Mathematisches Forschungsinstitut Oberwolfach

Report No. 30/2011

DOI: 10.4171/OWR/2011/30

Arithmetic Groups vs. Mapping Class Groups: Similarities, Analogies and Differences

Organised by
Benson Farb (Chicago)
Lizhen Ji (Ann Arbor)
Enrico Leuzinger (Karlsruhe)
Werner Müller (Bonn)

June 5th – June 11th, 2011

Abstract. Arithmetic groups arise naturally in many fields such as number theory, algebraic geometry, and analysis. Mapping class groups arise in both low dimensional topology and geometric group theory. They have been studied intensively by different groups of people. The purpose of this workshop is to bring experts and aspiring young mathematicians together to interact and develop further exchanges and new collaboration.

Mathematics Subject Classification (2000): 11F06, 30F60.

Introduction by the Organisers

Arithmetic groups such as $SL(n,\mathbb{Z})$ occur naturally in many subjects in mathematics such as number theory, representation theory, differential geometry, algebraic geometry and topology. They have been studied intensively. A closely related class of groups consists of mapping class groups of surfaces and has played a fundamental role in the low dimensional topology, algebraic geometry and mathematical physics. Many results about mapping class groups are inspired and motivated by corresponding results on arithmetic groups. For example, it is a known theorem of Borel and Serre that the virtual cohomological dimension of arithmetic groups can be explicitly computed by proving that they are duality groups (i.e., generalized Poincare duality group) in the sense of Bieri and Eckmann. Similar results were then proved by Harer for the mapping class groups.

In proving the above result, the natural action of the arithmetic groups on associated symmetric spaces and the Borel-Serre compactification of the locally symmetric spaces are used crucially. Similarly, the mapping class groups act on the associated Teichmuller spaces, and this action was also used in proving the above result.

The quotient of the Teichmuller space by the mapping class group is the moduli space of Riemann surfaces. In order to construct an analogue of the Borel-Serre compactification of the moduli space, Harvey introduced the notion of curve complex of a surface, which is an analogue of the spherical Tits building of algebraic group and has since played a fundamental role in the recent study of low dimensional topology and mapping class groups.

There are many other analogous results for arithmetic groups and mapping class groups. Furthermore, there are also fruitful interactions between them. For example, there is a Jacobian map from the moduli space of compact Riemann surfaces of genus $g \geq 2$ to the Siegel modular variety of degree g, obtained by associating to its Riemann surface its Jacobian. This Jacobian map has been intensively studied in algebraic geometry. For example, the celebrated Schottky problem is to characterize the image of the Jacobian map. This Jacobian map was used to first prove that the moduli space of Riemann surfaces is a quasi-projective variety. It also allows one to relate topological properties of the above two important classes of groups.

Closely related to the above two classes of groups is the class of outer automorphism groups of free groups. Together they form the most important three classes of groups in geometric group theory.

Some important recent results:

Many important results related to the topics of the proposed workshop have been obtained. We list some of them for a glimpse of the recent status:

(1) The positivity of Gromov norm for irreducible, closed, locally symmetric manifolds with no local $\mathbf{H}^2$ factors, by Lafont and Schmidt in *Simplicial volume of closed locally symmetric spaces of non-compact type*, Acta Math. 197 (2006), no. 1, 129–143.
(2) The proof of the Morita-Mumford-Miller Conjecture on the stable cohomology of the mapping class group by Madsen and Weiss in *The stable moduli space of Riemann surfaces: Mumford's conjecture*, Ann. of Math. (2) 165 (2007), no. 3, 843–941; and its $Out(F_n)$ analogue by Galatius.
(3) The computation of the abstract commensurator of $Out(F_n)$ by Farb and Handel in *Commensurations of* $\mathrm{Out}(\mathrm{F}_n)$, Publ. Math. Inst. Hautes Études Sci. No. 105 (2007), 1–48. This result is analogous to the arithmetic group case by Mostow and Borel and the mapping class group case by Ivanov.

Purpose of the workshop:

The purpose of this workshop is to bring together experts from the following different areas in order to learn from each other and to encourage further interactions between them:

(1) locally symmetric spaces and discrete subgroups of Lie groups, in particular arithmetic groups,
(2) Teichmuller spaces, moduli spaces of Riemann surfaces, and mapping class groups,
(3) Outer automorphism groups of free groups and other closely related groups.
(4) Geometric group theory.

This workshop is the first workshop of such a nature and is well-attended by over 50 people, consisting of both leading experts and aspiring young mathematicians. Most talks are of very high quality and the speakers have tried to make their talks accessible. This is particularly important in view of diversity of participants. All the above topics have been covered. The atmosphere has been very active throughout the workshop, and there have been a lot of interaction and discussion after the talks. Some joint projects between participants have started due to this workshop.

We believe that such a workshop has achieved its goal and will have a lasting impact for various reasons:

(1) Each of the subject has been intensively studied by different groups of people. Many exciting results have been obtained in all these subjects, and it is difficult for any single person to grasp them all.
(2) There have been many analogues and similar results for different classes of groups such as arithmetic groups, mapping class groups and outer automorphism groups. It will be valuable to understand better the underlying unity among them and hence motivate further interactions between them.
(3) In spite of many deep results already obtained, some aspects on interaction between the different groups and spaces described above have not been pursued sufficiently. For example, locally symmetric spaces are special and important examples of complete Riemannian manifolds, and their spectral theory has played a fundamental role in the celebrated Langlands program. On the other hand, the modulis space of curves have not been understood well as Riemannian manifolds, in particular its spectral theory. So far one does not know a natural and complete metric on the outer space yet such that the outer automorphism group $Out(F_n)$ acts isometrically and properly.

