Given a weight $W = \left\{w_S : \ S \in \binom{V}{k}\right\}$, we consider the *partition function*

$$P(W) = \sum_{S_1,\ldots,S_m} w_{S_1} \cdots w_{S_m}, \tag{3}$$

where the sum is taken over all sets $\{S_1, \ldots, S_m\}$ of pairwise disjoint k-sets $S_1, \ldots, S_m \subset V$ such that $S_1 \cup \ldots \cup S_m = V$. For $k = 2$ one can think of W as of a symmetric $2m \times 2m$ matrix with rows and columns indexed by the elements of V (the diagonal entries of W are defined arbitrarily), in which case the expression $P(W)$ is known as the *hafnian* of W.

We say that a weight W is *k-stochastic* if

$$\sum_{S:\ v \in S} w_S = 1 \quad \text{for all} \quad v \in V.$$

For an $\alpha \geq 1$, we say that a positive weight W is *α-balanced* if

$$\frac{w_{S_1}}{w_{S_2}} \leq \alpha \quad \text{for all} \quad S_1, S_2 \in \binom{V}{k}.$$

Theorem 1. *Let us fix an integer $k > 1$ and a real $\alpha \geq 1$. Then there exists a $\gamma = \gamma(k, \alpha) > 0$ such that for any α-balanced k-stochastic weight W on $\binom{V}{k}$ we have*

$$e^{-(k-1)m} m^{-\gamma} \ \leq \ P(W) \ \leq \ e^{-(k-1)m} m^{\gamma}$$

where $|V| = km$ for $m > 1$.

For a positive weight W on $\binom{V}{k}$, let us define a function

$$f_W(X) = \sum_{S \in \binom{V}{k}} x_S \ln \frac{w_S}{x_S}$$

on weights $X = \{x_S\}$ on $\binom{V}{k}$. Let Ω_k be the set of all k-stochastic weights on $\binom{V}{k}$. The set Ω_k is a polytope defined by $\binom{km}{k}$ inequalities and km equations and f_W is a strictly concave function of $X \in \Omega_k$. In particular, the maximum

$$\zeta = \max_{X \in \Omega_k} f_W(X) \tag{4}$$

is attained at a unique point and can be computed in polynomial time.

Theorem 2. *Let us fix an integer $k > 1$ and a real $\alpha \geq 1$. Then there exists a $\gamma = \gamma(k, \alpha) > 0$ such that for any α-balanced weight W on $\binom{V}{k}$ we have*

$$e^{\zeta-(k-1)m} m^{-\gamma} \leq P(W) \leq e^{\zeta-(k-1)m} m^{\gamma}$$

where $|V| = km$ for $m > 1$ and ζ is the solution of the optimization problem (4).

In particular, for any fixed $\alpha \geq 1$ and $k > 1$, the value of the partition function $P(W)$ on an α-balanced weight W on $\binom{V}{k}$ can be computed in polynomial time within an $m^{O(1)}$ factor, where $|V| = km$.

Recall that a *k-uniform hypergraph* with the set V of vertices is a collection H of k-subsets $S \subset V$, called *edges* of H. A family $\{S_1, \ldots, S_n\}$ of pairwise disjoint edges of H is called a *matching* of H of *size* n. A matching $\{S_1, \ldots, S_m\}$ is called *perfect* if $V = S_1 \cup \ldots \cup S_m$, in which case $|V| = km$.

As is known, for any $k \geq 3$ it is an NP-complete problem to determine whether a given k-uniform hypergraph has a perfect matching, whereas for $k = 2$ the problem admits a polynomial time algorithm. To count the perfect matchings in a given k-uniform hypergraph H is a $\#P$-hard problem for any $k \geq 2$.

Given a k-uniform hypergraph H with the set V of vertices, let us define a weight W on $\binom{V}{k}$ by

$$w_S = \begin{cases} 1 & \text{if } S \in H \\ 0 & \text{if } S \notin H. \end{cases}$$

Then $P(W)$ is the number of perfect matchings in H.

Let us fix an $\epsilon > 0$ and define a weight $\widehat{W}$ by

$$\widehat{w}_S = \begin{cases} 1 & \text{if } S \in H \\ \epsilon & \text{if } S \notin H. \end{cases} \tag{5}$$

Theorem 2 then guarantees that the value of $P(\widehat{W})$ can be computed in polynomial time within an $m^{O(1)}$ factor.

Let us fix $0 < \beta < 1$ and $0 < \delta < 1$ (for example, $\beta = 0.99$ and $\delta = 0.01$). Let

$$\Phi_k(m) = \frac{(km)!}{(m!)^k k!}$$

be the number of perfect matchings in the complete k-uniform hypergraph $\binom{V}{k}$ with $|V| = km$. It turns out that we can distinguish in polynomial time the hyperpgraphs with km vertices that do not have a nearly perfect matching, namely a matching of size $n \geq \beta m$, from the hypergraphs with km of vertices that have sufficiently many, namely at least $\delta^m \Phi_k(m)$, perfect matchings. Defining the weight $\widehat{W}$ by (5) we obtain $P(\widehat{W}) \leq \epsilon^{(1-\beta)m} \Phi_k(m)$ in the former case and $P(\widehat{W}) \geq \delta^m \Phi_k(m)$ in the latter case. Choosing a sufficiently small $\epsilon = \epsilon(\beta, \delta) > 0$ we can distinguish between the two cases by computing $P(\widehat{W})$ within an $m^{O(1)}$ factor.

Theorems 1 and 2 remain true if the definition of the partition function $P(W)$ is modified as follows. Suppose that the set of vertices V is partitioned into a pairwise disjoint union $V = V_1 \cup \ldots \cup V_k$, where $|V_1| = \ldots = |V_k| = m$. We take the sum in (3) over all partitions $V = S_1 \cup \ldots \cup S_m$, where $|S_i \cap V_j| = 1$ for all i and j. For $k = 2$ the weight W is interpreted as an $m \times m$ matrix $W = (w_{ij})$ with rows indexed by the vertices of V_1 and columns indexed by the vertices of V_2. The corresponding expression $P(W)$ is known as the *permanent* of matrix W. In that case, the van der Waerden and Bregman-Minc inequalities for permanents imply that Theorems 1 and 2 can be somewhat strengthened: instead of requiring that W is α-balanced, we may require that $\max_{i,j} w_{ij} = O\left(m^{-1}\right)$. However, in the case of $k \geq 3$ and in the case of hafnians the condition that W is α-balanced seems to be unavoidable.

Proofs and some other applications can be found in [1].

References

[1] A. Barvinok and A. Samorodnitsky, *Computing the partition function for perfect matchings in a hypergraph*, preprint `arXiv:1009.2397` (2010), pp 19.

Hypergraph packing

Peter Keevash

(joint work with Richard Mycroft)

We show that if G is a 3-graph on n vertices with n divisible by 4 such that every pair of vertices is contained in at least $3n/4 + o(n)$ edges then G has a perfect tetrahedron packing, i.e. $n/4$ vertex-disjoint tetrahedra.

We also show that if G is an r-partite graph with n vertices in each part such that every vertex has at least $(r-1)n/r + o(n)$ neighbours in each part other than its own then G has a perfect packing by complete graphs of size r.

Both results are asymptotically best possible.

We prove them as part of a general framework for finding perfect matchings in an object we call an r-system. This consists of j-graphs for each $0 \leq j \leq r$ on the same set of vertices, related by a minimum degree condition on the number of $(j+1)$-edges containing any j-edge. We obtain an asymptotically best possible result for the minimum degree sequence that guarantees a perfect matching in an r-system with no 'divisibility barrier' based on a 'lattice construction'.

Colouring tournaments

Paul Seymour

(joint work with Eli Berger, Krzysztof Choromanski, Maria Chudnovsky, Jacob Fox, Martin Loebl, Alex Scott, and Stephan Thomassé)

A *tournament* is a finite digraph such that for every two distinct vertices u, v there is exactly one edge with ends $\{u, v\}$ (so, either the edge uv or vu but not both). If G is a tournament, $X \subseteq V(G)$ is *transitive* if the subtournament $G|X$ induced on X has no directed cycle. If $k \geq 0$, a *k-colouring* of a tournament G means a partition of $V(G)$ into at most k transitive subsets. The *chromatic number* $\chi(G)$ of a tournament G is the minimum k such that G admits a k-colouring.

If G, H are tournaments, we say G *contains* H if H is isomorphic to a subtournament of G, and otherwise G is *H-free.* For $0 \leq \epsilon \leq 1$, let us say a tournament H is *ϵ-timid* if there exists c such that $\chi(G) \leq c|V(G)|^\epsilon$ for every H-free tournament G. There is a famous open question about induced subgraphs of graphs, the Erdős-Hajnal conjecture [3], and it can be reformulated [1] in terms of colouring tournaments as follows.

1. Conjecture. *For every tournament H, there exists $\epsilon < 1$ such that H is ϵ-timid.*

This remains open; indeed, it is open for the five-vertex tournament H in which every vertex has out-degree two. (Maria Chudnovsky and I have checked it for all other tournaments with at most five vertices.) Working on this conjecture, we observed that many small tournaments are 0-timid, and this led us to ask for a complete characterization of the 0-timid tournaments, which is the subject of this talk. Full details are given in [2].

Let us call the 0-timid tournaments "heroes". Thus, a tournament H is a *hero* if there exists c (depending on H) such that every H-free tournament has chromatic number at most c. For instance, the tournament with three vertices forming a cyclic triangle is a hero; every tournament not containing it is 1-colourable. In fact there are infinitely many heroes, because of the following easy observation:

2. *Let H be a hero, and let J be the tournament obtained from H by adding a new vertex v adjacent from every vertex of H. Then J is a hero.*

Proof. Let $c \geq 0$ such that every tournament not containing H has chromatic number at most c. Now let G be a tournament not containing J. We claim its chromatic number is at most $c|V(H)|$. For we may assume that $G|X$ is isomorphic to H for some choice of X. ($G|X$ denotes the subtournament of G induced on X.) For each $x \in X$, let Y_x be the set of vertices in G adjacent to x. Then $G|Y_x$ does not contain H (because G does not contain J), and so $G|Y_x$ has chromatic number at most c, and therefore so does $G|(Y_x \cup \{x\})$. Hence the union of these $|X|$ sets induces a tournament with chromatic number at most $c|X|$. But the union of these sets is $V(G)$, since G does not contain J. This proves 2. ∎

In fact a much more general statement is true, the following, but its proof is much more complicated.

3. *Let H_1, H_2 be heroes, and let H be obtained from the disjoint union of H_1 and H_2 by making every vertex of H_2 adjacent from every vertex of H_1. Then H is a hero.*

We would like to list all heroes. Statement 2 above shows that there are infinitely many; and statement 3 shows that it suffices to list all strongly-connected heroes, because a tournament is a hero if and only if all its strong components are heroes. Let us see that not all tournament are heroes. If $A, B \subseteq V(H)$, we write $A \Rightarrow$

B if every vertex in B is adjacent from every vertex in A. A *trisection* of a tournament H is a partition (A, B, C) of $V(H)$, such that A, B, C are all nonempty and $A \Rightarrow B \Rightarrow C \Rightarrow A$. If $H|A, H|B, H|C$ are isomorphic to tournaments P, Q, R respectively, we write $H = \Delta(P, Q, R)$. T_k denotes the transitive tournament with k vertices.

4. *If H is a strongly-connected hero with $|V(H)| > 1$, then H admits a trisection (A, B, C) such that $|C| = 1$; and one of A, B is transitive.*

Proof. Define a sequence S_i ($i \geq 1$) of tournaments as follows. S_1 is the one-vertex tournament. Inductively, for $i \geq 2$, let $S_i = \Delta(S_{i-1}, S_{i-1}, T_1)$. It is easy to check that $\chi(S_i) \geq i$, and so there exists i such that S_i contains H. Choose i minimum. Now $i \geq 2$, and $S_i = \Delta(S_{i-1}, S_{i-1}, T_1)$; and $V(H)$ meets at least two sets of the corresponding trisection of S_i (because S_{i-1} does not contain H) and hence meets all three sets (because H is strongly-connected). This induces a trisection (A, B, C) of $V(H)$ with $|C| = 1$, and so proves the first part of statement 4. The second, that one of A, B is transitive, needs a different argument that we omit. ∎

Statement 4 turns out to be best possible, because of the following converse.

5. *If H is a hero, then for $k \geq 1$ so are $\Delta(H, T_k, T_1)$ and $\Delta(H, T_1, T_k)$.*

Again, the proof cannot be given here, but in combination with statement 4, this yields a complete characterization of heroes.

One could also ask for a weaker property; let us say a tournament H is a *celebrity* if there exists $c > 0$ such that every H-free tournament G has a transitive subset of cardinality at least $c|V(G)|$. Evidently every hero is a celebrity, but in fact the converse holds. Thus we have:

6. *A tournament is a celebrity if and only if it is a hero.*

Proof. We need to show that if H is a celebrity then it is a hero, and we prove this by induction on $|V(H)|$. By statement 3 above, we may assume that H is strongly-connected. By adapting the proof of statement 4 to apply to celebrities (define $S_i = \Delta(S_{i-1}, S_{i-1}, S_{i-1})$), it follows that H admits a trisection (A, B, C). If A, B, C all have more than one element, then H contains $\Delta(T_2, T_2, T_2)$, which therefore is also a celebrity; and we prove it is not, by a probabilistic construction that is perhaps the technically most complicated part of the paper. Thus we may assume that $|C| = 1$. But then the same argument as before (that we omitted) shows that one of A, B is transitive, and the result follows by applying statement 5 and the inductive hypothesis. ∎

In conclusion, let us mention two open questions; are the following two statements true?

7. *For all $k \geq 0$ there exists c such that, if G is a tournament in which the set of out-neighbours of each vertex has chromatic number at most k, then $\chi(G) \leq c$.*

8. *Let $H = \Delta(T_2, T_2, T_2)$. For all $\epsilon > 0$ there exists $c > 0$ such that every H-free tournament G has a transitive subset of cardinality at least $c|V(G)|^{1-\epsilon}$.*

We were unable to prove statement 7 even when $k = 3$. We observe that statement 8 is false when $\epsilon = 0$.

References

[1] N. Alon, J. Pach and J. Solymosi, *Ramsey-type theorems with forbidden subgraphs*, Combinatorica **21** (2001), 155–170.

[2] E. Berger, K. Choromanski, M. Chudnovsky, J. Fox, M. Loebl, A. Scott, P. Seymour, S. Thomassé, *Colouring tournaments*, submitted for publication (manuscript November 2010).

[3] P. Erdős and A. Hajnal, *Ramsey-type theorems*, Discrete Appl. Math. **25** (1989), 37–52.

Lower bounds for geometric ϵ-nets

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(joint work with János Pach)

Let $\mathcal{R}$ be a set system (range space). A set S is said to be *shattered* by $\mathcal{R}$ if every subset $H \subseteq S$ can be obtained as $H = S \cap R$ for a suitable $R \in \mathcal{R}$. The VC-dimension of $\mathcal{R}$ (named after Vapnik and Chervonenkis) is the maximum size of a set shattered by $\mathcal{R}$. An *ϵ-net* (also called strong ϵ-net) for the finite set S with respect to $\mathcal{R}$ is a set $H \subseteq S$ satisfying that if $R \in \mathcal{R}$ has $|R \cap S| \geq \epsilon|S|$, then we have $R \cap H \neq \emptyset$.

According to a well known theorem of Haussler and Welzl [6], with respect to a range space of bounded VC-dimension, any set admits an ϵ-net of size $O(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$. Using probabilistic techniques, Pach and Woeginger (1990) showed that there exist range spaces of VC-dimension 2, with respect to which the smallest size of an ϵ-net required by some sets is $\Omega(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$.

Geometric range spaces consisting of ranges of *constant description complexity* (e.g., balls, boxes, half-spaces, etc. in a Euclidean space) have bounded VC-dimension, so the Haussler Welzl theorem applies to them. However, in many instances there are smaller ϵ-nets than the ones whose existence guaranteed by this general result. In fact, it was widely believed that, with respect to geometric range spaces, there always exist *linear size* ϵ-nets (that is, ϵ-nets of size $O(\frac{1}{\epsilon})$). This conjecture was confirmed is several special cases, for instance, for *half-spaces* in dimensions 2 and 3, [7]. In a recent paper, Aronov Ezra and Sharir [2] proved that, with respect to axis-parallel boxes in dimensions 2 and 3, $O(\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$ size ϵ-nets exist.

The linear ϵ-net conjecture had to be revised after Alon [1] discovered some geometric range spaces of small VC-dimension, in which the ranges are straight lines, rectangles, or infinite strips in the plane, and which do not admit linear size ϵ-nets. Alon's construction is based on the density version of the Hales-Jewett theorem, due to Furstenberg and Katznelson [5], and recently improved in [9]. However, his lower bound is only barely superlinear: $\Omega(\frac{1}{\epsilon}g(\frac{1}{\epsilon}))$, where g is an extremely slowly growing function, closely related to the inverse Ackermann function.

We showed that the results in both [7] and [2] are best possible in the sense that (1) neither of them generalizes to dimension 4 or higher; (2) the log log factor in the second bound cannot be removed. More precisely, we have the following.

Theorem 1. *There exist point sets in the 4-dimensional Euclidean space, for which every ϵ-net with respect to axis-parallel boxes is of size $\Omega(\frac{1}{\epsilon}\log\frac{1}{\epsilon})$. The same result holds for ϵ-nets with respect to half-spaces in dimension 4.*

Theorem 2. *There exist point sets in the plane, for which every ϵ-net with respect to axis-parallel rectangles is of size $\Omega(\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$.*

For the proof of Theorem 2, it is sufficient to consider random point sets of a certain size in the unit square. The proof of Theorem 1 relies on dualizing an explicit construction of a collection of axis-parallel rectangles in the plane that have no small "dual ϵ-nets". For Theorem 1, our starting point is a result in [8] for hypergraph coloring. For Theorem 2, we strengthen [3].

Alon's lower bounds in [1] can be extended to *weak ϵ-nets*, i.e., to the case when the points of a net do not necessarily belong to the original point set. Our results do not generalize to this case. In fact, according to a result of Ezra [4], Theorem 1 cannot be extended to weak ϵ-nets, but it is possible that Theorem 2 can.

References

[1] N. Alon, *A non-linear lower bound for planar epsilon-nets*, in: Proc. 51st Annu. IEEE Sympos. Found. Comput. Sci. (FOCS '10), 2010, 341–346.

[2] B. Aronov, E. Ezra, M. Sharir, *Small-size epsilon-nets for axis-parallel rectangles and boxes*, SIAM J. Comput. **39** (2010), 3248–3282.

[3] X. Chen, J. Pach, M. Szegedy, G. Tardos, *Delaunay graphs of point sets in the plane with respect to axis-parallel rectangles*, Random Structures and Algorithms **34** (2009), 11–23.

[4] E. Ezra, *A note about weak ϵ-nets for axis-parallel boxes in d-space*, Information Processing Letters **110** (2010), 835–840.

[5] H. Furstenberg, Y. Katznelson, *A density version of the Hales-Jewett theorem*, J. Anal. Math. **57** (1991), 64–119.

[6] D. Haussler, E.Welzl, *ϵ-nets and simplex range queries*, Discrete and Computational Geometry **2** (1987), 127–151.

[7] J. Matoušek, R. Seidel, E. Welzl, *How to net a lot with little: Small ϵ-nets for disks and halfspaces*, In: Proc. 6th Annu. ACM Sympos. Comput. Geom., 1990, 16–22.

[8] J. Pach, G. Tardos, *Coloring axis-parallel rectangles*, J. Combin. Theory Ser. A **117** (2010), 776–782.

[9] D. H. J. Polymath, *A new proof of the density Hales-Jewett theorem*, preprint, available at arxiv.org/abs/0910.3926.

Probabilistic and deterministic vertex-coloring games

Reto Spöhel
(joint work with Torsten Mütze and Thomas Rast)

Introduction. Consider the following probabilistic online problem: The vertices of an initially hidden random graph $G_{n,p}$ are revealed one by one in increasing order, and at each step of the process only the edges induced by the vertices revealed so far are visible. Each vertex has to be colored immediately and irrevocably with one of r available colors as soon as it is revealed, and the goal is to color all n vertices without creating a monochromatic copy of some given fixed graph F in the process.

It follows from standard arguments that this problem has a threshold $p_0(F, r, n)$ in the following sense: For any function $p(n) = o(p_0)$ there is an online strategy that succeeds with probabiliy $1-o(1)$ in coloring $G_{n,p}$ as desired, and for any function $p(n) = \omega(p_0)$ the success probability of *any* online strategy is $o(1)$. The main goal when studying this and similar online problems is to determine $p_0(F, r, n)$ explicitly.

Related online problems have been studied by various authors, most notably for the so-called Achlioptas process [1, 2, 3, 6]. In the work presented here, we establish a one-to-one correspondence between the above probabilistic problem and a fairly natural *deterministic variant* of the same problem. To the best of our knowledge, this is the first time that such a correspondence has been observed.

