Mathematisches Forschungsinstitut Oberwolfach

Report No. 29/2011

DOI: 10.4171/OWR/2011/29

Stochastic Analysis

Organised by
Alice Guionnet, Lyon
Wendelin Werner, Orsay
Ofer Zeitouni, Rehovot/Minnesota

May 29th – June 4th, 2011

Abstract. The meeting took place on May 30 - June 3, 2011, with over 55 people in attendance. Each day had 6 to 7 talks of varying length (some talks were 30 minutes long), except for Thursday: the traditional hike was moved to Thursday due to the weather (and weather on thursday was indeed fine).

The talks reviewed directions in which progress in the general field of stochastic analysis occurred since the last meeting of this theme in Oberwolfach three years ago. Several themes were covered in some depth, in addition to a broad overview of recent developments. Among these themes a prominent role was played by random matrices, random surfaces/planar maps and their scaling limits, the KPZ universality class, and the interplay between SLE (Schramm-Loewner equation) and the GFF (Gaussian free field).

Mathematics Subject Classification (2000): 60Gxx, 60Jxx,82Bxx.

Introduction by the Organisers

The workshop opened in a bang with a description, by J.-F. Le Gall and by G. Miermont, of the recent proofs of the universal convergence (in the Gromov-Hausdorff metric) of random planar maps (q-angulations, with $q = 3, 2k$ with k integer) toward the Brownian map, an object identified earlier by J.-F. Le Gall. The importance of confluence of geodesics was stressed in both talks. These talks were completed by a talk of M. Bousquet-Mélou on combinatorial aspects of the Potts model on planar maps, and later in the week, by B. Eynard who derived general (universal) equations for the enumeration of maps and other objects, and discussed the link with random matrices. Later in the week, N. Curien presented natural examples (joint work with Le Gall and with Werner) of triangulations of the unit disc (splittings of the unit disc into triangles that have their three corners on the boundary of the disc) of a different type than the “uniform” random triangulation that had been studied by Aldous and that plays an important role in the understanding of planar maps.

Several talks discussed random surfaces and models from the SLE perspective. S. Sheffield introduced the “quantum zipper”, that allows to sew together two random surfaces along an SLE curve – here the random surfaces are defined in a generalized sense via the Gaussian free field, that is a rather central object in the study of continuous two-dimensional random geometries; Sheffield also discussed the link with work in progress with J. Miller and with B. Duplantier. Later in the week, J. Miller reported on his work with Sheffield concerning the identification of the geometry of “altimeter-compass lines” and “light cones” within the geometry defined via the Gaussian free field. J. Dubédat addressed questions related to dimers. In the direction pioneered by Kenyon on dimer configurations in planar graphs, he explained how when one controls analytically the quantities involved, one can get powerful results by estimating the behavior of suitably perturbed Laplacians and their determinant, in the scaling limit when the mesh-size of the lattice vanishes. This is one of the cases where the scaling limit of discrete models on discrete graphs can be connected to continuous limitting structures such as SLE curves and the Gaussian free field. V. Vargas recalled results by Jean-Pierre Kahane on mulitplicative cascades and constructions of limiting measures such as the one appearing in Sheffield's lecture (the “exponential of the Gaussian free field”) and that is conjecturally related to the Brownian map, and his recent work with Allez and Rhodes, that generalizes the construction of characterization of these measures for continuous cascades.

C. Garban described scaling limits for magnetization in the Ising model at criticality (where non-trivial scaling exponents appear), and G. Pete used again SLE methods to study near critical dynamics for the planar FK Ising model.

More classical topics related to percolation and Ising models were also present: G. Grimmett presented his recent work with I. Manolescu that enables to bound crossing probabilities of boxes for a wide class of critical planar percolation models, and A. Holroyd described his joint work with Grimmett on aspects of the geometry of supercritical percolation clusters (can one embed in a Lipschitz way a two-dimensional plane into a three-dimensional cluster etc.). In a different direction, H. Lacoin described a derivation of an upper bound on relaxation times for the zero temperature stochastic Ising dynamics. C. Bordenave described his joint work with Lelarge and Salez on the understanding of random configurations of dimers on a discrete graph, in the scaling limit (questions like “what is the asymptotic density of holes in such configurations”?).

Another cluster of talks discussed recent progress around scaling limits for models inspired by first passage percolation and the KPZ universality class. T. Seppalainen described an explicitely solvable model of a directed polymer with gamma weights, and his talk was continued by I. Corwin who reported on a follow up joint work with O'Connell, Seppalainen and Zygouras that uses a geometric RSK correspondence, a criterion of Rogers and Pitman, and Whittaker functions, to give a Fredholm determinant representation for endpoint fluctuations of a family of directed polymers, including the gamma-weighted one. The KPZ theme was taken up by Sasamoto, who described his results on convergence to the KPZ equation with appropriate initial conditions, and by J. Quastell, who gave an overview of his results (joint with Corwin, Remenik and Moreno) on fluctuations of extrema of the Airy_2 process around a parabolic barrier. He discussed a model of continuous Brownian polymer, studied by him, Alberts and Khanin, and its relation with the KPZ equation. Later in the week, H. Widom discussed his fundamental result with C. Tracy concerning the asymmetric exclusion process (ASEP), explaining an earlier gap in the proof and the way it is fixed, allowing for multi-type ASEP. Back on the first passage percolation theme, S. Chatterjee discussed his recent geometric proof of a universal relation between different scaling exponents.

A third cluster of talks was centered around random matrices and random Schroedinger operators. M. Aizenman and S. Warzel described their recent results on the boundary of the delocalization region for the random Schroedinger operator on the regular tree; this work revises the conjectured picture and provides a rigorous description of the boundary at weak disorder. Recursions of Green functions play a fundamental role in the proof. Another aspect of the spectrum of RSE was discussed by B. Virag, who reported on results with Kritchevski and Valko concerning convergence to a Brownian caroussel process for a 1-D RSE problem with scaled down potential, and to GOE statistics for a particular scaling of the RSE on a strip. A. Knowles described recent work with Erdos, Yau and Yin on universality results for the eigenvalues of the adjacency matrix of random Erdos-Renyi graphs, in the regime where the row sum goes to infinity. He introduced the steps, developed earlier by Erdos, Schlein, Yau and Yin, to prove universality for random matrices by deriving a local semi-circle law, (modified) Dyson flow and a matching lemma. This was followed up by H.-T. Yau, who gave more details on the Dyson flow and explained how that step can be bypassed in universal beta-ensembles by proving a version of local equilibrium for Gibbs measures.

Other talks given during the week covered other stochastic analysis themes. E. Bolthausen described his his joint work with F. Rubin on the asymmetric weakly self-avoiding walk in high dimension, and the use of appropriate recursions and induction to prove a CLT. H. Duminil-Copin gave an essentially complete proof of a recent work in progress with Benjamini, Kozma and Yadin concerning the control of coupling (and hence, Harmonic functions) by entropic methods for random walks on a variety of graphs. J.-D. Deuschel described his work with Berger on the invariance principle (quenched) for certain non-elliptic environments. A. Hammond talked about his joint work with Fribergh on biased random walk in random environment (such as supercritical percolation clusters) that allows to describe and understand the transition between a ballistic regime (when the drift is not too large) to slow regime (when the drift is too large, the walk is slowed down by traps). A. Bovier discussed the limiting law of the particles in a branching Brownian motion viewed from the leading edge, obtained with Arguin and Kistler, and explained the spin-glass motivation behind this work. T. Kumagai discussed an approach, based on the notion of spectral Gromov-Hausdorff distance, that allows to prove convergence of (L^p) mixing times on a family of graphs to the mixing time of a diffusion on a limiting object.

Workshop: Stochastic Analysis

Table of Contents

Abstracts

Resonant delocalization for random Schrödinger operators on tree graphs

Michael Aizenman, Simone Warzel

Abstract: *We resolve an existing question concerning the location of the mobility edge for operators with a hopping term and a random potential on the Bethe lattice. For unbounded potential we find that extended states appear well beyond the spectrum of the operator's hopping term, including in a Lifshitz tail regime of very low density of states. The relevant mechanism is the formation of extended states through disorder enabled resonances, for which the exponential increase of the volume plays an essential role. The general results is shown to have the surprising implication that for bounded random potentials at weak disorder there is no mobility edge in the form that was envisioned before.*

A bit more than 50 years ago Anderson, Mott, Twose, and other physicists, have proposed that the incorporation of random potential in self-adjoint operators of condensed matter physics results in a transition in the nature of the eigenstates of a homogeneous operator from extended (e.g., plane waves) to localized, at least in certain energy ranges. The transition is accompanied in the reduction of conduction. The study focused on self-adjoint operators of the form

$$H_\lambda(\omega) \; = \; T + \lambda \, V(\omega) \, , \tag{1}$$

acting in the space of square-summable functions $\ell^2(\mathcal{T})$ over a homogeneous graph $\mathcal{T}$, with T the graph-adjacency operator, $V(\omega)$ a random potential, whose values at different sites are independent identically distributed, and $\lambda \geq 0$ a disorder-strength parameter. As linear operators play key roles in many fields, myriads of other implications, and other interesting aspects (such as changes in the spectral gap statistics) have since then been noted of this transition. This has led to the mathematical challenge of explaining the spectral and dynamical properties of such operators, a task which requires the combination of analysis with probability.

The situation which has emerged from the mathematical studies of the *Anderson localization*, is that in a number of different contexts we now have robust mathematical tools for proving and explaining localization, in particular, in the sense of existence of pure point spectrum. However, only limited progress was made in shedding light on the nature of *extended eigenstates* of operators with random potential. The only case for which existence of continuous spectrum, and extended eigenstates, could be established in the presence of random potential has been the case of homogeneous tree graphs. However, major gaps have remained between the regime for which extended states have been established and the regimes for which localization was proven. The present work has closed this gap. It also led to revision of the phase diagram for the case of bounded random potentials, for which it is found that the mobility edge sets in (at least in the previously envisioned form) only if the disorder in sufficiently large.

Of no lesser interest is the mechanism for the formation of extended states, which can be viewed as based on resonant tunneling between what would locally appear to be localized states. For this the disorder actually plays a constructive role, and the exponential increase in the volume is essential. (Such exponential increase is found not only on trees, but also in the configuration spaces of interacting particles.

For a more explicit statement of the theorem, one should introduce the Green function

$$G_\lambda(0,x;E) := \left\langle \delta_0, (H_\lambda - E - i0)^{-1}\delta_x \right\rangle$$

and its *moment generating function* (or the 'free energy' function) which provides information on the large deviations of $|G_\lambda(0,x;E+i0)|$, and which is defined for $|s| < 1$ by:

$$\varphi_\lambda(s;E) \ := \lim_{|x|\to\infty} \frac{\log \mathbb{E}\left[|G_\lambda(0,x;E+i0)|^s\right]}{|x|}$$

and for $s = 1$ as: $\varphi_\lambda(1;E) \ := \lim_{s\nearrow 1} \varphi_\lambda(s;E)$. Past work has produced the following statement.

Theorem 1 (Localization - Aizenman/Molchanov '93, Aizenman '94)**.** *For the random operators H_λ on regular tree graphs, with the unbounded random iid potential of an absolutely continuous distribution, satisfying $\operatorname{supp}\rho = \mathbb{R}$ and certain regularity assumptions: if for almost all energies E in some interval $I \subset \mathbb{R}$*

$$\varphi_\lambda(1;E) < -\log K$$

then H_λ has only pure point (localized) spectrum in that interval.

The key new, *complementary*, result is:

Theorem 2 (Delocalization - Aizenman/Warzel '11)**.** *Under the above assumptions, at energies at which*

$$\varphi_\lambda(1;E) > -\log K$$

one has: $\quad \Im G_\lambda(x,x;E+i0) \ > \ 0.$

It may be added that if this condition holds for a positive measure of energies $E \in I$, then H_λ has absolutely continuous (delocalized) spectrum in that interval. The dynamical implication is that the graph conducts, and its transmission coefficient for current injected at a site is positive at energies in that range.

The new criterion led to the following results for the two cases of bounded and unbounded random potentials.

Extended states in a Lifshitz tail regime for unbounded random potential:

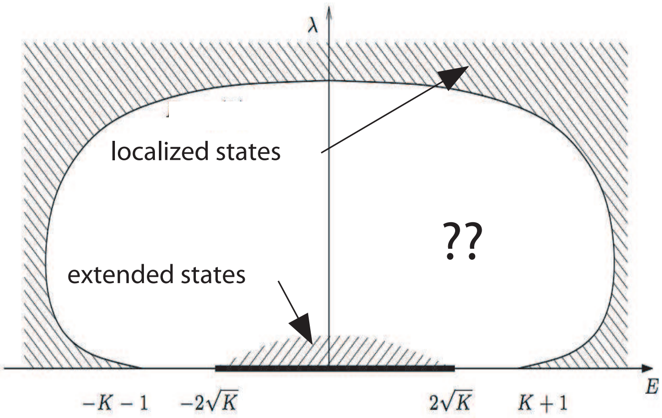

FIGURE 1. A sketch of the previously known parts of the phase diagram and the puzzle which was resolved by Theorem 1. The new result extends the regime of proven delocalization up to the outer curve, assuming $\varphi_\lambda(1; E) = -\log K$ holds only along a line.

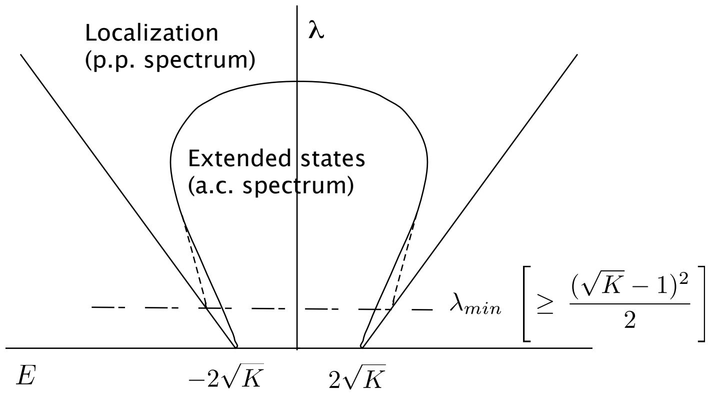

FIGURE 2. Sketch of the correction (dotted line) of the previously expected mobility edge for the Anderson model on the Bethe lattice (the solid line). Our analysis shows that for $\lambda \le (\sqrt{k}-1)^2/2)$ absolutely continuous spectrum can be found arbitrarily close to the spectral edge, suggesting that at week disorder there is no localization.

Using Theorem 2, we prove that in case of unbounded random potential (e.g., Gaussian or Cauchy distributions) under weak disorder ($\lambda \to 0$) the regime of absolutely continuous spectrum spreads discontinuously beyond the spectrum $\sigma(T) = [-2\sqrt{K}, 2\sqrt{K}]$ of the unperturbed operator T, see Figure 1. A notable aspect of the result is that extended states are proven to occur also in regimes where the density of states is extremely low (e.g. in the Gaussian case vanishing as e^{-C/λ^2}, for $\lambda \to 0$).

Absence of mobility edge for bounded random potentials at weak disorder:

For the Anderson model with bounded potential, Theorem 2 has the surprising implication that at weak disorder there is no transition to a spectral regime of Anderson localization. This corrects a picture of the phase diagram which has been widely quoted and not challenged before, as depicted in Figure 2. (The complete statement requires improved understanding of the regularity of the Lyapunov exponent, on which work is in progress.)

A more extended discussion and can be found in [1, 2].

References

[1] M. Aizenman, S. Warzel, Phys. Rev. Lett. **106**, 136804 (2011)
[2] M. Aizenman, S. Warzel, http://lanl.arxiv.org/abs/1104.0969

Asymmetric weakly self-avoiding random walks

Erwin Bolthausen
(joint work with F. Rubin)

We consider the weakly self-avoiding random walk on $\mathbb{Z}^d$ with jump distribution S, and self-avoidance parameter $\lambda \in [0,1]$. Here S is a probability distribution on $\mathbb{Z}^d$ with finite range, i.e. $S \in \mathcal{P}_R$ for some $R > 0$ where $\mathcal{P}_R$ denotes the set of probability distributions with support inside the ball of radius R. For a path $\omega = (\omega_0, \omega_1, \ldots, \omega_n)$ of length n, and $\omega_0 = 0$, we define

$$P_{S,\lambda,n}(\omega) \stackrel{\text{def}}{=} \frac{1}{c_n} \prod_{i=1}^{n} S(\omega_i - \omega_{i-1}) \prod_{0 \le i < j \le n} \left(1 - \lambda 1_{\omega_i = \omega_j}\right),$$

where c_n is the appropriate norming constant.

We also consider the unnormalized transition function

$$C_n(x) \stackrel{\text{def}}{=} c_n \sum_{\omega : \omega_n = x} P_{S,\lambda,n}(\omega).$$

The case of a symmetric one-jump distribution S in dimensions $d \ge 5$ has been treated by many authors, first by Brydges and Spencer [2], and later for instance by [3].

Theorem 1. *Assume $d \ge 9$, and that S_0 is an element in $\mathcal{P}_R$ which is invariant under lattice isometries. Then there exists $\varepsilon(d, S_0)$ such that for any $S \in \mathcal{P}_R$ satisfying $\sum_x |S(x) - S_0(x)| \le \varepsilon$, and $\lambda \le \varepsilon$, there exist $\kappa(S,\lambda) \in \mathbb{R}^d$, and a positive definite symmetric matrix $\Sigma(S,\lambda)$, such that*

$$\frac{C_n(\cdot - n\kappa)}{c_n} \to N(0, \Sigma),$$

weakly, as $n \to \infty$, where $N(0,\Sigma)$ is the centered normal distribution with covariance matrix Σ.

The method of proof is a modification of the contraction method introduced in the thesis of Christine Ritzmann.

By the lace expansion of Brydges-Spencer, one has a representation

$$C_n = S * C_{n-1} + \lambda \sum_{k=1}^{n} \Pi_k * C_{n-k}$$

with complicated kernels Π_k which however should be small for dimensions $d \geq 5$. Writing $\Pi_k = c_k B_k$, and gets

$$C_n = S * C_{n-1} + \lambda \sum_{k=1}^{n} c_k \left(B_k * C_{n-k} \right).$$

The approach consists in proving a general theorem for solutions $\{C_n\}$ of these equations, with an "input" sequence $\{B_k\}$, and then later to prove that the "true" B_k from the self-avoiding walk satisfy the necessary conditions for a central limit theorem.

In the non-symmetric case, there are considerable additional difficulties when compared to the symmetric situation which made it necessary (up to now) to assume $d \geq 9$.

References

[1] Bolthausen, E., and Ritzmann, Ch.: *A Central Limit Theorem for Convolution Equations and Weakly Self-Avoiding Walks.* arxiv: math.PR/0103218

[2] Brydges, D., and Spencer, Th.: *Self-avoiding walk in 5 or more dimensions.* Commun. Math. Phys. **97**, 125-148 (1985).

[3] Hara, T., and Slade, G. *The lace expansion for self-avoiding walk in five or more dimensions.* Reviews in Math. Phys. **4**, 235-327 (1992).

[4] Rubin, F.: *Two Limit Laws in Random Matrix Theory and Statistical Mechanics.* Dissertation, University of Zürich 2010

Matchings on infinite graphs

Charles Bordenave

(joint work with M. Lelarge and J. Salez)

A *matching* on a finite graph $G = (V, E)$ is a subset of pairwise non-adjacent edges $M \subseteq E$. The $|V| - 2|M|$ isolated vertices of (V, M) are said to be *exposed* by M. We let $\mathbb{M}(G)$ denote the set of all possible matchings on G. The *matching number* of G is defined as

$$\nu(G) = \max_{M \in \mathbb{M}(G)} |M|,$$

and those M which achieve this maximum – or equivalently, have the fewest exposed vertices – are called *maximum matchings.*

Our first result belongs to the theory of convergent graph sequences. Convergence of bounded degree graph sequences was defined by Benjamini and Schramm. The notion of local weak convergence has then inspired a lot of work. We have shown that for any sequence of graphs converging locally, the corresponding sequence of normalized matching numbers converges, and this limit can be expressed only on the limit of the graph sequence.