Workshop: Arithmetic Groups vs. Mapping Class Groups: Similarities, Analogies and Differences

Table of Contents

Abstracts

Polynomial Pick differentials differentials and affine spheres over polygons

Michael Wolf
(joint work with David Dumas)

We report on joint work (in progress) with David Dumas [DW].

Let S be a closed orientable surface of genus $g \leq 2$. We are interested in studying aspects of the space $Conv(S)$ of convex real projective structures on S, a particular component of $Rep(\pi_1(S), \mathrm{SL}(3, \mathbb{R})$. There is already much known about this space, beginning from Goldman's [Gol90] description of this space as being a topological ball of dimension $16g - 16$. See also Choi-Goldman [CG93], and Kim.

We will be interested in the point of view developed by Labourie [Lab07] and Loftin [Lof01], [Lof04], [Lof07]. These authors show that the space $Conv(S)$ may be parametrized by the bundle $\mathcal{C}$ of cubic differentials over Teichmüller space, where, for a given convex projective structure D, the cubic differential serves as the Pick differential for the unique affine sphere of mean curvature one whose convex hull is D. A good survey of this point of view is available in [Lof10].

Fix a Riemann surface X and consider a family C_t of cubic differentials which leave compacta in $\mathcal{C}$ (for example a ray tC_0, where C_0 is a particular cubic differential on X). In the metric $|C_t|^{\frac{2}{3}}$, the geometric limit of a subsequence of C_t will be a polynomial cubic differential on the place $\mathbb{C}$.

The point of this talk is to investigate the affine spheres associated to polynomial cubic differentials $P(z)dz^3$ on $\mathbb{C}$. We prove that these differentials are precisely the Pick differentials of affine spheres over convex polygons. The proof of that relies on an estimate we hope will be important in more careful studies of $Conv(S)$. We next state these results more precisely.

We prove

Theorem 1. *Let $\mathcal{S}$ be the affine sphere determined by a polynomial cubic differential Cdz^3 on the complex plane $\mathbb{C}$. Then the convex hull of $\bar{\mathcal{S}}$ is a cone over a k-gon with $k = \deg C + 3$ edges.*

We state a converse in terms of a map between a space of polynomial cubic differentials and a space of polygons. Let $\mathcal{C}_n$ denote the space of polynomial cubic differentials in the plane of degree n up to holomorphic automorphisms. Of course, we can specify $n+1$ complex coefficients, but the complex dimension of the automorphism group of $\mathbb{C}$ is two, so $\mathcal{C}_n$ has real dimension $2n-2$. Let $\P_{n+3}$ denote the space of polygons in $\mathbb{RP}^2$ with $n+3$ vertices, up to projective automorphism. As $\mathbb{RP}^2$ has two real dimensions, but the real dimension of $\mathrm{PGL}(3, \mathbb{R})$ is eight, we see that $\P_{n+3}$ also has dimension $2(n+3) - 8 = 2n - 2$.

We can interpret Theorem 1 as defining a map $\varphi_n : \mathcal{C}_n \to \P_{n+3}$ from the space $\mathcal{C}_n$ of degree n Pick differentials to the space $\P_{n+3}$ of projective polygons: given a cubic polynomial differential $C = P(z)dz^3$ in the plane, (we prove) there is a unique affine sphere with C as its Pick differential. Theorem 1 then asserts that the convex hull of the projection of this affine sphere is bounded by a polygon with $n+3$ vertices.

Theorem 2. *For each $n \geq 0$, the map $\varphi_n : \mathcal{C}_n \to \P_{n+3}$ is a surjective homeomorphism.*

The proofs depend on an estimate of independent interest. A particularly interesting affine sphere is due to Tzitzeica in the early part of the twentieth century, and is defined as the locus $\{xyz = 1\}$ in $\mathbb{R}^3$. It is defined over the plane $\mathbb{C}$, has Pick differential $-\frac{1}{4}dz^3$, and obviously has convex hull of a cone over a triangle.

Our central estimate states that as one leaves the zero set of a Pick differential, the affine sphere becomes rapidly nearly isometric to the Tzitzeica example: the rate of decay of the quasi-isometry is exponential (with uniform exponent) in the $|C|^{\frac{2}{3}}$ distance from the zeroes of C.

References

[CG93] Suhyoung Choi and William M. Goldman. Convex real projective structures on closed surfaces are closed. *Proc. Amer. Math. Soc.*, 118(2):657–661, 1993.

[DW] David Dumas and Michael Wolf. In preparation.

[Gol90] William M. Goldman. Convex real projective structures on compact surfaces. *J. Differential Geom.*, 31:791–845, 1990.

[Lab07] François Labourie. Flat projective structures on surfaces and cubic holomorphic differentials. *Pure Appl. Math. Q.*, 3(4, part 1):1057–1099, 2007.

[Lof01] John Loftin. Affine spheres and convex rp^n structures. *Amer. J. Math*, 123:255–275, 2001.

[Lof04] John Loftin. The compactification of the moduli space of convex $\mathbb{RP}^2$ surfaces. I. *J. Differential Geom.*, 68(2):223–276, 2004.

[Lof07] John Loftin. Flat metrics, cubic differentials and limits of projective holonomies. *Geom. Dedicata*, 128:97–106, 2007.

[Lof10] John Loftin. Survey on affine spheres. In *Handbook of geometric analysis, No. 2*, volume 13 of *Adv. Lect. Math. (ALM)*, pages 161–191. Int. Press, Somerville, MA, 2010.

Abelian quotients of subgroups of the mapping class group and higher Prym representations

Andrew Putman

(joint work with Ben Wieland)

A well-known conjecture asserts that the mapping class group of a surface (possibly with punctures/boundary) does not virtually surject onto $\mathbb{Z}$ if the genus of the surface is large. We prove that if this conjecture holds for some genus, then it also holds for all larger genera. We also prove that if there is a counterexample to this conjecture, then there must be a counterexample of a particularly simple form. We prove these results by relating the conjecture to a family of linear representations of the mapping class group that we call the higher Prym representations. They generalize the classical symplectic representation.