Our result. Our main result characterizes the threshold for the online coloring problem described above in terms of the following *deterministic two-player game* played by two players called Builder and Painter. The board is a graph that grows in each step of the game. Starting with an empty board, in each step Builder presents a new vertex and a number of edges leading from previous vertices to this new vertex. Painter has to color the new vertex with one of r available colors immediately, and as before her goal is to avoid monochromatic copies of some given graph F. Note that so far this is the same setting as before, except that we replaced 'randomness' by the second player Builder. However, we additionally impose the restriction that Builder is not allowed to present an edge that would create a (not necessarily monochromatic) subgraph H with $e(H)/v(H) > d$ on the board, for some fixed real number d known to both players. We will refer to this game as the *deterministic F-avoidance game with r colors and density restriction d.*

We say that *Builder has a winning strategy* in this game (for a fixed graph F, a fixed number of colors r, and a fixed density restriction d) if he can force Painter to create a monochromatic copy of F within a finite number of steps. For any graph F and any integer $r \geq 2$ we define the *online vertex-Ramsey density* $m_1^*(F, r)$ as

$$(6) \qquad m_1^*(F,r) := \inf \left\{ d \in \mathbb{R} \;\middle|\; \begin{array}{l} \text{Builder has a winning strategy in the} \\ \text{deterministic } F\text{-avoidance game with } r \\ \text{colors and density restriction } d \end{array} \right\} .$$

It is not hard to see that $m_1^*(F,r)$ is indeed well-defined for any F and r. With these definitions in hand, our results can be stated as follows.

Theorem 1. *For any graph F with at least one edge and any integer $r \geq 2$, the online vertex-Ramsey density $m_1^*(F,r)$ is a computable rational number, and the infimum in* (6) *is attained as a minimum.*

Theorem 2. *For any fixed graph F with at least one edge and any fixed integer $r \geq 2$, the threshold for finding an r-coloring of $G_{n,p}$ that does not contain a monochromatic copy of F in the online setting is*

$$p_0(F,r,n) = n^{-1/m_1^*(F,r)} ,$$

where $m_1^*(F,r)$ is defined in (6).

Theorem 2 reduces the problem of determining the threshold of the probabilistic problem to the purely deterministic combinatorial problem of determining $m_1^*(F,r)$ or, informally speaking, of 'solving' the deterministic two-player game. According to Theorem 1, the latter is possible by a finite computation.

The proof of the upper bound in Theorem 2 is fairly elementary, and generic in the sense that it does not require any real understanding of what happens in the deterministic game. In contrast, our proof of the lower bound in Theorem 2 is deeply intertwined with the proof of Theorem 1 and relies very much on structural properties of Painter's and Builder's optimal strategies in the deterministic game.

The online vertex-Ramsey density $m_1^*(F,r)$ and the corresponding optimal Builder and Painter strategies show a surprisingly complex (only partially understood) behaviour even for the innocent-looking case where F is a long path [4]. In view of this, we see little hope of replacing the abstract definition (6) by a more explicit formula.

An open question. The obvious question raised by our results is whether analogues of Theorem 1 and Theorem 2 hold in other similar settings, in particular for the edge-coloring setting first studied by Friedgut et al. [2]. The methods used in this work do not suffice to answer this question, as they are based on structural properties of the deterministic vertex-coloring game that do no hold for its edge-coloring counterpart.

We mention that such analogues are indeed true for the problem of avoiding small subgraphs in the Achlioptas process, which was first studied by Krivelevich et al. [3] and solved completely in our earlier work [5].

References

[1] T. Bohman and A. Frieze. Avoiding a giant component. *Random Structures Algorithms*, 19(1):75–85, 2001.

[2] E. Friedgut, Y. Kohayakawa, V. Rödl, A. Ruciński, and P. Tetali. Ramsey games against a one-armed bandit. *Combin. Probab. Comput.*, 12(5-6):515–545, 2003. Special issue on Ramsey theory.

[3] M. Krivelevich, P.-S. Loh, and B. Sudakov. Avoiding small subgraphs in Achlioptas processes. *Random Structures Algorithms*, 34:165–195, 2009.

[4] T. Mütze and R. Spöhel. On the path-avoidance vertex-coloring game. In preparation.

[5] T. Mütze, R. Spöhel, and H. Thomas. Small subgraphs in random graphs and the power of multiple choices. *J. Combin. Theory Ser. B*, to appear.
[6] J. Spencer and N. Wormald. Birth control for giants. *Combinatorica*, 27(5):587–628, 2007.

Disjoint paths in tournaments

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(joint work with Alex Scott and Paul Seymour)

The question of linking pairs of terminals by disjoint paths is a standard and well-studied question in graph theory. The setup is: given a graph G and vertices $s_1, \ldots, s_k$ and $t_1, \ldots, t_k$, is there a set of disjoint path $P_1, \ldots, P_k$ in G such that P_i is a path from s_i to t_i for every $i \in \{1, \ldots, k\}$? This question makes sense in both directed and undirected graphs, and the paths may be required to be edge- or vertex-disjoint.

For undirected graphs, a polynomial-time algorithm for solving both the edge-disjoint and the vertex-disjoint version of the problem (where the number k of terminals is fixed) was first found by Robertson and Seymour [6], and is a part of their well-known Graph Minors project. For directed graphs, both problems are NP-complete, even when $k = 2$ (by a result of Fortune, Hopcroft and Wyllie [4]). However, if we restrict our attention to tournaments (these are directed graphs with exactly one arc between every two vertices), the situation improves. Polynomial time algorithms for solving the edge-disjoint and the vertex-disjoint paths problems when $k = 2$ have been known for a while (these are results of Bang-Jensen [1], and Bang-Jensen and Thomassen [2], respectively).

Last year, Fradkin and Seymour [5] were able to design a polynomial-time algorithm to solve the edge-disjoint paths problem in tournaments for general (fixed) k, using a new parameter for tournaments, developed by Chudnovsky and Seymour, called "cut-width" [3]. However, the vertex-disjoint paths problem seemed to be resistant to similar methods.

The goal of this talk was to describe a polynomial-time algorithm to solve the vertex-disjoint paths problem in tournaments for general (fixed) k. We can actually solve a more general question, as follows (a digraph is *semicomplete* if for all distinct u, v, at least one of uv, vu is an edge):

Theorem 1. *For all k, there is a polynomial-time algorithm as follows:*

- **Input:** *A semicomplete digraph G, vertices $s_1, t_1 \ldots s_k, t_k$ of G, and integers $x_1 \ldots x_k \geq 0$.*
- **Output:** *Decides whether there exist pairwise vertex-disjoint directed paths $P_1 \ldots P_k$ of G such that for $1 \leq i \leq k$, P_i is from s_i to t_i and has at most x_i vertices.*

A *linkage* in a digraph G is a family $(P_i : 1 \leq i \leq k)$ of vertex-disjoint directed paths, and it is a linkage *for* $(G, s_1, t_1 \ldots s_k, t_k)$ if P_i is from s_i to t_i for each i. Let us say that a $2k+1$-tuple $(G, s_1, t_1 \ldots s_k, t_k)$ is *critical* if there exists a linkage for $(G, s_1, t_1 \ldots s_k, t_k)$, but for every $v \in V(G)$ there is no linkage for

$(G\backslash v, s_1, t_1 \ldots s_k, t_k)$ (where $G\backslash v$ is the graph obtained from G by deleting v). The first subroutine of our algorithm uses dynamic programming to find a linkage given a critical $2k+1$-tuple $(G', s_1', t_1' \ldots s_k', t_k')$ (this subroutine has two possible outputs: a linkage, or a determination that the $2k+1$-tuple is not critical). Simplifying a few technical steps, the main algorithm uses the this subroutine to construct an auxiliary directed graph $\mathcal{A}$, so that there is a linkage for $(G, s_1, t_1 \ldots s_k, t_k)$ if and only if there is a path between two specified vertices of $\mathcal{A}$.

References

[1] J. Bang-Jensen, "Edge-disjoint in- and out-branchings in tournaments and related path problems", *J. Combin. Theory Ser. B*, 51 (1991), 1-23.

[2] Jørgen Bang-Jensen and Carsten Thomassen , "A polynomial algorithm for the 2-path problem for semicomplete digraphs", *SIAM Journal on Discrete Mathematics* 5 (1992), 366–376.

[3] M. Chudnovsky and P. Seymour, "A well-quasi-order for tournaments", *J. Combinatorial Theory, Ser. B*, 101 (2011), 47-53.

[4] S.Fortune, J.Hopcroft and J.Wyllie, "The directed subgraphs homeomorphism problem", *Theoret. Comput. Sci.* 10 (1980), 111–121.

[5] Alexandra Fradkin and Paul Seymour, "The edge-disjoint paths problem in tournaments", in preparation.

[6] N. Robertson and P.D. Seymour, "Graph minors. XIII. The disjoint paths problem", *J. Combinatorial Theory, Ser. B*, 63 (1995), 65–110.

Edge-isoperimetric inequalities in the discrete cube

Alex Samorodnitsky

The edge-isoperimetric inequality for the Hamming cube $\{0,1\}^n$ [6] states that for any subset A of the cube holds

$$|E\left(A, A^c\right)| \geq |A| \log_2 \left(\frac{2^n}{|A|}\right) \tag{7}$$

Here $|E\left(A, A^c\right)|$ counts the edges between A and its complement.
Non-uniform versions of edge-isoperimetry were considered in [2], who introduced the notion of influence of variables on boolean functions. For $1 \leq i \leq n$, let $I_i(A)$ count the number of edges in direction i between A and its complement. (Then $|E\left(A, A^c\right)| = \sum_{i=1}^n I_i$.) Ben-Or and Linial conjectured that any set A, with $|A| \leq 2^{n-1}$, has a variable with large influence, more precisely $\max_i I_i \geq \Omega\left(\frac{\log n}{n}\right) \cdot |A|$. Note that a straightforward application of the edge-isoperimetric inequality (7) gives only $\max_i I_i \geq \Omega\left(\frac{1}{n}\right) \cdot |A|$.
This conjecture was proved in [7] using Fourier analysis and the hypercontractive inequality of [1, 3, 5]. In fact, Kahn, Kalai, and Linial proved more, namely, for $|A| \leq 2^{n-1}$:

$$\sum_{i=1}^n I_i^2 \geq \Omega\left(\frac{\log^2(n)}{n}\right) \cdot |A|^2 \tag{8}$$

An even stronger inequality was proved by Talagrand [9]. A special case of this inequality states (for $|A| \leq 2^{n-1}$):

$$\sum_{i=1}^{n} \frac{I_i}{\ln\left(\frac{2^n}{I_i}\right)} \geq \Omega(|A|) \tag{9}$$

In [4] we observed that (8) can be derived from a functional version of (7). Let $f: \{0,1\}^n \to \mathbb{R}$ be a real-valued function on the cube, and let $\mathcal{E}(f,f) = \sum_{x\sim y}(f(x)-f(y))^2$ be the Dirichlet quadratic form on the cube (the summation is over pairs of connected vertices). Then

$$\mathcal{E}(f,f) \geq 2\ln(2) \cdot \sum_x f^2(x) \cdot \log_2\left(\frac{2^n \cdot \sum_x f^2(x)}{\left(\sum_x |f(x)|\right)^2}\right) \tag{10}$$

This inequality was independently proved in [10]. Let us also mention that it can be derived from the logarithmic Sobolev inequality [5]:

$$\mathcal{E}(f,f) \geq 2 \cdot \left(\sum_x f^2(x)\ln\left(f^2(x)\right) - \sum_x f^2(x) \cdot \ln\left(\frac{1}{2^n} \cdot \sum_x f^2(x)\right)\right) \tag{11}$$

Substituting $f = 1_A$ in (10) or in (11) recovers (7), *up to a constant.* This disparity in constants is the main point we address in this work. We are looking for functional inequalities on the cube, which reduce to (7) when substituting $f = 1_A$. In particular, we conjecture the following inequality to hold for any subset $A \subseteq \{0,1\}^n$ and for any function $f: \{0,1\}^n \to \mathbb{R}$:

$$\mathcal{E}(f,f) + 4\sum_x f^2(x) \geq 2 \cdot \left(\frac{\log_2\left(\frac{2^n}{|A|}\right)}{|A|} \cdot \left(\sum_{x\in A}|f(x)|\right)^2\right) \tag{12}$$

This conjectured inequality is a slightly modified dual form of a special case of an inequality in [9], which was used to prove (9). It is possible to prove with $2\ln 2$ replacing 2 on the RHS, deriving it from (11). However, we require a tight constant, which, we conjecture, should be 2. If this holds, then it would imply a tight version of (9):

$$\sum_{i=1}^{n} \frac{I_i}{\ln\left(\frac{2^{n-1}}{I_i}\right)} \geq 2 \cdot |A|$$

and of (8)

$$\sum_{i=1}^{n} I_i^2 \geq 4 \cdot \left(\frac{\log_2^2(n)}{n}\right) \cdot |A|^2$$

We can only prove a special case of (12), in which f is required to be supported on A. In this case, a slightly stronger statement is true [8]:

$$\mathcal{E}(f,f) \geq 2 \cdot \left(\frac{\log_2\left(\frac{2^n}{|A|}\right)}{|A|} \cdot \left(\sum_{x\in A}|f(x)|\right)^2\right) \tag{13}$$

Substituting $f = 1_A$ recovers (7).
We note that (13) is an inequality between quadratic forms. This allows a random walk interpretation. Let Y be a random variable defined as follows: choose a uniformly random point $a \in A$ and consider the random walk in $\{0,1\}^n$ starting from a. Then Y measures the time it takes the walk to exit A for the first time. We refer to $\mathbb{E}Y$ as the *expected exit time* of A. This is a parameter of a subset A of the cube.
The following claim is equivalent to (13): Subcubes maximize the expected exit time among all subsets of the cube of the same cardinality.
More precisely, for any subset A of $\{0,1\}^n$,

$$\mathbb{E}Y \le \frac{n}{\log_2\left(\frac{2^n}{|A|}\right)}$$

If A is a subcube, this is an equality.

References

[1] W. Beckner, *Inequalities in Fourier Analysis*, Annals of Math., 102(1975), pp. 159-182.
[2] M. Ben-Or, N. Linial, *Collective Coin Flipping*, in **Randomness and Computation** (S. Micali, ed.), Academic Press.
[3] A. Bonami, *Etude des coefficients Fourier des fonctiones de $L^p(G)$*, Ann. Inst. Fourier (Grenoble) 20:2 (1970), pp. 335-402.
[4] D. Falik, A. Samorodnitsky, *Edge-isoperimetric inequalities and influences*, Comb.,Prob., and Comp. 16 (2007), pp. 693-712.
[5] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. of Math., 97 (1975), pp. 1061-1083.
[6] L. H. Harper, *Optimal numberings and isoperimetric problems on graphs*, J. Comb. Theory 1, 1966, pp. 385-393.
[7] J. Kahn, G. Kalai, and N. Linial, *The influence of variables on boolean functions*, FOCS 1988, pp. 68-80.
[8] A. Samorodnitsky, *An isometric-type inequality in the Hamming cube*, manuscript, 2009.
[9] M. Talagrand, *On Russo's approximate $0-1$ law*, The Annals of Prob., 22, 3 (1994), pp. 1576-1587.
[10] R. Montenegro, P. Tetali, **Mathematical Aspects of Mixing Times in Markov Chains**, Foundations and Trends in Theoretical Computer Science, 2006.

Exponentially many perfect matchings in cubic graphs

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(joint work with Louis Esperet, František Kardoš, Andrew D. King, and Serguei Norine)

Given a graph G, let $\mathcal{M}(G)$ denote the set of perfect matchings in G. A classical theorem of Petersen [10] states that every cubic bridgeless graph has at least one perfect matching, i.e. $\mathcal{M}(G) \neq \emptyset$. Indeed, it can be proven that any edge in a cubic bridgeless graph is contained in some perfect matching [11], which implies that $|\mathcal{M}(G)| \ge 3$.

In the 1970s, Lovász and Plummer conjectured that the number of perfect matchings of a cubic bridgeless graph G should grow exponentially with its order (see [8, Conjecture 8.1.8]). It is a simple exercise to prove that G contains at most $2^{|V(G)|}$ perfect matchings, so we can state the conjecture as follows:

Conjecture 1. *There exists a universal constant $\epsilon > 0$ such that for any cubic bridgeless graph G,*

$$2^{\epsilon|V(G)|} \leq |\mathcal{M}(G)| \leq 2^{|V(G)|}.$$

The problem of computing $|\mathcal{M}(G)|$ is connected to problems in molecular chemistry and statistical physics (see e.g. [8, Section 8.7]). In general graphs, this problem is $\sharp P$-complete [13]. Thus we are interested in finding good bounds on the number of perfect matchings for various classes of graphs such as the bounds in the conjecture above.

For bipartite graphs, $|\mathcal{M}(G)|$ is precisely the permanent of the graph biadjacency matrix. Voorhoeve proved the conjecture for cubic bipartite graphs in 1979 [14]; Schrijver later extended this result to all regular bipartite graphs [12]. We refer the reader to [7] for an exposition of this connection and of an elegant proof of Gurvits generalizing Schrijver's result. For *fullerene graphs*, a class of planar cubic graphs for which the conjecture relates to molecular stability and aromaticity of fullerene molecules, the problem was settled by Kardoš, Král', Miškuf and Sereni [5]. Chudnovsky and Seymour recently proved the conjecture for all cubic bridgeless planar graphs [1].

The general case has until now remained open. Edmonds, Lovász and Pulleyblank [2] proved that any cubic bridgeless G contains at least $\frac{1}{4}|V(G)| + 2$ perfect matchings (see also [9]); this bound was later improved to $\frac{1}{2}|V(G)|$ [6] and then $\frac{3}{4}|V(G)| - 10$ [4]. The order of the lower bound was not improved until Esperet, Kardoš, and Král' proved a superlinear bound in 2009 [3]. The first bound, proved in 1982, is a direct consequence of a lower bound on the dimension of the perfect matching polytope, while the more recent bounds combine polyhedral arguments with analysis of brick and brace decompositions.

We solve the general case. To avoid technical difficulties when contracting sets of vertices, we henceforth allow graphs to have multiple edges, but not loops. Let $m(G)$ denote $|\mathcal{M}(G)|$, and let $m^\star(G)$ denote the minimum, over all edges $e \in E(G)$, of the number of perfect matchings containing e. Our result is the following:

Theorem 2. *For every cubic bridgeless graph G we have $m(G) \geq 2^{|V(G)|/3656}$.*

We actually prove that at least one of two sufficient conditions applies:

Theorem 3. *For every cubic bridgeless graph G, at least one of the following holds:*

- [**S1**] $m^\star(G) \geq 2^{|V(G)|/3656}$, *or*
- [**S2**] *there exist $M, M' \in \mathcal{M}(G)$ such that $M \triangle M'$ has at least $|V(G)|/3656$ components.*

To see that Theorem 3 implies Theorem 2, we can clearly assume that [**S2**] holds since $m^\star(G) \leq m(G)$. Choose $M, M' \in \mathcal{M}(G)$ such that the set $\mathcal{C}$ of components of $M \triangle M'$ has cardinality at least $|V(G)|/3656$, and note that each of these components is an even cycle alternating between M and M'. Thus for any subset $\mathcal{C}' \subseteq \mathcal{C}$, we can construct a perfect matching $M_{\mathcal{C}'}$ from M by flipping the edges on the cycles in $\mathcal{C}'$, i.e. $M_{\mathcal{C}'} = M \triangle \bigcup_{C \in \mathcal{C}'} C$. The $2^{|\mathcal{C}|}$ perfect matchings $M_{\mathcal{C}'}$ are

References

[1] M. Chudnovsky and P. Seymour, *Perfect matchings in planar cubic graphs*, Combinatorica, to appear.

[2] J. Edmonds, L. Lovász, W. R. Pulleyblank, *Brick decompositions and the matching rank of graphs*, Combinatorica **2** (1982), 247–274.

[3] L. Esperet, D. Král', F. Kardoš, *A superlinear bound on the number of perfect matchings in cubic bridgeless graphs*, manuscript (2010).

[4] L. Esperet, D. Král', P. Škoda, and R. Škrekovski, *An improved linear bound on the number of perfect matchings in cubic graphs*, European J. Combin., **31** (2010), 1316–1334.

[5] F. Kardoš, D. Král', J. Miškuf, and J.-S. Sereni, *Fullerene graphs have exponentially many perfect matchings*, J. Math. Chem. **46** (2009), 443–447.

[6] D. Král', J.-S. Sereni, and M. Stiebitz, *A new lower bound on the number of perfect matchings in cubic graphs*, SIAM J. Discrete Math. **23** (2009), 1465–1483.

[7] M. Laurent and L. Schrijver, *On Leonid Gurvits' proof for permanents*, Amer. Math. Monthly **117(10)** (2010), 903–911.

[8] L. Lovász and M. D. Plummer, Matching theory, Elsevier Science, Amsterdam, 1986.

[9] D. Naddef, *Rank of maximum matchings in a graph*, Math. Programming **22** (1982), 52–70.

[10] J. Petersen, *Die Theorie der regulären graphs*, Acta Math. **15** (1891), 193–220.