Our second contribution concerns sequences of graphs converging locally to Galton-Watson trees. A classical example in this framework is the sequence of Erdős-Rényi graphs with connectivity c denoted by $G(n, c/n)$: the limiting tree is then a Galton-Watson tree with degree distribution a Poisson distribution with parameter c. In this case, Karp and Sipser showed in 1981 that almost surely

$$\frac{\nu(G(n,c/n))}{n} \xrightarrow[n\to\infty]{} 1 - \frac{t_c + e^{-ct_c} + ct_c e^{-ct_c}}{2},$$

where $t_c \in (0,1)$ is the smallest root of $t = e^{-ce^{-ct}}$. This explicit formula rests on the analysis of a heuristic algorithm now called Karp-Sipser algorithm. In the general case of any sequence of graphs converging locally to a Galton-Watson tree, this analysis does not carry over. Our first result shows that the normalized matching number converges. The computation of the limit requires another set of tools to solve a *recursive distributional equation* (a usual ingredient of the Aldous and Steele's objective method). This has allowed us to derive an explicit formula for the limit that considerably generalizes the aforementioned result.

The Potts model on planar maps

Mireille Bousquet-Mélou
(joint work with O. Bernardi)

Let q be an integer. We address the enumeration of q-colored planar maps, counted by the total number of edges and the number of *monochromatic* edges (those that have the same colour at both ends). We prove that the associated generating function is algebraic when $q \neq 0, 4$ is of the form $2 + 2\cos(j\pi/m)$, for integers j and m. This includes the two integer values $q = 2$ and $q = 3$, for which we give explicit algebraic equations.

For a generic value of q, we prove that the generating function satisfies a system of differential equations.

Both results hold as well for planar triangulations, with a strikingly similar system of differential equations.

The starting point of our approach is a recursive construction of q-coloured maps, in the spirit of what Tutte did in the seventies and eighties for properly coloured triangulations.

References

[1] O. Bernardi, M. Bousquet-Mélou, *Counting colored planar maps: algebraicity results*, arXiv:0909.1695.

The extremal process of branching Brownian motion

Anton Bovier
(joint work with L.-P. Arguin and N. Kistler)

We prove that the extremal process of branching Brownian motion, in the limit of large times, converges weakly to a cluster point process. The limiting process is a (randomly shifted) Poisson cluster process, where the positions of the clusters is a Poisson process with exponential density. The law of the individual clusters is characterized as branching Brownian motions conditioned to perform "unusually large displacements", and its existence is proved. The proof combines three main ingredients. First, the results of Bramson on the convergence of solutions of the Kolmogorov-Petrovsky-Piscounov equation with general initial conditions to standing waves. Second, the integral representations of such waves as first obtained by Lalley and Sellke in the case of Heaviside initial conditions. Third, a proper identification of the tail of the extremal process with an auxiliary process, which fully captures the large time asymptotics of the extremal process. The analysis through the auxiliary process can be seen as a rigorous formulation of the *cavity method* developed in the study of mean field spin glasses.

1. Branching Brownian motion

Branching Brownian Motion (BBM) is a continuous-time Markov branching process that is constructed as follows.

Start with a single particle which performs standard Brownian Motion $x(t)$ with $x(0) = 0$, which it continues for an exponential holding time T independent of x, with $\mathcal{P}[T > t] = e^{-t}$. At time T, the particle splits independently of x and T into k offsprings with probability p_k, where $\sum_{k=1}^{\infty} p_k = 1$, $\sum_{k=1}^{\infty} kp_k = 2$, and $K \equiv \sum_k k(k-1)p_k < \infty$. Then continue the same process for each particle independently and iterate. At time $t > 0$, there will be $n(t)$ particles located at positions $x_1(t), \dots, x_{n(t)}(t)$, with $\mathbb{E}n(t) = e^t$).

The link between BBM and partial differential equations is provided by the following observation due to McKean [16]: if one denotes by

$$u(t,x) \equiv \mathcal{P}\left[\max_{1\leq k\leq n(t)} x_k(t) \leq x\right] \tag{1}$$

the law of the maximal displacement, a renewal argument shows that $u(t,x)$ solves the Kolmogorov-Petrovsky-Piscounov or Fisher [F-KPP] equation [12, 13],

$$\begin{aligned} u_t &= \frac{1}{2}u_{xx} + \sum_{k=1}^{\infty} p_k u^k - u, \\ u(0,x) &= \begin{cases} 1, \text{ if } x \geq 0, \\ 0, \text{ if } x < 0. \end{cases} \end{aligned} \tag{2}$$

The F-KPP equation admits traveling waves: Bramson [7, 8] showed that there exists a unique solution satisfying

(3) $$u\big(t, m(t)+x\big) \to \omega(x) \qquad \text{uniformly in } x \text{ as } t \to \infty,$$

with the centering term given by

(4) $$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t,$$

and $w(x)$ the distribution function which solves the o.d.e.

(5) $$\frac{1}{2}\omega_{xx} + \sqrt{2}\omega_x + \omega^2 - \omega = 0.$$

Lalley and Sellke [14] provided a characterization of the limiting law of the maximal displacement in terms of a *random shift* of the Gumbel distribution. Let

(6) $$Z(t) \equiv \sum_{k=1}^{n(t)} \left(\sqrt{2}t - x_k(t)\right) \exp -\sqrt{2}\left(\sqrt{2}t - x_k(t)\right),$$

the so-called *derivative martingale*, Lalley and Sellke proved that $Z(t)$ converges almost surely to a strictly positive random variable Z, and established the integral representation

(7) $$\omega(x) = \mathbb{E}\left[\exp\left(-CZe^{-\sqrt{2}x}\right)\right],$$

for some specific constant C.

Understanding the extremal process of BBM is a longstanding problem of fundamental interest. The classical extremal process in the case of families of independent random variables are Poisson point processes, and it is well know that this feature persists even under relatively strong correlations citeleadbetter. Bramson's result shows that this cannot be the case for BBM. A class of models where a more complex structure of *Poisson cascades* was shown to emerge are the *generalized random energy models* of Derrida [9, 5]. These models, however, have a rather simple hierarchical structure involving a finite number of levels only which greatly simplifies the analysis, which cannot be carried over to models with infinite levels of branching such as BBM or the *continuous random energy models* studied in [6]. BBM is a case right at the borderline where correlations just start to effect the extremes and the structure of the extremal process. Our results thus allows to peek into the world beyond the simple Poisson structures and hopefully open the gate towards the rigorous understanding of complex extremal structures.

2. Main result: the extremal process of branching Brownian motion

Define the random measure:

(8) $$\mathcal{E}_t \equiv \sum_{k \leq n(t)} \delta_{x_k(t) - m(t)}.$$

Few papers have addressed so far the large time limit of the extremal process of branching Brownian motion.

On the physical literature side, we mention the contributions by Brunet and Derrida [10, 11], who reduce the problem of the statistical properties of particles "at the edge" of BBM to that of identifying the finer properties of the delay of traveling waves.

On the mathematical side, properties of the large time limit of the extremal process have been established in three papers of ours [2, 3, 4]. In a first paper we obtained a precise description of the *paths* of extremal particles which in turn imply a somewhat surprising restriction of the correlations of particles at the edge of BBM. These results were instrumental in our second paper on the subject where we proved that a certain process obtained by a correlation-dependent thinning of the extremal particles converges to a random shift of a Poisson Point Process (PPP) with exponential density. In [4] we presented the full characterisation of the extremal process which we explain below [1]

Let us now describe the main result from [4]. Let Z be the limiting derivative martingale. Conditionally on Z, we consider the Poisson point process (PPP) of density $CZe^{-\sqrt{2}x}\mathtt{d}x$:

$$P_Z \equiv \sum_{i\in\mathbb{N}} \delta_{p_i} \equiv \mathrm{PPP}\left(CZe^{-\sqrt{2}x}\mathtt{d}x\right), \tag{9}$$

with C as in (7). Now let $\{x_k(t)\}_{k\leq n(t)}$ be a BBM of length t. Consider the point process of the gaps $\sum_k \delta_{x_k(t)-\max_j x_j(t)}$ conditioned on the event $\{\max_j x_j(t) - \sqrt{2}t > 0\}$. Remark that, in view of (4), the probability that the maximum of BBM shifted by $-\sqrt{2}t$ does not drift to $-\infty$ is vanishing in the large time limit. In this sense, the BBM is conditioned to perform "unusually large displacements". The law of this process converges as $t \uparrow \infty$. Write $\Delta = \sum_i \delta_{\Delta_j}$ for a point process with this law and consider iid copies $(\Delta^{(i)}_{i\in\mathbb{N}})$.

Theorem 1. *Let P_Z and $\Delta^{(i)}$ be defined as above. Then the family of point processes $\mathcal{E}_t$, defined in* (8)*, converges in distribution to a point process, $\mathcal{E}$, given by*

$$\mathcal{E} \equiv \lim_{t\to\infty} \mathcal{E}_t \xrightarrow[N\to\infty]{law} \sum_{i,j} \delta_{p_i+\Delta^{(i)}_j}. \tag{10}$$

The key ingredient in the proof of Theorem 1 is an identification of the extremal process of BBM with an auxiliary process constructed from a Poisson process,with an explicit density of points in the tail. This is essentially a rigorous implementation of the *cavity approach* developed in the study of mean field spin glasses [17] for the case of BBM, and might be of interest to determine extreme value statistics for other processes.

[1] After our paper [4] was posted on the arxive, Aidekon, Berestycki, Brunet, and Shi posted a paper [1] containing essentially the same result as ours.

References

[1] E. Aidekon, J. Berestycki, É. Brunet, Z. Shi, *The branching Brownian motion seen from its tip*, arXiv:1104.3738.

[2] L.-P. Arguin, A. Bovier, and N. Kistler, *The genealogy of extremal particles in branching Brownian Motion*, arXiv:1008.4386

[3] L.-P. Arguin, A. Bovier, and N. Kistler, *Poissonian Statistics in the Extremal Process of Branching Brownian Motion*, arXiv:1010.2376

[4] L.-P. Arguin, A. Bovier, and N. Kistler, *The extremal process of branching Brownian motion*, arXiv:1103.2322.

[5] A. Bovier and I. Kurkova *Derrida's Generalized Random Energy Models. 1. Models with finitely many hierarchies.* Ann. Inst. H. Poincare. Prob. et Statistiques (B) Prob. Stat. 40, pp. 439-480 (2004)

[6] A. Bovier and I. Kurkova *Derrida's generalized random energy models. 2. Models with continuous hierarchies.* Ann. Inst. H. Poincare. Prob. et Statistiques (B) Prob. Stat. 40, pp. 481-495 (2004)

[7] M. Bramson, *Maximal displacement of branching Brownian motion*, Comm. Pure Appl. Math. 31, pp. 531-581 (1978)

[8] M. Bramson, *Convergence of solutions of the Kolmogorov equation to travelling waves*, Mem. Amer. Math. Soc. 44, no. 285, iv+190 pp. (1983)

[9] B. Derrida, *A generalization of the random energy model that includes correlations between the energies*, J.Phys.Lett. **46**, pp. 401-407 (1985)

[10] E. Brunet and B. Derrida, *Statistics at the tip of a branching random walk and the delay of traveling waves*, Europhys. Lett. 87: 60010, 5 pp. (2009)

[11] E. Brunet and B. Derrida, *A branching random walk seen from the tip*, arXiv:1011.4864 (2010)

[12] R.A. Fisher, *The wave of advance of advantageous genes*, Ann. Eugen. 7: 355-369 (1937)

[13] A. Kolmogorov, I. Petrovsky, and N. Piscounov, *Etude de l' équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Moscou Universitet, Bull. Math. 1, pp. 1-25 (1937)

[14] S.P. Lalley and T. Sellke, *A conditional limit theorem for the frontier of a branching brownian motion*, Ann. Probab. 15, no.3, pp. 1052-1061 (1987)

[15] M.R. Leadbetter, G. Lindgren and H. Rootzen, *Extremes and related properties of random sequences and processes*, Springer Series in Statistics. Springer-Verlag, New York-Berlin (1983)

[16] H.P. McKean, *Application of Brownian Motion to the equation of Kolmogorov-Petrovskii-Piskunov*, Comm.Pure. Appl. Math. 28, pp. 323-331 (1976)

[17] M. Mézard, G. Parisi, and M. Virasoro, *Spin Glass theory and beyond*, World scientific, Singapore (1987)

[18] M. Talagrand, *Spin Glasses: A Challenge for Mathematicians. Cavity and Mean Field Models*, Springer Verlag (2003)

The universal relation between scaling exponents in first-passage percolation

Sourav Chatterjee

To each edge of the integer lattice $\mathbb{Z}^d$, attach a non-negative random variable, and call it the 'passage time' through that edge, or alternatively, the 'edge-weight'. Assume that these passage times (edge-weights) are independent and identically distributed The first-passage time $T(x, y)$ from a point x to a point y is the minimum total passage time among all lattice paths from x to y. This is the classical model of first-passage percolation, introduced by Hammersley and Welsh [5].

Assume that the edge-weights are continuous random variables. Then almost surely there is a unique 'geodesic' between any two points x and y. Let $D(x, y)$ be the maximum deviation (in Euclidean distance) of this path from the straight line segment joining x and y.

First-passage percolation and related polymer models have attracted considerable attention in the theoretical physics literature (see [9] for a survey). Among other things, the physicists are particularly interested in two 'scaling exponents', sometimes denoted by χ and ξ in the mathematical physics literature. The *fluctuation exponent* χ is a number that quantifies the order of fluctuations of the first-passage time $T(x, y)$. Roughly speaking, for any x, y,

$$\text{the typical value of } T(x,y) - \mathbb{E}T(x,y) \text{ is of the order } |x-y|^{\chi}.$$

The *wandering exponent* ξ quantifies the magnitude of $D(x, y)$. Again, roughly speaking, for any x, y,

$$\text{the typical value of } D(x,y) \text{ is of the order } |x-y|^{\xi}.$$

There are many conjectures related to χ and ξ. The main among these, to be found in numerous physics papers, including the famous paper of Kardar, Parisi and Zhang [7], is that although χ and ξ may depend on the dimension, they always satisfy the relation

$$\chi = 2\xi - 1.$$

I've heard in private conversations the above relation being referred to as the 'KPZ relation' between χ and ξ.

There are a number of rigorous results for χ and ξ, mainly from the late eighties and early nineties. One of the first non-trivial results is due to Kesten [8], who proved that $\chi \le 1/2$ in any dimension. The only improvement on Kesten's result till date is due to Benjamini, Kalai and Schramm [2], who proved that for first-passage percolation in $d \ge 2$ with binary edge-weights,

$$\sup_{v \in \mathbb{Z}^d,\ |v|>1} \frac{\mathrm{Var} T(0,v)}{|v|/\log|v|} < \infty.$$

Benaïm and Rossignol [1] extended this result to a large class of edge-weight distributions that they call 'nearly gamma' distributions. The definition of a nearly gamma distribution is as follows. A positive random variable X is said to have a nearly gamma distribution if it has a continuous probability density function h supported on an interval I (which may be unbounded), and its distribution function H satisfies, for all $y \in I$, $\Phi' \circ \Phi^{-1}(H(y)) \le A\sqrt{y}h(y)$ for some constant A, where Φ is the distribution function of the standard normal distribution. Although the definition may seem a bit strange, Benaïm and Rossignol [1] proved that this class is actually quite large, including e.g. exponential, gamma, beta and uniform distributions on intervals.

The only non-trivial lower bound on the fluctuations of passage times is due to Newman and Piza [11] and Pemantle and Peres [12], who showed that in $d = 2$, $\mathrm{Var}T(0, v)$ must grow at least as fast as $\log |v|$. Better lower bounds can be proved if one can show that with high probability, the geodesics lie in 'thin cylinders' [4].

For the wandering exponent ξ, the main rigorous results are due to Licea, Newman and Piza [10] who showed that $\xi^{(2)} \geq 1/2$ in any dimension, and $\xi^{(3)} \geq 3/5$ when $d = 2$, where $\xi^{(2)}$ and $\xi^{(3)}$ are exponents defined in their paper which may be equal to ξ.

Besides the bounds on χ and ξ mentioned above, there are some rigorous results relating χ and ξ through inequalities. Wehr and Aizenman [13] proved the inequality $\chi \geq (1 - (d - 1)\xi)/2$ in a related model, and the version of this inequality for first-passage percolation was proved by Licea, Newman and Piza [10]. The closest that anyone came to proving $\chi = 2\xi - 1$ is a result of Newman and Piza [11], who proved that $\chi' \geq 2\xi - 1$, where χ' is a related exponent which may be equal to χ. This has also been observed by Howard [6] under different assumptions. Incidentally, in the model of Brownian motion in a Poissonian potential, Wüthrich [14] proved the equivalent of the KPZ relation assuming that the exponents exist.

The following theorem, which is the main result of the preprint [3], establishes the relation $\chi = 2\xi - 1$ assuming that the exponents χ and ξ exist in a certain sense, and that the distribution of edge-weights is nearly gamma.

Theorem 1. *Consider the first-passage percolation model on $\mathbb{Z}^d$, $d \geq 2$, with i.i.d. edge-weights. Assume that the distribution of edge-weights is 'nearly gamma' in the sense of Benaïm and Rossignol* [1] *(which includes exponential, gamma, beta and uniform distributions, among others), and has a finite moment generating function in a neighborhood of zero. Let χ_a and ξ_a be the smallest real numbers such that for all $\chi' > \chi_a$ and $\xi' > \xi_a$, there exists $\alpha > 0$ such that*

$$\text{(A1)} \qquad \sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} \exp\left(\alpha \frac{|T(0,v) - \mathbb{E}T(0,v)|}{|v|^{\chi'}} \right) < \infty,$$

$$\text{(A2)} \qquad \sup_{v \in \mathbb{Z}^d \setminus \{0\}} \mathbb{E} \exp\left(\alpha \frac{D(0,v)}{|v|^{\xi'}} \right) < \infty.$$

Let χ_b and ξ_b be the largest real numbers such that for all $\chi' < \chi_b$ and $\xi' < \xi_b$, there exists $C > 0$ such that

$$\text{(A3)} \qquad \inf_{v \in \mathbb{Z}^d,\ |v| > C} \frac{\mathrm{Var}(T(0,v))}{|v|^{2\chi'}} > 0,$$

$$\text{(A4)} \qquad \inf_{v \in \mathbb{Z}^d,\ |v| > C} \frac{\mathbb{E}D(0,v)}{|v|^{\xi'}} > 0.$$

Then $0 \leq \chi_b \leq \chi_a \leq 1/2$, $0 \leq \xi_b \leq \xi_a \leq 1$ and $\chi_a \geq 2\xi_b - 1$. Moreover, if it so happens that $\chi_a = \chi_b$ and $\xi_a = \xi_b$, and these two numbers are denoted by χ and ξ, then they must necessarily satisfy the relation $\chi = 2\xi - 1$.

Note that if $\chi_a = \chi_b$ and $\xi_a = \xi_b$ and these two numbers are denoted by χ and ξ, then χ and ξ are characterized by the properties that for every $\chi' > \chi$ and

$\xi' > \xi$, there are some positive α and C such that for all $v \neq 0$,

$$\mathbb{E}\exp\biggl(\alpha\frac{|T(0,v)-\mathbb{E}T(0,v)|}{|v|^{\chi'}}\biggr) < C \quad\text{and}\quad \mathbb{E}\exp\biggl(\alpha\frac{D(0,v)}{|v|^{\xi'}}\biggr) < C,$$

and for every $\chi' < \chi$ and $\xi' < \xi$ there are some positive B and C such that for all v with $|v| > C$,

$$\mathrm{Var}(T(0,v)) > B|v|^{2\chi'} \quad\text{and}\quad \mathbb{E}D(0,v) > B|v|^{\xi'}.$$

It seems reasonable to expect that if the two exponents χ and ξ indeed exist, then they should satisfy the above properties.

References

[1] Benaïm, M. and Rossignol, R. (2008). Exponential concentration for first passage percolation through modified Poincaré inequalities. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** no. 3, 544–573.

[2] Benjamini, I., Kalai, G. and Schramm, O. (2003). First passage percolation has sublinear distance variance. *Ann. Probab.* **31** no. 4, 1970–1978.

[3] Chatterjee, S. (2011). The universal relation between scaling exponents in first-passage percolation. *Preprint.* Available at http://arxiv.org/pdf/1105.4566

[4] Chatterjee, S. and Dey, P. S. (2009). Central limit theorem for first-passage percolation time across thin cylinders. *Preprint.* Available at http://arxiv.org/pdf/0911.5702

[5] Hammersley, J. M. and Welsh, D. J. A. (1965). First-passage percolation, subadditive processes, stochastic networks and generalized renewal theory. In *Bernoulli, Bayes, Laplace Anniversary Volume,* 61–110. (J. Neyman, L. Lecam, eds.), Springer-Verlag, Berlin.