The Steinberg module of the mapping class group

Nathan Broaddus

Harer has shown that the mapping class group is a virtual duality group mirroring the work of Borel-Serre on arithmetic groups in semisimple $\mathbb{Q}$-groups. Just as the homology of the rational Tits building provides the dualizing module for any torsion free arithmetic group, the homology of the curve complex is the dualizing module for any torsion free, finite index subgroup of the mapping class group. The homology of the curve complex was previously known to be an infinitely generated free abelian group, but to date, its structure as a mapping class group module has gone unexplored. Here we summarize results from [Broa07] on the homology of the curve complex as a mapping class group module.

1. Introduction

Let Σ_g^1 be the surface of genus g with one marked point. The mapping class group $\mathrm{Mod}(\Sigma_g^1)$ is a virtual duality group [Hare86] (see §2.1) . Moreover, the dualizing module for any torsion-free subgroup of $\mathrm{Mod}(\Sigma_g^1)$ is the reduced homology of the curve complex which we call the Steinberg module $\mathrm{St}(\Sigma_g^1)$ (see Definition 2 below). In [Broa07] a (virtually) free $\mathrm{Mod}(\Sigma_g^1)$-module resolution for the reduced homology of the curve complex is given.

Theorem 6 (Broaddus [Broa07]) *The Steinberg module of the mapping class group* $\mathrm{St}(\Sigma_g^1)$ *has a finite* $\mathrm{Mod}(\Sigma_g^1)$*-module resolution*

$$0 \to \mathcal{F}_{4g-3} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathrm{St}(\Sigma_g^1).$$

If one restricts coefficients to $\mathbb{Z}[\Gamma]$ *for a torsion free finite index subgroup* $\Gamma < \mathrm{Mod}(\Sigma_g^1)$ *then the above resolution is a free* Γ*-module resolution.*

Using the first two terms of this resolution we derive a $\mathrm{Mod}(\Sigma_g^1)$-module presentation for the reduced homology of the curve complex.

Theorem 5 (Broaddus [Broa07]) *The Steinberg module of the mapping class group* $\mathrm{St}(\Sigma_g^1)$ *is finitely presented with a generator for each* $\mathrm{Mod}(\Sigma_g^1)$*-orbit of filling arc system with* $2g$ *arcs and a relation for each* $\mathrm{Mod}(\Sigma_g^1)$*-orbit of filling arc system with* $2g+1$ *arcs. (See* [Broa07, Proposition 3.5] *for the precise generators and relations.)*

We then use this presentation to show that as a $\mathrm{Mod}(\Sigma_g^1)$-module the reduced homology of the curve complex is generated by a single element.

Theorem 7 (Broaddus [Broa07]) *The Steinberg module of the mapping class group* $\mathrm{St}(\Sigma_g^1)$ *is generated by a single element as a* $\mathrm{Mod}(\Sigma_g^1)$*-module. (See* [Broa07, Theorem 4.2] *for a description of the particular generator.)*

2. Background on the Steinberg module

Harvey defined the complex of curves [Harv81, §2] to serve as an analog for the mapping class group of the Tits building for semisimple $\mathbb{Q}$-groups. This analogy has proved to be stunningly successful. Some of the early fruits of this comparison concerned the homology and cohomology of the mapping class group.

2.1. **Virtual duality groups.** Poincaré Duality relates the homology and cohomology of a group Γ when Γ has a $K(\Gamma, 1)$ which is a compact manifold. More generally a (necessarily torsion-free) group Γ is called a Bieri-Eckmann Duality Group (*cf.* [BieEc73], [Bie76], [Brow82]) or simply a *duality group* if there is a Γ-module Ω and a number d such that for any Γ-module A

$$\mathrm{H}^k(\Gamma; A) \cong \mathrm{H}_{d-k}(\Gamma; \Omega \otimes_{\mathbb{Z}} A)$$

holds for all k where Γ acts on $\Omega \otimes_{\mathbb{Z}} A$ via the diagonal action: $\gamma \cdot (\omega \otimes a) = (\gamma\omega \otimes \gamma a)$. If Γ is a duality group then

$$\widetilde{\mathrm{H}}^k(\Gamma; \mathbb{Z}\Gamma) = \begin{cases} 0, & k \neq d \\ \Omega, & k = d \end{cases}$$

so both d (the *cohomological dimension*) and Ω (the *dualizing module*) are intrinsic to the group Γ. In fact, if Γ is a duality group and the torsion-free group Γ' is commensurable with Γ (*i.e.* they share a subgroup of finite index) then Γ' is a duality group with the same cohomological dimension [Se71] and dualizing module (restricting coefficients to the group ring of their intersection) [BieEc73, §3]. Thus we say that W is a *virtual duality group* if some (and hence every) torsion-free, finite-index subgroup $\Gamma < W$ is a duality group. Moreover, taking care with coefficients, one may view the dualizing module of Γ as an invariant of W.

Bieri-Eckmann [BieEc73, §6.3] provide a useful technique for showing that a group is a duality group. If Γ has a $K(\Gamma, 1)$ which is a compact m-manifold M with boundary and the pullback of the boundary $\partial\widetilde{M} = p^{-1}(\partial M)$ under the covering map $p : \widetilde{M} \to M$ to the universal cover $\widetilde{M}$ has the homotopy type of a wedge of n-spheres then Γ is a duality group with dualizing module

$$\Omega \cong \widetilde{\mathrm{H}}_n(\partial\widetilde{M}; \mathbb{Z})$$

and cohomological dimension $d = m - n + 1$.