[11] J. Plesník, *Connectivity of regular graphs and the existence of 1-factors*, Mat. Časopis Sloven. Akad. Vied **22** (1972), 310–318.

[12] A. Schrijver, *Counting* 1*-factors in regular bipartite graphs*, J. Combin. Theory Ser. B **72** (1998), 122–135.

[13] L. Valiant, *The complexity of computing the permanent*, Theor. Comput. Sci. **8** (1979), 189–201.

[14] M. Voorhoeve, *A lower bound for the permanents of certain* $(0,1)$*-matrices*, Nederl. Akad. Wetensch. Indag. Math. **41** (1979), 83–86.

A proof of the Simonovits-Sós conjecture on triangle-intersecting families of graphs

Ehud Friedgut

(joint work with David Ellis, Yuval Filmus)

A family of graphs $\mathcal{F}$ is *triangle-intersecting* if for every $G, H \in \mathcal{F}$, $G \cap H$ contains a triangle. A conjecture of Simonovits and Sós from 1976 states that the largest triangle-intersecting families of graphs on a fixed set of n vertices are those obtained by fixing a specific triangle and taking all graphs containing it, resulting in a family of size $\frac{1}{8}2^{\binom{n}{2}}$. We prove this conjecture and some generalizations.

The generalizations include the following features:

- One may replace the assumption of intersecting in a triangle with "agreeing" on a triangle, i.e. for every G, H in the family the complement of their symmetric difference contains a triangle.

- One may replace the assumption of agreeing on a triangle to agreeing on a non-bipartite graph (any two graphs in the family agree on some odd cycle).
- One may replace graphs by not-necessarily-uniform hypergraphs. The precise formulation may be found in the paper which is now available online.
- One may replace the counting measure (the uniform measure) with which the size of the family is measured by the product measure $G(n, p)$, for any $p \leq 1/2$, and obtain a bound of p^3 instead of $1/8$.

In addition we prove a stability version of all these generalizations. In other words, if a triangle-intersecting family is close enough in size to the size of a maximal one, then it is also close in structure to a maximal one.

The methods we use are spectral methods, and discrete Fourier analysis. We define a Cayley graph on the group $\mathbb{Z}^{\binom{n}{2}}$ which translates the notion of a triangle-intersecting family to that of an independent set. We then wish to use eigenvalue bounds in order to bound the size of the largest independent sets, and characterize them - however, for this to be useful we need to define an appropriate weighting on the edges of the graph. This enables us not only to obtain the bound of 1/8, but also to characterize the Fourier transform of the maximal families, and use this knowledge to characterize them as desired.

Asymptotic enumeration of sparse 2-connected graphs and strongly connected digraphs

Nicholas Wormald

(joint work with Graeme Kemkes, Cristiane Sato, and Xavier Pérez-Giménez)

This talk reports joint work with Graeme Kemkes and Cristiane Sato on graphs, and with Xavier-Pérez Giménez on digraphs.

Firstly, we determine an asymptotic formula for the number of labelled 2-connected (simple) graphs on n vertices and m edges, provided that $m - n \to \infty$ and $m = O(n \log n)$ as $n \to \infty$. This is the entire range of m not covered by previous results. The proof involves determining properties of the core and kernel of random graphs with minimum degree at least 2. The case of 2-edge-connectedness is treated similarly. We also obtain formulae for the number of 2-connected graphs with given degree sequence for most ('typical') sequences.

Secondly, we derive an asymptotic formula for the number of strongly connected digraphs with n vertices and m arcs (directed edges), valid for $m - n \to \infty$ as $n \to \infty$ provided $m = O(n \log n)$. This fills the gap between other results of Wright's, which apply to $m = n + O(1)$, and the long-known threshold for m, above which a random digraph with n vertices and m arcs is likely to be strongly connected.

Call a (simple) graph on the vertex set $[n] = \{1, \ldots, n\}$ with m edges an (n, m)-graph (and similarly for digraphs). A number of authors have addressed the problem of counting connected (n, m)-graphs. After results by various authors for various ranges of m with various degrees of approximation, Bender, Canfield and McKay [1] provided an asymptotic formula for the number whenever $m-n \to \infty$ as $n \to \infty$. They obtained this formula by studying a differential equation related to a recurrence relation for the number of connected graphs. Pittel and Wormald [6] provided a somewhat simpler proof for this formula, with an improved error term for some ranges of m.

A natural next step would be to count k-connected (n, m)-graphs. This problems turns out to be essentially already solved for $k \geq 3$. Łuczak [3] showed that a random graph with given degree sequence, all degrees between 3 and d, a.a.s. (asymptotically almost surely) has connectivity equal to minimum degree. As observed in the introduction of [5], this implies that, for $m = O(n \log n)$, a random (n, m)-graph with minimum degree $k \geq 3$ is a.a.s. k-connected. (To deduce this, one needs to know that such a random (n, m)-graph has no large degree vertices, which can be deduced from the results of [5], or alternatively by a more direct argument if m/n is bounded.) Thus, using the above-mentioned result from [5], one immediately obtains an asymptotic formula for k-connected (n, m)-graphs. However, this argument does not apply for 2-connected graphs.

The new result is an asymptotic formula for the number $T(n, m)$ of 2-connected (n, m)-graphs when $m - n \to \infty$ with $m = O(n \log n)$. Above this range, for any fixed k, it is well known that almost all graphs are k-connected. This follows by the classic result of Erdős and Rényi [2], that for fixed $k \geq 0$ and $m = m(n) = \frac{1}{2}n(\log n + k \log\log n + x + o(1))$,

$$\mathbf{P}(\mathcal{G}(n, m) \text{ is } k\text{-connected}) \to 1 - e^{-e^{-x}/k!},$$

where $\mathcal{G}(n, m)$ denotes an (n, m)-graph chosen uniformly at random.

For the statement of our results we define the odd falling factorial $(2m-1)!! = (2m-1)(2m-3)\cdots 1$, and the average degree $c = 2m/n$. Define $g(\lambda) = \lambda(e^\lambda - 1)/(e^\lambda - 1 - \lambda)$. Then g is an increasing function with $g(\lambda) \to 2$ as $\lambda \to 0$. Since $c > 2$, we may let λ_c be the (unique) positive root of

$$g(\lambda) = \frac{\lambda(e^\lambda - 1)}{e^\lambda - 1 - \lambda} = c,$$

and we set

$$\bar{\eta}_c = \frac{\lambda_c e^{\lambda_c}}{e^{\lambda_c} - 1} \quad \text{and} \quad p_c = \frac{\lambda_c^2}{2\left(e^{\lambda_c} - 1 - \lambda_c\right)}.$$

Our main result (with Kemkes and Sato) is the following. Suppose $m = O(n \log n)$ and $m - n \to \infty$. Then

$$T(n, m) \sim (2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m}\sqrt{2\pi n c(1 + \bar{\eta}_c - c)}} \sqrt{\frac{c - 2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right).$$

We prove this using different arguments for various ranges of m. In particular if $m = O(n)$ we use the kernel configuration model of [6] to find the probability that a random graph with given degree sequence is 2-connected.

Wright [8] found an asymptotic formula for the case $m - n = o(\sqrt{n})$ with $m - n \to \infty$, and it was noted that the problem of finding a formula for $m - n$ growing faster than $\sqrt{n}$ seems difficult. His formula was

$$T(n, m) = a\sqrt{6\pi}n^{n+3k-1/2}e^{2k-n}(18k^2)^{-k}(1 + O(k^{-1}) + O(k^2/n)),$$

where a is a constant. Wright gave a method of estimating a, and computed it to be $0.058549831\ldots$. From our result above it is straightforward to determine a to be $1/(2e\pi)$.

Palásti [4] determined the threshold of strong connectivity of digraphs, as follows. Let α be fixed and define $m(\alpha, n) = \lfloor n \log n + \alpha n \rfloor$. Then, for a random directed graph having n vertices and m arcs (with loops permitted but no multiple arcs), so that each of the $\binom{n^2}{N}$ possible choices is equiprobable, the probability that the digraph is strongly connected tends to $\exp(-2e^{-\alpha})$ as $n \to \infty$. Multiplying this probability by $\binom{n^2}{N}$ consequently gives an asymptotic formula for the number $S(n, m)$ of strongly connected digraphs with n vertices and m arcs, for such m. This also easily implies that $S(n, m) \sim \binom{n^2}{N}$ if $m = m(\alpha_n, n)$ with $\alpha_n \to \infty$. On the other hand, Wright [7] obtained recurrences for the exact value of $S(n, m)$ when $m = n + O(1)$. (We must require $m \geq n$ to avoid the failure to be strongly connected for trivial reasons.) In this paper, we fill the entire gap between these results, deriving an asymptotic formula for $S(n, m)$, valid for $m - n \to \infty$ as $n \to \infty$ provided $m = O(n \log n)$. Our main result, with Pérez-Giménez, is as follows. Uniformly for $m = O(n \log n)$ and $m - n \to \infty$, the number of strongly connected digraphs with n vertices and m arcs is asymptotic to

$$\frac{(m-1)!(e^\lambda - 1)^{2n}}{2\pi(1 + \lambda - c)\lambda^{2m}} \exp(-\lambda^2/2) \frac{e^\lambda(e^\lambda - 1 - \lambda)^2}{(e^{2\lambda} - e^\lambda - \lambda)(e^\lambda - 1)}, \tag{14}$$

where $c = m/n > 1$ and λ is determined by the equation $c = \lambda e^\lambda/(e^\lambda - 1)$. We use different arguments for various ranges of m. In particular, for $m - n = o(n)$, we use an analogue of the kernel configuration model mentioned above.

References

[1] E.A. Bender, E.R. Canfield, B.D. McKay, *The asymptotic number of labeled connected graphs with a given number of vertices and edges*, Random Structures Algorithms **1** (1990), pp. 127–169.

[2] Erdős, P. and Rényi, A., *On the strength of connectedness of a random graph*, Acta Math. Acad. Sci. Hungar. **12** (1961), pp. 261–267.

[3] T. Łuczak, *Sparse random graphs with a given degree sequence*, Random graphs, Vol. 2 (Poznań, 1989). Wiley-Intersci. Publ., Wiley, New York, 1992, pp. 165–182.

[4] I. Palásti, On the strong connectedness of directed random graphs, *Studia Sci. Math. Hungar.* **1** (1966), 205–214.

[5] B.Pittel and N.C. Wormald, *Asymptotic enumeration of sparse graphs with a minimum degree constraint*, J. Combin. Theory Ser. A **101** (2003), 249–263.

[6] B. Pittel and N.C. Wormald, Counting connected graphs inside-out, *J. Combinatorial Theory, Series B* **93** (2005), 127–172.
[7] E.M. Wright, Formulae for the number of sparsely-edged strong labelled digraphs, *Quart. J. Math. Oxford Ser. (2)* **28** (1977), 363–367.
[8] E.M. Wright, *The number of connected sparsely edged graphs. IV. Large nonseparable graphs*, J. Graph Theory **7** (1983), 219–229.

An approximate version of Sidorenko's conjecture

David Conlon
(joint work with Jacob Fox and Benny Sudakov)

A fundamental problem in extremal graph theory is to determine or estimate the minimum number of copies of a graph H which must be contained in another graph G of a certain order and size. The special case where one wishes to determine the minimum number of edges in a graph on N vertices which guarantee a single copy of H has received particular attention. The case where H is a triangle was solved by Mantel more than a century ago. This was generalized to cliques by Turán and the Erdős-Stone-Simonovits theorem determines the answer asymptotically if H is not bipartite. For bipartite graphs H, a classical result of Kővári, Sós, and Turán implies that $O(N^{2-\epsilon_H})$ edges are sufficient for some $\epsilon_H > 0$, but, despite much effort by researchers, the asymptotics, and even good estimates for the largest possible ϵ_H, are understood for relatively few bipartite graphs.

The general problem can be naturally stated in terms of subgraph densities. The *edge density* of a graph G with N vertices and M edges is $M/\binom{N}{2}$. More generally, the H*-density* of a graph G is the fraction of all one-to-one mappings from the vertices of H to the vertices of G which map edges of H to edges of G. The general extremal problem asks for the minimum possible H-density over all graphs on N vertices with edge density p. For fixed H, the asymptotic answer as $N \to \infty$ is a function of p. Determining this function is a classical problem and notoriously difficult even in the case where H is the complete graph of order r. Early results in this case were obtained by Erdős, Goodman, Lovász, Simonovits, Bollobás, and Fisher. Recently, Razborov [4] using flag algebras and Nikiforov [3] using a combination of combinatorial and analytic arguments gave an asymptotic answer in the cases $r = 3$ and $r = 4$, respectively.

There is a simple upper bound on the minimum H-density in terms of the edge density. Suppose that H has m edges. By taking G to be a random graph with edge density p, it is easy to see that the minimum possible H-density is at most p^m. The beautiful conjectures of Erdős and Simonovits [6] and Sidorenko [5] suggest that this bound is sharp for bipartite graphs. That is, for any bipartite H there is a $\gamma(H) > 0$ such that the number of copies of H in any graph G on N vertices with edge density $p > N^{-\gamma(H)}$ is asymptotically at least the same as in the N-vertex random graph with edge density p. This is known to be true in a few very special cases, e.g., for complete bipartite graphs, trees, even cycles (see [5]) and, recently, for cubes [2].

The original formulation of the conjecture by Sidorenko is in terms of graph homomorphisms. A *homomorphism* from a graph H to a graph G is a mapping $f : V(H) \to V(G)$ such that, for each edge (u, v) of H, $(f(u), f(v))$ is an edge of G. Let $h_H(G)$ denote the number of homomorphisms from H to G. We also consider the normalized function $t_H(G) = h_H(G)/|G|^{|H|}$, which is the fraction of mappings $f : V(H) \to V(G)$ which are homomorphisms.

Conjecture 1 (Sidorenko)**.** *For every bipartite graph H with m edges and every graph G,*

$$t_H(G) \geq t_{K_2}(G)^m.$$

We extend the class of graphs for which Sidorenko's conjecture is true, as follows.

Theorem 2. *Sidorenko's conjecture holds for every bipartite graph H which has a vertex complete to the other part.*

From Theorem 2, we may easily deduce an approximate version of Sidorenko's conjecture for all graphs. For a connected bipartite graph H with parts V_1, V_2, define the bipartite graph $\bar{H}$ with parts V_1, V_2 such that $(v_1, v_2) \in V_1 \times V_2$ is an edge of $\bar{H}$ if and only if it is not an edge of H. Define the *width* of H to be the minimum degree of $\bar{H}$. If H is not connected, the width of H is the sum of the widths of the connected components of H. Note that the width of a connected bipartite graph is 0 if and only if it has a vertex that is complete to the other part. Also, the width of a bipartite graph with n vertices is at most $n/2$.

Corollary 3. *If H is a bipartite graph with m edges and width w, then $t_H(G) \geq t_{K_2}(G)^{m+w}$ holds for every graph G.*

Our methods also allow us to contribute to the theory of quasirandomness. A sequence $(G_n : n = 1, 2, \ldots)$ of graphs is called *quasirandom with density* p (where $0 < p < 1$) if, for every graph H,

$$t_H(G_n) = (1 + o(1))p^{|E(H)|}. \tag{15}$$

Note that (15) is equivalent to saying that the H-density of G_n is $(1+o(1))p^{|E(H)|}$, since the proportion of mappings from $V(H)$ to $V(G_n)$ which are not one-to-one tends to 0 as $|V(G_n)| \to \infty$. This property is equivalent to many other properties shared by random graphs. One such property is that the edge density between any two vertex subsets of G_n of linear cardinality is $(1 + o(1))p$. A surprising fact, proved in [1], is that it is enough that (15) holds for $H = K_2$ and $H = C_4$ for a graph to be quasirandom. That is, a graph with edge density p is quasirandom with density p if the C_4-density is approximately p^4. A question of Chung, Graham, and Wilson [1] which has received considerable attention asks for which graphs H is it true that if (15) holds for K_2 and H, then the sequence is quasi-random with density p. Such a graph H is called *p-forcing*. We call H *forcing* if it is p-forcing for all p. Chung, Graham, and Wilson prove that even cycles C_{2t} and complete bipartite graphs $K_{2,t}$ with $t \geq 2$ are forcing. Skokan and Thoma [7] generalize this result to all complete bipartite graphs $K_{a,b}$ with $a, b \geq 2$.

There are two simple obstacles to a graph being forcing. It is easy to show that a forcing graph must be bipartite. Further, for any forest H, (15) is satisfied for any sequence of nearly regular graphs of edge density tending to p. The property of being nearly regular is not as strong as being quasirandom. Hence, a forcing graph must be bipartite and have at least one cycle. Skokan and Thoma [7] ask whether these properties characterize the forcing graphs. We conjecture the answer is yes and refer to it as the *forcing conjecture*.

Conjecture 4. *A graph H is forcing if and only if it is bipartite and contains a cycle.*

It is not hard to see that the forcing conjecture is stronger than Sidorenko's conjecture, and it further gives a stability result for Sidorenko's conjecture. A *stability* result not only characterizes the extremal graphs for an extremal problem, but also shows that if a graph is close to being optimal for the extremal problem, then it is close in a certain appropriate metric to an extremal graph. In recent years, there has been a great amount of research done toward proving stability results in extremal combinatorics. The forcing conjecture implies that if H is bipartite with m edges and contains a cycle, then G satisfies $t_H(G)$ is close to $t_{K_2}(G)^m$ if and only if it is quasirandom with density $t_{K_2}(G)$.

By extending our methods, we prove the forcing conjecture in the following particular case.

Theorem 5. *The forcing conjecture holds for every bipartite graph H which has two vertices in one part complete to the other part, which has at least two vertices.*

References

[1] F. R. K. Chung, R. L. Graham, R. M. Wilson, Quasi-random graphs, *Combinatorica* **9** (1989), 345–362.

[2] H. Hatami, Graph norms and Sidorenko's conjecture, *Israel J. Math.*, **175** (2010), 125–150.

[3] V. Nikiforov, The number of cliques in graphs of given order and size, *Transactions of AMS*, to appear.

[4] A. Razborov, On the minimal density of triangles in graphs, *Combin. Probab. Comput.* **17** (2008), 603–618.

[5] A. F. Sidorenko, A correlation inequality for bipartite graphs, *Graphs Combin.* **9** (1993), 201–204.

[6] M. Simonovits, Extremal graph problems, degenerate extremal problems and supersaturated graphs, in *Progress in graph theory* (Waterloo, Ont., 1982), Academic Press, Toronto, ON, 1984, 419–437.

[7] J. Skokan and L. Thoma, Bipartite subgraphs and quasi-randomness, *Graphs Combin.* **20** (2004), 255–262.

Higher-order tournaments

Imre Leader
(joint work with Ta Sheng Tan)

A *tournament* is a complete graph in which each edge is assigned a direction. It is well known (see e.g. [5]) that there are at most $\frac{1}{4}\binom{n}{3}+O(n^2)$ directed triangles in a tournament on n vertices. The constant $\frac{1}{4}$ is easily seen to be best possible, since for example the random tournament (where the direction of each edge is assigned randomly and independently with probability $\frac{1}{2}$) has expected number of directed triangles precisely $\frac{1}{4}\binom{n}{3}$. Actually, any tournament in which all degrees are close to $\frac{n}{2}$ will have about this number of directed triangles (see e.g. [5]).

Our aim in this talk is to investigate some 'higher order' analogues of this result. Before we make our definitions, we give some geometric background, to explain how the question arose. However, our question is natural even without any motivation.

Let $T \subset \mathbb{R}^d$ be a set of n points in general position. What is the greatest possible number of d-simplices of T that contain (say in their interior) a given point of $\mathbb{R}^d$? In two dimensions, this question was asked by Kárteszi [4] and answered by Boros and Füredi [2, 3], who showed that for any set T of n points in the plane in general position and any point x the number of triangles of T containing x is at most $\frac{1}{4}\binom{n}{3}+O(n^2)$. (Note that this can be attained, for example by taking T to be a regular n-gon and x its centre). Their elegant proof was to note that there is a natural way to make T into a tournament: given a and b in T, direct the edge ab from a to b (respectively from b to a) in such a way that the triple abx (respectively bax) is clockwise. Then the triangles of T containing x correspond precisely to the directed triangles of this tournament.

In this talk we are usually interested in asymptotic bounds, but we remark in passing that Boros and Füredi actually proved the exact best possible bound on the number of triangles, because the exact tournament bound (namely $\frac{1}{24}(n^3-n)$ if n is odd and $\frac{1}{24}(n^3-4n)$ if n is even) can in fact be realised geometrically. Indeed, the above construction, with x moved slightly so as not to be collinear with any pair from T, achieves this value.

The general question (in d dimensions) was asked by Boros and Füredi, and answered by Bárány [1]. He showed that if $T \subset \mathbb{R}^d$ is a set of n points in general position and x is any point then the number of d-simplices of T containing x is at most $\frac{1}{2^d}\binom{n}{d+1}+O(n^d)$. The constant $\frac{1}{2^d}$ is best possible, as may be seen in [1].