[6] Howard, C. D. (2004). Models of first-passage percolation. In *Probability on discrete structures*, 125–173, Encyclopaedia Math. Sci., **110**, Springer, Berlin.

[7] Kardar, M., Parisi, G. and Zhang, Y.-C. (1986). Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56** 889–892.

[8] Kesten, H. (1993). On the speed of convergence in first-passage percolation. *Ann. Appl. Probab.* **3** no. 2, 296–338.

[9] Krug, J. and Spohn, H. (1991). Kinetic roughening of growing surfaces. In *Solids far from equilibrium: growth, morphology and defects,* 479–582. (C. Godréche, ed.), Cambridge University Press.

[10] Licea, C., Newman, C. M. and Piza, M. S. T. (1996). Superdiffusivity in first-passage percolation. *Probab. Theory Related Fields* **106** no. 4, 559–591.

[11] Newman, C. M. and Piza, M. S. T. (1995). Divergence of shape fluctuations in two dimensions. *Ann. Probab.* **23** no. 3, 977–1005.

[12] Pemantle, R. and Peres, Y. (1994). Planar first-passage percolation times are not tight. In *Probability and phase transition,* 261–264, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **420**, Kluwer Acad. Publ., Dordrecht.

[13] Wehr, J. and Aizenman, M. (1990). Fluctuations of extensive functions of quenched random couplings. *J. Stat. Phys.* **60** 287–306.

[14] Wüthrich, M. V. (1998). Scaling identity for crossing Brownian motion in a Poissonian potential. *Probab. Theory Related Fields,* **112** no. 3, 299–319.

Exact solvability of directed random polymers via gRSK

Ivan Corwin

The Kardar-Parisi-Zhang (KPZ) universality class encompasses a wide variety of stochastic growth models, interacting particle systems, polymer models and stochastic PDEs. Progress in providing exact statistics for this class of models has been due to the discovery of a few such models for which exact formulas can be derived and asymptotics can be taken. Recent advances have gone through three approaches: (1) The asymmetric simple exclusion process; (2) The replica approach for the continuum directed random polymer; (3) The geometric Robinson-Schensted-Knuth (gRSK) correspondence and finite temperature polymers.

This talk will focus on the gRSK approach.

1. The statistics of the KPZ renormalization fixed point and KPZ equation

How does one compute exact formulas for statistics of complex, non-linear stochastic processes? Such problems are generally intractable and one can only hope to compute asymptotic statistics in the large-scale/long-time limit, when minute differences in models are expected to wash out. In these limits, many disparate models attract to the same limit laws – or universality classes. Far and away the most important such class in $1+1$ dimensions is the Kardar-Parisi-Zhang (KPZ) class. Among the models in this KPZ class are the asymmetric simple exclusion process (ASEP), directed polymers in random environment (DPRE) and last passage percolation (LPP), and the KPZ equation (stochastic PDE).

The KPZ equation gives the evolution of a continuum growth profile $h(t,x)$:

$$\partial_t h = \tfrac{1}{2}\partial_x^2 h - \tfrac{1}{2}(\partial_x h)^2 + \dot{W} \tag{1}$$

where $\dot{W}$ is space-time white noise. As written this equation is ill-posed (due to the non-linearity) and the correct interpretation is that of the Hopf-Cole solution in terms of the multiplicative stochastic heat equation (SHE): $h(t,x) = -\log Z(t,x)$, where $\partial_t Z = \frac{1}{2}\partial_x^2 Z - Z\dot{W}$.

The (non-rigorous) prediction of Kardar, Parisi and Zhang [11] was that this equation scales with a dynamic scaling exponent $z = 3/2$ and that this same exponent should hence arise for a whole class of related (discrete) growth processes. This means that in a large time t, the (properly centered) fluctuations of the height function for such models should live in the scale $t^{1/z} = t^{2/3}$. Much more is true – setting $h_\epsilon(T,X) = \epsilon^{1/2}h(\epsilon^{-z}T, \epsilon^{-1}X)$, [6] provide a (presently non-rigorous) description of the properly centered limit of $h_\epsilon(T,X)$ as $\epsilon \to 0$ in terms of a random semi-group S_T with independent stationary increments constructed in terms of a process called the Airy sheet. This represents the renormalization fixed point of the entire KPZ universality class.

The KPZ fixed point and KPZ equation correspond to two different regimes of growth / polymer models. The first arises as the scaling limit of asymmetric or finite temperature models, while the second arises under weak asymmetry or high temperature scaling. The symmetric or near-infinite temperature regime is governed by the Edwards Wilkinson fixed point.

Since the seminal work of Baik, Deift and Johansson [2] much progress has been made in compute the exact statistics associated with the fixed T spatial marginal of the KPZ renormalization fixed point. In fact, for the six universality subclasses (corresponding to different growth regimes) the statistics are now known for the entire spatial process. What enabled the computation of these asymptotic statistics was the discovery of a class of finite models (last passage percolation with exponential weights or equivalently the TASEP) for which finite statistics were exactly solvable and asymptotics were accessible.

1.1. **The Tracy Widom ASEP formula.** Since the KPZ equation arises only under weak asymmetric, the statistics of the KPZ equation remained inaccessible until the work of Tracy and Widom [22, 23, 24] provided exact formulas for the one-point probability distribution of the height of the ASEP with finite asymmetry. Using this formula Sasamoto and Spohn [18, 19, 20] and Amir, Corwin and Quastel [1] simultaneously and independently derived the exact probability distribution for the Hopf-Cole solution to the KPZ equation.[1] The formulas of Tracy and Widom are formidable and have not yet yielded expressions for joint probability distributions of the height at distant two locations (same time), as would be necessary to solve for the multipoint (fixed time) distributions of the KPZ equation. Additionally, the work of Tracy and Widom presently only covers two of the six universality classes (see also [25, 5]).

Two other approaches to the solvability of the KPZ equation have arisen and appear to be able to yield more statistics than above: The replica approach and Bethe ansatz (non-rigorous), and the geometric Robinson-Schensted-Knuth correspondence (rigorous).

1.2. **The replica approach and Bethe ansatz.** The stochastic heat equation can be formulated in terms as a Feynman-Kacs path integral in a space-time white noise potential. The solution to the SHE then has the interpretation as the partition function for the continuum directed random polymer (CDRP), and the KPZ equation governs its free energy. The replica approach [10] uses the polymer formulation of Z to express the moments of $Z(T,X)$ (with respect to the disorder induced by the white noise potential) in terms of the solution to a quantum many body system governed by a the Lieb-Liniger Hamiltonian with two-body attractive delta interaction. As opposed to the repulsive delta interaction which was solved by Lieb and Liniger [13] by Bethe ansatz, the attractive case was only just solved last year by [8] (and later [3] with a different approach). This enabled [8] to write down expressions for the moments of Z. From this they sought to recover the large time asymptotics of the probability distribution of $\log Z$. Unfortunately the moments of Z grow far to rapidly to uniquely identify the distribution of $\log Z$, and in the course of resuming divergent series and analytically continuing functions

[1]While the derivation in [18, 19, 20] is formal and non-rigorous, [1] provides a rigorous proof of the formula.

only a priori defined on the integers, [8] derived the wrong asymptotic formula. The paper [8] was posted about a month ahead of [18, 19, 20, 1], and about two months afterwards [7] and [3] repaired the mistakes in the derivation and recovered the correct formulas.

Though non-rigorous and rather involved, the correctly applied replica trick has some benefits. Prolhac and Spohn [16, 17] used this approach to derive a conjectural form of the spatial process for the KPZ equation in the geometry corresponding to growth in a narrow wedge and confirm that the long time limit of this spatial process is the Airy_2 process.[2] In [6] this approach is used to derive a conjectural form for the transition probabilities of the KPZ renormalization fixed point operator random non-linear semi-group S_T.

1.3. **The geometric Robinson-Schensted-Knuth correspondence.** The solvability of LPP [9, 2] relies on the combinatorial Robinson-Schensted-Knuth (RSK) correspondence. RSK maps a matrix of positive entries onto a pair of *Gelfand Zetlin (GZ) patterns*, from which one can immediately read off information like the last passage time for the original matrix. When matrix entries are chosen as independent exponential random variables, the resulting measure on GZ patterns is given by the Schur measure. Thus, in this case it is possible to write exact formulas for the probability distribution for the last passage time – hence the solvability.

Last passage percolation represents a zero temperature polymer model, and thus in order to access the statistics of the KPZ equation, it is necessary to find a solvable finite temperature polymer. This is accomplished in the on-going work of [4] (building on recent work of [21, 14]) using a finite temperature version of RSK – the so called *geometric Robinson-Schensted-Knuth correspondence* introduced by Kirillov [12] in the context of tropical combinatorics. The RSK correspondence can be encoded as a combinatorial algorithm over the $(\max, +)$ algebra – gRSK amounts to formally replacing: $\max \mapsto +$ and $+ \mapsto \times$. The image of a matrix of positive entries under gRSK is now a pair of triangular arrays and from these one may immediately read off the polymer partition function associated with the original matrix. Exponential weights are no longer the distinguished solvable distribution – but rather inverse Gamma distributions. The solvability stems from an operator intertwining relation (combine with general theory of Markov functions [15]) and an integrate-out lemma originally developed in the study of automorphic forms.

The algebraic structure associated with the finite temperature polymer solvability is much better understood than in the context of the ASEP, and thus much more information about the KPZ equation should be accessible through this approach. Asymptotics of the resulting formulas are presently being computed.

[2] This and the work of [6] make a critical factorization assumption in the form of the moments of Z which may not be true at finite t but which appears to hold in the long-time limit.

References

[1] G. Amir, I. Corwin, J. Quastel. Probability Distribution of the Free Energy of the Continuum Directed Random Polymer in $1+1$ dimensions. *Comm. Pure Appl. Math.*, to appear.

[2] J. Baik, P.A. Deift, K. Johansson On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, **12**:1119–1178 (1999).

[3] P. Calabrese, P. Le Doussal, and A. Rosso. Free-energy distribution of the directed polymer at high temperature. *Euro. Phys. Lett.*, **90**, 20002, 2010.

[4] I. Corwin, N. O'Connell, T. Seppäläinen, N. Zygouras. Tropical combinatorics and exactly solvable directed polymers. In preparation.

[5] I. Corwin, J. Quastel. Universal distribution of fluctuations at the edge of the rarefaction fan. arXiv:1006.1338.

[6] I. Corwin, J. Quastel. Renormalization fixed point of the KPZ universality class. In preparation.

[7] V. Dotsenko. Replica Bethe ansatz derivation of the Tracy-Widom distribution of the free energy fluctuations in one-dimensional directed polymers. *J.Stat.Mech.* P07010, 2010.

[8] V. Dotsenko, B. Klumov. Attempted Bethe ansatz solution for one-dimensional directed polymers in random media. *J. Stat. Mech.*, P03022, 2010.

[9] K. Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.*, **209**:437–476, 2000.

[10] K. Kardar. Replica-Bethe Ansatz studies of two-dimensional interfaces with quenched random impurities. *Nucl. Phys. B* **290**, 582, 1987.

[11] K. Kardar, G. Parisi, Y.Z. Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, **56**:889–892, 1986.

[12] A. N. Kirillov. Introduction to tropical combinatorics. In: *Physics and Combinatorics. Proc. Nagoya 2000 2nd Internat.Workshop* (A. N. Kirillov and N. Liskova, eds.), World Scientific, Singapore, 82–150, 2001,

[13] E.H. Lieb, W. Liniger. Exact analysis of an interacting bose gas. I. The general solution and the ground state. Phys. Rev. Lett., **130**, 1605, 1963.

[14] N. O'Connell. Directed polymers and the quantum Toda lattice arXiv:0910.0069.

[15] J.W. Pitman, L.C.G. Rogers. Markov functions. *Ann. Probab.*, **9**:573–582, 1981.

[16] S. Prolhac, H. Spohn. Two-point generating function of the free energy for a directed polymer in a random medium. *J. Stat. Mech.*, P01031, 2011.

[17] S. Prolhac, H. Spohn. The one-dimensional KPZ equation and the Airy process. arXiv:1101.4622

[18] T. Sasamoto, H. Spohn. Exact height distributions for the KPZ equation with narrow wedge initial condition. *Nucl. Phys. B*, **834**:523–542, 2010.

[19] T. Sasamoto, H. Spohn. One-dimensional KPZ equation: an exact solution and its universality. *Phys. Rev. Lett.*, **104**, 23, 2010.

[20] T. Sasamoto, H. Spohn. The crossover regime for the weakly asymmetric simple exclusion process. *J. Stat. Phys.*, **140**:209–231, 2011.

[21] T. Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions. arXiv: 0911.2446.

[22] C. Tracy, H. Widom. Integral formulas for the asymmetric simple exclusion process. *Comm. Math. Phys.*, **279**:815–844, 2008.

[23] C. Tracy, H. Widom. A Fredholm determinant representation in ASEP. *J. Stat. Phys.*, **132**:291–300, 2008.

[24] C. Tracy, H. Widom. Asymptotics in ASEP with step initial condition. *Comm. Math. Phys.*, **290**:129-154, 2009.

[25] C. Tracy, H. Widom. Formulas for ASEP with Two-Sided Bernoulli Initial Condition. *J. Stat. Phys.*, **140**:619-634, 2010.

Some random continuous triangulations

Nicolas Curien
(joint work with J.-F. Le Gall and W. Werner)

Definition 1. *A* continuous triangulation *of the disk* $\overline{\mathbb{D}}$ *is a closed set* T *of* $\overline{\mathbb{D}}$ *that can be written as a union of non-crossing chords with endpoints located on* $\mathbb{S}_1$ *and such* $\overline{\mathbb{D}} \backslash T$ *is a disjoint union of (Euclidean) triangles.*

We present three different models of random continuous triangulations of the disk, relate them to discrete models and compare their properties.

The first model is the Brownian triangulation of Aldous [1, 2]. This random triangulation takes its name from its intimate link with the Brownian excursion and Aldous' Brownian CRT. It can also be obtained as the limit in distribution (for the Hausdorff distance on closed subsets of $\overline{\mathbb{D}}$) of uniform triangulations of the regular polygon with n edges inscribed in $\mathbb{D}$ as $n \to \infty$. This limit is universal in the sense that various models of uniform non-crossing configurations of convex polygons converge towards Aldous' triangulation. This random triangulation almost surely has Hausdorff dimension 3/2.

The second model is the recursive triangulation introduced in [3]. This random continuous triangulation can be constructed as follows: Consider $(X_i)_{i \geq 1}$ a sequence of independent variables uniformly distributed over $\mathbb{S}_1$. We imagine that the points fall one after the other and we pair the points as soon as possible by drawing a chord between two points, provided that this chord does not cross any of the existing chords. The closure of the set obtained after pairing all the points $(X_i)_{i \geq 1}$ is the random recursive triangulation. This object is also universal as it appears as a limit in distribution of various discrete triangulations of convex polygons that are built recursively. Its Hausdorff dimension is almost surely equal to $1 + \frac{\sqrt{17}-3}{2}$.

The last model is a model of hyperbolic triangulation: The chords of the triangulation T are drawn using the hyperbolic structure of $\mathbb{D}$ instead of the Euclidean one and we further suppose that T is of empty interior. The main result of [4] is then the following:

Theorem 2. *There exists a unique (law of a) hyperbolic triangulation that is invariant (in law) with respect to Möbius transformations, and possesses a natural spatial Markov property that can be roughly described as the conditional independence of the two parts of the triangulation on the two sides of the edge of one of its triangles.*

We show how to construct this object from a certain Poisson point process and present some open problems and conjectures about it.

References

[1] D. Aldous. Recursive self-similarity for random trees, random triangulations and brownian excursion. *Ann. Probab.*, 22(2):527–545, 1994.
[2] D. Aldous. Triangulating the circle, at random. *Amer. Math. Monthly*, 101(3), 1994.
[3] N. Curien and J.-F. Le Gall. Random recursive triangulations of the disk *via* fragmentation theory. *Ann. Probab.* to appear.
[4] N. Curien and W. Werner. The markovian hyperbolic triangulation. *arXiv:1105.5089.*

Quenched invariance principle for a balanced, non-elliptic, random walk in balanced random environment

Jean-Dominique Deuschel
(joint work with N. Berger)

Let M^d be all probability measures on $\{\pm e_i\}_{i=1}^d$. An **environment** is a point $\omega \in \Omega = \left(M^d\right)^{\mathbb{Z}^d}$

$$\omega = \{\omega(x, \pm e_i), i = 1, ..., d\}_{x \in \mathbb{Z}^d}$$

The law of environment P is an i.i.d. measure, i.e.

$$P = \mu^{\mathbb{Z}^d}$$

for some distribution μ on M^d.

For an environment $\omega \in \Omega$, the *Random Walk* on ω is a time-homogenous Markov chain with transition kernel

$$P_\omega \left(X_{n+1} = z + e \middle| X_n = z \right) = \omega(z, e).$$

The **quenched law** P_ω^z is defined to be the law on $\left(\mathbb{Z}^d\right)^{\mathbb{N}}$ induced by the kernel P_ω and $P_\omega^z(X_0 = z) = 1$. An environment ω is said to be *balanced* if for every $z \in \mathbb{Z}^d$ and neighbor e of the origin, $\omega(z, e) = \omega(z, -e)$.

An environment ω is said to be *genuinely d-dimensional* if for every neighbor e of the origin, there exists $z \in \mathbb{Z}^d$ such that $\omega(z, e) > 0$.

Throughout this work we make the following assumption. P-almost surely, ω is balanced and genuinely d-dimensional.

Set

$$X_t^N = \frac{1}{\sqrt{N}} X_{[tN]} + \frac{tN - [tN]}{\sqrt{N}} \left(X_{[tN]+1} - X_{[tN]} \right), \quad t \geq 0.$$

The **quenched invariance principle holds** if for P a.a. ω the law of $\{X_t^N\}_{t \geq 0}$ under P_ω^0 converges weakly to a Brownian motion with deterministic non-degenerate matrix.

Theorem 1. *Let $d \geq 2$ and assume that the environment is i.i.d., genuinely d-dimensional and balanced, then the quenched invariance principle holds with non-degenerate limiting covariance matrix.*

Let $d \geq 2$ and assume that the environment is i.i.d., genuinely d-dimensional

Lawler showed in [L] the quenched invariance principle for ergodic uniformly elliptic environments: that is, if there exits $\epsilon_0 > 0$ with

$$P(\forall i = 1, ..., d,\ \omega(z, e_i) > \epsilon_0) = 1.$$

Guo and Zeitouni showed in [GZ] the quenched invariance principle for i.i.d elliptic environments

$$P(\forall i = 1, ..., d,\ \omega(z, e_i) > 0) = 1.$$

and for ergodic environments under the moment condition

$$E[\big(\prod_{i=1}^{d} \omega(x, e_i)\big)^{-p/d}] < \infty \quad \text{for some} \quad p > d$$

One can find an example of ergodic elliptic balanced environment, where the invariance principle fails.

Note that, due to the balanced environment, $\{X_n\}$ is a martingale.

Let $\{\bar{\omega}_n\}_{n \in \mathbb{N}}$ be the environment viewed from the point of view of the particle:

$$\bar{\omega}_n = \tau_{X_n} \omega,$$

where τ is the shift on Ω This is a Markov chain on Ω under P with transition kernel

$$M(\omega', d\omega) = \sum_{i=1}^{d} \left[\omega'(0, e_i)\delta_{\tau_{e_i}\omega'} + \omega'(0, -e_i)\delta_{\tau_{-e_i}\omega'}\right]$$

The quenched invariant principle follows once we can find a probability measure $Q \ll P$ which is an invariant ergodic measure for $\{\bar{\omega}_n\}$ and such that P-almost surely, after some finite time the shifted environment is in the support of Q.

Note that in the elliptic case it follows immediately when $Q \ll P$ that $P \ll Q$, but in our case it is possible to have $Q \ll P$ but $P \not\ll Q$. Thus we need to be more careful.