2.2. **Buildings and duality for** $\mathrm{SL}(n, \mathbb{Z})$**.** Results concerning the virtual duality for $\mathrm{SL}(n, \mathbb{Z})$ serve as the inspiration for similar results and questions about the mapping class group. Borel-Serre [BoSe73] establish that $\mathrm{SL}(n, \mathbb{Z})$ is a virtual duality group by *bordifying* symmetric space

$$X = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n, \mathbb{R}).$$

That is, they attach a certain boundary set to symmetric space X to get bordified symmetric space $\widehat{X}$. They then show that $\widehat{X}$ is still contractible and that $\partial\widehat{X} = \widehat{X} - X$ has the homotopy type of a wedge of $(n-2)$-spheres. In fact they show that $\partial\widehat{X}$ is homotopic to the well-known Bruhat-Tits building Δ for $\mathrm{SL}(n,\mathbb{Q})$ (see [Ron92] or [Brow89]).

Definition 1 (Bruhat-Tits building and apartments)**.** Let K be a field. The *Bruhat-Tits building* $\Delta = \Delta(n, K)$ for $\mathrm{SL}(n, K)$ is the flag complex with a vertex for each proper nontrivial subspace of K^n and an m-simplex for each sequence $V_0 \subset V_1 \subset \cdots \subset V_m$ of proper nontrivial subspaces of K^n. For each basis $B = \{v_1, \cdots, v_n\}$ of K^n we get an *apartment* A_B of Δ by considering the subcomplex of Δ of all simplices whose vertices are all subspaces of K^n spanned by a nonempty proper subset of B. $\|$

Hence by [BieEc73, §6.3] the virtual dualizing module for $\mathrm{SL}(n,\mathbb{Z})$ is $\widetilde{\mathrm{H}}_{n-2}(\Delta;\mathbb{Z})$ which Borel-Serre call the Steinberg module[1].

Definition 2 (Steinberg module for $\mathrm{SL}(n,\mathbb{Z})$)**.** The *Steinberg module* [BoSe73, pg. 437] for $\mathrm{SL}(n,\mathbb{Z})$ is the $\mathrm{SL}(n,\mathbb{Z})$-module (in fact $\mathrm{SL}(n,\mathbb{Q})$-module)

$$\mathrm{St}(n) = \widetilde{\mathrm{H}}_{n-2}(\Delta(n,\mathbb{Q});\mathbb{Z}). \quad \|$$

The Steinberg module $\mathrm{St}(n)$ is the dualizing module of every torsion free finite index subgroup of $\mathrm{SL}(n,\mathbb{Z})$ (see [BoSe73, Theorem 11.4.2]).

Note that each apartment of Δ is the barycentric subdivision of the boundary of an $(n-1)$-simplex which is topologically an $(n-2)$-sphere. One may consider the homology classes of these spheres in $\mathrm{St}(n)$ and ask if they form a $\mathbb{Z}$-generating set. The Solomon-Tits Theorem [So69] (see also [Brow89, §IV.5 Theorem 2] and [CuLeh82]) provides the answer:

Theorem 3 (Solomon-Tits)**.** *The Steinberg module* $\mathrm{St}(n) = \widetilde{\mathrm{H}}_{n-2}(\Delta;\mathbb{Z})$ *satisfies the following.*

(1) $\mathrm{St}(n)$ *is generated by the homology classes of the spheres of all apartments of* Δ.
(2) $\mathrm{St}(n)$ *is generated by the homology class of a single apartment as an* $\mathrm{SL}(n,\mathbb{Q})$*-module.*
(3) $\mathrm{St}(n)$ *is has a* $\mathbb{Z}$*-basis consisting of the orbit of the homology class of the apartment coming from the standard basis under the subgroup* $U < \mathrm{SL}(n,\mathbb{Q})$ *consisting of unipotent upper triangular matrices.*

Ash-Rudolph [AsRu79] improve on part 2 of Theorem 3 above and show that upon restricting the coefficient ring from $\mathbb{Z}[\mathrm{SL}(n,\mathbb{Q})]$ to $\mathbb{Z}[\mathrm{SL}(n,\mathbb{Z})]$ the Steinberg module remains cyclic.

Theorem 4 (Ash-Rudolph)**.** *The Steinberg module* $\mathrm{St}(n)$ *is generated by the homology class of a single apartment as an*$\mathrm{SL}(n,\mathbb{Z})$*-module.*

[1]Steinberg himself (see [Ste51] and [Hum87] for a survey) was interested in the homology of the Bruhat-Tits building $\Delta(n,K)$ for K a finite field. In that case one has an irreducible representation of $\mathrm{PSL}(n,K)$.

2.3. **The curve complex and duality for mapping class groups.** Harer [Hare88] has shown that the mapping class group is a virtual duality group in a manner that mirrors that of Borel-Serre for $\mathrm{SL}(n, \mathbb{Z})$. Just as $\mathrm{SL}(n, \mathbb{Z})$ acts on symmetric space, the mapping class group $\mathrm{Mod}(\Sigma_g^1)$ of the surface of genus g with a marked point Σ_g^1 acts on Teichmüller space (see [Hub06])

$$\mathcal{T}(\Sigma_g^1) = \{ \text{ marked hyperbolic metrics on } \Sigma_g^1 \} \cong \mathbb{R}^{6g-4}.$$

Harer introduces the arc complex $\mathcal{A}(\Sigma_g^1)$ which acts a sort of bordification of $\mathcal{T}(\Sigma_g^1)$ (see [Hare86]).

Definition 3 (Arc complex)**.** The *arc complex* $\mathcal{A}(\Sigma_g^1)$ of the surface Σ_g^1 is the cell complex with vertices corresponding to *arcs* which are isotopy classes of embedded loops beginning and ending at the marked point. The arc complex has a k-simplex for each *arc system* $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}$ of $k+1$ disjoint arcs. ‖

An arc system *fills* the surface Σ_g^1 if the complement of the arcs is a union of disks. Note that any filling arc system must have at least $2g$ arcs. A subcomplex of the arc complex is the arc complex at infinity (see [Hare86]).