Now, Bárány's result uses the Upper Bound Theorem [6] (about facet counts in polytopes). In other words, it uses a geometric theorem, as opposed to the abstract tournament theorem used by Boros and Füredi. But what would the corresponding abstract result be? Just as in the case $d=2$, for a general d we would give an orientation to each d-set ($(d-1)$-simplex) in T, according to 'on which side of it' the point x lies. And then the d-simplices containing x would correspond exactly to the $(d+1)$-sets in T whose d-sets were 'oriented compatibly' (in other words, whose d-sets had orientations that could be induced from a fixed orientation of the $(d+1)$-set – this will be made more precise in a moment). Hence our abstract question is as follows: suppose that we orient (in some sense) every d-set of an n-set; what is the greatest number of directed $(d+1)$-sets that arise? In particular, do we get as small a bound as $\frac{1}{2^d}\binom{n}{d+1} + O(n^d)$?

We now give the precise (and non-geometric) definitions. We define an *orientation* of a d-set inductively. An orientation of a 1-set $\{x\}$ is just an assignment of ± 1 to x, and an orientation of a 2-set $\{a, b\}$ is a directed edge from a to b or vice versa. (We may, if we wish, think of a directed edge from a to b as assigning $+1$ to b and -1 to a). And for $d \geq 3$, an orientation of a d-set consists of an orientation for each of its $(d-1)$-subsets in such a way that these orientations are *compatible*, meaning that any two give different orientations to their common $(d-2)$-subset. Then, for $d \geq 2$, a *tournament of order* d, or d*-tournament*, consists of a set together with an orientation of each of its d-sets. Finally, in a d-tournament a d*-simplex* is a $(d+1)$-set, and we say that it is *directed* if its d-subsets are pairwise compatible.

For example, a 2-tournament is just a tournament, and its directed 3-sets are precisely its directed triples in the usual sense. And a 3-tournament is specified by giving each 3-set (from a given set) a cyclic ordering: then a 4-set is directed if any two of its 3-sets have cyclic orderings that go in opposite directions on their common 2-set.

Our question is then: what is the greatest number of directed $(d+1)$-sets for a d-tournament on n vertices? For $d = 2$ this is $\frac{1}{4}\binom{n}{3} + O(n^2)$; what can we say in general? And how does this bound compare with the 'geometric' version (when the d-tournament is induced from a set T in $\mathbb{R}^d$), where the bound is $\frac{1}{2^d}\binom{n}{d+1} + O(n^d)$?

To put it another way, define the constant c_d to be the limit, as $n \to \infty$, of this greatest number as a fraction of $\binom{n}{d+1}$ – an easy averaging argument shows that the limit does exist. In this language, the $d = 2$ result is that $c_2 = \frac{1}{4}$, and the geometric construction shows that $c_d \geq \frac{1}{2^d}$. In fact, another reason why it is obvious that $c_d \geq \frac{1}{2^d}$ is that a random d-tournament has expected number of directed $(d+1)$-sets exactly $\frac{1}{2^d}\binom{n}{d+1}$. How does c_d behave, for fixed small d and also as d gets large?

The plan of the talk is as follows. We start by considering the case $d = 3$. Here it turns out that $\frac{1}{8}$ is not the right answer. We give an upper bound of $\frac{1}{4}$, by a simple counting argument. And then we show that that in fact $c_3 = \frac{1}{4}$, by a slightly unexpected random argument.

Then we turn our attention to general d. Here we do not know what the exact value of c_d is. We give an upper bound of $\frac{1}{d+1}$, again by a simple counting argument. For the lower bound, the method for $d = 3$ seems unfortunately not to generalise, and indeed we do not know how to use any random methods to improve significantly on $\frac{1}{2^d}$. However, we give an explicit construction to show that $c_d \geq \frac{1}{d^2}$. Thus the abstract version of the problem exhibits genuinely different behaviour to the geometric version.

Finally, we give further results and open questions.

References

[1] I. Bárány, *A generalization of Carathéodory's theorem*, Discrete Math. **40** (1982), 141-152.
[2] E. Boros and Z. Füredi, *Su un teorema di Kárteszi nella geometria combinatoria*, Archimede **2** (1977), 71-76.
[3] E. Boros and Z. Füredi, *The number of triangles covering the center of an n-set*, Geom. Dedicata **17** (1984), 69-77.
[4] F. Kárteszi, *Extremalaufgaben über endliche Punktsysteme*, Publ. Math. Debrecen **4** (1955), 16-27.
[5] J.W. Moon, *Topics on Tournaments*, Holt, Rinehart and Winston, New York (1968).
[6] P. McMullen and G.C. Shephard, *Convex Polytopes and the Upper Bound Conjecture*, Cambridge Univ. Press (1971).

Right-convergence of sparse random graphs

David Gamarnik

Recently there was a lot of research devoted to the development of the theory of graph limits [2],[1],[7],[10]. The theory is in a very mature state for the case of dense graphs $\mathbb{G} = (V, E) = (V(\mathbb{G}), E(\mathbb{G}))$, namely graphs where the number of edges $|E| = \Theta(|V|^2)$. A certain limiting object *Graphon* was constructed in [2], which is a measure on $[0, 1]^2$. This object in some loose sense captures the notion of limiting density of edges between different parts of graphs $\mathbb{G}_n$. It was established that a sequence of dense graphs converges to such a limiting object if and only if left and right convergence of homomorphisms from and into test graphs takes place, and also if and only if the graph sequence converges with respect to a so-called cut metric. Also, a random dense graph $\mathbb{G}(n, p)$ with a constant edges probability p trivially converges to a Lebesgue measure, according to this definition. Many parts of the theory extend to the case when the graph is sparse, but with superlinear number of edges $|E| = \omega(|V|)$.

The situation with very sparse graphs, namely $|E| = O(|V|)$, appears to be more problematic. Before we summarize the state of the art, let us formally introduce some definitions. The notion of a left convergence was introduced first in the form of local convergence by Benjamini and Schramm [6] for bounded degree sparse graphs. In the form of left convergence the definition can be stated as follows. Given two graphs H and F, let $h(H, F)$ denote the number of homomorphisms from H to F.

Definition 1. *A sequence of graphs $\mathbb{G}_n$ is said to be left-converging if for every graph H, the limit $\lim_n |V(\mathbb{G}_n)|^{-1} h(H, \mathbb{G}_n)$ exists.*

For the case of right-convergence, we would like to extend the notion of homomorphisms to the case of weighted graphs. Consider a weighted graph $H = H(w)$ with a symmetric non-negative weight matrix $w = (w_{i,j}, 1 \le i, j \le m = |V(H)|)$. We think of H is a weighted graph in the sense that (i, j) is an edge if $w_{i,j} > 0$, and is not an edge if $w_{i,j} = 0$. Given an (unweighted) graph $\mathbb{G} = (V(\mathbb{G}), E(\mathbb{G}))$

and any mapping $\sigma : V(\mathbb{G}) \to V(H)$, let

$$w(\sigma) = \prod_{(u,v)\in E(\mathbb{G})} w_{\sigma(u),\sigma(v)},$$

and let $h(\mathbb{G}, H) = \sum_\sigma w(\sigma)$, where the sum is over all mappings $\sigma : V(\mathbb{G}) \to V(H)$. Observe that when w is a zero-one matrix, $h(\mathbb{G}, H)$ is the number of homomorphisms from $\mathbb{G}$ to H (hence the same notation). The object $h(\mathbb{G}, H)$ is also known in statistical physics as the partition function associated with potentials given by $w = w(H)$.

Definition 2. *A sequence of graphs $\mathbb{G}_n$ is said to be right-converging if for every graph H, the limit $\lim_n |V(\mathbb{G}_n)|^{-1} \log h(\mathbb{G}_n, H)$ exists.*

The use of log function in the definition turns out to be necessary for "correct" scaling. For example, suppose w is 2×2 given by $w_{11} = w_{12} = w_{21} = 1, w_{22} = 0$. Then $h(\mathbb{G}, H)$ is the number of independent sets in $\mathbb{G}$. For bounded degree graphs, this quantity is exponentially in $|V(\mathbb{G})|$ large, hence the log-scaling is justified.

A simple example was demonstrated in the paper by Borgs, Chayes, Kahn and Lovasz [3], which shows that left convergence does not imply the right-convergence: consider cycles C_n on n nodes. They are trivially left converging, according to the Benjamini-Schramm definition, but not right-converging, since C_n is (not) bipartite for even (odd) n. Namely for H which is complete graph without loops on 2 nodes, $h(C_n, H) \geq 2$ for even n and $h(C_n, H) = 0$ for odd n, implying non-convergence of the sequence C_n. The same paper shows that right-convergence implies left-convergence and establishes that left-converging graph sequences are also right-converging but with respect to a restricted class of test graphs H, which satisfy some density condition. A limiting object for a sequence of sparse graphs was constructed by Bollobas and Riordan [5]. The limiting object is a sequence indexed by $m = 1, 2, \ldots,$ of measures on the space of $m \times m$ matrices. They also defined a so-called partition metric with respect to which the converging sequence of sparse graphs has to converge.

A natural question to ask in the context of convergence of sparse graphs is when does a random graph converge with respect to either left/right convergence or the partition metric considered by Bollobas and Riordan? For example, does a sparse Erdös-Rényirandom graph $\mathbb{G}(n, c/n)$ converge with respect to the either of the definition, w.h.p.? The random graph $\mathbb{G}(n, c/n)$ is left-converging w.h.p., since its local structure is described by a branching process with a Poisson out-degree distribution. The situation with right-convergence is, however, far more problematic, and this is where the main contribution of the current work is. We establish the right-convergence of the graph sequence $\mathbb{G}(n, c/n)$ with respect to a certain subclass of weighted graphs H. Specifically,

Theorem 3. *Suppose a symmetric matrix $w = (w_{i,j}, 1 \leq i, j \leq m = |V(H)|)$ is such that $(w_{\max} - w_{i,j}, 1 \leq i, j \leq m)$ is positive semi-definite, where $w_{\max} = \max_{i,j} w_{i,j}$, and suppose $\max_i \min_j w_{i,j} > 0$. Then for every $c > 0$, the sequence $h(\mathbb{G}(n, c/n), H)$ is right-converging w.h.p. Namely the limit $\lim_n n^{-1} \log h(\mathbb{G}(n, c/n), H)$ exists in the high probability sense.*

This result extends and unifies several earlier specialized results obtained by Bayati, Gamarnik and Tetali [4], which uses the interpolation method as the main technique. The interpolation method originated in the statistical physics literature by Guerra and Toninelli [8], as a method of proving the existence of the scaling limits for some thermodynamic quantities, such as the logarithm of the partition function. Franz and Leone [9] used this method to show the existence of such a scaling limit for the partition function associated with the random K-SAT problem. Panchenko and Talagrand [11] extended and unified this further. The paper by Bayati, Gamarnik [4] proved the existence of scaling limits for several counting objects, such as the number of independent sets and weighted partial colorings of $\mathbb{G}(n, c/n)$. Theorem 3 unifies most of these results and also establishes the relevance to the problem of sparse graph limits convergence. The proof is based on the expansion of $\log h(\mathbb{G}(n, c/n), H)$ into a power series and using the fact that the tensor product of symmetric positive semidefinite matrices is also positive semidefinite. This can be used to show the following relation: for every $n = n_1 + n_2$

$$\begin{aligned}\mathbb{E}[\log h(\mathbb{G}(n, c/n), H)] \geq\ & \mathbb{E}[\log h(\mathbb{G}(n_1, c/n_1), H)] \\ & + \mathbb{E}[\log h(\mathbb{G}(n_2, c/n_2), H)] - O(\sqrt{n}).\end{aligned}$$

Namely, the sequence $\mathbb{E}[\log h(\mathbb{G}(n, c/n), H)]$ is nearly super-additive, from which its convergence follows. The convergence with high probability follows from standard concentration arguments.

References

[1] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi, *Convergent graph sequences II: Multiway cuts and statistical physics*, Submitted.

[2] ———, *Convergent graph sequences I: Subgraph frequencies, metric properties, and testing*, Advances in Math. **219** (208), 1801–1851.

[3] C. Borgs, J.T. Chayes, L. Lováasz, and J. Kahn, *Left and right convergence of graphs with bounded degree*, Preprint.

[4] M. Bayati, D. Gamarnik, and P. Tetali, *Combinatorial approach to the interpolation method and scaling limits in sparse random graphs*, Proc. 42nd Ann. Symposium on the Theory of Computing (STOC), http://arxiv.org/abs/0912.2444, 2010.

[5] B. Bollobás and O. Riordan, *Sparse graphs: metrics and random models*, Random Structures and Algorithms (To appear).

[6] I. Benjamini and O. Schramm, *Recurrence of distributional limits of finite planar graphs*, Electronic Journal of Probability **23** (2001), 1–13.

[7] G. Elek, *On limits of finite graphs*, Combinatorica **27** (2007), 503507.

[8] F. Guerra and F.L. Toninelli, *The thermodynamic limit in mean field spin glass models*, Commun. Math. Phys. **230** (2002), 71–79.

[9] S. Franz and M. Leone, *Replica bounds for optimization problems and diluted spin systems*, Journal of Statistical Physics **111** (2003), no. 3/4, 535–564.

[10] L. Lovász and B. Szegedy, Journal of Combinatorial Theory, Series B **96** (2006), 933–957.

[11] D. Panchenko and M. Talagrand, *Bounds for diluted mean-fields spin glass models*, Probability Theory and Related Fields **130** (2004), 312–336.

A full derandomization of Schöning's k-SAT algorithm

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(joint work with Dominik Scheder)

Let a set $V := \{x_1, x_2, \ldots, x_n\}$ of n *propositional variables* be given. A *literal* is a variable x_i or its negation $\bar{x}_i$. A *clause* is a finite set of literals such that no literal appears together with its negation. A CNF formula is a finite set of clauses. If $x \in V$ is a variable, we write $F^{[x \mapsto 0]}$ (or $F^{[x \mapsto 1]}$) to denote the formula arising from deleting all clauses that feature $\bar{x}$ (or x) and deleting all literals x (or $\bar{x}$) from the remaining ones. A k-CNF formula is a CNF formula where each clause contains exactly k literals. The k-SAT problem is to decide, for a given k-CNF formula F, whether there exists an assignment $\alpha : V \to \{0, 1\}$ that satisfies at least one literal in every clause (where x_i is satisfied if $x_i \mapsto 1$ and $\bar{x}_i$ is satisfied if $x_i \mapsto 0$).

This problem is well-known to be NP-complete, so barring a major breakthrough in theory, we will have to accept that any algorithm solving it in full generality will have a running time of $\lambda^{n+o(n)}$ for some $\lambda > 1$. While achieving $\lambda = 2$ is trivial, considerable effort has gone into finding faster methods. An exceptionally easy algorithm has been proposed by Uwe Schöning in [5]: select an assignment $\alpha_0 : V \to \{0, 1\}$ uniformly at random and then for $i = 1, 2, \ldots$, iteratively try to improve by selecting any clause $C_i \in F$ violated by α_{i-1}, randomly choosing a literal within C_i and flipping the value of the underlying variable to produce α_i. Schöning has delivered a beautifully simple analysis for the procedure. We suppose F is satisfiable (otherwise the algorithm must fail) and fix a satisfying assignment α^*. We then measure the progress of the algorithm in terms of the distances $D_i := |\{\, x \in V \mid \alpha_i(x) \neq \alpha^*(x) \,\}|$. Clearly, $D_0 \sim \mathrm{Bin}(n, 1/2)$ and for any subsequent i, $D_i \in \{D_{i-1} \pm 1\}$ and since α^* and α_i differ in at least one of the k variables that C_i features, $\Pr[D_i = D_{i-1} - 1] \geq 1/k$. Juxtaposing an idealized process $\{D'_t\}_{t \geq 0}$ with $D'_0 \sim \mathrm{Bin}(n, 1/2)$, $D'_i \in \{D'_{i-1} \pm 1\}$ and $\Pr[D'_i = D'_{i-1} - 1] = 1/k$ for all i and coupling the two experiments, it is not hard to deduce that the probability of reaching a state with $D_T \leq D'_T = 0$ within $T = \mathcal{O}(n)$ steps is at least $(k/(2(k-1)))^{n+o(n)}$. If we abort in case success in not forthcoming within linearly many steps and then repeat the procedure, a randomized algorithm with $\lambda = 2(k-1)/k$ results.

The key to finding a determinstic analogue of this algorithm is to make more specific observations concerning the behavior of the random process $\{D'_t\}_{t \geq 0}$. In fact, one can show the following facts quite easily. Let $a := n/k$, $b := n/(k-2)$, $c := \lceil \log \log n \rceil$, $d := b/c$, $e := a/d$ and $f := (e + c)/2$. If we condition on the event $\{\exists t : D'_t = 0\}$, then with probability at least $2^{-o(n)}$, we will start at $D'_0 \approx a$, we will reach $D'_T = 0$ within $T \approx b$ steps and the progress we make in between is regularly distributed over time, that is $D'_{jc} \approx a - je$ for $0 \leq j < d$. In other words, every c steps we have to get e positions closer to the solution, meaning that approximately f steps must have selected a correct and just $c - f$ steps a wrong literal. These are hence the 'typical paths' that lead to success of the algorithm and all other paths contribute only negligibly to the total success probability.

The task is thus first to find a starting assignment α_0 at distance at most a from some satisfying assignment and then to make roughly d sequences of c correction steps each, making at least f good and at most $c-f$ bad corrections per sequence.

It has been known ever since Dantsin et. al. [2] presented the first derandomization of Schöning's algorithm that α_0 can be found easily using a *covering code*. Indeed, there is a code $\mathcal{C} \subseteq \{0,1\}^V$ containing approximately $2^n/\binom{n}{a}$ assignments such that for all $\alpha^* \in \{0,1\}^V$, there exists $\alpha_0 \in \mathcal{C}$ at distance at most a from α^*. Moreover, $\mathcal{C}$ can be constructed efficiently, i.e. there is a polynomial-time algorithm producing the i-th assignment of $\mathcal{C}$ given i. If we try all assignments in $\mathcal{C}$, we will be successful in one of the attempts, given that the rest of the algorithm goes through. And the number of starting assignments we try is only negligibly larger than the expected number of times we try in the randomized version.

From here, Dantsin et. al. [2] proceeded by simply branching on all possibilities of flipping a literal in a violated clause, i.e. if C_i violates α_i, branch k times into the assignments $\alpha_i^{(1)}$ through $\alpha_i^{(k)}$ that arise from flipping one of the k literals in C_i. One of those assignments must be one position closer to α^* than α_{i-1}. This way, we spend at most k^a time (per starting assignment) until a solution is discovered. In comparison, the randomized version spends only $(k-1)^a$ time for the same task.

The reason for this gap is that the strategy of branching into all possible literals undermines the actual strength of Schöning's algorithm which bases on the fact that it allows for mistakes being made on the way to success and as we have illustrated above, the making of a specific constant fraction of erroneous steps is crucial to achieving optimal success probability. The loss in performance that results in this way has been tried to compensate for by a series of quite involved improvements first by Dantsin et. al. themselves [2], later by Brueggemann and Kern [1], then by Scheder [4] and most recently by Kutzkov and Scheder [3]. A significant gap always remained.

Our contribution is to use the idea of covering codes a second time when it comes to the iterative correction process. Generalizing the codes from [2], it is easy to see that there is a code $\mathcal{D} \subseteq \{1..k\}^c$ containing approximately $k^c/((k-1)^{c-f}\binom{c}{f})$ codewords such that for all $w^* \in \{1..k\}^c$, there is $w \in \mathcal{D}$ such that w^* and w agree in at least f positions. Now suppose we currently look at assignment α_{i-1} and suppose there are c *independent clauses* (i.e. over pairwise disjoint sets of variables) $C_i^{(1)}$ through $C_i^{(c)}$, all of which are violated by α_{i-1}. We know that α^* satisfies all these clauses, so α_i and α^* disagree in at least one variable in each clause. Instead of making these c correction steps one after the other (as the randomized variant does), we can as well make them at once. We must choose one out of the k literals in every clause for flipping, which corresponds to choosing a string from $\{1..k\}^c$. Instead of doing this randomly, we try each word from the code $\mathcal{D}$, resulting in $|\mathcal{D}|$ assignments $\alpha_i^{(1)}$ through $\alpha_i^{(|\mathcal{D}|)}$, at least one of which is guaranteed to flip a good literal in at least f of the c clauses, resulting in a distance that is at least e positions smaller than the one of α_{i-1}. On each possibility, we branch. Correctly combining the numbers, this results in an algorithm of exactly the performance desired, namely one that needs roughly $(k-1)^a$ time to walk from a good starting assignment to a solution.