Our proof is based on analytical methods, in particular on the maximum principle which we have to adapt to the non-elliptic setting. The estimates are based on the rescaled random walk, obtained from the original walk stopped after each coordinate has been upgraded. The maximum inequality allows us to control for $p > 1$ the L^p- norm of the density of the invariant measure of the walk on the reflected-periodized cube of size N, uniformly in large N. From this we get the existence of an invariant measure $Q \ll P$, however due to the non-ellipticity of the walk, the proof of the ergodicity of Q, which is related to the uniqueness of a maximal strongly connected component, is more delicate. In the 2 dimensional case a simple coupling argument is applicable, while in higher dimensions we need to adapt the Burton-Keane argument [BK], to our setting, where we only have a weak version of the finite energy condition. We compensate for the weaker finite energy condition by using density bounds on the support of the invariant measure.

References

[BK89] R. M. Burton and M. Keane, Density and uniqueness in percolation *Comm. Math. Phys.* Vol. 121, 501-505, (1989),

[GZ10] Xiaoqin Guo and Ofer Zeitouni. Quenched invariance principle for random walks in balanced random environment. *to appear Prob. Theo Rel. Fields*, 2011.

[KT90] Hung Ju Kuo and Neil S. Trudinger. Linear elliptic difference inequalities with random coefficients. *Math. Comp.*, 55(191):37–53, 1990.

[Law83] Gregory F. Lawler. Weak convergence of a random walk in a random environment. *Comm. Math. Phys.*, 87(1):81–87, 1982/83.

Dimers and analytic torsion

Julien Dubédat

The dimer (or perfect matching) model consists in uniformly sampling from perfect matchings of a fixed graph, ie subsets of the edge set such that each vertex is adjacent to exactly one of the selected edges. If the graph is planar, Kasteleyn showed that these matchings are enumerated by the Pfaffian of an appropriately signed adjacency matrix. If the graph is bipartite and planar, one can associate to each matching a height function defined on faces, following Thurston. The large scale behaviour of this height function exhibits a rich phenomelogy: limit shapes, Gaussian fluctuations, random matrix-type fluctuations at the edge of frozen regions, ... (Cohn-Kenyon-Propp, Kenyon, Johansson, ...).

In this talk we focus on Gaussian fluctuations in the case where the limit is conformally invariant (this is dictated by boundary conditions, or lack thereof). The point of view adopted here is that of families of Cauchy-Riemann operators. More specifically, the Kasteleyn operator can be thought as a finite difference version of a Cauchy-Riemann operator $\bar\partial : f \mapsto f_{\bar z} d\bar z$. One may evaluate the characteristic functional of the height field by modifying this reference operator by a degree 0 "potential", multiplied by a perturbation parameter. The analysis consists in showing that the logarithmic variation of the discrete determinants (of finite dimensional, finite difference operators) converges to the logarithmic variation of the regularised functional determinant along such a family of Cauchy-Riemann operators. This involves a detailed analysis of the inverting kernel near the diagonal and relates to Quillen's curvature formula.

This approach enables to treat various problems on height fluctuations including compactified free field limit for dimer coverings on a torus; a strong invariance principle; vertex correlators and monomer correlators. The latter involve CR operators operating on a line bundle with monodromies around singularities (the "operator insertions"). The variation considered here fixes the monodromy data and displaces the singularities, and is based on a study of discrete holomorphic functions with monodromy (ie multiplicative multivalued).

Entropy, random walks and harmonic functions

Hugo Duminil-Copin
(joint work with I. Benjamini, G. Kozma and A. Yadin)

Since the work of Yau in 1975, where a Liouville property for positive harmonic functions on complete manifolds with non-negative Ricci curvature is proved, the structure of various spaces of harmonic functions have been at the heart of geometric analysis. We extend this question to the random context, with an emphasize on the infinite cluster of percolation. We are especially interested in harmonic functions on it (meaning functions f with vanishing laplacian) with sublinear growth.

Theorem 1. *For almost every supercritical-cluster of percolation, there are no non-constant sublinear harmonic functions.*

In order to prove this result, we show that the total variation between random walks starting at two neighbors is controlled by the averaged entropy of the walk. The natural context in which the averaged entropy appears is the stationary random graphs. Let (G,ρ) be a random rooted graph. Consider the measure $\mathbb{P}$ on couples $(G,(x_n)_{n\in\mathbb{N}})$, where G is a graph and $(x_n)_{n\in\mathbb{N}}$ a semi-infinite path such that conditionally on (G,ρ), $(x_n)_{n\in\mathbb{N}}$ is distributed according to the simple random walks on G starting at ρ.

Definition 2. *The graph G is called* stationary *if (G,ρ) and (G,X_1) have the same distribution (under $\mathbb{P}$), where X_1 is the first step of the random walk.*

The entropy is the averaged Shannon entropy. Consider a stationary random graph (G,ρ) with law $\mathbb{P}$. Conditionally on (G,ρ), define the *entropy* of the random walk started at x at times n,m by

$$\mathbf{H}_n(G,x) \;=\; \sum_{y\in G}\phi\big(P_x(X_n=y)\big)$$

where $\phi(t) = -t\log t$ and $\phi(0) = 0$. The *mean entropy* is then defined by $h_n = \mathbb{E}[\mathbf{H}_n(G,\rho)]$. With this definition, we obtain the following theorem:

Theorem 3. *Let (G,ρ) be a stationary random graph. For every $n>0$, we have*

$$\mathbb{E}\left(\big\|P_\rho(X_n\in\cdot)-P_{\tilde{X}_1}(X_{n-1}\in\cdot\big\|_{TV}^2\right) \;\le\; h_n-h_{n-1}, \tag{1}$$

where $\tilde{X}_1$ is the first step of the random walk and $||\cdot||_{TV}$ is the total variation.

When the graph has polynomial growth, the entropy grows logarithmically and there exists an infinite number of n such that $h_n - h_{n-1}\le C/n$. A slight modification of Theorem 3 (note that percolation equipped with the stationary measure is a stationary random graph) allows then to control the gradient of a harmonic function on the infinite cluster and implies Theorem 1. We mention that this result has many other consequences, including the uniqueness of the corrector, Lipschitz regularity for the heat kernel, etc...

The question of harmonic functions in random environment is not closed. One can ask what is the dimension of linear growth harmonic functions, if there are higher order harmonic functions etc... Among the open questions, we mention that harmonic functions on the Uniform Infinite Planar Triangulation are not understood at all, and that investigating this subject could provide information on the random walk.

References

[1] I. Benjamini, H. Duminil-Copin, G. Kozma and A. Yadin. *Disorder, entropy and harmonic functions.* To appear.

Random matrices, combinatorics, algebraic geometry and the topological recursion

Bertrand Eynard

The generating functions of maps (possibly carrying an Ising model, or a O(n) model, or other decorations) of different topologies, are related by a recursion relation. This "topological recursion" relates the generating functions of maps of lower Euler characteristics χ to $\chi+1$. In other words, knowning the generating function of maps of the highest χ, i.e. planar rooted maps, allows to compute all the other topologies. Moreover, many other problems of enumerative geometry, for instance Gromov–Witten invariants, or knot invariants, do satisfy the same recursion.

Definition 1. *Let $\mathbb{M}_{g,n}(v)$ be the set of connected oriented maps of genus g, with v vertices, made of n_3 triangles, n_4 4-angles, ..., n_d $d-$angles, and n marked faces (each marked face having a marked edge, oriented by the orientation of the map), of respective lengths $l_1,\dots,l_n$ (umarked faces are at least 3-angles, but marked faces can have $l_i \geq 1$).*

We aslo define $\mathbb{M}_{0,1}(1)=\{.\}$ i.e. we say that there is one rooted map of genus 0 with a single vertex, and with no edges, it has $n_3=n_4=\cdots=0$ and $l_1=0$.

Remark: $\mathbb{M}_{g,n}(v)$ is a finite set (proof: write the Euler characteristics).

We then define the generating functions of maps of genus g with n marked faces, as formal power series in their number of vertices v:

$$W_n^{(g)}(x_1,\dots,x_n;t,t_3,\dots,t_d)=\sum_{v=1}^{\infty} t^v \sum_{\mathbb{M}_{g,n}(v)} \frac{t_3^{n_3}\dots t_n^{n_d}}{\#\text{Aut } x_1^{l_1+1}\dots x_n^{l_n+1}}. \tag{1}$$

The x_i's are catalytic variables associated to the lenghts of marked faces, we shall need to integrate on them, whereas the t_k's will play a spectator role, so most often, for shorter notations, we shall note write them:

$$W_n^{(g)}(x_1,\dots,x_n;t,t_3,\dots,t_d)\equiv W_n^{(g)}(x_1,\dots,x_n). \tag{2}$$

The planar maps with 1 marked edge, are in fact planar rooted maps (the root is the marked edge of the marked face, oriented so that the marked face is on the right). Their generating function $W_1^{(0)}$ was computed by Tutte and his collaborators in the 60's, for instance for quadrangulations (only $t_4 \neq 0$), is [8, 9]:

$$W_1^{(0)}(x) = \frac{1}{2t}\left(x - t_4 x^3 + t_4(x^2 + c(t))\sqrt{x^2 - 4\gamma(t)^2}\right) \tag{3}$$

where $c(t)$ is a formal series in t which we shall not write, and $\gamma(t)$ is given by

$$\gamma(t)^2 = \frac{1}{6t_4}\left(1 - \sqrt{1 - 12tt_4}\right). \tag{4}$$

In fact, it is much simpler to use another coordinate than x, and write

$$x(z) = \gamma\,(z + 1/z) \tag{5}$$

with this new variable z, we have $\sqrt{x^2 - 4\gamma^2} = \gamma(z - 1/z)$, and thus $W_1^{(0)}$ is a rational function of z, namely:

$$W_1^{(0)}(x(z)) = y(z) = \frac{t}{\gamma z} - \frac{t_4\,\gamma^3}{z^3}. \tag{6}$$

The parameter z is called the uniformizing parameter, we see that $W_1^{(0)}$ is a multivalued function of x, but when we use variable z, it becomes a monovalued function, in fact rational.

This fact is general, for all sorts of maps, it is more convenient to introduce a better variable z instead of x, so that $x(z)$ and $y(z) = W_1^{(0)}(x(z))$ are nice functions of z. For many examples of maps, the parameter z is a complex number $\in \mathbb{C}$, but sometimes, like the O(n) model, in fact z can live on a torus, or on some higher genus Riemann surface.

Universal 2-point function

Then, it was proved, that for all cases where z lives on the Riemann sphere (i.e. maps, maps with an Ising model,...):

$$W_2^{(0)}(x(z_1), x(z_2))\,x'(z_1)x'(z_2) + \frac{x'(z_1)x'(z_2)}{(x(z_1) - x(z_2))^2} = \frac{1}{(z_1 - z_2)^2} \tag{7}$$

For the O(n) model, z lives on a torus, and this function is replaced by a n-deformed Weierstrass function $\wp_n(z_1 - z_2)$, and in all cases, the function in the RHS, is the "fundamental 2-form of 2nd kind" on the Riemann surface defined by $W_1^{(0)}$, i.e. the cannonical function with a double pole.

→ The 2-point function $W_2^{(0)}$ thus takes a universal form.

Other topologies, topological recursion

Then, it was proved [3] that all other generating functions satisfy the following recursion (valid if $2g - 2 + n \geq 0$):

$$\begin{aligned} W_{n+1}^{(g)}&(x_0, x_1, \dots, x_n) \\ &= \sum_{a_i = \text{branch points}} \operatorname*{Res}_{z \to a_i} K(x_0, x(z)) \Big[W_{n+2}^{(g-1)}(x(z), x(\bar{z}), x_1, \dots, x_n) \\ &\qquad + \sum_h \sideset{}{'}\sum_{I \uplus J = \{x_1, \dots, x_n\}} W_{1+\#I}^{(h)}(x(z), I)\, W_{1+\#J}^{(g-h)}(x(\bar{z}), J) \Big] \end{aligned} \tag{8}$$

where the recursion kernel is given by

$$K(x_0, z) = \frac{\int_{z'=\bar z}^{z} W_2^{(0)}(x_0, x(z'))x'(z')dz'}{2(W_1^{(0)}(x(z)) - W_1^{(0)}(x(\bar z))} \tag{9}$$

and where $\bar z$ means the value of z which corresponds to the other branch of the multivalued function $W_1^{(0)}(x(z))$, i.e. it is such that $x(\bar z) = x(z)$ (in the example of quadrangulations above, it is $\bar z = 1/z$ i.e. it corresponds to changing the sign of the square root). The branch points are those at which $x'(z) = 0$ (in the example of quadrangulations, there are two, $z = 1$ and $z = -1$). The resiudes are computed in the z variable, not in the x variable. The prime in $\sum_h \sum'_{I,J}$, means that we exclude from that sum, the terms which are in the left hand side of the recursion, i.e. namely, we exclude $(h = 0, I = \emptyset)$ and $h = g, I = \{x_1, \ldots, x_n\}$.

We can also compute $W_0^{(g)}$, often denoted F_g by the relation (for $g \geq 2$):

$$\begin{aligned} W_0^{(g)} &= F_g \\ &= \frac{1}{2-2g} \sum_{a_i=\text{branch points}} \operatorname*{Res}_{z\to a_i} W_1^{(g)}(x(z))\, x'(z)\, dz \int_{z'=z_0}^{z} W_1^{(0)}(z')\, x'(z')dz' \end{aligned}$$

which is independent of the z_0 chosen. There is also a formula for F_1 and F_0, which we don't write here, see [4, 5].

$\rightarrow$ This recursion means that knowing $W_1^{(0)}$ allows to find all the other $W_n^{(g)}$'s by a systematic algorithm !

This recursion is universal, it holds for many models of maps, like maps, Ising maps, O(n) model maps, it is natural to conjecture that it holds for Potts model maps,... Notice that there is no free parameter in that relation, it depends on nothing.

Other applications. It is remarkable that many different sorts of maps all obey the same recursion, but even more remarkable, is that many other problems, not related to maps, also obey the same recursion.

- Gromov–Witten invariants

Let $\mathfrak{X}$ be a 3dimensional Calabi–Yau manifold with a toric symmetry, and $L \subset \mathfrak{X}$ a toric sublagrangian manifold. The Gromov–Witten invariants $N_{g,d,l}(\mathfrak{X}, L)$ count the number (divided by automorphisms) of analytical embeddings of a Riemann surface $\Sigma_{g,n}$ of genus g with n boundaries in $\mathfrak{X}$, so that the boundaries end on L. $d = (d_1, \ldots, d_{b_2}) \in \mathbb{Z}^{b_2}$ is the degree of the embedding (i.e. the homology class in $H_2(\mathfrak{X}, \mathbb{Z})$, which has dimension b_2) and $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ are the winding numbers of the boundaries in L. We define their generating series:

$$W_n^{(g)}(x_1, \ldots, x_n; t) = \sum_{d,l} N_{g,d,l}(\mathfrak{X}, L)\ \mathrm{e}^{-\sum_i d_i t_i}\ \mathrm{e}^{-\sum_i l_i x_i} \prod_i l_i. \tag{10}$$

Then, it was conjectured [6, 1], and verified on many examples at low g, and proved on a few cases for all g, that those $W_n^{(g)}$'s satisfy the same topological recursion (8).

For example, this result was proved in [2, 10] for $\mathfrak{X} = \mathbb{C}^3$ with framing f, the recursion starts with $y = W_1^{(0)}(x)$ given by the relation:

$$\mathrm{e}^{-x} = \mathrm{e}^{-fy}\,(1-\mathrm{e}^{-y}). \tag{11}$$

which is the mirror curve of $\mathfrak{X}$.

• Weil–Petersson volumes

The moduli space of Riemann surfaces of genus g with n boundaries, is a finite dimensional manifold (in fact orbifold, because surfaces with a symmetry group are counted modulo their automorphism group) of dimension $3g-3+n$, endowed with a natural hyperbolic geometry, and has a natural symplectic volume form, and its volume is called "Weil-Pettersson volume". Let

$$W_{g,n}(L_1,\dots,L_n) = \int_0^\infty \dots \int_0^\infty \prod_i L_i dL_i\, \mathrm{e}^{-L_i\sqrt{x_i}}\ \mathrm{Volume}(\mathcal{M}_{g,n}(L_1,\dots,L_n)) \tag{12}$$

be the Laplace–transform of the volume of the moduli space of Riemann surfaces of genus g with n boundaries of lenghts $L_1,\dots,L_n$. Then, it was proved that these $W_n^{(g)}$ satisfy the same topological recursion (8), starting with

$$y = W_1^{(0)}(x) = \sin\sqrt{x} \tag{13}$$

(the uniformizing parameter z is such that $x(z) = z^2$ and $y(z) = \sin z$). The topological recursion in that case, turns out to coincide with Mirzakhani's recursion [7].

• There are many other examples where the topological recursion is satisfied (some proved, some only conjectured), for instance Hurwitz numbers, knot invariants, sums over plane partitions (like Mac–Mahon formula), and of course matrix models,...

Conclusion. • We have a universal recursion which is satisfied by an important set of combinatorial problems.

• That recursion allows to compute everything as soon as we know the first term, i.e. $W_1^{(0)}$.

• We don't even need to have a combinatorial problem, we may choose an arbitrary function $W_1^{(0)}(x)$ (it must be multivalued), and see what the recursion does. It computes some functions $W_n^{(g)}$, and some numbers $F_g = W_0^{(g)}$'s, and independently of the choice of $y = W_1^{(0)}$, these F_g have remarkable properties:

- The F_g's are invariant under symplectic transformations of the curve $y(x)$, for instance they are invariant under the exchange of x and y.

- The F_g's are almost modular forms, like Eisenstein series, they can easily be turned into modular forms by adding a simple non–analytical term. They thus provide a natural basis of modular forms.

- They obey the relations of special geometry, related to form–cycle duality, i.e. if we make an infinitesimal deformation $y \to y + \epsilon\,\delta y$ such that $\omega = \delta y\, dx$ is a meromorphic differential form, then we have

$$(14)\qquad \frac{\partial}{\partial\epsilon}W_n^{(g)}(x_1,\dots,x_n) = \int_{z\in\omega^*} W_{n+1}^{(g)}(x(z),x_1,\dots,x_n)\,x'(z)\,dz$$

where ω^* is the cycle dual to the form ω.

- They behave well under taking limits, this allows to easily study the limits of large maps for instance. This can be used to give an easy rigorous proof that the generating function of numbers of large maps, satisfy some Painlevé equation.

- They define an integrable system (for example Hirota equations, or determinantal formulae)

- and much more ..., see [4, 5].

• Now it remains to understand why (at the combinatoric level) these equations are satisfied (the only available proofs are very technical, and very far from the combinatorics).

Aknowledgments

The author wants to thank the Oberwolfach mathematics institute and the CERN for their hospitality, and this work is partly supported by the ANR project Grandes Matrices Ale´atoires ANR-08-BLAN-0311-01, the European Science Foundation through the Misgam program, and the Quebec government with the FQRNT.

References

[1] V. Bouchard, A. Klemm, M. Mariño, S. Pasquetti, "Remodelling the B-model", hep-th/0709.1453.
[2] L. Chen, "Bouchard-Klemm-Marino-Pasquetti Conjecture for $\mathbb{C}^3$", arXiv:0910.3739.
[3] B. Eynard, "Topological expansion for the 1-hermitian matrix model correlation functions," arXiv:hep-th/0407261.
[4] B. Eynard and N. Orantin, "Invariants of algebraic curves and topological expansion", arXiv:math-ph/0702045.
[5] B. Eynard, N. Orantin, "Algebraic methods in random matrices and enumerative geometry", arXiv:0811.3531.
[6] M. Mariño, "Open string amplitudes and large order behavior in topological string theory," JHEP **0803**, 060 (2008). [hep-th/0612127].
[7] M. Mirzakhani, "Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces", Invent. Math. 167, 179-222 (2007).
[8] W.T. Tutte, "A census of planar triangulations", *Can. J. Math.* **14** (1962) 21-38.
[9] W.T. Tutte, "A census of planar maps", *Can. J. Math.* **15** (1963) 249-271.
[10] J. Zhou, "Local Mirror Symmetry for the Topological Vertex", arXiv:0911.2343.

Magnetization field at criticality in the Ising model

Christophe Garban
(joint work with F. Camia and C. Newman)

If one considers an $N \times N$ grid with independent coin flips $\sigma_x \in \{-1, 1\}$ at each vertex, it is well known that the renormalized field $\frac{1}{N}\sum_x \sigma_x\,\delta_{x/N}$ converges as N goes to infinity to a Gaussian white noise in the square $[0,1]^2$. More precisely for each "nice" subset A of this square, the field measured in A is a Gaussian random variable with variance the area of A.