Definition 4 (Arc complex at infinity)**.** The *arc complex at infinity* $\mathcal{A}_\infty(\Sigma_g^1)$ of the surface Σ_g^1 is the subcomplex of simplices of $\mathcal{A}(\Sigma_g^1)$ whose vertices correspond to arc systems which do not fill Σ_g^1. ‖

Using a construction of Strebel (see [Str84] or [Hub06]) one may identify

$$\mathcal{T}(\Sigma_g^1) \cong \mathcal{A}(\Sigma_g^1) - \mathcal{A}_\infty(\Sigma_g^1).$$

Harer shows that $\mathcal{A}(\Sigma_g^1)$ is contractible, that $\mathcal{A}_\infty(\Sigma_g^1)$ is homotopic to the boundary of a true bordification of $\mathcal{T}(\Sigma_g^1)$, and that $\mathcal{A}_\infty(\Sigma_g^1)$ has the homotopy type of a wedge of $(2g-2)$-spheres. It then follows that the virtual dualizing module of the mapping class group $\mathrm{Mod}(\Sigma_g^1)$ is $\widetilde{\mathrm{H}}_{2g-2}(\mathcal{A}_\infty; \mathbb{Z})$.

Definition 5 (Curve complex [Harv81])**.** The *curve complex* $\mathcal{C}(\Sigma_g^1)$ of the surface Σ_g^1 is the cell complex with vertices corresponding to isotopy classes of essential simple closed curves embedded in the complement of the marked point in Σ_g^1 which do not bound a disk containing the marked point. The arc complex has a k-simplex for each *arc system* $\{\gamma_0, \gamma_1, \cdots, \gamma_k\}$ of $k+1$ disjoint simple closed curves. ‖

Harer shows that the more familiar curve complex $\mathcal{C}(\Sigma_g^1)$ is homotopic to the arc complex at infinity $\mathcal{A}_\infty(\Sigma_g^1)$.

Definition 6 (Steinberg module of the mapping class group [Hare88])**.** The *Steinberg module of the mapping class group* is the $\mathrm{Mod}(\Sigma_g^1)$-module

$$\mathrm{St}(\Sigma_g^1) = \widetilde{\mathrm{H}}_{2g-2}(\mathcal{C}; \mathbb{Z}). \quad \|$$

3. Problems on the Steinberg module

One would like to understand the following.

Problem 1 (Module structure of the Steinberg module)**.** Understand the structure of the Steinberg module

$$\mathrm{St}(\Sigma_g^1) = \widetilde{\mathrm{H}}_{2g-2}(\mathcal{C};\mathbb{Z})$$

of the mapping class group $\mathrm{Mod}(\Sigma_g^1)$ as a mapping class group module. ‖

Theorems 5, 6 and 7 below represent progress in addressing Problem 1.

3.1. **Module structure.** Initial results the structure of the $\mathrm{Mod}(\Sigma_g^1)$-module structure of $\mathrm{St}(\Sigma_g^1)$ are described in [Broa07] starting with a finite presentation for $\mathrm{St}(\Sigma_g^1)$.

Theorem 5 (Broaddus)**.** *The Steinberg module of the mapping class group* $\mathrm{St}(\Sigma_g^1)$ *is finitely presented with a generator for each* $\mathrm{Mod}(\Sigma_g^1)$*-orbit of filling arc system with* $2g$ *arcs and a relation for each* $\mathrm{Mod}(\Sigma_g^1)$*-orbit of filling arc system with* $2g+1$ *arcs. (See* [Broa07, Proposition 3.5] *for the precise generators and relations.)*

This result can be seen as part 1 of the Solomon-Tits Theorem (Theorem 3 above). In fact the presentation of Theorem 5 is derived from a (virtually) free $\mathrm{Mod}(\Sigma_g^1)$-module resolution of $\mathrm{St}(\Sigma_g^1)$. For $k \geq 0$ let

$$\mathcal{F}_k = \mathrm{C}_{2g-1+k}(\mathcal{A}/\mathcal{A}_\infty;\mathbb{Z})$$

where $\mathrm{C}_*(\mathcal{A}/\mathcal{A}_\infty;\mathbb{Z})$ is the chain complex for cellular homology of the quotient space $\mathcal{A}/\mathcal{A}_\infty$ with the cell complex structure inherited from $\mathcal{A}$.

Theorem 6 (Broaddus)**.** *The Steinberg module of the mapping class group* $\mathrm{St}(\Sigma_g^1)$ *has a finite* $\mathrm{Mod}(\Sigma_g^1)$*-module resolution*

$$0 \to \mathcal{F}_{4g-3} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathrm{St}(\Sigma_g^1).$$

If one restricts coefficients to $\mathbb{Z}[\Gamma]$ *for a torsion free finite index subgroup* $\Gamma <$ $\mathrm{Mod}(\Sigma_g^1)$ *then the above resolution is a free* Γ*-module resolution.*

Finally we show that the Steinberg module is a cyclic (generated by a singleton set) $\mathrm{Mod}(\Sigma_g^1)$-module.

Theorem 7 (Broaddus)**.** *The Steinberg module of the mapping class group* $\mathrm{St}(\Sigma_g^1)$ *is generated by a single element as a* $\mathrm{Mod}(\Sigma_g^1)$*-module. (See* [Broa07, Theorem 4.2] *for a description of the particular generator.)*

One should view this result as an analog for part 2 of the Solomon-Tits Theorem (Theorem 3 above). In fact, it is more accurately analogous to Ash and Rudolph's stronger result (Theorem 4 above).