The algorithm might still get stuck, however, if the assignment α_{i-1} currently under consideration does not admit any set of c independent clauses that are all violated by α_{i-1}. But this situation turns out to be even easier to handle. Instead of going on evolving the assignment, we freeze $\alpha := \alpha_{i-1}$. From now on, we substitute values directly into the formula F. Choose a maximal set $M \subseteq F$ of independent clauses violated by α. We know $|M| < c$ in the present case. Try all $l < 7^{|M|}$ admissible assignments $\beta_1, \beta_2, \ldots, \beta_l$ over the variables that occur in M, producing formulas $F_i = F^{[\beta_i]}, i = 1..l$. As M was maximal, these formulas have the property that all clauses that are violated by α have at most $k-1$ literals, and we know that one of them must have preserved satisfiability. Starting from any such formula F_j, we pick any such violated clause and branch on the at most $k-1$ literals, producing formulas $F_j^{(1)}$ through $F_j^{(k-1)}$, one of which is still satisfiable. All formulas produced in this manner preserve the property that all clauses violated by α have at most $k-1$ literals, so we can continue recursively. Correctly combining the numbers, this mode of the algorithm has the same performance as the other one, such that we have now covered all the cases. This way, we have reached a deterministic algorithm for k-SAT with $\lambda = 2(k-1)/k$.

In the randomized setting, Schöning is not anymore the fastest algorithm for k-SAT known. Depending on k, either the DPLL-based procedure by Paturi, Pudlák, Saks and Zane (PPSZ) from [6] or an amalgamation of it with Schöning beat pure Schöning. The main open question in the field would be to determine whether these faster algorithms admit deterministic versions which best the one we presented. However, the corresponding analyses being several orders of magnitude more complicated than Schöning's, derandomizing them could prove far more intricate.

References

[1] T. Brueggemann and W. Kern, *An improved deterministic local search algorithm for 3-SAT*, Theor. Comput. Sci. **329**, 303-313, 2004.

[2] E. Dantsin, A. Goerdt, E. A. Hirsch, R. Kannan, J. Kleinberg, C. Papadimitriou, O. Raghavan, and U. Schöning, *A deterministic* $(2-2/(k+1))^n$ *algorithm for k-SAT based on local search*, Theor. Comput. Sci. **289**, 69-83, 2002.

[3] K. Kutzkov and D. Scheder, *Using CSP to improve deterministic 3-SAT*, CoRR, abs/1007.1166, 2010.

[4] D. Scheder, *Guided search and a faster deterministic algorithm for 3-SAT*, In Proc. of the 8th Latin American Symposium on Theoretical Informatics (LATIN08), Lecture Notes In Computer Science **4957**, 60-71, 2008.

[5] U. Schöning, *A probabilistic algorithm for k-SAT and constraint satisfaction problems*, In Proceedings of the 40th Annual Symposium on Foundations of Computer Science (FOCS'99), 410, Washington, DC, USA, 1999. IEEE Computer Society.

[6] R. Paturi, P. Pudlák, M. E. Saks, and F. Zane, *An improved exponential-time algorithm for k-SAT*, J. ACM, **52**(3), 337–364 (electronic), 2005.

On a conjecture of Erdős and Simonovits on bipartite Turán numbers

Jacques Verstraëte

(joint work with Peter Keevash and Benny Sudakov)

Let $\mathcal{F}$ be a family of graphs. A graph is *$\mathcal{F}$-free* if it contains no copy of a graph in $\mathcal{F}$ as a subgraph. A cornerstone of extremal graph theory is the study of the *Turán number* $\mathrm{ex}(n, \mathcal{F})$, the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. The *Zarankiewicz number* $\mathrm{z}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free bipartite graph on n vertices. Let C_k denote a cycle of length k, and let $\mathcal{C}_k$ denote the set of cycles C_ℓ, where $3 \leq \ell \leq k$ and ℓ and k have the same parity. Erdős and Simonovits conjectured that for any family $\mathcal{F}$ consisting of bipartite graphs there exists an odd integer k such that

$$\mathrm{ex}(n, \mathcal{F} \cup \mathcal{C}_k) \sim \mathrm{z}(n, \mathcal{F}).$$

While this conjecture remains open in general, Erdős and Simonovits proved the conjecture when $\mathcal{F} = \{C_4\}$ by showing that $\mathrm{ex}(n, \{C_4, C_5\}) \sim \mathrm{z}(n, C_4)$. In this talk we will outline the proof of the conjecture for various other families $\mathcal{F}$, in particular, we will show for any odd $k \geq 5$ that

$$\mathrm{ex}(n, \{C_4, C_k\}) = \mathrm{z}(n, \{C_4\})$$

for infinitely many n, and that extremal $\{C_4, C_k\}$-free graphs are bipartite. In contrast, it is an open conjecture of Erdős as to the value of $\mathrm{ex}(n, \{C_3, C_4\})$ relative to $\mathrm{z}(n, \{C_4\})$. We will show in this direction that for there is a constant $\delta > 1$ such that for large enough n,

$$\mathrm{ex}(n, \{C_3, K_{2,3}\}) > \delta \mathrm{ex}(n, \{C_3, K_{2,3}\})$$

which may indicate that the same might be true with $K_{2,3}$ replaced by C_4. The proofs make use of pseudorandomness properties of nearly extremal graphs that are of independent interest, and a general approach to the Erdős-Simonovits conjecture for $\mathcal{F}$-free graphs is currently work in progress.

Sumner's universal tournament conjecture

Deryk Osthus

(joint work with Daniela Kühn and Richard Mycroft)

A tournament is an orientation of a complete graph. Obviously one cannot guarantee any substructures which contain a cycle within an arbitrary tournament. On the other hand, Sumner's universal tournament conjecture (made in 1971) states that one can find any directed tree T within an arbitrary tournament G, even if the order of T is rather large compared to that of G. More precisely, the conjecture states that any tournament on $2n-2$ vertices contains any directed tree on n vertices. Many partial results towards this conjecture have been proved – some of them are described below. We proved this conjecture for all large n [9].

Theorem 1. *There exists n_0 such that the following holds. Let T be a directed tree on $n \geq n_0$ vertices, and G a tournament on $2n-2$ vertices. Then G contains a copy of T.*

To see that the bound is best possible, let T be a star with all edges directed inwards, and let G be a regular tournament on $2n-3$ vertices. Then every vertex of G has $n-2$ inneighbours and $n-2$ outneighbours, and so G does not contain a copy of T, whose central vertex has $n-1$ inneighbours. There are also 'near-extremal' examples which have a different structure to the one given above.

In [8], we used a randomised embedding algorithm to prove an approximate version of Sumner's universal tournament conjecture, and also a stronger result for directed trees of bounded degree. Both of these results are important tools in the proof of Theorem 1.

Theorem 2. *Let $\alpha > 0$. Then the following properties hold.*

(i) *There exists n_0 such that for any $n \geq n_0$, any tournament G on $2(1+\alpha)n$ vertices contains any directed tree T on n vertices.*
(ii) *Let Δ be any positive integer. Then there exists n_0 such that for any $n \geq n_0$, any tournament G on $(1+\alpha)n$ vertices contains any directed tree T on n vertices with $\Delta(T) \leq \Delta$.*

Let $f(n)$ denote the smallest integer such that any tournament on $f(n)$ vertices contains any directed tree on n vertices. So Sumner's conjecture states that $f(n) = 2n-2$. Chung (see [13]) observed that $f(n) \leq n^{1+o(1)}$, and Wormald [13] improved this to $f(n) \leq O(n \log n)$. The first linear bound on $f(n)$ was established by Häggkvist and Thomason [6]. Havet [2] then showed that $f(n) \leq 38n/5$, and later Havet and Thomassé [4] used their notion of median orders to improve this to $f(n) \leq 7n/2$. Finally El Sahili used the same notion to prove the best known for general n, namely that $f(n) \leq 3n-3$.

Sumner's conjecture is also known to hold for special classes of trees. In particular, Havet and Thomassé [4] proved it for 'outbranchings', again using median orders. Here an *outbranching* is a directed tree T in which we may choose a root vertex $t \in T$ so that for any vertex $t' \in T$, the path between t and t' in T is directed from t to t'. (Outbranchings are also known as arborescences.)

For many types of trees, Sumner's conjecture holds with room to spare. (For instance, Theorem 2 implies that it holds with room to spare for bounded degree trees.) A classical result of this type is Redei's theorem [11], which states that every tournament contains a spanning directed path (where all edges are directed consistently). This was generalised considerably by Thomason [12] who showed that whenever n is sufficiently large, every tournament on n vertices contains every orientation of the path on n vertices (this was a conjecture of Rosenfeld). Havet and Thomassé [5] proved that this even holds for all $n \neq 3, 5, 7$. They also proposed the following generalisation of Sumner's conjecture (see [3]):

Conjecture 3. *Let T be a directed tree on n vertices with k leaves. Then every tournament on $n+k-1$ vertices contains a copy of T.*

Some special cases are known (see e.g. [7]). It would be interesting to know whether our methods can be used to prove this conjecture.

Our proof of Theorem 1 relies on many of the theorems mentioned above as well as a directed version of Szemerédi's regularity lemma and several structural results proved in [8]. In turn, the proofs in [8] rely on a recent result by Kühn, Osthus and Treglown [10] on the existence of Hamilton cycles in so-called 'robust expander digraphs'.

References

[1] A. El Sahili, Trees in tournaments, *Journal of Combinatorial Theory, Series B* **92** (2004), 183–187.

[2] F. Havet, Trees in tournaments, *Discrete Mathematics* **243** (2002), 121–134.

[3] F. Havet, On unavoidability of trees with k leaves, *Graphs and Combinatorics* **19** (2003), 101–110.

[4] F. Havet and S. Thomassé, Median orders of tournaments: a tool for the second neighbourhood problem and Sumner's conjecture, *Journal of Graph Theory* **35** (2000), 244–256.

[5] F. Havet and S. Thomassé, Oriented Hamiltonian paths in tournaments: a proof of Rosenfeld's conjecture, *Journal of Combinatorial Theory, Series B* **78** (2000), 243–273.

[6] R. Häggkvist and A.G. Thomason, Trees in tournaments, *Combinatorica* **11** (1991), 123–130.

[7] S. Céroi and F. Havet, Trees with three leaves are $(n+1)$-unavoidable, *Discrete Applied Mathematics* **141** (2004), 19–39.

[8] D. Kühn, R. Mycroft and D. Osthus, An approximate version of Sumner's universal tournament conjecture, *Journal of Combinatorial Theory, Series B*, to appear.

[9] D. Kühn, R. Mycroft and D. Osthus, A proof of Sumner's conjecture for large tournaments, *Proceedings of the London Mathematical Society*, to appear.

[10] D. Kühn, D. Osthus and A. Treglown, Hamiltonian degree sequences in digraphs, *Journal of Combinatorial Theory, Series B* **100** (2010), 367–380.

[11] L. Redei, Ein kombinatorischer Satz, *Acta Lit. Szeged* **7** (1934), 39–43.

[12] A. Thomason, Paths and cycles in tournaments, *Transactions of the American Mathematical Society* **296** (1986), 167–180.

[13] N.C. Wormald, Subtrees of large tournaments, *Combinatorial Mathematics X, Springer Lecture Notes in Mathematics* **1036** (1983) 417–419.

On Gromov's method of selecting heavily covered points

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(joint work with Uli Wagner)

Let $P \subset \mathbb{R}^2$ be a set of n points in general position (i.e., no three points collinear). Boros and Füredi [2] showed that there always exists a point $a \in \mathbb{R}^2$ contained in a positive fraction of all the $\binom{n}{3}$ triangles spanned by P, namely, in at least $\frac{2}{9}\binom{n}{3} - O(n^2)$ triangles. (Generally we cannot assume $a \in P$, as the example of points in convex position shows.)

This result was generalized by Bárány [1] to point sets in arbitrary fixed dimension: *For every n-point set P in general position in $\mathbb{R}^d$ there exists a point in $\mathbb{R}^d$ that is contained in at least $c_d \cdot \binom{n}{d+1} - O(n^d)$ d-dimensional simplices spanned by* *the points in P, where the constant $c_d > 0$, as well as the constant implicit in the O-notation, depend only on d.*

From now on, let c_d be the largest constant for which this statement holds. The value of c_d has been the subject of ongoing research. The best current upper bound, due to Bukh et al. [3], is

$$c_d \leq \frac{(d+1)!}{(d+1)^{(d+1)}} \sim \frac{\sqrt{2\pi d}}{e^d} = e^{-\Theta(d)}; \tag{16}$$

for $d = 2$, this yields $c_2 \leq \frac{2}{9}$, which is tight. For general d, Bárány's proof yields $c_d \geq \frac{1}{(d+1)^d}$, which is smaller than (16) by a factor of $d!$. Wagner [10] improved the lower bound by a factor of roughly d to $c_d \geq \frac{d^2+1}{(d+1)^{d+1}}$; then [3] showed that his method in [10] cannot be pushed farther. An improvement of the lower bound for c_3 by a clever elementary geometric argument was recently achieved by Basit et al. [4].

Recently, Gromov [5] introduced a new, topological proof method, which improves on the previous lower bounds considerably and which, moreover, applies to a more general setting, described next.

An n-point set $P \subseteq \mathbb{R}^d$ determines an affine map T from the $(n-1)$-dimensional simplex $\Delta^{n-1} \subseteq \mathbb{R}^{n-1}$ to $\mathbb{R}^d$ as follows. Label the vertices of Δ^{n-1} by $V = \{v_1, \dots, v_n\}$, and the points as $P = \{p_1, \dots, p_n\}$. Then T is given by mapping v_i to p_i, $1 \leq i \leq n$, and by interpolating linearly on the faces Δ^{n-1}. Thus, Bárány's result can be restated by saying that for any *affine* map $T\colon \Delta^{n-1} \to \mathbb{R}^d$, there exists a point in $\mathbb{R}^d$ that is contained in the ψ-images of at least $c_d \cdot \binom{n}{d+1} - O(n^d)$ many d-dimensional faces of Δ^{n-1}. Gromov proved an analogous statement for an arbitrary *continuous* map $T\colon \Delta^{n-1} \to \mathbb{R}^d$, with some constant $c_d^{\text{top}} > 0$. His method gives $c_d \geq c_d^{\text{top}} \geq \frac{2d}{(d+1)!(d+1)}$. For $d = 2$, this yields the tight bound $c_2^{\text{top}} = c_2 = \frac{2}{9}$. For general d, it improves on the earlier bounds by a factor exponential in d, but it is still of order $e^{-\Theta(d \log d)}$ and thus far from the upper bound.

In [9] we provide an exposition of the combinatorial component of Gromov's approach, in terms accessible to combinatorialists and discrete geometers. Here we give the basic definitions, state our results, and formulate a combinatorial problem (the "pagoda problem") whose solution might possibly lead to determining the best possible value of c_d.

After our paper was written, Karasev [6] found a two-page proof of the Gromov bound for c_d, which is inspired by Gromov's method but uses only quite elementary topological considerations. It applies only to the affine case. We believe that our improvements to Gromov's bound can also be implanted into Karasev's proof and they yield the correspoding improvements for the values of the c_d, but the details are yet to be checked.

Let V be a fixed set of n elements, w.l.o.g., $V = [n] := \{1, 2, \dots, n\}$. Let $E \subseteq \binom{V}{d}$ be a system of (unordered) d-tuples. We write $\|E\| := |E|/\binom{n}{d}$ for the *normalized size* of E; one can also interpret $\|E\|$ as the probability that a random d-tuple lies in E.

The *coboundary* δE is the system of those $(d+1)$-tuples in $f \in \binom{V}{d+1}$ that contain an odd number of $e \in E$.[1]

Many different E's may have the same coboundary. We call E *minimal* if $\|E\| \le \|E'\|$ for every E' with $\delta E' = \delta E$. We define the *cofilling profiles* as follows:

$$\varphi_d(\alpha) := \liminf_{|V|=n\to\infty} \min\{\|\delta E\| : E \in \tbinom{V}{d} \text{ minimal}, \|E\| \ge \alpha\}.$$

We remark that there is no minimal E with $\|E\| > 1/2$, so formally, $\varphi_d(\alpha) = \infty$ for $\alpha > 1/2$.

It is easily shown that $\varphi_1(\alpha) = 2\alpha(1-\alpha)$, $0 \le \alpha \le \frac{1}{2}$. Indeed, we can view an $S \in \binom{V}{1}$ as a subset of $S \subseteq V$, and δS is the *edge cut* determined by S in the complete graph on V (and the minimality of S means $|S| \le n/2$).

Gromov's argument yields the following general lower bound:

$$c_d^{\rm top} \ge \varphi_d(\tfrac{1}{2}\varphi_{d-1}(\tfrac{1}{3}\varphi_{d-2}(\ldots \tfrac{1}{d}\varphi_1(\tfrac{1}{d+1})\ldots))). \tag{17}$$

A simple argument, observed by Gromov, and independently by Linial, Meshulam, and Wallach [7, 8], yields the lower bound $\varphi_d(\alpha) \ge \alpha$. Substituting this in (17), except for φ_1 where we use the tight bound mentioned above, yields Gromov's lower bound for $c_d^{\rm top}$.

One possible way of improving that bound is by improving the basic lower bound on (some of) the φ_d in appropriate ranges. However, a simple example (a suitable complete d-partite system) shows that $\varphi_d(\alpha) \le \frac{d+1}{d}\alpha$ (for all $\alpha \le \frac{1}{d+1}$). Thus, the bound on $c_d^{\rm top}$ cannot be improved by more than a factor of roughly d using (17) alone. Still, we consider determining the value of the φ_d a fascinating combinatorial problem. We have the following lower bounds for small values of α:

Theorem 1. *For $d = 2$ and all $\alpha \le \frac{1}{4}$, we have $\varphi_2(\alpha) \ge \frac{3}{4}\left(1-\sqrt{1-4\alpha}\right)(1-4\alpha) = \frac{3}{2}\alpha - O(\alpha^2)$. For $d = 3$ and α sufficiently small, $\varphi_3(\alpha) \ge \frac{4}{3}\alpha - O(\alpha^2)$ (with constants that could be made explicit).*

We believe that a suitable extension of the proof of this theorem should provide a bound of the form $\varphi_d(\alpha) \ge \frac{d+1}{d}\alpha - o(\alpha)$ as $\alpha \to 0$ for all d. At present it seems that such an extension would be highly technical and complicated.

These new lower bounds do not improve on Gromov's lower bounds for c_3, for example, since they do not beat the basic bound for the values of α needed in (17) for d small. However, they do apply if we take d sufficiently large, and so they at least show that Gromov's lower bound on $c_d^{\rm top}$ is not tight for large d.

We conjecture that the upper bound example mentioned above, which yields $\varphi_d(\alpha) \le \frac{d+1}{d}\alpha$, is actually optimal, and that it is essentially the only possible extremal example. However, a proof may be challenging even for the $d = 2$ case (where we have an extremal question about graphs).

Next, we formulate a combinatorial extremal problem whose solution might perhaps lead to a tight lower bound for the $c_d^{\rm top}$. At present we can obtain only a

[1]For $d = 2$, this notion and some of the following considerations are related to *Seidel switching* and *two-graphs*, which are notions studied in combinatorics.

slightly improved lower bound for c_3^{top} from it (namely, $c_3^{\mathrm{top}} \geq 0.06332$ as opposed to Gromov's $c_3^{\mathrm{top}} \geq 0.0625$), but this at least shows that one can go beyond (17).

To formulate the problem, we introduce the notation $X \approx Y$ for sets X, Y of k-tuples, meaning that $|X + Y| = o(n^k)$. The problem deals with a structure called a *pagoda*; for concreteness, we define it only for the first case of interest, with $d = 3$. A (3-dimensional) *pagoda* over a vertex set V consists of vertex sets $V_1, V_2, V_3, V_4 \subseteq V$, edge sets $E_{12}, E_{13}, \ldots, E_{34} \subseteq \binom{V}{2}$, sets F_{123}, F_{124}, F_{134}, $F_{234} \subseteq \binom{V}{3}$ of triples, and a set $G = G_{1234} \subseteq \binom{V}{4}$ of 4-tuples (the *top* of the pagoda). The sets V_i, E_{ij} and F_{ijk} are *minimal* (in the sense introduced above the definition of φ_d), and they satisfy the following relations (here i, j, k denote mutually distinct indices):

$$V_1 + V_2 + V_3 + V_4 \approx V,\ \delta V_i \approx \sum_j E_{ij},\ \delta E_{ij} \approx \sum_k F_{ijk},\ \delta F_{ijk} \approx G.$$

It can be deduced from Gromov's argument that $c_3 \geq c_3^{\mathrm{top}} \geq \liminf_{|V| \to \infty} \min \|G\|$, where the minimum is over tops G of pagodas over V. We know of *no example* of a pagoda whose top is smaller than the best known upper bound for c_3, i.e., $\frac{3}{32}$.

References

[1] I. Bárány. A generalization of Carathéodory's theorem. *Discrete Math.*, 40(2-3):141–152, 1982.

[2] E. Boros and Z. Füredi. The number of triangles covering the center of an n-set. *Geom. Dedicata*, 17:69–77, 1984.

[3] B. Bukh, J. Matoušek, and G. Nivasch. Stabbing simplices by points and flats. *Discrete Comput. Geom.*, 43(2):321–338, 2010.

[4] A. Basit, N. H. Mustafa, S. Ray, and S. Raza. Improving the first selection lemma in $\mathbb{R}^3$. In *Proc. 26th Annu. ACM Sympos. Comput. Geom., Snowbird, Utah*, 2010.