The aim of this talk is to study what happens when the coin flips are no longer independent of each other. This situation has been considered in various contexts and one cannot hope for a "universality" result as in the iid case. In particular, one has to precise what type of dependency structure one is interested in. In this talk, I will focus on some famous distributions which arise in statistical mechanics and in particular on the case where the coin flips σ_x are defined to be the spins of an Ising model on the $N \times N$ grid. In this context, the sum over the spins corresponds to the so called **magnetization field**. Away from the critical point, it is known that this magnetization field (properly renormalized) converges also towards a Gaussian white noise. It remains to understand the magnetization field at criticality. In a joint work with Federico Camia and Chuck Newman, we prove the following facts:

(i) at $T = T_c$, the discrete magnetization fields have a unique scaling limit.
(ii) This limit is **non-Gaussian**.
(iii) The limit has an explicit conformally covariant structure.
(iv) The tail probabilities behave like $e^{-cx^{16}}$.

Universality for bond percolation in two dimensions

Geoffrey Grimmett
(joint work with I. Manolescu)

The star–triangle transformation has a long history. It dates back at least as far as the 19th century in the study of electrical networks; it was used by Onsager and Kramers–Wannier in their work on the Ising model; it was exploited by Sykes and Essam in their predictions of values of critical points for bond percolation on triangular and hexagonal lattices; it has proved a standard tool in statistical physics, known as the Yang–Baxter equation, and so on. The purpose of this talk is to explain how the star–triangle transformation may be used to show that inhomogeneous bond percolation on square, triangular, and hexagonal lattices belong to the same universality class, and that the critical processes have the so-called 'box-crossing property'. This work is joint with Ioan Manolescu, see [2, 3].

Consider a 'lattice' $\mathbb{L} = (V, E)$ embedded in $\mathbb{R}^2$. While the arguments of this talk may be applied to a variety of graphs $\mathbb{L}$, for simplicity here we take $\mathbb{L}$ to be one of the square, triangular, or hexagonal lattices. The configuration space of bond percolation is $\Omega = \{0, 1\}^E$, and we take as probability measure a product measure $\mathbb{P}_\mathbf{p}$ in which each class of parallel edges has the same parameter. Thus the measure $\mathbb{P}_\mathbf{p}$ is parametrized by $\mathbf{p} = (p_0, p_1) \in [0, 1)^2$ in the square case, and by $\mathbf{p} = (p_0, p_1, p_2) \in [0, 1)^3$ otherwise.

It is standard (see [1]) that these processes are critical when $\kappa(\mathbf{p}) = 0$, where

$$\kappa(\mathbf{p}) = \begin{cases} p_0 + p_1 - 1 & \text{for the square lattice,} \\ p_0 + p_1 + p_2 - p_0 p_1 p_2 - 1 & \text{for the triangular lattice,} \end{cases}$$

and with a related formula for the hexagonal lattice. There is a singularity as $\mathbf{p}$ passes through the so-called *critical surface* $\kappa(\mathbf{p}) = 0$.

Russo and Seymour–Welsh discovered the importance of box-crossing probabilities for the control of the geometry of critical and near-critical percolation. We say that a measure $\mathbb{P}$ on Ω has the *box-crossing property* if: for all $a > 1$, there exists $b > 0$ such that, for all rectangles R with aspect-ratio a, the probability that R possesses a long-way open crossing is at least b. Our first theorem is that the above models have the box-crossing property when $\kappa(\mathbf{p}) = 0$. This is proved by studying the transportation of open paths under the star–triangle transformation.

Theorem 1. *The inhomogeneous bond percolation models on the square, triangular, and hexagonal lattices have the box-crossing property when* $\kappa(\mathbf{p}) = 0$.

The singularity that occurs at a critical point of a percolation model is of power-law type, and is described by a collection of *critical exponents*. These exponents may be divided into two classes: those arising *at criticality*, and those arising *near criticality*. The former class includes the one-arm exponent ρ, the volume exponent δ, the connectivity exponent η, and the $2k$ alternating arm exponents ρ_{2k}. The latter class includes, for example, the correlation-length exponent ν, the percolation exponent β, and the cluster-size exponent γ. A discussion of the phase transition and of scaling theory may be found in [1, Chap. 10].

The hypothesis of *universality* asserts for these systems that all bond percolation models on two-dimensional lattices have equal exponents. Very little universality indeed has been proved so far for percolation. The two further theorems of this talk are as follows. No proof is claimed here for the existence of any of the above exponents, and the statements of the theorems are to be interpreted as being conditional on such existence. In the context of two-dimensional models, the existence of critical exponents has been proved essentially only for site percolation on the triangular lattice.

Theorem 2. *The exponents* ρ *and* ρ_k, $k \geq 1$, *are constant within the class of inhomogeneous bond percolation models on the square, triangular, and hexagonal lattices.*

A result of Kesten may now be used to extend the class of such invariant exponents to include δ and η. Further arguments and results of Kesten allow us to consider also near-critical exponents. Some further symmetries of the models under study are helpful at this point, and for simplicity we restrict ourselves to the following statement.

Theorem 3. *The near-critical exponents* β, γ, ν *are constant for homogeneous bond percolation on the square, triangular, and hexagonal lattices.*

The methods used in the proofs may be applied also to isoradial graphs and to critical random-cluster measures.

References

[1] G. R. Grimmett. *Percolation*. Springer, Berlin, 1999.
[2] G. R. Grimmett, and I. Manolescu. Inhomogeneous bond percolation on square, triangular, and hexagonal lattices. 2011. arxiv:1105.5535.
[3] G. R. Grimmett and I. Manolescu. Universality for bond percolation in two dimensions. 2011. In preparation.

The sharpness of the phase transition for speed for biased walk in supercritical percolation

Alan Hammond
(joint work with A. Fribergh)

I will discuss a joint work with Alex Fribergh in which we study the biased random walk on the infinite cluster of supercritical percolation. Fixing any $d \geq 2$ and supercritical parameter $p > p_c$, the model has a parameter $\lambda > 0$ for the degree of bias of the walker in a certain preferred direction (which is another parameter, in S^{d-1}). We prove that the model has a sharp phase transition, that is, that there exists a critical value $\lambda_c > 0$ of the bias such that the walk moves at positive speed if $\lambda < \lambda_c$ and at zero speed if $\lambda > \lambda_c$. This means that a stronger preference for the walker to move in a given direction actually causes the walk to slow down. The reason for this effect is a trapping phenomenon, and, as I will explain, our result is intimately tied to understanding the random geometry of the local environment that is trapping the particle at late time in the case when motion is sub-ballistic.

Multi-dimensional Percolation

Alexander E. Holroyd
(joint work with Dirr, Dondl, Grimmett and Scheutzow)

Percolation is concerned with the existence of an infinite path in a (Bernoulli) random subgraph of the lattice $\mathbb{Z}^D$. We can rephrase this as the existence of a Lipschitz embedding (or equivalently an injective graph homomorphism) of the infinite line $\mathbb{Z}$ into the random subgraph. What happens if we replace the line $\mathbb{Z}$ with another lattice $\mathbb{Z}^d$? I'll answer this for all values of the two dimensions d and D, and the Lipschitz constant. Based on joint works with Dirr, Dondl, Grimmett and Scheutzow.

Spectral and Eigenvector Statistics of Random Matrices

Antti Knowles
(joint work with L. Erdős, H.T. Yau and J. Yin)

I review recent results on the spectral and eigenvector statistics of random matrices. In particular, I cover the bulk and edge universalities of generalized Wigner matrices. I also outline the universality of eigenvectors associated with eigenvalues near the spectral edge. In addition to generalized Wigner matrices, I consider the Erdős-Rényi graph and discuss the complete delocalization of its eigenvectors as well as its bulk and edge universalities. Finally, I sketch the main ingredients of the proofs. (Joint work with L. Erdős, H.T. Yau and J. Yin.)

What do the eigenvalues and eigenvectors of a typical large matrix look like? A naive attempt to give meaning to the word "typical" is to consider the *Gaussian unitary ensemble* (GUE), defined as an $N \times N$ Hermitian matrix $H = (h_{ij})$ whose entries are given by

$$h_{ij} \;=\; \overline{h}_{ji} \;=\; \frac{1}{\sqrt{N}}\, x_{ij}\,, \qquad h_{jj} \;=\; \frac{\sqrt{2}}{\sqrt{N}}\, x_{ii}\,, \tag{1}$$

where $(x_{ij} : i \leq j)$ is a family of independent standard Gaussian random variables. The law of H can also be expressed as

$$\mathbb{P}(\mathrm{d}H) \;=\; \frac{1}{Z}\mathrm{e}^{-N\,\mathrm{Tr}H^2/2}\,\mathrm{d}H\,, \tag{2}$$

where $\mathrm{d}H$ denotes the Lebesgue measure on the space of Hermitian matrices. Similarly, one may define the *Gaussian orthogonal ensemble* (GOE) on the space of real symmetric matrices.

The Gaussian ensembles GUE and GOE are invariant under conjugation by a unitary (respectively orthogonal) matrices. As a consequence, it is relatively easy to compute the joint distribution of their eigenvalues $\rho_N(\lambda_1, \ldots, \lambda_N)$. For example for GUE one finds

$$\rho_N(\lambda_1, \ldots, \lambda_N) \;=\; C \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_i \mathrm{e}^{-N\lambda_i^2/2}\,.$$

The *k-point correlation function* is defined as

$$\rho_N^{(k)}(\lambda_1, \ldots, \lambda_k) \;:=\; \int \rho_N(\lambda_1, \ldots, \lambda_N)\,\mathrm{d}\lambda_{k+1} \cdots \mathrm{d}\lambda_N\,.$$

In a seminal paper published in 1955, Wigner showed that the macroscopic statistics follow the *semicircle law*:

$$\lim_{N\to\infty} \int_{E-\ell}^{E+\ell} \big(\rho_N^{(1)}(x) - \rho_{sc}(x)\big)\,\mathrm{d}x \;=\; 0\,, \qquad \rho_{sc}(x) \;:=\; \frac{1}{2\pi}\sqrt{[4-x^2]_+}\,,$$

for any fixed $\ell > 0$. This provides *macroscopic information* about the eigenvalue statistics, in the sense that one considers an interval $[E-\ell, E+\ell]$ that typically contains an order N eigenvalues.

A much finer question concerns the *microscopic* eigenvalue statistics. Gaudin, Mehta, and Dyson proved in the 1960s that the local correlation structure of GUE is given by a simple determinantal point process. More precisely, for any fixed k we have

$$\lim_{N\to\infty} \frac{1}{\rho_{sc}(E)^k}\rho_N^{(k)}\left(E+\frac{\alpha_1}{N\rho_{sc}(E)},\dots,E+\frac{\alpha_k}{N\rho_{sc}(E)}\right) \;=\; \det\bigl[S(\alpha_i-\alpha_j)\bigr]_{i,j=1}^k$$

for any fixed $E\in(-2,2)$, where $S(\alpha):=\frac{\sin\pi\alpha}{\pi\alpha}$ is the *sine kernel.*

The question of the distribution of the largest eigenvalue of GUE was settled in 1994, when Tracy and Widom proved that

$$\lim_{N\to\infty}\mathbb{P}(N^{2/3}(\lambda_{\max}-2)\le s) \;=\; F_2(s)\,,$$

where F_2 can be explicitly computed using Painlevé equations.

Wigner's original vision was that the microscopic spectral statistics of GUE are *universal.* He postulated that the spectrum of any strongly correlated physical model should exhibit GUE or GOE (depending on symmetries) statistics at a microscopic level. While the macroscopic statistics may vary from model to model, the microsopic statistics only depend on the details of the model through its symmetries. A very simple illustration of such universality is the central limit theorem: the sum of a large number of centred and normalized random variables is Gaussian, independent of the distribution of the individual random variables.

A mathematical justification of universality entails the analysis of a large class of random matrices, for which one seeks to establish GUE/GOE microscopic statistics. There are two natural ways to generalize the Gaussian ensembles. The first is to replace the quadratic dependence on H in the exponent of (2) with a more general function V. Thus, one considers the *invariant β-ensemble* with law

$$\mathbb{P}(H) \;=\; \frac{1}{Z}\mathrm{e}^{-N\,\mathrm{Tr}V(H)/2}\,\mathrm{d}H\,.$$

As for GUE, one may easily compute the joint probability density of the eigenvalues of H,

$$\rho_N(\lambda_1,\dots,\lambda_N) \;=\; C\prod_{i<j}|\lambda_i-\lambda_j|^\beta\prod_i \mathrm{e}^{-N\beta V(\lambda_i)/2}\,,$$

where β is a parameter that describes the symmetry ($\beta=2$ for Hermitian matrices and $\beta=1$ for real symmetric matrices). More details about the universality of the β-ensemble are given in Yau's talk.

The second way of generalizing the Gaussian ensembles is to keep the random variables x_{ij} in (1) independent, but to change their laws. For instance, one may consider the following class of random matrices, known as *generalized Wigner matrices.* Let $H=(h_{ij})$ be an $N\times N$ Hermitian or real symmetric matrix with $h_{ij}=\sigma_{ij}x_{ij}$, where $\sigma_{ij}>0$ is deterministic, $(x_{ij}:i\le j)$ are independent, and

$$\mathbb{E}x_{ij}=0\,,\qquad \mathbb{E}\,|x_{ij}|^2 \;=\; 1\,,\qquad \sum_j\sigma_{ij}^2 \;=\; 1\,,\qquad \frac{c}{N}\le\sigma_{ij}^2\le\frac{C}{N}\,.$$

Theorem 1 (Erdős, K, Schlein, Yau, Yin)**.** *Bulk universality holds for generalized Wigner matrices provided that*

$$\mathbb{E}\,|x_{ij}|^{4+\epsilon} \;\leq\; C\,,$$

i.e. for $-2 < E < 2$ *and* $b = N^{-1+\delta}$ *we have*

$$\lim_{N\to\infty}\int_{E-b}^{E+b}\frac{\mathrm{d}E'}{2b}\big(\rho_N^{(k)} - \rho_{\mu,N}^{(k)}\big)\bigg(E' + \frac{\alpha_1}{N},\dots,E' + \frac{\alpha_k}{N}\bigg) \;=\; 0\,,$$

where μ *stands for GUE or GOE depending on the symmetry of* H*.*

We note that bulk universality for *Wigner matrices* (satisfying $\sigma_{ij}^2 = N^{-1}$) was proved by Tao and Vu assuming that $\mathbb{E}\,|x_{ij}|^K < C$ for some large enough K and the first four moments of the entries of H match those of GUE/GOE. (Together with a result of Johansson, this result implies that bulk universality holds for Hermitian matrices if the first three moments match.)

Similarly, one may establish the edge universality of generalized Wigner matrices. In order to state the result, we order the eigenvalues of H so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$.

Theorem 2 (Erdős, K, Yau, Yin)**.** *Suppose that* $H^{\mathbf{v}}$ *and* $H^{\mathbf{w}}$ *are generalized Wigner matrices. Assume that two moments match, i.e.*

$$\mathbb{E}(x_{ij}^{\mathbf{v}})^2 \;=\; \mathbb{E}(x_{ij}^{\mathbf{w}})^2\,,$$

and that for both ensembles we have

$$\mathbb{E}\,|x_{ij}|^{12} \;\leq\; C\,. \tag{3}$$

Then for all $s \in \mathbb{R}$ *we have*

$$\mathbb{P}^{\mathbf{v}}\big(N^{2/3}(\lambda_N - 2) \leq s\big) - \mathbb{P}^{\mathbf{w}}\big(N^{2/3}(\lambda_N - 2) \leq s\big) \;\to\; 0\,.$$

Similarly: convergence of correlation functions of eigenvalues near edge.

The moment condition (3) is not optimal. (In fact, with some additional work, the number 12 may be improved to 7.) Edge universality is believed to hold down $4+\epsilon$ moments of the matrix entries. In fact, Auffinger, Ben Arous, and Péché have shown that if $\mathbb{E}\,|x_{ij}|^{4-\epsilon} = \infty$, edge universality does not hold. Previously, Sinai, Soshnikov, Ruzmaikina, and Sodin have shown that edge universality holds if the law of x_{ij} is symmetric and has a finite moment of sufficiently high order. Moreover, Tao and Vu have showed that edge universality holds under the additional assumption that the first three moments moments vanish and x_{ij} has a sufficiently fast decay.

One may also ask whether the distribution of the eigenvectors of H is universal. The following result establishes this for eigenvectors associated with eigenvalues close to the spectral edges ± 2. Let $\mathbf{u}_1,\dots,\mathbf{u}_N$ be the ℓ^2-normalized eigenvectors associated with the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$.

Theorem 3 (K, Yin). *Let H be a Wigner matrix whose entries have subexponential decay, and V be a GUE/GOE matrix. Then for any test function function θ we have*

$$\lim_{N\to\infty}\big(\mathbb{E}^H-\mathbb{E}^V\big)\theta\Big(N\overline{u}_{\alpha_1}(i_1)u_{\alpha_1}(j_1),\dots,N\overline{u}_{\alpha_k}(i_k)u_{\alpha_k}(j_k)\Big) \;=\; 0\,,$$

provided that $\alpha_1,\dots,\alpha_k\le N^{\epsilon}$ for some small $\epsilon>0$.

Note that this result characterizes the distribution of the eigenvector components completely (since eigenvectors are only defined up to a global phase). We may also include eigenvalues and prove an analogous result about the joint eigenvalue-eigenvector distribution function.

For bulk eigenvectors, the same result holds under the much stronger assumption that four, instead of two, moments match. This was also recently established by Tao and Vu.

Instead of generalized Wigner matrices, one may consider many other classes of random matrices. One such class of particular interest is the *Erdős-Rényi graph*, a random graph on N vertices in which each edge of the complete graph is chosen with probability p independently from all other edges. It may be characterized through its adjacency matrix $A=(a_{ij})$, a real symmetric matrix satisfying

$$a_{ij} \;=\; \frac{\gamma}{q}\begin{cases}1 & \text{with probability } p\\ 0 & \text{with probability } 1-p\,,\end{cases}$$

where $q:=\sqrt{pN}$ and $\gamma=(1-p)^{-1/2}$ so that $\operatorname{Var}a_{ij}=N^{-1}$. Each column typically has $pN=q^2$ nonvanishing entries. Typically, one is interested in the case where the graphs is sparse, i.e. $1\le q\ll N^{1/2}$. Note that the entries of A are not centred like those of Wigner matrices. Moreover, their fluctuations are much stronger in the sense that the k-th moment of a_{ij} decays much slower than in the case of a Wigner matrix.

The following result establishes the *complete delocalization* of the eigenvectors of A. By *delocalization* of an eigenvector $\mathbf{u}_\alpha$ at scale ℓ we mean that $\|\mathbf{u}_\alpha\|_\infty = O(\ell^{-1/2})$. Informally, this means that $\mathbf{u}_\alpha$ is supported on at least ℓ sites.

Theorem 4 (Erdős, K, Yau, Yin). *Let $\mathbf{u}_1,\dots,\mathbf{u}_N$ denote the eigenvectors of the Erdős-Rényi graph. Fix $\epsilon>0$ and let $q\ge(\log N)^{1+\epsilon}$. Then*

$$\mathbb{P}\bigg(\max_\alpha\|\mathbf{u}_\alpha\|_\infty\le\frac{(\log N)^4}{\sqrt{N}}\bigg)\;\ge\;1-\mathrm{e}^{-(\log N)^{1+\epsilon}}\,.$$

In their seminal paper of 1960, Erdős and Rényi proved that if $q\le(1-\epsilon)\sqrt{\log N}$, then there are a.s. isolated vertices, and consequently not all eigenvectors can be delocalized. Previously, Tran, Vu, and Wang proved that if λ_α is away from the spectral edges ±2, then the weaker estimate $\|\mathbf{u}_\alpha\|_\infty\le q^{-1}$ holds with high probability.

Finally, one we address the universality of the Erdős-Rényi graph. It is well known that its eigenvalues satisfy

$$\lambda_1,\dots,\lambda_{N-1}\in[-2-o(1),2+o(1)]\,.$$

The largest eigenvalue λ_N satisfies $\lambda_N \approx q + q^{-1}$; this outlier eigenvalue arises from fact that the entries a_{ij} have nonzero mean. Moreover, λ_N is known to have Gaussian fluctuations on scale $N^{-1/2}$.