Problem 2. Give a presentation for the Steinberg module $\mathrm{St}(\Sigma_g^1)$ based on the single generator from [Broa07, Theorem 4.2]. ‖

A major component missing from our understanding of $\mathrm{St}(\Sigma_g^1)$ is a $\mathbb{Z}$-basis which would give an analog of part 3 of the Solomon-Tits Theorem (Theorem 3 above).

Problem 3 (Give a basis for the Steinberg module)**.** Give a $\mathbb{Z}$-basis for $\mathrm{St}(\Sigma_g^1)$. ‖

For $g \geq 2$ the Steinberg module $\mathrm{St}(\Sigma_g^1)$ is not a faithful $\mathrm{Mod}(\Sigma_g^1)$-module since it is stabilized by the point pushing subgroup $P < \mathrm{Mod}_g^1$ which is the image of $\pi_1(\Sigma_g^1)$ in the Birman exact sequence

$$1 \to \pi_1(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_g) \to 1.$$

Question 1 (Is the Steinberg module faithful?)**.** Is $\mathrm{St}(\Sigma_g^1)$ a faithful $\mathrm{Mod}(\Sigma_g)$-module for $g \geq 1$? If not what is the kernel of the action? ‖

A $\mathbb{Z}$-basis for $\mathrm{St}(\Sigma_g^1)$ coming from a solution to Problem 3 might help resolve Question 1.

References

[AsRu79] A. Ash and L. Rudolph, The modular symbol and continued fractions in higher dimensions, Invent. Math. **55** (1979), no. 3, 241–250.

[Bie76] R. Bieri, *Homological dimension of discrete groups*, Mathematics Department, Queen Mary College, London, 1976.

[BieEc73] R. Bieri and B. Eckmann, Groups with homological duality generalizing Poincaré duality, Invent. Math. **20** (1973), 103–124.

[BoSe73] A. Borel and J.-P. Serre, Corners and arithmetic groups, Comment. Math. Helv. **48** (1973), 436–491.

[Broa07] N. Broaddus, The homology of the curve complex and the Steinberg module for the mapping class group. arXiv:0711.0011 [math.GT] *(submitted preprint).*

[Brow82] K. S. Brown, *Cohomology of groups*, Springer, New York, 1982.

[Brow89] K. S. Brown, *Buildings*, Springer, New York, 1989.

[CuLeh82] C. W. Curtis and G. I. Lehrer, A new proof of a theorem of Solomon-Tits, Proc. Amer. Math. Soc. **85** (1982), no. 2, 154–156.. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson, W. P. Thurston, *Word processing in groups*, Jones and Bartlett, Boston, MA, 1992.

[Hare86] J. L. Harer, The virtual cohomological dimension of the mapping class group of an orientable surface, Invent. Math. **84** (1986), no. 1, 157–176.

[Hare88] J. L. Harer, The cohomology of the moduli space of curves, in *Theory of moduli (Montecatini Terme, 1985)*, 138–221, Lecture Notes in Math., 1337, Springer, Berlin.

[Harv81] W. J. Harvey, Boundary structure of the modular group, in *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, 245–251, Ann. of Math. Stud., 97, Princeton Univ. Press, Princeton, N.J.

[Hum87] J. E. Humphreys, The Steinberg representation, Bull. Amer. Math. Soc. (N.S.) **16** (1987), no. 2, 247–263.

[Hub06] J. H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*, Matrix Editions, Ithaca, NY, 2006.

[Ron92] M. Ronan, Buildings: main ideas and applications. I. Main ideas, Bull. London Math. Soc. **24** (1992), no. 1, 1–51.

[Se71] J.-P. Serre, Cohomologie des groupes discrets, in *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, 77–169. Ann. of Math. Studies, 70, Princeton Univ. Press, Princeton, N.J.

[So69] L. Solomon, The Steinberg character of a finite group with BN-pair, in *Theory of Finite Groups (Symposium, Harvard Univ., Cambridge, Mass., 1968)*, 213–221, Benjamin, New York.
[Ste51] R. Steinberg, A geometric approach to the representations of the full linear group over a Galois field, Trans. Amer. Math. Soc. **71** (1951), 274–282.
[Str84] K. Strebel, *Quadratic differentials*, Springer, Berlin, 1984.

Deformations and rigidity of lattices in solvable Lie groups

Oliver Baues
(joint work with Benjamin Klopsch)

Let G be a simply connected, solvable real Lie group. Let $\mathcal{X}(\Gamma, G)$ denote the space of all homomorphic embeddings of Γ as a lattice into G. By a result of Weil, $\mathcal{X}(\Gamma, G)$ is an open subset in the space of all homomorphisms $\Gamma \to G$ and Wang showed subsequently that its components are smooth manifolds. The rigidity properties of lattices in G are reflected in the natural left-action of $\mathrm{Aut}(G)$ on $\mathcal{X}(\Gamma, G)$.

Rigidity of lattices. The lattice Γ is *rigid* in G if and only if $\mathrm{Aut}(G)$ acts transitively on $\mathcal{X}(\Gamma, G)$. More generally, the orbit space

$$\mathrm{Aut}(G)\backslash\mathcal{X}(\Gamma, G)$$

provides a quantitative measure for the degree of non-rigidity of Γ in G. We are particularly interested in describing principal situations where Γ is *deformation rigid* in G, that is, situations, where the space $\mathrm{Aut}(G)\backslash\mathcal{X}(\Gamma, G)$ is finite or countable, and has a totally disconnected topology.

A finiteness theorem. Let Γ be a Zariski-dense lattice in G. Under the additional hypothesis that G is unipotently connected we prove that the orbit space $\mathrm{Aut}(G)\backslash\mathcal{X}(\Gamma, G)$ is finite. This generalises a classical theorem of Mal'tsev–Saitô about the rigidity of lattices in solvable Lie groups of real type.