[5] M. Gromov. Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry. *Geom. Funct. Anal.*, 20(2):416–526, 2010.

[6] R. N. Karasev. A simpler proof of the Boros–Füredi–Bárány–Pach–Gromov theorem. Preprint, arXiv:1012.5890v1, 2010.

[7] N. Linial and R. Meshulam. Homological connectivity of random 2-complexes. *Combinatorica*, 26(4):475–487, 2006.

[8] R. Meshulam and N. Wallach. Homological connectivity of random k-dimensional complexes. *Random Structures Algorithms*, 34(3):408–417, 2009.

[9] J. Matoušek and U. Wagner. On Gromov's method of selecting heavily covered points. Manuscript, 2011.

[10] U. Wagner. *On k-sets and applications.* PhD thesis, ETH Zürich, 2003.

Random planar graphs (and more)

Konstantinos Panagiotou

Let $\mathcal{P}_n$ be the class of simple labeled planar graphs with n vertices, and denote by P_n a graph drawn uniformly at random from this set. Basic properties of P_n were first investigated by Denise, Vasconcellos, and Welsh [3]. Since then, the random planar graph has attracted considerable attention, and is nowadays an important and challenging model for evaluating methods that are developed to study properties of random graphs from classes with structural side constraints. In particular, the planarity condition (or any other global structural condition) makes almost all tools and methods that have been used in the past decades for the analysis of classical random graphs models fail in this context. Consequently, the development of new approaches is necessary and essential.

One attempt to resolve this issue was taken by the author and Steger in [6] and by the author in [8]. The precise setting considered there is as follows. Let $\mathcal{C}$ be a class of labeled connected graphs, and let C_n be a uniform random graph from $\mathcal{C}$ with n vertices. We assume that a graph belongs to $\mathcal{C}$ if and only if all its blocks (i.e., its maximal biconnected subgraphs) also belong to $\mathcal{C}$. Here as well as in the rest of this note, we will be calling a graph *biconnected*, if it is either 2-connected or a single edge. The main idea in their work is to consider the blocks of C_n. In this context, they showed, among other results, that under certain assumptions C_n belongs a.a.s.[2] to exactly one of the following categories:

(i) There is a unique *giant* block in C_n. More precisely, there is a $0 < c = c(\mathcal{C}) < 1$ such that the largest block contains $\sim cn$ vertices, while every other block contains $o(n)$ vertices.
(ii) All blocks contain $O(\log n)$ vertices.

Additionally, in [6] it was shown that random planar graphs belong to the former category, whereas e.g. random outerplanar graphs belong to the latter. Observe that for graphs that belong to category (ii), almost all pairs of vertices lie in different blocks, while this is not the case for graphs from the first category. A consequence of these facts is the following important observation. Random graphs from classes that belong to category (ii) "contain", in a well-defined sense, plenty of independence. In particular, any such graph can be generated by choosing independently every one of its blocks, and gluing them together at the cut-vertices. As the blocks contain few vertices, and as they intersect each other only at single vertices, the *impact* of each block to the whole graph is small. Such graphs resemble in a certain way the behavior of classical random graphs, where each edge is included independently with a specified probability, with the difference that here we choose the blocks independently of each other. However, random graphs from classes that belong to category (i) do not have this property: a lot of structure that we cannot control is "hidden" in the giant block, which contains a constant fraction of the vertices.

Using the above observation, Bernasconi, the author, and Steger gave in [1] and [2] an almost complete characterization of the degree distribution of random graphs that are drawn from *any* class that belongs to the second category. Moreover, by applying the result to special classes, like labeled trees, cacti, outerplanar graphs, and series-parallel graphs, they obtained exponential tail estimates for the probability that the number of vertices of degree k deviates significantly from its expectation, where k can be essentially as large as the maximum degree.

[2] asymptotically almost surely, i.e. with probability tending to 1 when $n \to \infty$

The above mentioned results left open the case of graph classes belonging to the first category, and in particular planar graphs. This motivates a finer analysis regarding the typical structure of large random biconnected graphs. In [4], which is joint work with N. Fountoulakis, we investigated how and under which conditions a biconnected graph can be decomposed into building blocks of higher complexity. Such a decomposition in the so-called 3-connected cores is well-known and goes back to the pioneering work of Tutte [9]. We showed that again a fundamental dichotomy is encountered: depending on some critical condition, which was determined explicitly, we proved that large biconnected graphs have either only "small" cores, or a constant fraction of the vertices is contained in a *single* such core. Hence, we discovered a picture that is completely analogous to the distribution of the sizes of the blocks in random connected graphs. Moreover, random planar graphs belong to the case in which there exists a giant core.

With the above results at hand, it seems that it is necessary to study the typical structure of large random 3-connected graphs that are drawn from the class in question. In this context, together with D. Johannsen, we studied in [5] the degree sequence of random 3-connected planar graphs, and showed sharp concentration results for the number of vertices of a given degree. In the subsequent paper [7], which is joint work with A. Steger, we addressed the problem of obtaining bounds on the degree sequence of a random member from a class that contains a large 3-connected core. We described a general framework for obtaining the degree-sequence for random connected objects from that of a random 2-connected object, and, similarly, for a random 2-connected object from that of a random 3-connected object. Applied to the class of planar graphs, this finally allowed us to obtain the bounds on the degree sequence of a random planar graph.

References

[1] N. Bernasconi, K. Panagiotou and A. Steger, On the Degree Sequences of Random Outerplanar and Series-Parallel Graphs, In *Proceedings of the 12th International Workshop on Randomized Techniques in Computation (RANDOM'08)*, 303–316, 2008.

[2] N. Bernasconi, K. Panagiotou and A. Steger, The Degree Sequence of Random Graphs from Subcritical Classes, *Combinatorics, Probability and Computing* 18(5), 647–681, 2009.

[3] A. Denise, M. Vasconcellos and D.J.A. Welsh, The random planar graph, *Congressus Numerantium* 113, 61–79, 1996.

[4] N. Fountoulakis and K. Panagiotou, 3-connected Cores in Random Planar Graphs, *Combinatorics, Probability and Computing*, accepted for publication in 2010.

[5] D. Johannsen and K. Panagiotou,Vertices of Degree k in Random Maps, In *Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '10)*, 1436–1447, 2010.

[6] K. Panagiotou and A. Steger, Maximal biconnected subgraphs of random planar graphs, In *Proceedings of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '09)*, pp. 432–440, 2009, and *Transactions on Algorithms*, 6(2), p. 1–21, 2010.

[7] K. Panagiotou and A. Steger, On the Degree Distribution of Random Planar Graphs, In *Proceedings of the 22nd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '11)*, accepted for publication, 2011.

[8] K. Panagiotou, Blocks in Constrained Random Graphs with Fixed Average Degree, In *21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC '09)*, DMTCS Proceedings AK, 733–744, 2009.

[9] W. T. Tutte, *Connectivity in graphs*, University of Toronto Press, Toronto, 1966.

The chromatic threshold of graphs

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(joint work with Peter Allen, Simon Griffiths, Yoshiharu Kohayakawa, and Robert Morris)

A classic theorem of Andrásfai, Erdős and Sós states that K_r-free graphs with high minimum degree resemble in structure extremal K_r-free graphs.

Theorem 1 (Andrásfai, Erdős & Sós [1]). *Any K_r-free graph G with minimum degree $\delta(G) > \frac{3r-7}{3r-4}n$ satisfies $\chi(G) \leq r-1$.*

More generally one can ask, given a graph H and a positive $d \leq 1$,

(1) is there a natural number χ
(2) what is the smallest χ

such that all H-free graphs G with $\delta(G) > dn$ satisfy $\chi(G) \leq \chi$. To our knowledge the only non-trivial graph H for which the answer to the second question is known in full generality is the triangle (see, e.g., [2]). In this work we address the first question. More precisely, we consider the following graph parameter.

Let H be any graph with $\chi(H) = r \geq 2$. Let $\delta_\chi(H)$ be the infimum of the numbers δ such that there is a constant $C = C(H, \delta)$ such that every H-free graph G on n vertices with $\delta(G) \geq \delta n$ satisfies $\chi(G) \leq C$. That is, for $\delta < \delta_\chi(H)$ the chromatic number of H-free graphs with minimum degree δn is unbounded, while for $\delta > \delta_\chi(H)$ it is bounded. We call $\delta_\chi(H)$ the *chromatic threshold* of H. Clearly, we have $0 \leq \delta_\chi(H) \leq \pi(H)$, where $\pi(H) := \mathrm{ex}(n, H)/\binom{n}{2} = \frac{r-2}{r-1}$ is the *Turán density* $\pi(H)$ of H.

Recently, the question of determining chromatic thresholds received increased attention. In particular, rather general results for 3-chromatic graphs H were obtained by Lyle [4] and by Łuczak and Thomassé [3].

The *decomposition family* $\mathcal{H}$ of H is the set of 2-chromatic induced subgraphs H' of H such that the vertices of H which are not in H' induce an $(r-2)$-chromatic subgraph of H. In other words, $\mathcal{H}$ contains exactly those graphs H' which are obtained from H by deleting $r-2$ arbitrary colour classes in some r-colouring of H.

Theorem 2 (Lyle [4]). *For a 3-chromatic graph H we have $\delta_\chi(H) < \frac{1}{2}$ iff the decomposition family of H contains a forest.*

Let H be a 3-chromatic graph. We say that H is a *near acyclic* graph, if there is a forest F in the decomposition family of H, such that the independent set $S := H - F$ has the following property. Every odd cycle of H meets S in at least two vertices. Equivalently, for each tree T in F with bipartition classes $V_1 \dot\cup V_2$ we have that the neighbourhood sets of V_1 and V_2 in S, that is, the sets $\bigcup_{v_1 \in V_1} N_H(v_1, S)$ and $\bigcup_{v \in V_2} N_H(v_2, S)$ are disjoint.

A construction of Łuczak and Thomassé [3, Section 8] shows that graphs H which are not near acyclic have chromatic threshold at least $\frac{1}{3}$.

Theorem 3 (Łuczak & Thomassé [3])**.** *For all integers h and C and all $\gamma > 0$ there is an n-vertex graph G with $\delta(G) \geq (\frac{1}{3} - \gamma)n$ and $\chi(G) \geq C$ such that all induced subgraphs of G on at most h vertices are near acyclic.*

Łuczak and Thomassé [3, Conjecture 1] conjecture that near acyclic graphs on the other hand have chromatic threshold 0. They prove that this is true in the case that the forest F in the definition of near acyclic graphs above is a matching.

We generalise the results of Lyle and of Łuczak and Thomassé and determine the chromatic threshold of each graph H.

Theorem 4. *If H is an r-chromatic graph with $r > 2$ then*

$$\delta_\chi(H) \in \left\{\frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1}\right\}.$$

The graphs H with $\delta_\chi(H) \leq \frac{2r-5}{2r-3}$ are exactly those which have a forest in the decomposition family. The graphs H with $\delta_\chi(H) = \frac{r-3}{r-2}$ are exactly those which have an r-colouring such that by deleting $r-3$ colour classes we obtain a near acyclic graph.

In order to prove this result we combine the approach of Lyle, which uses *Szemerédi's regularity lemma*, and techniques developed by Łuczak and Thomassé, who introduce a concept called *paired VC-dimension*, with new ideas.

References

[1] B. Andrásfai, P. Erdős, and V. T. Sós, *On the connection between chromatic number, maximal clique and minimal degree of a graph*, Discrete Math. **8** (1974), 205–218.

[2] S. Brandt and S. Thomassé, *Dense triangle-free graphs are four colorable: A solution to the Erdős-Simonovits problem*, To appear in J. Combin. Theory Ser. B.

[3] T. Łuczak and S. Thomassé, *Coloring dense graphs via VC-dimension*, Preprint (arXiv:1007.1670v1 [math.CO]).

[4] J. Lyle, *On the chromatic number of H-free graphs of large minimum degree*, Graphs and Combinatorics (2010), 1–14.

The quasi-randomness of hypergraph cut properties

Asaf Shapira

(joint work with Raphy Yuster)

We study quasi-random hypergraphs (and graphs), that is, hypergraphs which have the properties one would expect to find in "truly" random hypergraphs. We focus on k-uniform hypergraphs $H = (V, E)$ in which every edge contains precisely k distinct vertices of V. Quasi-random graphs were first explicitly studied by Thomason [6, 7] and then followed by Chung, Graham, and Wilson [3]. We start with discussing quasi-random graphs. One of the most natural questions that arise when studying quasi-random objects, is which properties "force" an object to behave like a truly-random one. The cornerstone result of this type is the theorem of quasi-random graphs due to Chung, Graham and Wilson [3]. We start with some notation. For a subset of vertices U in a graph G we denote by $e(U)$ the number of edges spanned by U in G. For a pair of sets U, U' we denote by $e(U, U')$ the number of edges with one vertex in U and the other in U'. Note that in a random graph $G(n, p)$ we expect every $U \subseteq V$ to satisfy $\frac{1}{2}p|U|^2 - o(n^2) \le e(U) \le \frac{1}{2}p|U|^2 + o(n^2)$. We say that a graph G is p-quasi-random if $e(U) = \frac{1}{2}p|U|^2 \pm o(n^2)$ for all $U \subseteq V(G)$. We say that a graph property P is quasi-random if any graph satisfying P must be quasi-random.

Our main focus is on the quasi-randomness of graph (and hypergraph) properties that involve the number of edges in certain cuts in a graph (hypergraph). Namely, for an $0 < a < 1$, we say that a graph satisfies property P_a if for any $U \subseteq V(G)$ of size $|U| = an$ we have $e(U, V \setminus U) = pa(1-a)n^2 + o(n^2)$. These properties were first studied by Chung and Graham [1, 2]. The main result of [2, 1] was a precise characterization of the cut properties P_a which are quasi-random. The somewhat surprising characterization states that P_a is quasi-random if and only if $a \neq 1/2$. To see that $P_{1/2}$ is not quasi-random, Chung and Graham [1] observed that the graph obtained by taking a random graph $G(n/2, 2p)$ on $n/2$ of the vertices, an independent set on the other $n/2$ vertices and then connecting these two graphs with a random bipartite graph with edge probability p gives a non-quasi-random graph that satisfies $P_{1/2}$. For later reference, we call this graph $C_2(n, p)$.

One of the open problems raised by Chung and Graham in their paper on quasi-random hypergraphs [1], was if one can obtain a hypergraph analog of their characterization of the cut properties of graphs which are quasi-random. Our first result in this paper answers this question positively by obtaining a precise characterization of the hypergraph cut properties that are quasi-random. Our second result in this paper will show that one can "describe" the graphs (and hypergraphs) that satisfy the cut properties P_a which are not quasi-random. In particular, it will turn out that the example of Chung-Graham [1] (the graph $C_2(n, p)$ described above) showing that $P_{1/2}$ is not quasi-random is (essentially) the only graph that satisfies P_a and is not quasi-random.

Let us first define a property of k-uniform hypergraphs which is analogous to property P_1. We say that a k-uniform hypergraph $H = (V, E)$ satisfies D_1 if $e(U) = \frac{p}{k!}|U|^k \pm o(n^k)$ for every $U \subseteq V(H)$. So property D_1 is perhaps the most intuitive notion of what it means for a hypergraph to be quasi-random. Let us briefly mention that D_1 is sometimes called *weak quasi-randomness*. We now define the appropriate generalization of the graph properties studied by Chung and Graham. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a vector of positive reals satisfying $\sum_i \alpha_i = 1$. We say that a k-uniform hypergraph on n vertices satisfies P_α if for any partition of its vertices into k-sets $V_1, \ldots, V_k$, where $|V_i| = \alpha_i n$, we have $e(V_1, \ldots, V_k) = (p + o(1))n^k \prod_{i \in S} \alpha_i$. Here $e(V_1, \ldots, V_k)$ denotes the number of edges that cross the cut $(V_1, \ldots, V_k)$ (that is, the number of edges that intersect each V_i in at most one point). Chung and Graham [1] asked whether one can find a characterization of the properties P_α which are quasi-random, that is, equivalent to D_1. Our first result in this paper answers this question by proving that property P_α is equivalent to D_1 if and only if $\alpha \neq (1/r, \ldots, 1/r)$.

Given the graph $C_2(n,p)$ which shows that $P_{1/2}$ is not quasi-random, it seems natural to try and show that when $\alpha = (1/r, \ldots, 1/r)$ property P_α is not quasi-random for k-uniform hypergraph by defining an appropriate k-partite k-uniform hypergraph. This approach does not seem to work. Instead, we define the following k-uniform hypergraph. Let $C_k(n,p)$ be the n-vertex hypergraph constructed randomly as follows. We partition the vertex set into two sets A, B of size $n/2$ each. Each set of k vertices $\{v_{i_1}, \ldots, v_{i_k}\}$ is put in $C_k(n,p)$ with probability $2pj/k$ where $j = |\{v_{i_1}, \ldots, v_{i_k}\} \cap A|$.

Observe that when $k = 2$ the graph $C_k(n,p)$ defined above is (indeed) equivalent to the (randomly constructed) graph $C_2(n,p)$ we described earlier. As we show in the paper, this random hypergraph satisfies P_α (for $\alpha = (1/r, \ldots, 1/r)$) with high probability but is not quasi-random, that is, does not satisfy D_1 defined above. This will establish that P_α is not quasi-random. Our second result in this paper shows that the hypergraphs $C_k(n,p)$ are essentially the only non quasi-random hypergraphs satisfying P_α.

Let's consider first the case of graphs. In this case the non-quasi-random cut property is $P_{1/2}$ which corresponds to counting the number of edges in balanced $(n/2, n/2)$-cuts. To describe our structure result about the graphs satisfying $P_{1/2}$ it will be more convenient to consider the following non-discrete version of $P_{1/2}$ which we denote $P^*_{1/2}$; in this problem we are asked to assign arbitrary real weights to the edges of the complete graph on n vertices in a way that for any partition of its vertices into two sets of equal size $n/2$, the total weight of edges crossing the cut is $p(n/2)^2$. Note that since $P^*_{1/2}$ allows for non-integer weights, we require the total weight crossing the cuts to be *exactly* $p(n/2)^2$, while in $P_{1/2}$ the requirement is only up to an error of $o(n^2)$.

Considering the fractional property $P^*_{1/2}$ we now ask which weight assignments satisfy $P^*_{1/2}$? Observe that this problem can be stated as trying to solve a set of linear equations, where for every $i < j$ we have an unknown $x_{i,j}$ and where for every partition of the n vertices into two sets of equal size $n/2$, we have a linear equation $\ell_{A,B}$ which checks whether $\sum_{i \in A, j \in B} x_{i,j} = p(n/2)^2$. So this set has $\binom{n}{2}$ unknowns and $\binom{n-1}{n/2-1}$ equations. One solution to this set of equations is the one corresponding to the random graph $G(n,p)$ in which all $x_{i,j} = p$. Another solution corresponds to the graph $C_2(n,p)$ define above. In this case, we obtain a solution by partitioning the vertices into two sets A and B of size $n/2$ each, and setting $x_{i,j} = 2p$ if $i,j \in A$, setting $x_{i,j} = 0$ if $i,j \in B$ and setting $x_{i,j} = p$ otherwise. Note that we thus obtain $\binom{n-1}{n/2-1}$ solutions which correspond to the possible ways of picking the sets A, B. However, observe that all these solutions are isomorphic to $C_2(n,p)$, if we consider them as weighted complete graphs.

So we can restate our question and ask if there are any other solutions to $P^*_{1/2}$ besides the above $1 + \binom{n-1}{n/2-1}$ solutions? Since we are trying to solve a set of linear equations, then one can trivially obtain other solutions by taking affine combinations of the above solutions. That is, if one considers each of the above solutions as an $\binom{n}{2}$ dimensional vector, then any affine combination of these vectors is also a solution. Our second result in this paper is that the only solutions to $P^*_{1/2}$ are the affine combinations of $G(n,p)$ and $C_2(n,p)$. We then show that given this theorem one can show (via the regularity lemma) that any graph satisfying $P_{1/2}$ can be approximated by an affine combination of $G(n,p)$ and $C_2(n,p)$, thus supplying a structural characterization of the graphs satisfying $P_{1/2}$.

When considering the hypergraph cut properties P_α, we can of course define P^*_α to be their non-discrete analog. That is, we now try to assign weights to the edges of the complete k-uniform hypergraph. We also prove the that when $\alpha = (1/k,\dots,1/k)$, the only solutions to P^*_α are the affine combinations of $G_k(n,p)$ and $C_k(n,p)$, where $G_k(n,p)$ is the random k-uniform hypergraph. This is the most challnegin part of the paper. This result was recently used by Huang and Lee [5] in order to resolve a conjecture raised in a preliminary version of this paper, and which was also raised independently by Janson [4].

References

[1] F. R. K. Chung and R. L. Graham, *Quasi-random set systems*, Journal of the AMS 4 (1991), 151–196.

[2] F. R. K. Chung and R. L. Graham, Maximum cuts and quasi-random graphs, Random Graphs, (Poznan Conf., 1989) Wiley-Interscience Publishers vol 2, 23–33.