Theorem 5 (Erdős, K, Yau, Yin)**.** *If $q \gg N^{1/3}$ (i.e. $p \gg N^{-1/3}$) then the bulk and edge universalities hold for the Erdős-Rényi graph. Edge universality means that*

$$\lim_{N\to\infty} \Big[\mathbb{P}^A\Big(N^{2/3}(\lambda_{N-1}-2) \le s\Big) - \mathbb{P}^{\mathrm{GOE}}\Big(N^{2/3}(\lambda_N - 2) \le s\Big)\Big] \;=\; 0\,.$$

We conclude this summary by outlining the strategy behind the proofs.

Step 1.: The *local semicircle law.* Control of the Green functions down to spectral windows of size N^{-1}, which implies localization estimates for the eigenvalues.

Proof: System of self-consistent equations for the Green functions; errors controlled using large deviations methods.

Step 2.: *Bulk universality of Gaussian divisible ensembles*

$$H \;=\; \sqrt{1-t}\,H_0 + \sqrt{t}\,V\,, \qquad H_0 \text{ is a Wigner matrix and } V \text{ is GUE.}$$

(Matrix entries have small Gaussian components, i.e. they are *Gaussian divisible.*)

Proof: Estimate the speed of convergence to *local equilibrium* of Dyson Brownian motion. More details are given in Yau's talk.

Step 3.: Density argument: approximation by Gaussian divisible ensembles.

For edge universality: resolvent expansion and moment matching condition.

The main tool behind Step 1 is the resolvent (or *Green function*) of H. Let $z = E + \mathrm{i}\eta$ with imaginary part $\eta > 0$. Define the Stieltjes transform

$$m_{sc}(z) \;=\; \int \mathrm{d}x\, \frac{\rho_{sc}(x)}{x-z}$$

and the resolvent

$$G(z) \;=\; (H-z)^{-1}.$$

The Stieltjes transform of the *empirical eigenvalue density* is

$$m(z) \;:=\; \frac{1}{N}\sum_{\alpha=1}^N \frac{1}{\lambda_\alpha - z} \;=\; \frac{1}{N}\mathrm{Tr}\, G(z)\,.$$

The parameter $\eta = \mathrm{Im}\, z$ describes the spectral resolution: $\mathrm{Im}\, m(E+\mathrm{i}\eta)$ is the density at E averaged over an interval of size η. Indeed,

$$\mathrm{Im}\, m(z) \;=\; \frac{\pi}{N}\sum_{\alpha=1}^N \delta_\eta(\lambda_\alpha - E)\,,$$

where $\delta_\eta(x) := \frac{1}{\pi}\frac{\eta}{x^2+\eta^2}$ is an approximate delta function on scale η. In this language, the local semicircle may be formulated as follows.

Theorem 6 (Erdős, K, Yau, Yin). *Let $H = (h_{ij})$ be a Hermitian or symmetric matrix, either generalized Wigner or Erdős-Rényi. Let $z = E + \mathrm{i}\eta$ with $\eta \gtrsim N^{-1}$. With high probability we have*

$$\big|G_{ij}(z) - m_{sc}(z)\delta_{ij}\big| \;\lesssim\; \frac{1}{q} + \frac{1}{\sqrt{N\eta}}\,.$$

The averaged quantity $m(z)$ satisfies the stronger bound

$$\big|m(z) - m_{sc}(z)\big| \;\lesssim\; \frac{1}{q^2} + \frac{1}{N\eta}\,.$$

Here $q = \sqrt{N}$ if H is a Wigner matrix. Both of these estimates hold in the bulk; analogous estimates hold near the edges.

Informally, we prove that $G(z) \approx m_{sc}(z)\mathbb{1}$ for η down to the *optimal scale* $\eta \approx N^{-1}$

We conclude by mentioning two consequences of this local semicircle law. First, it immediately implies the complete delocalization of the eigenvectors. Indeed, setting $\eta \approx N^{-1}$ and using the spectral decomposition of H yields

$$C \;\geq\; \operatorname{Im} G_{ii}(\lambda_\alpha + \mathrm{i}\eta) \;=\; \sum_\beta \frac{\eta\,|u_\beta(i)|^2}{(\lambda_\beta - \lambda_\alpha)^2 + \eta^2} \;\geq\; \frac{|u_\alpha(i)|^2}{\eta}\,,$$

which is the claim. A second corollary of the local semicircle law is the *eigenvalue rigidity estimate*

$$|\lambda_\alpha - \gamma_\alpha| \;\lesssim\; \big(N^{-2/3}\alpha^{-1/3} + q^{-2}\big)\,, \qquad (\alpha \leq N^{1/2})\,, \tag{4}$$

which may be derived using the Helffer-Sjöstrand functional calculus. Here γ_α is the *classical location of the α-th eigenvalue*, defined through

$$\int_{-\infty}^{\gamma_\alpha} \rho_{sc}(x)\,\mathrm{d}x = \frac{\alpha}{N}\,.$$

(The form (4) is true for $q \geq N^{1/3}$; a more complicated formula holds for $q \leq N^{1/3}$.)

Convergence of mixing times for sequences of random walks on finite graphs

Takashi Kumagai

(joint work with D.A. Croydon and B.M. Hambly)

We report the main results in our recent preprint [2].

Let $G = (V(G), E(G))$ be a finite connected graph, and $(X^G_m)_{m\geq 0}$ be an irreducible Markov chain with transition probability $P_G(x,y)$, and the invariant probability measure $\pi^G(\cdot)$. Let $p^G_m(x,y) := \mathbf{P}^G_x(X_m = y)/\pi^G(\{y\})$ be the transition density of X^G with respect to π^G.

For $p \in [1,\infty]$, we define **the L^p-mixing time** of G by

$$t^p_{\rm mix}(G) := \inf\{m>0 : \sup_{x\in V(G)} D^G_p(x,m) \le 1/4\},$$

where $D^G_p(x,m) := \|(p^G_m(x,\cdot)+p^G_{m+1}(x,\cdot))/2 - 1\|_{L^p(\pi^G)}$.

We will discuss the following problem.

(Q) Given a sequence of graphs $(G^N)_{N\ge 1}$ with irreducible Markov chains, when does (suitably rescaled) $t^p_{\rm mix}(G^N)$ converge as $N\to\infty$?

1. Framework and theorem

In order to discuss the convergence of mixing times, we will introduce a notion of spectral Gromov-Hausdorff convergence.

Let (F,d_F) be a compact metric space, and assume there is a conservative irreducible Hunt process on F with the invariant Borel probability measure (full support) on F. We also assume that there exists a jointly continuous heat kernel $(q_t(x,y))_{x,y\in F, t>0}$ of the process on F, and assume

$$\lim_{t\to\infty} \|q_t(x,\cdot)-1\|_{L^p(\pi)} = 0, \qquad \forall x\in F. \tag{1}$$

Then the L^p-mixing time of F is finite, i.e.

$$t^p_{\rm mix}(F) := \inf\{t>0 : \sup_{x\in F} \|q_t(x,\cdot)-1\|_{L^p(\pi)} \le 1/4\} < \infty.$$

Let $\mathcal{M}_I$ be the collection of (an equivalence class of) triples of the form (F,π,q). Now, for a compact interval $I\subset(0,\infty)$ and $(F,\pi,q),(F',\pi',q')\in\mathcal{M}_I$, set

$$\Delta_I((F,\pi,q),(F',\pi',q')) := \inf_{Z,\phi,\phi',\mathcal{C}} \{d^Z_H(\phi(F),\phi'(F')) + d^Z_P(\pi\circ\phi^{-1},\pi'\circ\phi'^{-1})$$
$$+ \sup_{(x,x'),(y,y')\in\mathcal{C}} (d_Z(\phi(x),\phi'(x')) + d_Z(\phi(y),\phi'(y')) + \sup_{t\in I}|q_t(x,y)-q'_t(x',y')|)\},$$

where the infimum is taken over all metric spaces $Z=(Z,d_Z)$, isometric embeddings $\phi: F\to Z$, $\phi': F'\to Z$, and correspondences $\mathcal{C}$ between F and F'. Here d^Z_H is the Hausdorff distance in Z, and d^Z_P is the Prohorov distance between Borel probabilities on Z. $\mathcal{C}$ is a correspondence between F and F', i.e. it is a subset of $F\times F'$ such that for each $x\in F$, there exists $x'\in F'$ with $(x,x')\in\mathcal{C}$, and conversely for all $x'\in F'$, there exists $x\in F$ with $(x,x')\in\mathcal{C}$.

Lemma 1. *$(\mathcal{M}_I,\Delta_I)$ is a separable metric space for each compact interval I.*

Now we are ready to define the spectral Gromov-Hausdorff convergence. (F_n,π_n,q_n) converges to (F,π,q) in a spectral Gromov-Hausdorff sense if

$$\lim_{n\to\infty} \Delta_I((F_n,\pi_n,q_n),(F,\pi,q)) = 0, \quad \forall I: \text{ compact interval.}$$

Remark. We note that similar notion of spectral convergences were introduced in the setting of compact Riemannian manifolds by Bérard-Besson-Gallot ('94) and by Kasue-Kumura ('94).

We now give the theorem.

Assumption 2. Let $(G^N)_{N\geq 1}$ be a sequence of finite connected graphs for which there exists a sequence $(\gamma(N))_{N\geq 1}$ such that for any compact interval $I \subset (0,\infty)$,

$$\left(\left(V(G^N), d_{G^N}\right), \pi^N, \left(q^N_{\gamma(N)t}(x,y)\right)_{x,y\in V(G^N),t\in I}\right) \to \left((F,d_F), \pi, (q_t(x,y))_{x,y\in F, t\in I}\right)$$

in a spectral Gromov-Hausdorff sense.

Theorem 2. *Assume Assumption 2. If* $\lim_{t\to\infty} \|q_t(x,\cdot)-1\|_{L^p(\pi)} = 0$, *for all* $x \in F$, *where* $p \in [1,\infty]$ *and* $q_t(\cdot,\cdot)$ *is the heat kernel, then*

$$\lim_{N\to\infty} \gamma(N)^{-1} t^p_{\mathrm{mix}}(G^N) = t^p_{\mathrm{mix}}(F) \in (0,\infty). \tag{2}$$

We have the following sufficient condition for Assumption 2.

Proposition 3. *Suppose that* $(V(G^N), d_{G^N})$, $N \geq 1$, *and* (F, d_F) *can be isometrically embedded into a metric space* (E, d_E) *in such a way that*

$$\lim_{N\to\infty} d^E_H(V(G^N), F) = 0, \quad \lim_{N\to\infty} d^E_P(\pi^N, \pi) = 0. \tag{3}$$

Assume further there exists a dense set F^* *in* F *such that for any compact interval* $I \subset (0,\infty)$, *any* $x \in F^*$, $y \in F$ *and* $r > 0$,

$$\lim_{N\to\infty} \mathbf{P}^{G^N}_{g_N(x)}\left(X^{G^N}_{\lfloor\gamma(N)t\rfloor} \in B_E(y,r)\right) = \int_{B_E(y,r)} q_t(x,y)\pi(dy) \ \textit{uniformly for } t\in I,$$

$$\lim_{\delta\to 0}\limsup_{N\to\infty} \sup_{\substack{x,y,z\in V(G^N):\\ d_{G^N}(y,z)\leq\delta}} \sup_{t\in I}\left|q^N_{\gamma(N)t}(x,y) - q^N_{\gamma(N)t}(x,z)\right| = 0,$$

then Assumption 2 holds.

2. Example

We have various examples of sequences of random and non-random graphs where we can verify Assumption 2. Here we give one interesting example.

Consider the Erdös-Rényi random graph $G(N,p)$ at the critical window, i.e. $p = 1/N + \lambda N^{-4/3}$ for fixed $\lambda \in \mathbb{R}$. Let $\mathcal{C}^N$ be the largest connected component. It is known ([1]) that $N^{-1/3}\mathcal{C}^N \overset{d}{\to} \mathcal{M}$ in Gromov-Hausdorff sense, where $\mathcal{M}$ can be constructed from a random real tree by gluing a (random) finite number of points. We take distinguished points $\rho^N \in \mathcal{C}^N$ and $\rho \in \mathcal{M}$. Let X^N be a simple random walk on $\mathcal{C}^N$ started at ρ^N. Then,

$$(N^{-1/3}X^N_{[Nt]})_{t\geq 0} \overset{d}{\to} (B^{\mathcal{M}}_t)_{t\geq 0},$$

where $B^{\mathcal{M}}$ is the Brownian motion on $\mathcal{M}$ started from ρ ([3]). We can verify Assumption 2 and obtain the following, which improves the result in [4].

Theorem 4. *For* $p \in [1,\infty]$, *let* $t^p_{\mathrm{mix}}(\rho^N)$ *is the* L^p*-mixing time of* X^N. *Then*

$$N^{-1}t^p_{\mathrm{mix}}(\rho^N) \to t^p_{\mathrm{mix}}(\rho), \ \textit{in distribution},$$

where $t^p_{\mathrm{mix}}(\rho) \in (0,\infty)$ *is the* L^p*-mixing time of* $B^{\mathcal{M}}$.

References

[1] L. Addario-Berry, N. Broutin, and C. Goldschmidt, *The continuum limit of critical random graphs*, Probab. Theory Related Fields, to appear.

[2] D. A. Croydon, B. M. Hambly and T. Kumagai, *Convergence of mixing times for sequences of random walks on finite graphs*, Preprint 2011.

[3] D. A. Croydon, *Scaling limit for the random walk on the largest connected component of the critical random graph*, Preprint 2010.

[4] A. Nachmias and Y. Peres, *Critical random graphs: diameter and mixing time*, *Ann. Probab.* **36** (2008), 1267–1286.

Approximate Lifshitz law for the zero-temperature stochastic Ising Model in any dimension

Hubert Lacoin

We study the zero-temperature stochastic Ising model in a finite hypercube of sidelength L in $\mathbb{Z}^d$ with plus boundary condition. Initially the whole cube is filled with minus spins. The spins evolve following the majority rule: with rate one each spin spin takes the same sign of the majority of its neighbors (including those at the boundary) if the latter is well defined. Otherwise its sign is determined by tossing a fair coin. Eventually, all minus spins disappear, as a result of the pressure imposed by plus spins from the boundary.

Our aim is to study the time $\mathcal{T}_+$ needed for the spins in the hypercube to become entirely plus. According to a heuristics from Lifshitz [3], on a macroscopic scale, each point of the interface between plus and minus spins should move feeling a local drift proportional to its local mean curvature. This readily implies that, with high probability, $\mathcal{T}_+ = O(L^2)$.

The result we present is that with high probability, $\mathcal{T}_+ = O(L^2(\log L)^{10})$ for all dimension $d \geq 4$. Our result complements existing analogous bounds for dimension $d = 2$ and $d = 3$ obtained in [2, 1]. It is important to keep in mind that the cases $d = 2$ and $d = 3$ are completely different, mainly because the equilibrium fluctuations of Ising interfaces in $d = 2$ and in $d = 3$ occur on very different scales ($O(L^{1/2})$ for $d = 2$ and $O(\log(L))$ for $d = 3$). As such, they have been analyzed by very different approaches. The case $d \geq 4$ should be more similar to the case $d = 3$ which therefore plays a role of critical dimension. However, contrary to what has been done in [1], our proof does not attempt to control the local mean drift of the interface. Such an approach, in fact, would at least require two main missing tools: (i) a detailed analysis of the local equilibrium fluctuations of Ising interfaces[1]; (ii) good estimates on mixing times for $(d-1)$-monotone surfaces, following e.g. the method developed by Wilson [5] for the three dimensional case. Instead, we use an ad-hoc construction that allows us to bound the hitting time $\mathcal{T}_+$ for the d-dimensional dynamics using known sharp estimates for the same hitting time in three dimensions [1]. One of the reasons why this approach is successful is

[1] In dimension $d \geq 4$ the fluctuations of Ising hypersurfaces are believed to be order one. Recently a result of this sort has been proved under a Lifshitz condition [4].

that in dimension three, Ising interfaces are flat with only logarithmic fluctuations (see e.g. Proposition 4 in [1] for more details).

References

[1] P. Caputo, F. Martinelli, F. Simenhaus and F. Toninelli, *"Zero" temperature stochastic 3D Ising model and Dimer covering fluctuation: a first step towards mean curvature motion*, to appear in Comm. Pure Appl. Math.

[2] L. R. Fontes, R. H. Schonmann, V. Sidoravicius, *Stretched Exponential Fixation in Stochastic Ising Models at Zero Temperature*, Commun. Math. Phys. **228** (2002) 495-518.

[3] I. M. Lifshitz, *Kinetics of ordering during second order phase transitions*, Soviet Physics JETP **15** (1962), 939-942.

[4] R. Peled *High-Dimensional Lipschitz Functions are Typically Flat*, (2010)preprint, arXiv:1005.4636 (math-ph).

[5] D.B. Wilson *Mixing times of Lozenge tiling and card shuffling Markov chains*, Ann. Appl. Probab. **14** (2004) 274–325.

Uniqueness and universality of the Brownian map

Jean-François Le Gall

We discuss the convergence in distribution in the Gromov-Hausdorff sense of random planar maps viewed as random metric spaces. Recall that a planar map is a proper embedding of a finite connected graph in the two-dimensional sphere, viewed up to orientation-preserving homeomorphisms of the sphere. The faces of the map are the connected components of the complement of edges, and the degree of a face counts the number of edges that are incident to it. Special cases of planar maps are triangulations, where each face has degree 3, quadrangulations, where each face has degree 4, and more generally q-angulations, where each face has degree q. For technical reasons, one often considers rooted planar maps, meaning that there is a distinguished oriented edge. Planar maps have been studied thoroughly in combinatorics, and they also play an important role in other areas of mathematics and in theoretical physics, in particular in the theory of two-dimensional quantum gravity.

Let $q \geq 3$ be an integer. We assume that either $q = 3$ or q is even. The set of all rooted planar q-angulations with n faces is denoted by $\mathcal{A}_n^q$. For every integer $n \geq 1$ (if $q = 3$ we must restrict our attention to even values of n, since $\mathcal{A}_n^3$ is empty if n is odd), we consider a random planar map M_n that is uniformly distributed over $\mathcal{A}_n^q$. We denote the vertex set of M_n by $V(M_n)$. We equip $V(M_n)$ with the graph distance d_{gr}, and we view $(V(M_n), d_{gr})$ as a random variable taking values in the space $\mathbb{K}$ of all isometry classes of compact metric spaces, which is equipped with the Gromov-Hausdorff distance.

Theorem 1. *Set*

$$c_q = \Big(\frac{9}{q(q-2)}\Big)^{1/4}$$

if q is even, and

$$c_3 = 6^{1/4}.$$

There exists a random compact metric space $(\mathbf{m}_\infty, D^*)$ *called the Brownian map, which does not depend on* q*, such that*

$$(V(M_n), c_q\, n^{-1/4} d_{gr}) \xrightarrow[n\to\infty]{\text{(d)}} (\mathbf{m}_\infty, D^*)$$

where the convergence holds in distribution in the space $\mathbb{K}$*.*

The case of triangulations ($q = 3$) solves a problem stated by Schramm [6]. The case $q = 4$ has been obtained independently by Miermont [5] using very different methods.

Let us give a precise definition of the Brownian map. We first introduce the random real tree called the CRT. Let $(\mathbf{e}_s)_{0\le s\le 1}$ be a normalized Brownian excursion, i.e. a positive excursion of linear Brownian motion conditioned to have duration 1, and set, for every $s, t \in [0, 1]$,

$$d_{\mathbf{e}}(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2 \min_{s\wedge t\le r\le s\vee t} \mathbf{e}_r.$$

Then $d_{\mathbf{e}}$ is a (random) pseudometric on $[0, 1]$, and we consider the associated equivalence relation $\sim_{\mathbf{e}}$: for $s, t \in [0, 1]$,

$$s \sim_{\mathbf{e}} t \quad \text{if and only if} \quad d_{\mathbf{e}}(s,t) = 0.$$

Since $0 \sim_{\mathbf{e}} 1$, we may as well view $\sim_{\mathbf{e}}$ as an equivalence relation on the unit circle $\mathbb{S}^1$. The CRT is the quotient space $\mathcal{T}_{\mathbf{e}} := \mathbb{S}^1/\sim_{\mathbf{e}}$, which is equipped with the distance induced by $d_{\mathbf{e}}$. We write $p_{\mathbf{e}}$ for the canonical projection from $\mathbb{S}^1$ onto $\mathcal{T}_{\mathbf{e}}$, and $\rho = p_{\mathbf{e}}(1)$. If $u, v \in \mathbb{S}^1$, we let $[u, v]$ be the subarc of $\mathbb{S}^1$ going from u to v in clockwise order, and if $a, b \in \mathcal{T}_{\mathbf{e}}$, we define $[a, b]$ as the image under the canonical projection $p_{\mathbf{e}}$ of the smallest subarc $[u, v]$ of $\mathbb{S}^1$ such that $p_{\mathbf{e}}(u) = a$ and $p_{\mathbf{e}}(v) = b$. Roughly speaking, $[a, b]$ corresponds to the set of vertices that one visits when going from a to b around the tree in clockwise order.