Examples show that both Zariski-denseness of lattices Γ, and unipotent connectedness of the ambient group G are necessary conditions for our finiteness result. The class of simply connected solvable Lie groups such that the orbit space $\mathrm{Aut}(G)\backslash\mathcal{X}(\Gamma, G)$ is finite for all Zariski-dense lattices Γ seems to be only 'slightly' larger.

The finiteness theorem also shows that any lattice in some simply connected solvable Lie group H has a finite index subgroup which can be embedded as a Zariski-dense and deformation rigid lattice in a simply connected solvable Lie group G, in such a way that $\mathrm{Aut}(G)\backslash\mathcal{X}(\Gamma, G)$ is finite.

Topology of $\mathcal{X}(\Gamma, G)$. The group $\mathrm{Aut}(G)$ has a natural structure of a linear algebraic group and its action on $\mathcal{X}(\Gamma, G)$ is continuous. Our results therefore apply to study topological properties of the space $\mathcal{X}(\Gamma, G)$. In particular, we can estimate the number of its connected components.

Non-Zariski-dense lattices. In the context of non-Zariski-dense lattices, there are other situations, where the deformation space of a lattice Γ is a finite-dimensional, non-finite, variety. This variety then admits a non-trivial continuous action of the *arithmetic* group $\mathrm{Aut}(\Gamma)$ of automorphisms of Γ.

Strongly rigid lattices. As another corollary of our work we obtain a characterisation of groups Γ which are *strongly rigid* with respect to Zariski-dense embeddings. In fact, a lattice is shown to be strongly rigid if and only it it can be embedded as a lattice into a group G which is of real type.

Methods. Our methods are based on a strong relation which links lattice embeddings of Γ into Lie groups to *algebraically rigid* embeddings of Γ into arithmetic subgroups of linear algebraic groups in a functorial manner. This construction is originally due to Mostow.

This report is based on joint work with Benjamin Klopsch, Royal Holloway, University of London.

References

[1] O. Baues, B. Klopsch, Deformations and rigidity of lattices in solvable Lie groups, preprint 2011, in preparation
[2] G.D. Mostow, Representative functions on discrete groups and solvable arithmetic subgroups, Amer. J. Math. **92** (1970), 1-32.
[3] A.N. Starkov, Rigidity problem for lattices in solvable Lie groups, Proc. Indian Acad. Sci. (Math. Sci.) **104** (1994), 495–514.
[4] A. Weil, On discrete subgroups of Lie groups, Ann. Math. **72** (1960), 369–384.
[5] D. Witte, Superrigidity of lattices in solvable Lie groups, Invent. Math. **122** (1995), 147–193.

Finiteness Properties of Arithmetic Groups: the Rank Theorem

Kai-Uwe Bux
(joint work with Ralf Köhl née Gramlich, Stefan Witzel)

The arithmetic group $\mathrm{SL}_2(\mathbf{Z})$ is finitely generated: one can see that by tweaking the Euclidean algorithm. It is also finitely presented: one can see that using its action on the hyperbolic plane, which has an invariant tessellation. Generalizing the latter line of thought, one can deduce that $\mathrm{SL}_2(\mathbf{Z})$ is of type F_∞. Recall that a group Γ is of type F_m if there it acts freely on a contractible CW-complex X such that the m-skeleton is cocompact. Γ is of type F_∞ if it is of type F_m for all m. The largest m for which Γ is of type F_m, is called the finiteness length of Γ. We denote it by $\phi(\Gamma)$.

Arithmetic groups are relatives of $\mathrm{SL}_2(\mathbf{Z})$, and we shall be interested in their finiteness length. Let K be a global field, S be a finite set of places including all archimedean places, and $\mathcal{O}_S$ be the ring of S-integers in K. Let $\mathcal{G}$ be a linear algebraic group defined over K faithfully represented as a matrix group. An S-arithmetic group is a group Γ commensurable to the group $\mathcal{G}(\mathcal{O}_S)$ of $\mathcal{O}_S$-points of $\mathcal{G}$. Although the notion of an $\mathcal{O}_S$-point depends on the chosen matrix representation for $\mathcal{G}$, the notion of an S-arithmetic group does not: this is the effect of passing to the commensurability class of $\mathcal{G}(\mathcal{O}_S)$. Finiteness length is constant on commensurability classes [4].

The group Γ depends on two parameters S and $\mathcal{G}$, which can vary independently. Hence, we can more specifically ask how the finiteness length of Γ depends on S and $\mathcal{G}$. The case of reductive groups $\mathcal{G}$ has attracted most research. Borel and Serre [7] have shown that S-arithmetic groups over number fields are of type F_∞ if the group scheme $\mathcal{G}$ is reductive. This is rare in the function field case. Here, an arithmetic group is of type F_∞ only in the cocompact case: Assume that K is a global function field. Then, to each place $v \in S$ one can associate a euclidean building X_v. The group Γ acts on the product $X := \prod_{v\in S} X_v$; and Γ is of type F_∞ if and only if the orbit space for this action is compact. The "if" part was shown by Serre [13], the "only if" part was treated in [9].

The dimension of the building X_v is the local rank of $\mathcal{G}$ at v. Evidence was mounting that the sum of the local ranks determines the finiteness length of Γ: see, e.g., [12], [5], [14], [2], [1], [3], [6], [10], [11], and [8]. All these results pointed towards the following:

Theorem. *Let $\mathcal{G}$ be a connected non-commutative absolutely almost simple linear algebraic group defined and isotropic over K. Then the finiteness length $\phi(\Gamma)$ of the S-arithmetic group $\Gamma = \mathcal{G}(\mathcal{O}_S)$ is $m-1$ where $m := \sum_{v\in S} \dim X_v$ is the sum of the local ranks of $\mathcal{G}$.*

We remark that the finiteness length of any reductive group can be reduced to the special case treated in the theorem. The proof of the theorem uses Behr-Harder reduction theory and the euclidean metric structure of X to find a Γ-invariant cocompact filtration of X with good relative links. Then, standard techniques allow one to deduce the finiteness length. The paper can be found at arxiv:1102.0428.