[3] F. R. K. Chung, R. L. Graham, and R. M. Wilson, *Quasi-random graphs*, Combinatorica 9 (1989), 345–362.

[4] S. Janson, Quasi-random graphs and graph limits, manuscript, 2009 (arXiv:0905.3241).

[5] H. Huang and C. Lee, Quasi-randomness of graph balanced cut properties, submitted.

[6] A. Thomason, Pseudo-random graphs, Annals of Discrete Mathematics 33 (1987), 307-331

[7] A. Thomason, Random graphs, strongly regular graphs and pseudo-random graphs, In: *Surveys in Combinatorics*, C. Whitehead, ed., LMS Lecture Note Series 123 (1987), 173–195.

Primality of trees

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(joint work with Penny Haxell and Anusch Taraz)

The *coprime graph* S_n has vertex set $[n] := \{1,\dots,n\}$ in which two vertices are adjacent if and only if they are coprime (as numbers). For example, S_5 is isomorphic to K_5 minus one edge. Various questions and results about combinatorial properties of S_n can be found in Erdős [6, 7, 8], Erdős, Sárközy, and Szemerédi [9, 11], Szabó and Tóth [24], Erdős and Sárközy [10, 12], Ahlswede and Khachatrian [1, 2, 3], Sárközy [22], and others.

A graph G of order n is called *prime* if it is a subgraph of S_n, that is, if there is a bijection $f : V(G) \to [n]$ such that any two adjacent vertices of G are assigned coprime numbers. This notion was introduced by Entringer who, according to [14], conjectured around 1980 that every tree is prime. The earliest statement of this conjecture that we could find in the literature comes from the 1982 paper of Tout, Dabbouchy, and Howalla [27], where its formulation is preceded by the names of Entringer and Tout. Therefore we shall refer to it as the *Entringer–Tout Conjecture.*

One popular direction of research was to verify this conjecture for some very special classes of trees (small trees, caterpillars, spiders, complete binary trees, olive trees, palm trees, banana trees, twigs, binomial trees, bistars, etc). We refer the reader to the dynamic survey by Gallian [14, Section 7.2] for references to these and related results.

Here we prove this conjecture for all large n.

Theorem 1. There exists n' such that every tree with $n \geq n'$ vertices is prime.

In fact, we can show a more general result, extending Theorem 1 to a larger class of bipartite graphs. In order to state it, we have to present some definitions first. We say that a graph G is *d-degenerate* if every non-empty subgraph of G has a vertex of degree at most d. For example, a graph is 1-degenerate if and only if it is acyclic. Let us a call a graph G *s-separable* if for every subgraph $G' \subseteq G$ there is a set $S \subseteq V(G')$ such that $|S| \leq s$ and each component of $G' - S$ has at most $|V(G')|/2$ vertices. The choice of the constant $1/2$ is rather arbitrary; we choose it for the convenience of calculations and because of the well-known fact that trees are 1-separable. Also, in order to make our results stronger, we use a weaker version of separability where the upper bound s depends only on $|V(G)|$ and not on $|V(G')|$.

Lipton and Tarjan [18] showed that every order-n planar graph G contains a set X with $|X| \leq 2\sqrt{2n}$ such that no component of $G - X$ has more than $2n/3$ vertices. Clearly, by applying this theorem twice to any given subgraph $G' \subseteq G$, we can eliminate all components of order larger than $|V(G')|/2$. Thus G is $4\sqrt{2n}$-separable. Likewise, the result of Alon, Seymour, and Thomas [4] implies that any order-n graph without a K_h-minor is $2h^{3/2}n^{1/2}$-separable.

Given an integer $d \geq 1$, define a function $s = s(n)$ by

$$s(n) := n^{1-\frac{10^6 \cdot d}{\ln \ln n}}.$$

Here is the main result of this paper.

Theorem 2. For every $d \geq 1$ there exists n'' such that every $s(n)$-separable bipartite d-degenerate graph F of order $n \geq n''$ is prime.

Mader [19] showed that every K_h-minor free graph F has average degree at most $f(h)$, with more precise estimates on the function $f(h)$ given by Kostochka [15] and Thomason [25, 26]. Since not containing a K_h-minor is a hereditary property, such a graph F is necessarily $f(h)$-degenerate. This and the above-mentioned result of Alon, Seymour, and Thomas [4] allow us to deduce that for every h, all sufficiently large bipartite graphs without a K_h-minor are prime.

Based on an earlier version of this manuscript that had a slightly simpler proof just for trees and using the results of Dusart [5] on the distribution of primes, Spiess [23] estimated that taking $n' = 10^{10^{100}}$ in Theorem 1 is a suitable choice.

Although the Entringer–Tout Conjecture for trees of small order n seems quite amenable (see [13, 20, 21, 17] with the current record $n \leq 206$ claimed in a recent manuscript of Kuo and Fu [16]), closing this gap is beyond any small-order approaches.

Two main difficulties in proving Theorem 2 are that we have to use every element of $[n]$ as a label (that is, we look for spanning subgraphs in S_n) and that S_n has a large independent set $\{2, 4, 6, \ldots\}$. On the other hand, every vertex in the set

$$P_1 := \{p \in [n] \mid p > n/2 \text{ and } p \text{ is prime}\}$$

is *universal*, that is, it is adjacent to all other vertices of S_n. Likewise, every vertex $2p$ in the set

$$P_0 := \{2p \in [n] \mid p > n/3 \text{ and } p \text{ is prime}\}$$

is adjacent to all vertices in S_n with odd labels, except p. The existence of these two sets, each of order $\Theta(n/\ln n)$, crucially helps in our proof.

Our proof consists of three parts. First, we split the given graph F satisfying the assumptions of Theorem 2 into tiny components by removing a small set M of vertices using the separability property. It also arranges these components into groups in order to balance more evenly the distribution of vertices among groups. Then we specify where each group is to be mapped inside $[n]$. Since we do not have much control over the vertices in M, they are mapped into $P_0 \cup P_1$. As we have already mentioned, one has to be careful to ensure that every group has a sufficiently large independent set to host all even labels that are assigned to it. Apart from the Prime Number Theorem, we use only very basic results about divisibility and primality of integers. Finally, we show how to embed each group into its assigned part of S_n; this is the point when we need the d-degeneracy property.

References

[1] R. Ahlswede and L. H. Khachatrian, *On extremal sets without coprimes*, Acta Arith. **66** (1994), 89–99.

[2] ———, *Maximal sets of numbers not containing $k+1$ pairwise coprime integers*, Acta Arith. **72** (1995), 77–100.

[3] ———, *Sets of integers and quasi-integers with pairwise common divisor*, Acta Arith. **74** (1996), 141–153.

[4] N. Alon, P. Seymour, and R. Thomas, *A separator theorem for nonplanar graphs*, J. Amer. Math. Soc. **3** (1990), 801–808.

[5] P. Dusart, *The kth prime is greater than $k(\ln k + \ln\ln k - 1)$ for $k \geq 2$*, Math. Comp. **68** (1999), 411–415.

[6] P. Erdős, *Remarks on number theory. IV. Extremal problems in number theory. I*, Mat. Lapok **13** (1962), 228–255.

[7] P. Erdős, *A survey of problems in combinatorial number theory*, Ann. Discrete Math. **6** (1980), 89–115.

[8] P. Erdős, *Some of my new and almost new problems and results in combinatorial number theory*, Number theory (Eger, 1996), de Gruyter, Berlin, 1998, pp. 169–180.

[9] P. Erdős, A. Sárközi, and E. Szemerédi, *On some extremal properties of sequences of integers*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **12** (1969), 131–135.

[10] P. Erdös and A. Sárközy, *On sets of coprime integers in intervals*, Hardy-Ramanujan J. **16** (1993), 1–20.
[11] P. Erdős, A. Sárközy, and E. Szemerédi, *On some extremal properties of sequences of integers. II*, Publ. Math. Debrecen **27** (1980), 117–125.
[12] P. Erdős and G. N. Sárközy, *On cycles in the coprime graph of integers*, Electronic J. Combin. **4** (1997), 11 pp., The Wilf Festschrift (Philadelphia, PA, 1996).
[13] H.-L. Fu and K.-C. Huang, *On prime labellings*, Discrete Math. **127** (1994), 181–186.
[14] J. A. Gallian, *A dynamic survey of graph labeling*, Electronic J. Combin. **DS6** (2009), 219pp.
[15] A. V. Kostochka, *A lower bound for the Hadwiger number of a graph as a function of the average degree and its vertices (in Russian)*, Diskret. Analiz. Novosibirsk **38** (1982), 37–58.
[16] J. Kuo and H.-L. Fu, *Modified prime labelling on small trees*, Manuscript, 2010.
[17] S.-H. Lin, *A study of prime labeling*, M.S. Thesis, National Chiao Tung University, Taiwan, 1999.
[18] R. J. Lipton and R. E. Tarjan, *A separator theorem for planar graphs*, SIAM J. Appl. Math. **36** (1979), 177–189.
[19] W. Mader, *Homomorphiesätze für Graphen*, Mathematische Annalen **178** (1968), 154–168.
[20] O. Pikhurko, *Every tree with at most* 34 *vertices is prime*, Utilitas Math **62** (2002), 185–190.
[21] ______, *Trees are almost prime*, Discrete Math. **307** (2007), 1455–1462.
[22] G. N. Sárközy, *Complete tripartite subgraphs in the coprime graph of integers*, Discrete Math. **202** (1999), 227–238.
[23] J. Spiess, *Embedding of spanning trees into special classes of graphs*, Bachelor's thesis, TU München, 2010.
[24] C. Szabó and G. Tóth, *Maximal sequences not containing four pairwise coprime integers*, Mat. Lapok **32** (1985), 253–257.
[25] A. G. Thomason, *An extremal function for contractions of graphs*, Math. Proc. Camb. Phil. Soc. **95** (1984), 261–265.
[26] A. G. Thomason, *The extremal function for complete minors*, J. Combin. Theory (B) **81** (2001), 318–338.
[27] A. Tout, A. N. Dabboucy, and K. Howalla, *Prime labeling of graphs*, Nat. Acad. Sci. Letters **11** (1982), 365–368.

Extremal results for random discrete structures

Mathias Schacht

Introduction. Extremal problems are widely studied in discrete mathematics. Given a finite set Γ and a family $\mathcal{F}$ of subsets of Γ an *extremal result* asserts that any sufficiently large (or dense) subset $G \subseteq \Gamma$ must contain an element from $\mathcal{F}$. Often all elements of $\mathcal{F}$ have the same size, i.e., $\mathcal{F} \subseteq \binom{\Gamma}{k}$ for some integer k, where $\binom{\Gamma}{k}$ denotes the family of all k-element subsets of Γ. For example, if $\Gamma_n = [n] = \{1, \dots, n\}$ and $\mathcal{F}_n$ consists of all k-element subsets of $[n]$ which form an arithmetic progression, then Szemerédi's celebrated theorem [7] asserts that every subset $Y \subseteq [n]$ with $|Y| = \Omega(n)$ contains an arithmetic progression of length k.

A well known result from graph theory, which fits this framework, is Turán's theorem [8] and its generalization due to Erdős and Stone [5] (see also [4]). Here $\Gamma_n = E(K_n)$ is the edge set of the complete graph with n vertices and $\mathcal{F}_n$ consists of the edge sets of copies of some fixed graph F (say with k edges) in K_n. Here the Erdős-Stone theorem implies that every subgraph $H \subseteq K_n$ which contains at least $\left(1+\frac{1}{\chi(F)-1}+o(1)\right)\binom{n}{2}$ edges must contain a copy of F, where $\chi(F)$ denotes the chromatic number of F (see, e.g. [1]).

We are interested in "random versions" of such extremal results. We study the *binomial model* of random substructures. For a finite set Γ_n and a probability $p \in [0,1]$ we denote by $\Gamma_{n,p}$ the random subset where every $x \in \Gamma_n$ is included in $\Gamma_{n,p}$ independently with probability p. For example, if Γ_n is the edge set of the complete graph on n vertices, then $\Gamma_{n,p}$ denotes the usual binomial random graph $G(n,p)$ (see, e.g., [2, 6]).

The deterministic extremal results mentioned earlier can be viewed as statements which hold with probability 1 for $p=1$ and it is natural to investigate the asymptotic of the smallest probabilities for which those results hold. In the context of Szemerédi's theorem for every $k \geq 3$ and $\varepsilon > 0$ we are interested in the smallest sequence $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ of probabilities such that the binomial random subset $[n]_{p_n}$ has asymptotically almost surely (a.a.s., i.e. with probability tending to 1 as $n \to \infty$) the following property: Every subset $Y \subseteq [n]_{p_n}$ with $|Y| \geq \varepsilon|[n]_{p_n}|$ contains an arithmetic progression of length k. Similarly, in the context of the Erdős-Stone theorem, for every graph F and $\varepsilon > 0$ we are interested in the asymptotic of the smallest sequence $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}}$ such that the random graph $G(n,p_n)$ a.a.s. satisfies: every $H \subseteq G(n,p)$ with

$$e(H) \geq \left(1 - \frac{1}{\chi(F)-1} + \varepsilon\right) e(G(n,p_n))\,,$$

contains a copy of F.

We determine the asymptotic growth of the smallest such sequence $\boldsymbol{p}$ of probabilities for those and some related extremal properties including multidimensional versions of Szemerédi's theorem, for extremal problems for hypergraphs, and several other extremal results from combinatorics. In other words, we determine the *threshold* for those properties. Similar results were obtained by Conlon and Gowers [3].

The new results follow from a general result (Theorem 3), which allows us to transfer certain extremal results from the classical deterministic setting to the probabilistic setting.

Main result. The main result will be phrased in the language of hypergraphs. An ℓ-uniform hypergraph H is a pair (V,E), where the vertex set V is some finite set and the edge set $E \subseteq \binom{V}{\ell}$ is a subfamily of the ℓ-element subsets of V. As usual we call 2-uniform hypergraphs simply graphs. For some hypergraph H we denote by $V(H)$ and $E(H)$ its vertex set and its edge set and we denote by $v(H)$ and $e(H)$ the cardinalities of those sets. For an integer n we denote by $K_n^{(\ell)}$ the complete ℓ-uniform hypergraph on n vertices, i.e., $v(K_n^{(\ell)}) = n$ and $e(K_n^{(\ell)}) = \binom{n}{\ell}$. An ℓ-uniform hypergraph H' is a sub-hypergraph of H, if $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$ and we write $H' \subseteq H$ to denote that. For a subset $U \subseteq V(H)$ we denote by $E(U)$ the edges of H contained in U and we set $e(U) = |E(U)|$. Moreover, we write $H[U]$ for the sub-hypergraph induced on U, i.e., $H[U] = (U, E(U))$.

We will study sequences of hypergraphs $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$. In the context of Szemerédi's theorem one may think of $V_n = [n]$ and E_n being the arithmetic progressions of length k. and for the Erdős–Stone theorem one should think of $V_n = E(K_n^{(\ell)})$ being the edge set of the complete hypergraph $K_n^{(\ell)}$ and edges of E_n correspond to copies of F in $K_n^{(\ell)}$.

An ℓ-uniform hypergraph H' is a sub-hypergraph of H, if $V(H') \subseteq V(H)$ and $E(H') \subseteq E(H)$ and we write $H' \subseteq H$ to denote that. For a subset $U \subseteq V(H)$ we denote by $E(U)$ the edges of H contained in U and we set $e(U) = |E(U)|$. Moreover, we write $H[U]$ for the sub-hypergraph induced on U, i.e., $H[U] = (U, E(U))$.

In order to transfer an extremal result from the classical, deterministic setting to the probabilistic setting we will require that a stronger quantitative version of the extremal result holds (see Definition 1 below). Roughly speaking, we will require that a sufficiently dense sub-structure not only contains one copy of the special configuration (not only one arithmetic progression or not only one copy of F), but instead the number of those configurations should be of the same order as the total number of those configurations in the given underlying ground set.

Definition 1. *Let $\boldsymbol{H} = (H_n)_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs and $\alpha \geq 0$. We say $\boldsymbol{H}$ is α-**dense** if the following is true.*

For every $\varepsilon > 0$ there exist $\zeta > 0$ and n_0 such that for every $n \geq n_0$ and every $U \subseteq V(H_n)$ with $|U| \geq (\alpha + \varepsilon)|V(H_n)|$ we have $|E(H_n[U])| \geq \zeta |E(H_n)|$.

The second condition in Theorem 3 imposes a lower bound on the smallest probability for which we can transfer the extremal result to the probabilistic setting (see Definition 2). For a k-uniform hypergraph $H = (V, E)$, $i \in [k-1]$, $v \in V$, and $U \subseteq V$ we denote by $\deg_i(v, U)$ the number of edges of H containing v and having at least i vertices in $U \setminus \{v\}$. More precisely,

$$\deg_i(v, U) = |\{e \in E\colon\ |e \cap (U \setminus \{v\})| \geq i \text{ and } v \in e\}| \ .$$

For $q \in (0, 1)$ we let $\mu_i(H, q)$ denote the expected value of the sum over all such degrees squared with $U = V_q$ being the binomial random subset of V

$$\mu_i(H, q) = \mathbb{E}\left[\sum_{v\in V} \deg_i^2(v, V_q)\right] .$$

Definition 2. *Let $\boldsymbol{H} = (H_n)_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}} \in (0, 1)^{\mathbb{N}}$ be a sequence of probabilities, and let $K \geq 1$. We say $\boldsymbol{H}$ is $(K, \boldsymbol{p})$-**bounded** if the following is true.*

For every $i \in [k-1]$ there exists n_0 such that for every $n \geq n_0$ and $q \geq p_n$ we have $\mu_i(H_n, q) \leq Kq^{2i}|E(H_n)|^2/|V(H_n)|$.

With those definitions at hand, we can state the main result.

Theorem 3. *Let $\boldsymbol{H} = (H_n = (V_n, E_n))_{n\in\mathbb{N}}$ be a sequence of k-uniform hypergraphs, let $\boldsymbol{p} = (p_n)_{n\in\mathbb{N}} \in (0, 1)^{\mathbb{N}}$ be a sequence of probabilities satisfying*

$p_n^k|E_n| \to \infty$ as $n \to \infty$, and let $\alpha \geq 0$ and $K \geq 1$. If $\boldsymbol{H}$ is α-dense and $(K, \boldsymbol{p})$-bounded, then the following holds.

For every $\delta > 0$ and $(\omega_n)_{n \in \mathbb{N}}$ with $\omega_n \to \infty$ as $n \to \infty$ there exists $C \geq 1$ such that for every $q_n \geq Cp_n$ the following holds a.a.s. for V_{n,q_n}: If $W \subseteq V_{n,q_n}$ with $|W| \geq (\alpha + \delta)|V_{n,q_n}|$, then $E(H_n[W]) \neq \emptyset$.

Several 1-statements of the thresholds for extremal problems in combinatorics follow from Theorem 3. In particular, Theorem 3 can used to obtain the threshold for the probabilistic versions of Szemerédi's theorem,its multidimensional generalizations, and for the Erdős–Stone theorem.

References

[1] B. Bollobás, *Extremal graph theory*, London Mathematical Society Monographs, vol. 11, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1978.

[2] ———, *Random graphs*, second ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001.

[3] D. Conlon and W. T. Gowers, *Combinatorial theorems in sparse random sets*, submitted.

[4] P. Erdős and M. Simonovits, *A limit theorem in graph theory*, Studia Sci. Math. Hungar **1** (1966), 51–57.

[5] P. Erdős and A. H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091.

[6] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000.

[7] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245, Collection of articles in memory of Juriĭ Vladimirovič Linnik.

[8] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452.

Hypergraph list coloring and Euclidean Ramsey theory

Noga Alon

(joint work with Alexandr Kostochka)

The *list chromatic number* (or *choice number*) $\chi_\ell(G)$ of a graph $G = (V, E)$ is the minimum integer s such that for every assignment of a list L_v of s colors to each vertex v of G, there is a proper vertex coloring of G in which the color of each vertex is in its list. This notion was introduced independently by Vizing and by Erdős, Rubin and Taylor. In both papers the authors realized that this is a variant of usual coloring that exhibits several new properties, and that in general $\chi_\ell(G)$, which is always at least as large as the chromatic number of G, may be arbitrarily large even for graphs G of chromatic number 2.

It is natural to extend the notion of list coloring to hypergraphs. The list chromatic number $\chi_\ell(H)$ of a hypergraph H is the minimum integer s such that for every assignment of a list of s colors to each vertex of H, there is a vertex coloring of H assigning to each vertex a color from its list, with no monochromatic edges.

An intriguing property of list coloring of graphs, which is not shared by ordinary vertex coloring, is the result proved in [1] that the list chromatic number of any (simple) graph with a large average degree is large. Indeed, it is shown in [1] that the list chromatic number of any graph with average degree d is at least $(\frac{1}{2} - o(1)) \log_2 d$, where the $o(1)$-term tends to zero as d tends to infinity. Our main combinatorial result is an extension of this fact to **simple** uniform hypergraphs.

Recall that a hypergraph is called *simple* if every two of its distinct edges share at most one vertex. We prove that the result of [1] can be extended to simple r-graphs. This is stated in the following theorem.