We then introduce Brownian labels on the CRT. We consider a real-valued process $Z = (Z_a)_{a\in\mathcal{T}_{\mathbf{e}}}$ indexed by the CRT, such that, conditionally on $\mathcal{T}_{\mathbf{e}}$, Z is a centered Gaussian process with $Z_\rho = 0$ and $E[(Z_a - Z_b)^2] = d_{\mathbf{e}}(a, b)$ (this presentation is slightly informal as we are considering a random process indexed by a random set). We define, for every $a, b \in \mathcal{T}_{\mathbf{e}}$,

$$D^\circ(a,b) = Z_a + Z_b - 2\max\Big(\min_{c\in[a,b]} Z_c, \min_{c\in[b,a]} Z_c\Big),$$

and we put $a \simeq b$ if and only if $D^\circ(a, b) = 0$. Although this is not obvious, it turns out that $\simeq$ is an equivalence relation on $\mathcal{T}_{\mathbf{e}}$, and we let

$$\mathbf{m}_\infty := \mathcal{T}_{\mathbf{e}}/\simeq$$

be the associated quotient space. We write Π for the canonical projection from $\mathcal{T}_{\mathbf{e}}$ onto $\mathbf{m}_\infty$. We then define the distance on $\mathbf{m}_\infty$ by setting, for every $x, y \in \mathbf{m}_\infty$,

$$D^*(x,y) = \inf\Big\{\sum_{i=1}^{k} D^\circ(a_{i-1}, a_i)\Big\}, \tag{1}$$

where the infimum is over all choices of the integer $k \geq 1$ and of the elements $a_0, a_1, \ldots, a_k$ of $\mathcal{T}_{\mathbf{e}}$ such that $\Pi(a_0) = x$ and $\Pi(a_k) = y$. It follows from [1, Theorem 3.4] that D^* is indeed a distance, and the resulting random metric space $(\mathbf{m}_\infty, D^*)$ is the Brownian map.

Theorem 1 is the main result of [3], which is a continuation and in a sense a conclusion to the preceding papers [1] and [2]. In [1], we proved the existence of sequential Gromov-Hausdorff limits for rescaled uniformly distributed rooted $2p$-angulations with n faces, and we called Brownian map any random compact metric space that can arise in such limits (the name "Brownian map" first appeared in the paper of Marckert and Mokkadem [4]). The main result of [1] used a compactness argument that required the extraction of suitable subsequences in order to get the desired convergence. The reason why this extraction was needed is the fact that the limit could not be characterized completely. It was proved in [1] that any Brownian map can be written in the form $(\mathbf{m}_\infty, D)$, where the set $\mathbf{m}_\infty$ is as described above, and D is a distance on $\mathbf{m}_\infty$, for which only upper and lower bounds were available in [1, 2]. In particular, the paper [1] provided no characterization of the distance D and it was conceivable that different sequential limits, or different values of q, could lead to different metric spaces. This uniqueness problem is solved by establishing the explicit formula (1), which had been conjectured in [1]. As a consequence, we obtain the uniqueness of the Brownian map, and we get that this random metric space is the scaling limit of uniformly distributed q-angulations with n faces, for the values of q discussed above. Our proofs strongly depend on the study of geodesics in the Brownian map that was developed in [2].

References

[1] J.F. Le Gall, *The topological structure of scaling limits of large planar maps*, Invent. Math., **169** (2007), 621-670.

[2] J.F. Le Gall, J.F., *Geodesics in large planar maps and in the Brownian map*, Acta Math., **205** (2010), 287-360.

[3] J.F. Le Gall, *Uniqueness and universality of the Brownian map*, preprint, arXiv:1105.4842

[4] J.F. Marckert, J.F., A. Mokkadem, *Limit of normalized quadrangulations. The Brownian map.* Ann. Probab. **34** (2006), 2144-2202.

[5] G. Miermont, The Brownian map is the scaling limit of uniform random plane quadrangulations, preprint, arXiv:1104.1606

[6] O. Schramm, *Conformally invariant scaling limits: an overview and a collection of problems*, in Proceedings of the International Congress of Mathematicians (Madrid 2006), Vol.I, pp. 513-543. European Math. Soc., Zürich, 2007.

The Brownian map is the scaling limit of uniform random plane quadrangulations

Grégory Miermont

A (plane) map is a proper embedding of a finite, connected graph in the 2-dimensional sphere, considered up to orientation-preserving homeomorphisms. A map is called *rooted* if it has a distinguished oriented edge, and is called a *quadrangulation* if every face (a connected component of the complement of the edges) is a quadrangle, meaning that it is incident to exactly 4 oriented edges.

Let $\mathcal{Q}_n$ be the (finite) set of rooted plane quadrangulations with n faces. Let Q_n be a random variable with uniform distribution on $\mathcal{Q}_n$. We consider the set $V(Q_n)$ of its vertices as a finite metric space by endowing it with the usual graph distance d_{Q_n}. We prove that the sequence $(V(Q_n), n^{-1/4}d_{Q_n})$ converges in distribution as $n \to \infty$ to a limiting metric space, in the sense of the Gromov-Hausdorff topology [3]. We also show that the limit is, up to a scale constant, the so-called *Brownian map*, which was introduced by Marckert & Mokkadem [12] and Le Gall [7] as the most natural candidate for the scaling limit of many models of random plane maps. The Brownian map is defined in terms of the *Brownian snake* [6], which can be understood as Brownian motion indexed by the Brownian continuum random tree.

The theme of scaling limits of random plane maps has attracted a lot of interest in the recent years, motivated in part by the physical theory of 2-dimensional quantum gravity [1]. The mathematical study of this problem was initiated by Chassaing and Schaeffer [4], who identified, among other things, the 2-point function of the map, i.e. the limiting distribution of $n^{-1/4}d_{Q_n}(v_1, v_2)$, where v_1 and v_2 are two points chosen uniformly at random in $V(Q_n)$. Since then, important steps towards the understanding of the scaling limits of maps have been accomplished. In particular, Le Gall [7] showed the existence of subsequential limits of $(V(Q_n), n^{-1/4}d_{Q_n})$, and identified with Paulin the topology of any of these limits, which is that of the 2-sphere [11] (see also [13]). The 3-point function of quadrangulations, that is, the limit in law of the vector $n^{-1/4}(d_{Q_n}(v_1, v_2), d_{Q_n}(v_1, v_3), d_{Q_n}(v_2, v_3))$ where v_1, v_2, v_3 are independent uniform points in $V(Q_n)$, was also determined by Bouttier and Guitter [2]. Nevertheless, the problem of the determination of the limit of $(V(Q_n), n^{-1/4}d_{Q_n})$ ("uniqueness of the Brownian map") had remained open. We mention, however, that simultaneously with our work, an alternative proof of the uniqueness of the Brownian map has been obtained by Le Gall [9], using different methods.

The proof relies strongly on the concept of *geodesic stars*, which are configurations made of several geodesics that only share a common endpoint and do not meet elsewhere. More precisely, let x_1, x_2, x_3 be randomly chosen points in a distributional limit of $(V(Q_n), n^{-1/4}d_{Q_n})$ along a subsequence. Then the set of points x on the geodesic γ from x_1 to x_2 such that there exists a geodesic γ' from x to x_3 that intersects γ only at x, is a set of Hausdorff dimension stricly less than 1. Imagining that x_3 is fixed once and for all as a distinguished point in the limiting space, this allows to show the (somewhat surprising) property that geodesic paths between "typical" pairs of vertices in the map (say x_1, x_2), can be approximated by gluing pieces of geodesic paths originating from x_3. It turns out that structure of the latter paths is well-understood [8], because of the celebrated Cori-Vauquelin-Schaeffer bijection [5, 16, 4], that allows to code a quadrangulation with a distinguished vertex by a labeled tree, in which the geodesics to the distinguished vertex have a natural interpretation that is preserved after taking scaling limits. The study of these "star" configurations is performed by making use of a variant [14] of the Cori-Vauquelin-Schaeffer bijection that encodes quadrangulations with several distinguished points using simpler, "locally tree-like" objects, namely, labeled maps with a fixed number of faces.

References

[1] J. Ambjørn, B. Durhuus, and T. Jonsson. *Quantum geometry. A statistical field theory approach*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1997.

[2] J. Bouttier, E. Guitter. The three-point function of planar quadrangulations. *J. Stat. Mech.* P07020, 2008.

[3] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[4] P. Chassaing and G. Schaeffer. Random planar lattices and integrated superBrownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.

[5] R. Cori and B. Vauquelin. Planar maps are well labeled trees. *Canad. J. Math.*, 33(5):1023–1042, 1981.

[6] J.-F. Le Gall. *Spatial branching processes, random snakes and partial differential equations.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1999.

[7] J.-F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169(3):621–670, 2007.

[8] J.-F. Le Gall. Geodesics in large planar maps and in the Brownian map. *Acta Mathematica*, 205(2):287–360, 2010.

[9] J.-F. Le Gall. Uniqueness and universality of the Brownian map. arXiv:1105.4842, 2011.

[10] J.-F. Le Gall and G. Miermont. Scaling limits of random trees and planar maps. 2011. arXiv:1101.4856. Submitted to the Proceedings of the Clay Mathematics Institute 2010 Summer School.

[11] J.-F. Le Gall and F. Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18(3):893–918, 2008.

[12] J.-F. Marckert and A. Mokkadem. Limit of normalized random quadrangulations: the Brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006.

[13] G. Miermont. On the sphericity of scaling limits of random planar quadrangulations. *Electron. Commun. Probab.*, 13:248–257, 2008.

[14] G. Miermont. Tessellations of random maps of arbitrary genus. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(5):725–781, 2009.

[15] G. Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. arXiv:1104.1606, 2011.

[16] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires.* PhD thesis, Université Bordeaux I, 1998.

AC Geometry of the Gaussian Free Field

Jason Miller

(joint work with S. Sheffield)

Fix a constant $\chi > 0$ and let h be an instance of the Gaussian free field on a planar domain. We study flow lines of the vector fields $e^{i(h/\chi+\theta)}$ starting at a fixed boundary point of the domain. Letting θ vary, one obtains a family of non-crossing curves that look locally like SLE_κ processes with $\kappa \in (0,4)$ (where $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$) and can be interpreted as the rays of a random geometry. So-called *counterflow lines* ($\mathrm{SLE}_{16/\kappa}$) are constructed within the same geometry as "light cones" of points accessible by angle-restricted trajectories.

Although h is a distribution-valued random variable, we show that these paths are well-defined and are in fact path-valued functions of h. In contrast to what happens when h is smooth, these paths bounce off of each other and interact in interesting ways. As one consequence of our methods, we prove in general that $\mathrm{SLE}_\kappa(\rho)$ processes are almost surely continuous random curves, even when they intersect the boundary.

Critical versus near-critical dynamics in the planar FK Ising model

Gábor Pete

(joint work with C. Garban)

We study the natural heat-bath dynamics of the Fortuin-Kasteleyn random cluster measures $\mathrm{FK}(p,q)$ on $\mathbb{Z}^2$ at the critical density $p = p_c(q)$, primarily in the Ising case $q = 2$, where the conformal invariance of the spin Ising and FK Ising models are proved by Smirnov and Chelkak, and the alternating 4-arm (pivotal) exponents are determined by Garban. Extending our previous joint work with Oded Schramm on the $q = 1$ case, Bernoulli percolation, we prove the existence and conformal covariance of the scaling limit of the properly rescaled heat-bath dynamics for the critical FK(2) model.

We also consider a kind of "asymmetric dynamics", a natural monotone coupling of the $\mathrm{FK}(p,q)$ models as p varies, introduced by Grimmett, in the near-critical regime. Contrary to what happens in the percolation case and what seems to have been expected by most people in the community, the near-critical window and the correlation length (found by Onsager) are not governed anymore by the amount of pivotal edges at criticality. Instead, changes are much faster than in the symmetric heat-bath dynamics, due to a fascinating self-organized mechanism that controls how new edges arrive as one raises p near p_c. We prove some simple qualitative properties of the near-critical monotone coupling, but the more interesting quantitative analysis remains to be understood.

For the Glauber dynamics of the critical spin Ising model, the existence of a scaling limit that would describe the evolution of macroscopic spin clusters remains completely mysterious, mainly due to the alternating 4-arm exponent being larger than 2, hence having more pivotals on small scales than on large ones. On the other hand, we can use the same exponent to prove that in the dynamics on the infinite lattice there are no exceptional times where an infinite spin cluster appears (answering a question of Broman and Steif).

Finally, we conjecture the following interesting phenomenon in the heat-bath dynamics of critical FK(q) models: there is a regime $q \in (q^*, 4)$, or $\kappa \in (4, \kappa^*)$ in the SLE_κ world, where there exist macroscopic pivotals, but there are no exceptional times with an infinite cluster. Probably these are the first natural models that are expected to be noise- but not dynamically sensitive.

End point distribution for directed random polymers in 1+1 dimensions

Jeremy Quastel

(joint work with I. Corwin, D. Remenik and G. Moreno-Flores)

The Airy_2 process $A(x)$ is a stationary process whose finite dimensional distributions are given in terms of certain determinants. In particular, the one point marginals are the GUE Tracy-Widom distribution from random matrices. The Airy_2 process minus a parabola is important because it approximates the time rescaled free energy of directed polymer models. In particular, its maximum will have the GOE Tracy-Widom distribution. Johansson proved this in a very indirect way, going through polynuclear growth models. With Corwin and Remenik we obtain a direct proof of this fact by taking a fine mesh limit of the determinental formulas to obtain an asymptotic formula in terms of a Fredholm determinant of the solution operator of a boundary value problem. The method is extended in work with Moreno-Flores and Remenik to obtain an exact formula for the joint density of the max and the argmax of the Airy_2 process minus a parabola. In particular, the distribution of the argmax is the universal asymptotic distribution of the endpoint of one dimensional directed polymers in random environment.

Replica Bethe ansatz approach to the KPZ equation with half Brownian motion initial condition

Tomohiro Sasamoto

(joint work with T. Imamura)

The Kardar-Parisi-Zhang (KPZ) equation is a well known equation to describe stochastic surface growth phenomena. Its one-dimensional version is

$$\partial_t h(x,t) = \tfrac{1}{2}(\partial_x h(x,t))^2 + \tfrac{1}{2}\partial_x^2 h(x,t) + \eta(x,t), \tag{1}$$

where $\eta(x,t)$ is white noise with covariance $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$. Applying the dynamical renormalization group analysis, they showed that the exponent for the height fluctuations is 1/3. In 2010, the first exact solution for this equation was obtained. The explicit distribution function was determined for the narrow wedge initial condition, based on those for ASEP.

By the Cole-Hopf transformation

$$h(x,t) = \log\left(Z(x,t)\right),$$

the KPZ equation is turned into a problem of directed polymer in random environment. It turned out that the above distribution function can be computed using the Bethe ansatz for a replicated system of this directed polymer. An advantage of this approach is that one can discuss other initial conditions. In this presentation, we apply this replica Bethe ansatz to study the KPZ equation for a half-Brownian motion initial condition, given by

$$h(x,t=0) = \begin{cases} x/\delta, \ \delta \to 0, & x<0, \\ B(x), & x \geq 0. \end{cases}$$

where $B(x)$ is a standard Brownian motion.

Our main result is that the generating function of the moments $\langle Z^N(x,t)\rangle$,

$$G_{\gamma_t}(s;X) = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N\left(2\gamma_t^2 X, t\right)\right\rangle e^{N\frac{\gamma_t^3}{12} + N\gamma_t X^2}, \quad \gamma_t = \left(\frac{t}{2}\right)^{1/3},$$

is given by a Fredholm determinant with the kernel,

$$K_X(\xi_j,\xi_k) = \int_{\mathbb{R}} dy \mathrm{Ai}^{\Gamma}\left(\xi_j + y, \frac{1}{\gamma_t}, -\frac{X}{\gamma_t}\right) \mathrm{Ai}_{\Gamma}\left(\xi_k + y, \frac{1}{\gamma_t}, -\frac{X}{\gamma_t}\right) \frac{e^{\gamma_t y}}{e^{\gamma_t y} + e^{\gamma_t s}}.$$

Here $\mathrm{Ai}^{\Gamma}(a,b,c)$, $\mathrm{Ai}_{\Gamma}(a,b,c)$ are deformed Airy functions

$$\mathrm{Ai}^{\Gamma}(a,b,c) = \frac{1}{2\pi} \int_{\Gamma_{i\frac{c}{b}}} dz e^{iza + i\frac{z^3}{3}} \Gamma\left(ibz + c\right),$$

$$\mathrm{Ai}_{\Gamma}(a,b,c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz e^{iza + i\frac{z^3}{3}} \frac{1}{\Gamma\left(-ibz + c\right)}$$

where the contour $\Gamma_{i\frac{c}{b}}$ is from $-\infty$ to ∞ and passes below the pole at $i\frac{c}{b}$. From this one can also obtain the height distribution and take the long time limit. Using the replica one can also study the multi-point distributions. We obtained again a Fredholm determinant expression and considered the long time limit.

More details can be found in arxiv:1105.4659.

Quantum Gravity Zippers

Scott Sheffield

I present an overview of the paper "Conformal weldings of random surfaces: SLE and the quantum gravity zipper", recently posted to the arXiv. The main result is that if one welds together two independent Liouville quantum gravity surfaces along their boundaries and conformally maps the result to the half-plane, then the image of the interface is a Schramm-Loewner evolution.

Fluctuation exponents for a 1+1 dimensional directed random polymer

Timo Seppäläinen

This talk describes a 1+1 dimensional directed polymer in a random environment whose random weights are log-gamma distributed. The directed polymer is a statistical mechanical model of a random walk path in a random potential. To describe the finite model in a rectangle, fix the lower left and upper right corners of the rectangle at $(0,0)$ and (m,n). Here are the elements of the model.

- The set of admissible paths is $\Pi_{m,n} = \{$up-right paths $x_\cdot = (x_k)_{k=0}^{m+n}$ from $(0,0)$ to $(m,n)\}$
- The environment $(Y_{i,j} : (i,j) \in \mathbb{Z}_+^2)$ consists of independent weights $Y_{i,j}$ with joint distribution $\mathbb{P}$.
- The quenched polymer measure on paths $x_\cdot \in \Pi_{m,n}$ and the partition function are given by

$$Q_{m,n}(x_\cdot) = \frac{1}{Z_{m,n}} \prod_{k=1}^{m+n} Y_{x_k} \quad \text{and} \quad Z_{m,n} = \sum_{x_\cdot \in \Pi_{m,n}} \prod_{k=1}^{m+n} Y_{x_k}.$$

- The averaged measure is $P_{m,n}(x_\cdot) = \mathbb{E}Q_{m,n}(x_\cdot)$.

The key assumption is on the distributions of the weights. Fix two parameters $0 < \theta < \mu < \infty$. The weights are reciprocals of gamma distributed random variables, with these parameters for $i,j \geq 1$: $Y_{i,0}^{-1} \sim \text{Gamma}(\theta)$, $Y_{0,j}^{-1} \sim \text{Gamma}(\mu - \theta)$, and $Y_{i,j}^{-1} \sim \text{Gamma}(\mu)$. Explicitly, a Gamma$(\theta)$-distributed random variable is supported on the positive reals where it has density $\Gamma(\theta)^{-1} x^{\theta-1} e^{-x}$.

For the asymptotic results the endpoint (m,n) of the polymer is taken to infinity in a particular characteristic direction as $N \nearrow \infty$:

$$|\, m - N\Psi_1(\mu-\theta) \,| \leq \gamma N^{2/3} \quad \text{and} \quad |\, n - N\Psi_1(\theta) \,| \leq \gamma N^{2/3}$$

for some fixed constant γ.