Theorem 1. *For every fixed $r \geq 2$ and $s \geq 6r$, there is $d = d(r, s)$, such that the list chromatic number of any simple r-graph with n vertices and nd edges is greater than s.*

A similar result for the special case of d-regular 3-uniform simple hypergraphs has been obtained independently by Haxell and Verstraete [2].

It is worth noting that the theorem provides a linear time algorithm for computing, for a given input simple r-graph, a number s such that its list chromatic number is at least s and at most $f(s)$ for some explicit function f. There is no such known result for ordinary coloring, and it is known that there cannot be one under some plausible hardness assumptions in Complexity Theory.

The above result has an intriguing geometric application. A well known problem of Hadwiger and Nelson is that of determining the minimum number of colors required to color the points of the Euclidean plane so that no two points at distance 1 have the same color. Hadwiger showed already in 1945 that 7 colors suffice, and L. Moser and W. Moser noted that 3 colors do not suffice. These bounds have not been improved, despite a considerable amount of effort by various researchers.

A more general problem has been considered by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus under the name Euclidean Ramsey Theory. The main question is the investigation of finite point sets K in the Euclidean space for which any coloring of an Euclidean space of a sufficiently high dimension $d \geq d_0(K, r)$ by r colors must contain a monochromatic copy of K. The main conjecture is that this holds for any set K that can be embedded in a sphere.

The situation is different for list coloring. Indeed, as a corollary of our results here, we prove the following.

Theorem 2. *For any finite set X in the Euclidean plane and for any positive integer s, there is an assignment of a list of size s to every point of the plane, such that whenever we color the points of the plane from their lists, there is a monochromatic isometric copy of X.*

References

[1] N. Alon, *Degrees and choice numbers*, Random Structures & Algorithms **16** (2000), 364–368.

[2] P. E. Haxell and J. Verstraete, *List coloring hypergraphs*, Elect. J. Combinatorics **17** (2010), #R129, 12pp.

Convexity spaces and extremal set theory

Boris Bukh

Radon's lemma [Rad21] states that every set P of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two classes $P = P_1 \cup P_2$ so that the convex hulls of P_1 and P_2 intersect. Birch [Bir59] (for $d = 2$) and Tverberg [Tve66] (for general d) extended Radon's theorem to the analogous statement for partitions of a set into more than two parts: For a set $P \subset \mathbb{R}^d$ of $|P| \geq (k-1)(d+1)+1$ points there is a partition $P = P_1 \cup \cdots \cup P_k$ into k parts, such that the intersection of the convex hulls $\operatorname{conv} P_1 \cap \cdots \cap \operatorname{conv} P_k$ is non-empty. The bound of $(k-1)(d+1)+1$ is sharp, as witnessed by any set of points in sufficiently general position.

Calder [Cal71] conjectured and Eckhoff [Eck79] speculated that Tverberg's theorem is a consequence of Radon's theorem in the context of abstract convexity spaces. The conjecture, which we now present, is commonly referred as "Eckhoff's conjecture", and we will maintain this tradition to avoid additional confusion. If true, the conjecture would have provided a purely combinatorial proof of Tverberg's theorem. However, we will show that the conjecture is false.

A *convexity space* on the ground set X is a family $\mathcal{F} \subset 2^X$ of subsets of X, called *convex sets*, such as both $\emptyset$ and X are convex, and intersection of any collection of convex sets is convex. For example, the familiar convex sets in $\mathbb{R}^d$ form a convexity space on $\mathbb{R}^d$. Among the other examples are axis-parallel boxes in $\mathbb{R}^d$, finite subsets on any ground set, closed sets in any topological space (see the book [vdV93] for a through overview of convexity spaces). If the ground set X in the convexity space $(X, \mathcal{F})$ is clear from the context, we will speak simply of a convexity space $\mathcal{F}$. The *convex hull* of a set $P \subset X$, denoted $\operatorname{conv} P$, is the intersection of all the convex sets containing P. We write $\operatorname{conv}_{\mathcal{F}} P$ if the convexity space is not clear from the context. The k-th *Radon number* of $(X, \mathcal{F})$ is the minimum natural number r_k, if it exists, so that every set $P \subset X$ of at least r_k points admits a partition $P = P_1 \cup \cdots \cup P_k$ into k parts whose convex hulls have an element in common. It is not hard to show[3] that if r_2 is finite, then so is r_k. Eckhoff's conjecture states that $r_k \leq (k-1)(r_2-1)+1$ in every convexity space. The conjecture has been proved for $r_2 = 3$ by Jamison [JW81], and for convexity space with at most $2r_2$ points by Sierksma and Boland [SB83].

The best bounds on r_k are

$$r_{k_1 k_2} \leq r_{k_1} r_{k_2} \qquad \text{(due to Jamison [JW81])},$$
$$r_{2k+1} \leq (r_2-1)(r_{k+1}-1) + r_k + 1 \qquad \text{(due to Eckhoff [Eck00])}.$$

In particular,

$$r_k \leq k^{\lceil \log_2 r_2 \rceil}. \tag{18}$$

We present a new bound that improves on (18).

[3]According to [Eck00] it was first shown by R.E.Jamison (1976). The first published proofs appear to be in [DRS81] and [JW81].

Theorem 1. *Let $(X,\mathcal{F})$ be a convexity space, and assume that r_2 is finite. Then*

$$r_k \le c(r_2)k^2 \log^2 k,$$

where $c(r_2)$ is a constant that depends only on r_2.

Though this bound is not far from Eckhoff's conjecture, the conjecture itself is false.

Theorem 2. *For each $k \ge 3$ there is a convexity space $(X,\mathcal{F})$ such that $r_2 = 4$, but $r_k \ge 3(k-1)+2$.*

Informally, the idea behind these results is to study the nerve of a family of a convex sets rather than the sets themselves. The *nerve* associate to a set $[n] = \{1,2,\ldots,n\}$ in a convexity space is the collection consisting of all set families $\mathcal{F} \subset 2^{[n]}$ such that $\operatorname{conv}_{S\in\mathcal{F}} \operatorname{conv}(S)$ is non-empty. It can be shown that there is a correspondence between such collections, and the convexity spaces. The advantage of viewing the problem from this angle is that one can apply results from extremal set theory. For example, the main ingridient in the Theorem 1 is a multidimensional generalization of a Kruskal–Katona theorem [Kru63, Kat68], which we now describe.

A *d-dimensional r-uniform family* is a collection of d-tuples of r-element sets. In other words, if we denote by $\binom{X}{r}$ the family of all r-element subsets of X, then d-dimensional r-uniform family is a subset of $\binom{X}{r}^d$. A *shadow* of such a family $\mathcal{F} \subset \binom{X}{r}^d$ is defined to be

$$\partial\mathcal{F} \overset{\text{def}}{=} \{(S_1 \setminus \{x_i\},\ldots,S_d \setminus \{x_d\}) : (S_1,\ldots,S_d) \in \mathcal{F}, \text{and } x_i \in S_i \text{ for } i = 1,\ldots,d\}.$$

Note that in the case $d = 1$, the definition reduces to the familiar definition of a shadow of a set family. The following is thus a generalization of Lovász's version [Lov79, Ex. 13.31(b)] of Kruskal–Katona theorem.

Theorem 3. *Suppose $\mathcal{F} \subset \binom{X}{r}^d$ is a d-dimensional r-uniform family of size*

$$|\mathcal{F}| = \binom{x}{r}^d,$$

where $x \ge r$ is a real number. Then

$$|\partial\mathcal{F}| \ge \binom{x}{r-1}^d.$$

Moreover, the equality holds only if $\mathcal{F}$ is of the form $\binom{Y_1}{r} \times \cdots \times \binom{Y_d}{r}$ for some sets $Y_1,\ldots,Y_d \subset X$.

References

[Bir59] B. J. Birch. On $3N$ points in a plane. *Proc. Cambridge Philos. Soc.*, 55:289–293, 1959.

[Cal71] J. R. Calder. Some elementary properties of interval convexities. *J. London Math. Soc. (2)*, 3:422–428, 1971.

[DRS81] Jean-Paul Doignon, John R. Reay, and Gerard Sierksma. A Tverberg-type generalization of the Helly number of a convexity space. *J. Geom.*, 16(2):117–125, 1981.

[Eck79] Jürgen Eckhoff. Radon's theorem revisited. In *Contributions to geometry (Proc. Geom. Sympos., Siegen, 1978)*, pages 164–185. Birkhäuser, Basel, 1979.
[Eck00] Jürgen Eckhoff. The partition conjecture. *Discrete Math.*, 221(1-3):61–78, 2000. Selected papers in honor of Ludwig Danzer.
[JW81] Robert E. Jamison-Waldner. Partition numbers for trees and ordered sets. *Pacific J. Math.*, 96(1):115–140, 1981. http://projecteuclid.org/getRecord?id=euclid.pjm/1102734951.
[Kat68] G. Katona. A theorem of finite sets. In *Theory of graphs (Proc. Colloq., Tihany, 1966)*, pages 187–207. Academic Press, New York, 1968.
[Kru63] Joseph B. Kruskal. The number of simplices in a complex. In *Mathematical optimization techniques*, pages 251–278. Univ. of California Press, Berkeley, Calif., 1963.
[Lov79] L. Lovász. *Combinatorial problems and exercises.* North-Holland Publishing Co., Amsterdam, 1979.
[Rad21] Johann Radon. Mengen konvexer Körper, die einen gemeinsamen Punkt enthalten. *Math. Ann.*, 83(1-2):113–115, 1921.
[SB83] Gerard Sierksma and Jan Ch. Boland. On Eckhoff's conjecture for Radon numbers; or how far the proof is still away. *J. Geom.*, 20(2):116–121, 1983.
[Tve66] H. Tverberg. A generalization of Radon's theorem. *J. London Math. Soc.*, 41:123–128, 1966.
[vdV93] M. L. J. van de Vel. *Theory of convex structures*, volume 50 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam, 1993.

Problem session

Among others, the following problems were presented.

Noga Alon: Call a directed graph $D = (V, E)$ s-dominating if for any subset U of at most s vertices of V there is a vertex v such that (v, u) is a directed edge for every $u \in U$. Is it true that any 100-dominating finite directed graph contains a directed cycle of length at most 100?

Anders Björner: Consider the following graph $G_{q,d}$.

- Vertices: The maximal chains $S_0 \subset S_1 \subset \cdots \subset S_{d+1}$ in the subspace lattice of all linear subspaces of a $(d+1)$-dimensional vector space over the finite field GF_q ordered by inclusion.
- Edges: Pairs of such chains that are identical in all dimensions except one.

For instance, $G_{q,2}$ is the line graph of the point-line incidence graph of a projective plane.

Question: What is the chromatic number $\chi(G_{q,d})$?

It is known that $q + 1 \leq \chi(G_{q,d}) \leq dq$, where the lower bound comes from the clique size and the upper bound from Brooks' theorem.

Remarks: 1. The question arose in connection with chamber graphs of Tits buildings. In recent work with Kathrin Vorwerk [2] we determined the degree of connectivity of such graphs. It is natural in this connection to inquire about other important graph parameters for chamber graphs, such as chromatic number. The graphs $G_{q,d}$ are the chamber graphs of buildings of type A, so this is a reasonable starting point.

2. Note the structural similarity with the graphs figuring in the Erdös-Faber-Lovász conjecture. Namely, $G_{q,d}$ consists of a collection of $(q+1)$-cliques that pairwise intersect in at most one vertex. In addition, here every vertex belongs to exactly d such cliques.

3. Gábor Kun has shown that $\chi(G_{q,d}) \leq (q+1)^2$ (private communication the day after the problem session).

Maria Chudnovsky: For a graph G, denote by $\omega(G)$ the size of the largest clique in G, and by $\chi(G)$ the chromatic number of G. A *wheel* in a graph G is a pair (C, v) where C is an induced cycle of length at least 4 in G, and $v \in V(G) \setminus V(C)$ has at least three neighbors in $V(C)$. Let $\mathcal{C}$ be the class of graphs that do not contains a wheel. Does there exist a function $f : \mathcal{N} \to \mathcal{N}$ such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{C}$? This problem was shown to me by Nicolas Trotignon.

Michael Krivelevich: Prove that for any $p = p(n)$, if G is distributed as the random graph $G(n, p)$, then with high probability G contains $\lfloor \delta(G)/2 \rfloor$ edge disjoint Hamilton cycles.

Comments: posed explicitly by Frieze and Krivelevich [5]. Known to be true for $p = (1 + o(1) \ln n/n$ (Bollobás and Frieze [3], Frieze and Krivelevich [6]). Known to hold asymptotically for $p \gg \log n/n$ (Frieze and Krivelevich [5], Knox, Kühn and Osthus [7]).

Imre Leader: In the following problem, we think of elements of S_n (the symmetric group of order n) as words that are rearrangements of the symbols $1, ..., n$, and we write xy to denote the concatenation of words x and y. Thus for example if x is the word 1345 and y is the word 26 then xy is the word 134526, which is a member of S_6.

Question: Given k, does there exist an n such that whenever S_n is k-coloured there exist words x, y, z such that all of the words xyz, xzy, yxz, yzx, zxy, zyx (are in S_n and) have the same colour? One could view this as a 'monochromatic copy of S_3'. Similarly, one would like a monochromatic copy of S_m, for each value of m.

Nati Linial: Let $f : \{0,1\}^n \to \{0,1\}$ be a boolean function and let $S \subseteq \{1, \ldots, n\}$. Let us conduct the following experiment: Set all variables x_j for $j \notin S$ to 0 or 1 independently at random. This partial assignment of values to the variables may or may not determine the value of f. The probability that f remains undetermined is called the *influence* of S on f. This notion was defined by Ben-Or and Linial. A well-known result of Kahn Kalai and Linial says that if the expectation of f is around 1/2 (In fact, it's enough that it's bounded away from both zero and one.), then there is a set S of cardinality $o(n)$ and influence $1 - o(1)$. We ask: Is it true that under the same conditions there is a set S of cardinality $\leq n/3$ and influence

$\geq 1 - c^n$ for some absolute $1 > c > 0$? We note that the constant 1/3 was chosen arbitrarily and any coefficient $< 1/2$ should work just as well.

László Lovász: Let $\mathcal{H}$ be a hypergraph with Vapnik–Červonenkis dimension k. We set $\mathcal{H}(\triangle)\mathcal{H} = \{A \triangle B : A, B \in \mathcal{H}\}$.

Question: How large can be the VC-dimension of $\mathcal{H}(\triangle)\mathcal{H}$? It is not hard to prove that $\dim_{VC}(\mathcal{H}(\triangle)\mathcal{H}) \leq 10k$ [8], and (at least for an even k) one can construct an example with $\dim_{VC}(\mathcal{H}(\triangle)\mathcal{H}) = 3k$.

Jaroslav Nešetřil: The efforts to extend the characterizations of Nowhere Dense Classes in logical terms put some of the old problems in new light. For example the following was isolated recently by Nešetřil and Ossona de Mendez [9] in the context of characterization of bounded expansion classes by means of First Order Logic definability.

Question: Is it true that for any two positive integers k, ℓ there exist $f(k, \ell)$ and $s(g)$ such that every graph G with chromatic number $\geq f(k, g)$ contains a subgraph G' such that one of the following two conditions holds:

- either $\chi(G') \geq k$ and G' contains no odd cycle of length $\leq \ell$,
- or G' contains as a subgraph complete graph K_k with every edge being subdivided by at most $s(k)$ vertices.

Without the second alternative and with girth instead of odd-girth, this is a famous old problem of Erdős and Hajnal [4]. This has been proved for $\ell = 3$ by Rödl [10] and this remains presently the only known case for this conjecture. Shallow topological minors may shed some light here.

Oleg Pikhurko: Let F be a 3-graph. Its *saturation* function $\mathrm{sat}(n, F)$ is the smallest size of a maximal F-free 3-graph on n vertices. Suppose that F has an edge D such that any other edge intersects D in at most one vertex. Prove that $\mathrm{sat}(n, F) = O(n)$. This is the first open case of a conjecture of Tuza [11].

If the above statement is true, then we would know the order of magnitude of $\mathrm{sat}(n, F)$ for every 3-graph F; it will be $\Theta(n^2)$, $\Theta(n)$, or eventually constant.

József Solymosi: (C_4 Removal Lemma for sparse graphs.) Is it true that for any $\epsilon > 0$ there is a threshold, $n_0 = n_0(\epsilon)$ such that the following holds? If a graph on $n \geq n_0$ vertices is the edge-disjoint union of at least $\epsilon n^{3/2}$ quadrilaterals, then it contains another quadrilateral.

Comments: It was noticed by Maria Axenovich that a positive answer would contradict a conjecture of Felix Lazebnik and Jacques Verstraëte about generalized Sidon sets.

I conjecture that the following stronger version holds: For any $\epsilon > 0$ there is a $\delta > 0$ so that if a graph on n vertices is the edge-disjoint union of at least $\epsilon n^{3/2}$ quadrilaterals, then it contains at least δn^2 quadrilaterals.

Benny Sudakov: Let G be an n-vertex graph with minimum degree larger than $3n/4$. Is it true that the largest triangle-free subgraph of G is bipartite? If yes, then this is tight, see example in [1].

Tibor Szabó: A (k,d)-tree is a binary tree, where each vertex has either two or zero children, the depth of each leaf is at least k, and every vertex has at most d leaf-descendants of distance at most k. Let

$$f_2(k) = \max\{d : \text{there exists no } (k,d)\text{-tree}\}.$$

A (k,d)-$MU(1)$-system is a binary tree $T = (V,E)$ where each vertex has either two or zero children, together with a family $\mathcal{F} = \{F_l \subseteq V : \; l \text{ is a leaf of } T\}$ such that each set F_l contains exactly k non-leaf vertices on the path from l to the root of T and every vertex is in at most d sets of the family. Let

$$f_1(k) = \max\{d : \text{there exists no } (k,d)\text{-}MU(1)\text{-system}\}.$$

Question: Is $f_1(k) = f_2(k)$?

Motivation: A (k,s)-CNF formula is one with exactly k distinct variables in each clause, such that every variable appears in at most s clauses. The function

$$f(k) = \max\{s : \text{every } (k,s)\text{-CNF is satisfiable}\}$$

is not known to be computable, while $f_1(k)$ and $f_2(k)$ are computable. It is not hard to see that $f(k) \le f_1(k) \le f_2(k)$, and we know that all three functions are asymptotically $\left(\frac{2}{e} + o(1)\right)\frac{2^k}{k}$. It would be very intersting to decide whether any two of them are equal.

Gábor Tardos: Is either of the following two conflicting statements true?

Statement 1: For any finite set S of points in the plane one can find another set H of cardinality at most $|S|/2$ such that any axis-parallel rectangle R contains a point of H or contains at most 1000 points of S.

Statement 2: Question 1 fails badly for almost all S. More concretely, for a uniform random set S of n points in the unit square with high probability the following holds. For any set H of at most $n/2$ points there exists an axis parallel rectangle R containing no points from H and $\Omega(\log\log n)$ points of S.

Robin Thomas: Is there a polynomial-time algorithm to test membership in the linear hull of xx^T over $GF(2)$, where x ranges over all incidence vectors of perfect matchings of a graph G? A positive answer would give a polynomial-time algorithm to test whether an input graph has a Pfaffian orientation.

References

[1] J. Balogh, P. Keevash and B. Sudakov. On the minimal degree implying equality of the largest triangle-free and bipartite subgraphs, Journal of Combinatorial Theory Series B 96 (2006), 919–932.

[2] A. Björner and K. Vorwerk. Connectivity of chamber graphs of buildings and related complexes, European Journal of Combinatorics 31 (2010), 2149–2160.

[3] B. Bollobás and A. Frieze. On matchings and hamiltonian cycles in random graphs. In Random Graphs (Poznań 1983), volume 28 of Annals of Discrete Mathematics, pages 23-46. North-Holland, Amsterdam, 1985.
[4] P. Erdős and A. Hajnal. On chromatic number of graphs and set-systems, Acta Math. Acad. Sci. Hungar. 17 (1966), 61–99.
[5] A. Frieze and M. Krivelevich. On packing Hamilton cycles in ϵ-regular graphs, Journal of Combinatorial Theory Series B 94 (2005), 159–172.
[6] A. Frieze and M. Krivelevich. On two Hamilton cycle problems in random graphs, Israel Journal of Mathematics 166 (2008), 221–234.
[7] F. Knox, D. Kühn and D. Osthus. Approximate Hamilton decompositions of random graphs, Random Structures and Algorithms, to appear.
[8] L. Lovász and B. Szegedy. Regularity partitions and the topology of graphons, `http://arxiv.org/abs/1002.4377`.
[9] J. Nešetřil and P. Ossona de Mendez. Sparsity (Graph, Structures and Algorithms), Springer 2011+.
[10] V. Rödl. On the chromatic number of subgraphs of a given graph, Proc. Amer. Math. Soc. 64 (1977), no. 2, 370–371.
[11] Z. Tuza. Asymptotic growth of sparse saturated structures is locally determined, Discrete Math. 108 (1992), no. 1-3, 397-402.

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