The order of magnitude of the random fluctuations in the model are described by two exponents ζ and χ, informally defined as follows. In a system of size N,

- fluctuations of the path $x_\cdot$ are of order N^ζ, and
- fluctuations of the partition function $\log Z_{m,n}$ are of order N^χ.

The conjectured values in 1+1 dimensions are $\zeta = 2/3$ and $\chi = 1/3$. Under the assumptions listed above we have results that verify these conjectured values for the exponents.

Theorem. There exist constants $0 < C_1, C_2 < \infty$ such that, for $N \geq 1$,

$$C_1 N^{2/3} \leq \mathbb{V}\text{ar}(\log Z_{m,n}) \leq C_2 N^{2/3}.$$

Let $v_0(j)$ and $v_1(j)$ denote the left- and rightmost points of the path on the horizontal line with ordinate j:

$$v_0(j) = \min\{i \in \{0,\ldots,m\} : \exists k \text{ such that } x_k = (i,j)\}$$

and

$$v_1(j) = \max\{i \in \{0,\dots,m\} : \exists k \text{ such that } x_k = (i,j)\}.$$

Theorem. Let $0 < \tau < 1$. Then there exist constants $C_1, C_2 < \infty$ such that for $N \geq 1$ and $b \geq C_1$,

$$P\big\{v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \quad \text{or} \quad v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3}\big\} \leq C_2 b^{-3}.$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\overline{\lim_{N\to\infty}} P\{\ \exists k \text{ such that } |\, x_k - (\tau m, \tau n)| \leq \delta N^{2/3}\ \} \leq \varepsilon.$$

Further details can be found in [1].

References

[1] Timo Seppäläinen. Scaling for a one-dimensional directed polymer with boundary conditions. *To appear in Ann. Probab.*, arXiv:0911.2446, 2009.

Lognormal ⋆-scale invariant measures

Vincent Vargas

(joint work with R. Allez and R. Rhodes)

We consider random measures M in $\mathbb{R}$ that satisfy the following conditions:

- **Stationarity**: for all y, $M(dx) \overset{(Law)}{=} M(dx+y)$.
- **Lognormal ⋆-scale invariance**: for all $\epsilon < 1$, there exists some Gaussian process $(\omega_\epsilon(x))_{x\in\mathbb{R}}$ such that:

$$M(dx) \overset{(Law)}{=} e^{\omega_\epsilon(x)} \epsilon M(\frac{dx}{\epsilon}). \tag{1}$$

- **Moment condition**: there exists $\delta > 0$ such that $E[M[0,1]^{1+\delta}] < \infty$.

Remark 1. *With no assumption on the law of ω_ϵ, one can show the following: if there exists x such that $\epsilon \to \omega_\epsilon(x)$ is continuous, then relation (1) implies that $(\omega_\epsilon(x))_{x\in\mathbb{R}}$ is a Gaussian process.*

In order to state our main theorem, we need the theory of Gaussian multiplicative chaos developped by J.P. Kahane: we refer to [1] for an introduction to this theory. We can know state the main theorem of the talk:

Theorem 2. *[Allez, Rhodes, Vargas] Let M be a random measure satisfying the above conditions. Then M is the product of a random variable $Y \in L^{1+\delta}$ and an independent Gaussian multiplicative chaos:*

$$\forall A \in \mathcal{B}(\mathbb{R}),\ M(A) = Y \int_A e^{\omega(x) - \frac{E[\omega(x)^2]}{2}} dx$$

with associated kernel given by:

$$K(x,y) = \int_{|y-x|}^{\infty} \frac{k(u)}{u} du,$$

where k is some continuous covariance function such that $k(0) \leq \frac{2}{1+\delta}$.

Remark 3. *The above result can be generalized to all dimensions.*

References

[1] Robert R., Vargas V.: Gaussian Multiplicative Chaos revisited, to appear in the Annals of Probability, available on arxiv at the URL http://arxiv.org/abs/0807.1036v1.

Random matrices and random Schrödinger operators

Bálint Virág

(joint work with E. Kritchevski and B. Valkó)

We consider the one-dimensional discrete random Schrödinger operators

$$(H_n\psi)_\ell = \psi_{\ell-1} + \psi_{\ell+1} + v_\ell\psi_\ell,$$

$\psi_0 = \psi_{n+1} = 0$ where $v_k = \sigma\omega_k/\sqrt{n}$. and the ω_k are independent random variables with mean 0, variance 1 and bounded third absolute moment.

The matrix H_n is a perturbation of the adjacency matrix of a path. When the variance of v_k does not depend on n, eigenvectors are localized and the local statistics of eigenvalues are Poisson (see [1, 3], from which this abstract was distilled, for detailed references). Our regime, where the variance of the random variables v_ℓ are of order $n^{-1/2}$ captures the transition between localization an delocalization.

If there is no noise (i.e. $\sigma = 0$) then the eigenvalues of the operator are given by $2\cos(\pi k/(n+1))$ with $k = 1, \dots, n$. The asymptotic density near $E \in (-2, 2)$ is given by $\frac{\rho}{2\pi}$ with $\rho = \rho(E) = 1/\sqrt{1 - E^2/4}$. We will study the spectrum Λ_n of the scaled operator $\rho n(H_n - E)$. By the well-known transfer matrix description the eigenvalue equation $H_n\psi = \mu\psi$ is written as

$$\begin{pmatrix}\psi_{\ell+1}\\ \psi_\ell\end{pmatrix} = T(\mu - v_\ell)\begin{pmatrix}\psi_\ell\\ \psi_{\ell-1}\end{pmatrix} = M_\ell^\lambda\begin{pmatrix}\psi_1\\ \psi_0\end{pmatrix}, \tag{1}$$

where $T(x) := \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$ and with $\mu = E + \frac{\lambda}{\rho n}$ and $\varepsilon_\ell = \frac{\lambda}{\rho n} - \frac{\sigma\omega_\ell}{\sqrt{n}}$, we have

$$M_\ell^\lambda = T(E + \varepsilon_\ell)T(E + \varepsilon_{\ell-1}) \cdots T(E + \varepsilon_1) \text{ for } 0 \leq \ell \leq n. \tag{2}$$

Then μ is an eigenvalue of H_n if and only if $M_n\binom{1}{0} = c\binom{0}{1}$. The scaling of $v_\ell = \sigma\omega_\ell/\sqrt{n}$ ensures that, with high probability, the transfer matrices M_ℓ^λ are bounded and the eigenfunctions are delocalized.

The starting observation is that M_ℓ^λ cannot have a continuous limit, since for large n the transfer matrix $T(E + \varepsilon_k)$ in (2) is not close to I but to $T(E)$. Thus we are led to consider, instead of M_ℓ^λ, the regularly-evolving matrices

$$X_\ell^\lambda = T^{-\ell}(E)M_\ell^\lambda, \qquad 0 \leq \ell \leq n. \tag{3}$$

To control the correction factor $T^{-\ell}(E)$, we diagonalize $T(E) = ZDZ^{-1}$ with

$$D = \begin{pmatrix} \overline{z} & 0 \\ 0 & z \end{pmatrix}, \quad Z = \begin{pmatrix} \overline{z} & z \\ 1 & 1 \end{pmatrix}, \quad z = E/2 + i\sqrt{1 - (E/2)^2}. \tag{4}$$

Theorem 1. *Assume $0 < |E| < 2$. Let $\mathcal{B}(t), \mathcal{B}_2(t), \mathcal{B}_3(t)$ be independent standard Brownian motions in $\mathbb{R}$, $\mathcal{W}(t) = \frac{1}{\sqrt{2}}(\mathcal{B}_2(t) + i\mathcal{B}_3(t))$. Then the stochastic differential equation*

$$dX^\lambda = \frac{1}{2}Z(\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix} dt + \begin{pmatrix} id\mathcal{B} & d\mathcal{W} \\ d\overline{\mathcal{W}} & -id\mathcal{B} \end{pmatrix})Z^{-1}X^\lambda, \qquad X^\lambda(0) = I \tag{5}$$

has a unique strong solution $X^\lambda(t) : \lambda \in \mathcal{C}, t \geq 0$, which is analytic in λ. Moreover with $\tau = (\sigma\rho)^2$

$$(X^\lambda_{\lfloor nt/\tau \rfloor}, 0 \leq t \leq \tau) \Rightarrow (X^\lambda(t), 0 \leq t \leq \tau),$$

in the sense of finite dimensional distributions for λ and uniformly in t. Also, for any given $0 \leq t \leq \tau$ the random analytic functions $X^\lambda_{\lfloor nt/\tau \rfloor}$ converge in distribution to $X^\lambda(t)$ with respect to the local uniform topology.

Moreover the shifted eigenvalue process $\Lambda_n - \arg(z^{2n+2})$ converges in distribution to a point process Sch_τ.

The point process Sch_τ is only invariant under translation by integer multiples of 2π. A translation-invariant version (shifted by an independent uniform random variable) $\mathsf{Sch}^*_\tau = \mathsf{Sch}_\tau + U[0, 2\pi]$ can be described through a variant of the the Brownian carousel introduced in [2].

The Brownian carousel. Let $(\mathcal{V}(t), t \geq 0)$ be Brownian motion on the hyperbolic plane $\mathbb{H}$. Pick a point on the boundary $\partial\mathbb{H}$ and let $x^\lambda(0)$ equal to this point for all $\lambda \in \mathbb{R}$. Let $x^\lambda(t)$ be the trajectory of this point rotated continuously around $\mathcal{V}(t)$ at speed λ. Recall that Brownian motion in $\mathbb{H}$ converges to a point $\mathcal{V}(\infty)$ in the boundary $\partial\mathbb{H}$. Then we have $\mathsf{Sch}^*_\tau \stackrel{d}{=} \{\lambda : x^{\lambda/\tau}(\tau) = \mathcal{V}(\infty)\}$.

The following properties of Sch_τ help compare it to random matrices.

Theorem 2 (Eigenvalue repulsion). *For $\mu \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$\{\mathsf{Sch}_\tau[\mu, \mu + \varepsilon] \geq 2\} \leq 4\exp\left(-(\log(\tau/\varepsilon) - \tau)^2/\tau\right). \tag{6}$$

whenever the squared expression is nonnegative.

Theorem 3 (Probability of large gaps). *The probability that Sch_τ has a large gap is*

$$\mathbb{P}(\mathsf{Sch}_\tau[0, \lambda] = 0) = \exp\left\{-\frac{\lambda^2}{4\tau}(1 + o(1))\right\}$$

where $o(1) \to 0$ for a fixed τ as $\lambda \to \infty$.

The above results show that the eigenvalue statistics of 1D random Schrödinger operators are *not* universal. However, GOE statistics appear for very thin boxes in $\mathbb{Z}^2$. The proof first establishes a fixed higher dimensional version of Theorem 1 and then uses recent results in universality of Wigner matrices.

Theorem 4. [3] *There exists a sequence of weighted boxes on $\mathbb{Z}^2$ with diameter converging to ∞ so that the rescaled eigenvalue process of the adjacency matrix plus diagonal noise converges to the bulk point process limit of the GOE ensemble.*

References

[1] E. Kritchevski, B. Valkó, and B. Virág. The scaling limit of the critical one-dimensional random Schrödinger operator. In preparation.
[2] B. Valkó and B. Virág (2009). Continuum limits of random matrices and the Brownian carousel. *Inventiones Math.*, 177:463–508.
[3] B. Valkó and B. Virág (2010). Random Schrödinger operators on long boxes, noise explosion and the GOE. *arXiv:0912.0097*

Recent Results on the Asymmetric Simple Exclusion Process

Harold Widom
(joint work with C. A. Tracy)

In the usual asymmetric simple exclusion process (ASEP) particles are at integer sites on the line. Each particle waits exponential time, then with probability p it moves one step to the right if the site is unoccupied, otherwise it stays put; and with probability $q = 1 - p$ it moves one step to the left if the site is unoccupied, otherwise it stays put.

In *multispecies ASEP* particles belong to different species, labelled $1, 2, \ldots, n$. Particles of a higher species have priority over those of a lower species. Thus, if a particle of species α tries to move to a neighboring site occupied by a particles of species β it is blocked if $\alpha \le \beta$, but if $\alpha > \beta$ the particles interchange positions.

A configuration in ASEP with N particles is the set of occupied sites

$$X = \{x_1, \ldots, x_N\}, \quad (x_1 < \cdots < x_N).$$

In earlier work the authors found a formula for $P_Y(X;\, t)$, the probability that the system is in configuration X at time t, given the initial configuration was $Y = \{y_1, \ldots, y_N\}$. It is given as a sum over the permutation group $\mathbb{S}_N$ of N-dimensional integrals with explicitly given integrands.

In multispecies ASEP a configuration $\mathcal{X}$ is a pair $(X,\, \pi)$ where $X = \{x_1, \ldots, x_N\}$ as before and π is a function from $[1,\, N]$ to $[1,\, n]$. If the system is in configuration $\mathcal{X}$ then the ith particle from the left is at x_i and belongs to species π_i. A special case is that of first and second class particles, a first class particle having priority over a second class particle. For example, if $\pi = (1\ 2\ 2\ 2)$ the left-most particle is second class and the other three are first class.

The new result establishes for multispecies ASEP a formula for $P_{\mathcal{Y}}(\mathcal{X};\, t)$, the probability that the system is in configuration $\mathcal{X} = (X,\, \pi)$ at time t, given that the initial configuration is $\mathcal{Y} = (Y,\, \nu)$. There is a formula analogous to the one for ordinary ASEP mentioned above. The main difference is that the integrands are not (except in special cases) given explicitly. They are determined by a family of relations whose consistency is verified by establishing the *Yang-Baxter equations* for these relations.

Universality of β-ensemble

Horng-Tzer Yau

We prove the universality of the β-ensembles with convex analytic potentials and for any $\beta > 0$, i.e. we show that the spacing distributions of log-gases at any inverse temperature β coincide with those of the Gaussian β-ensembles.

Reporter: Igor Kortchemski

Participants

Prof. Dr. Michael Aizenman
Department of Mathematics
Princeton University
P.O. Box 708
Jadwin Hall
Princeton , NJ 08544-0708
USA

Prof. Dr. Gidi Amir
Department of Mathematics
Bar-Ilan University
52900 Ramat Gan
ISRAEL

Dr. Omer Angel
Department of Mathematics
University of British Columbia
1984 Mathematics Road
Vancouver B.C. V6T 1Z2
CANADA

Dr. Vincent Beffara
Mathematiques
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

Prof. Dr. Gerard Ben Arous
Courant Institute of Math. Sciences
New York University
251, Mercer Street
New York NY 10012-1185
USA

Prof. Dr. Erwin Bolthausen
Institut für Mathematik
Universität Zürich
Winterthurerstr. 190
CH-8057 Zürich

Dr. Charles Bordenave
Institut de Mathematiques de Toulouse
Universite Paul Sabatier
118, route de Narbonne
F-31062 Toulouse Cedex 9

Dr. Mireille Bousquet-Melou
LABRI
Universite Bordeaux 1
351, cours de la Liberation
F-33405 Talence Cedex

Dr. Cedric Boutillier
Laboratoire de Probabilites et Modeles Aleatoires
Universite Pierre et Marie Curie
4, place Jussieu
F-75005 Paris

Prof. Dr. Anton Bovier
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Sourav Chatterjee
New York University
Courant Institute of Mathematical Sciences
251 Mercer Street
New York , NY 10012
USA

Ivan Corwin
Courant Institute of
Mathematical Sciences
New York University
251, Mercer Street
New York , NY 10012-1110
USA

Nicolas Curien
Departement de Mathematiques et
Applications
Ecole Normale Superieure
45, rue d'Ulm
F-75005 Paris Cedex

Prof. Dr. Amir Dembo
Department of Mathematics
Stanford University
Stanford , CA 94305
USA

Prof. Dr. Jean Dominique Deuschel
Institut für Mathematik
Sekr. MA 7-4
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin

Dr. Julien Dubedat
Department of Mathematics
Columbia University
2990 Broadway
New York , NY 10027
USA

Hugo Duminil-Copin
Section de Mathematiques
Universite de Geneve
Case postale 240
CH-1211 Geneve 24

Prof. Dr. Bertrand Duplantier
Institut de Physique Theorique
CEA Saclay
F-91191 Gif-sur-Yvette Cedex

Prof. Dr. Laszlo Erdös
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Bertrand Eynard
Institut de Physique Theorique
CEA Saclay
F-91191 Gif-sur-Yvette Cedex

Prof. Dr. Patrik Ferrari
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Nina Gantert
Zentrum Mathematik
TU München
Boltzmannstr. 3
85748 Garching b. München

Christophe Garban
ENS - UMPA
46, allee d'Italie
F-69007 Lyon

Prof. Dr. Geoffrey R. Grimmett
Statistical Laboratory
Centre for Mathematical Sciences
Wilberforce Road
GB-Cambridge CB3 OWB

Prof. Dr. Alice Guionnet
Dept. de Mathematiques, U.M.P.A.
Ecole Normale Superieure de Lyon
46, Allee d'Italie
F-69364 Lyon Cedex 07

Dr. Alan Michael Hammond
Department of Statistics
University of Oxford
1 South Parks Road
GB-Oxford OX1 3TG

Prof. Dr. Alexander E. Holroyd
Microsoft
1 Microsoft Way
Redmond , WA 98052
USA

Prof. Dr. Dmitri Ioffe
Faculty of Industrial Engineering &
Management
Technion
Israel Institute of Technology
Haifa 32000
ISRAEL

Dr. Antti Knowles
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
USA

Igor Kortchemski
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Takashi Kumagai
Research Institute of Math. Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Hubert Lacoin
Laboratoire de Probabilites et
Modeles Aleatoires
Universite Paris VII
175 rue du Chevaleret
F-75013 Paris Cedex

Prof. Dr. Michel Ledoux
Institut de Mathematiques de Toulouse
Universite de Toulouse
F-31062 Toulouse

Prof. Dr. Jean-Francois Le Gall
Departement de Mathematiques
Universite de Paris-Sud
Bat. 425
F-91405 Orsay Cedex

Dr. Gregory Miermont
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Jason Miller
Microsoft Research
One Microsoft Way
Redmond , WA 98052-6399
USA

Prof. Dr. Gabor Pete
Department of Mathematics
University of Toronto
40 St George Street
Toronto , Ont. M5S 2E4
CANADA

Prof. Dr. Jeremy Quastel
Department of Mathematics
University of Toronto
40 St George Street
Toronto , Ont. M5S 2E4
CANADA

Prof. Dr. Tomohiro Sasamoto
Department of Mathematics and
Informatics, Faculty of Science
Chiba University
1-33 Yayoi-cho, Inage
Chiba-Shi 263-8522
JAPAN

Prof. Dr. Timo Seppäläinen
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison , WI 53706-1388
USA

Prof. Dr. Scott Sheffield
Department of Mathematics
MIT
Cambridge , MA 02139
USA

Prof. Dr. Vladas Sidoravicius
Instituto Nacional de Matematica
Pura e Aplicada; IMPA
Estrada Dona Castorina 110
Rio de Janeiro, RJ - CEP: 22460-320
BRAZIL

Prof. Dr. Alain-Sol Sznitman
Departement Mathematik
ETH-Zentrum
Rämistr. 101
CH-8092 Zürich

Dr. Beatrice de-Tiliere
Laboratoire de Probabilites et Modeles
Aleatoires
Universite Pierre et Marie Curie
4, place Jussieu
F-75005 Paris

Prof. Dr. Balint Toth
Institute of Mathematics
Technical University of Budapest
Egry Jozsef u. 1
H-1111 Budapest

Prof. Dr. Craig A. Tracy
Department of Mathematics
University of California, Davis
1, Shields Avenue
Davis , CA 95616-8633
USA

Dr. Benedek Valko
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison , WI 53706-1388
USA

Dr. Vincent Vargas
CEREMADE
Universite Paris Dauphine
Place du Marechal de Lattre de
Tassigny
F-75775 Paris Cedex 16

Prof. Dr. Balint Virag
Department of Mathematics
University of Toronto
40 St George Street
Toronto , Ont. M5S 2E4
CANADA

Prof. Dr. Simone Warzel
Fakultät für Mathematik
Technische Universität München
Boltzmannstr. 3
85748 Garching bei München

Prof. Dr. Wendelin Werner
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Harold Widom
Department of Mathematics
University of California
Santa Cruz , CA 95064
USA

Hao Wu
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Prof. Dr. Horng-Tzer Yau
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
USA

Prof. Dr. Ofer Zeitouni
Department of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis , MN 55455
USA