References

[1] H. Abels, *Finiteness Properties of certain arithmetic groups in the function field case*, Israel J. Math. 76 (1991) 113 – 128.

[2] P. Abramenko, *Endlichkeitseigenschaften der Gruppen* $\mathrm{SL}_n(\mathbf{F}_q[t])$, Thesis Frankfurt am Main (1987).

[3] P. Abramenko, *Twin Buildings and Applications to S-Arithmetic Groups*, Springer LNM 1641 (1996).

[4] J. Alonso, *Finiteness conditions on groups and quasi-isometries*, J. Pure Appl. Algebra 95 (1994) 121 – 129.

[5] H. Behr, *Endliche Erzeugbarkeit arithmetischer Gruppen über Funktionenkörpern*, Invent. Math. 7 (1969) 1 – 32.

[6] H. Behr, *Arithmetic groups over function fields I. A complete characterization of finitely generated and finitely presented arithmetic subgroups of reductive algebraic groups*, J. Reine Angew. Math. (Crelles Journal) 495 (1998) 79–118.
[7] A. Borel, J.-P. Serre, *Cohomologie d'immeubles et de groupes S-arithmétiques*, Topology 15 (1976) 211 – 232.
[8] K.-U. Bux, R. Gramlich, S. Witzel, *Finiteness Properties of Chevalley Groups over a Polynomial Ring over a Finite Field*, Preprint (2009) arXiv:0908.4531.
[9] K.-U. Bux, K. Wortman, *Finiteness properties of arithmetic groups over function fields*, Invent. Math. 167 (2007) 355 – 378.
[10] K.-U. Bux, K. Wortman, *Connectivity Properties of Horospheres in Euclidean Buildings and Applications to Finiteness Properties of Discrete Groups*, Preprint (2008) arXiv:0808.2087. Preprint (2008) arXiv:0808.2087; to appear in Invent. Math..
[11] A. Devillers, R. Gramlich, B. Mühlherr, *The sphericity of the complex of nondegenerate subspaces*, J. Lond. Math. Soc. (2) 79 (2009) 684 – 700.
[12] H. Nagao, *On* $GL(2, K[X])$, J. Inst. Polytech. Osaka City Univ. Ser. A 10 (1959) 117 – 121.
[13] J.-P. Serre, *Cohomologie des groupes discrets*, Prospects in mathematics. Princeton Univ. Press, Princeton, N.J. (1971) 77 – 169.
[14] U. Stuhler, *Homological properties of certain arithmetic groups in the function field case*, Invent. Math. 57 (1980) 263 – 281.

Surfaces of large genus

Hugo Parlier

Much of my recent research has been focussed on studying surfaces of large genus and related objects. I'm interested in inequalities on Riemannian surfaces, the geometry of hyperbolic surfaces, and the geometry of Teichmüller and moduli spaces. The type of questions that arise in my research are:

(1) What does a surface of large genus look like? What does a "typical" surface look like?

(2) Which hyperbolic surface of given topological type has the largest possible systole?

(3) What can you say about metrics on Teichmüller space, such as the Teichüller metric or the Weil-Petersson metric, by studying the geometry of hyperbolic surfaces?

1. Systolic type inequalities

The classical systolic inequality for surfaces, due to Gromov [6, 2.C], states that the shortest non-trivial curve on a closed Riemannian surface of genus g and area normalized to $4\pi(g-1)$, is of length at most $C\log(g)$ for some constant $C > 0$. This result is optimal: indeed their exist families of hyperbolic surfaces, one in each genus, whose systoles grow like $\sim \log(g)$. The first of these were constructed by P. Buser and P. Sarnak in their seminal article [4], and there have been other constructions since by R. Brooks [2] and M. Katz, M. Schaps and U. Vishne [8].

The same results hold if one restricts oneself to homological systoles, the length of the shortest homologically non-trivial curve on a surface. As an example of an application of such results, P. Buser and P. Sarnak also derived new bounds on the minimal norm of nonzero period lattice vectors of Riemann surfaces. This result paved the way for a geometric approach of the Schottky problem which consists in characterizing Jacobians (or period lattices of Riemann surfaces) among abelian varieties. There are other types of inequalities that exist and are due to different authors, such as bounds on lengths of pants decompositions of surfaces or homology bases (see [3] for instance).

As an example of work I've done on these types of problems: with F. Balacheff and S. Sabourau [1], we show that on every closed Riemannian surface of genus g with normalized area there exist almost g homologically independent loops of lengths at most $\sim \log(g)$. More precisely, we prove the following.

Theorem 8. *Let $\eta : \mathbb{N} \to \mathbb{N}$ be a function such that $\lambda := \sup_g \frac{\eta(g)}{g} < 1$. Then there exists a constant C_λ such that for every closed Riemannian surface M of genus g there are at least $\eta(g)$ homologically independent loops $\alpha_1, \dots, \alpha_{\eta(g)}$ which satisfy*

$$\mathrm{length}(\alpha_\mathrm{i}) \leq \mathrm{C}_\lambda \frac{\log(\mathrm{g}+1)}{\sqrt{\mathrm{g}}} \sqrt{\mathrm{area}(\mathrm{M})}$$

for every $i \in \{1, \dots, \eta(g)\}$.

Typically, this result applies to $\eta(g) = [\lambda g]$ where $\lambda \in (0, 1)$. As a corollary, we are able to extend Buser and Sarnak's results on Jacobians.

2. The geometry of the Weil-Petersson diameter of moduli space

The Teichmüller space Teich(Σ) of an orientable surface Σ of negative Euler characteristic with genus g and n punctures is the set of marked hyperbolic metrics on Σ. The moduli space of curves $\mathcal{M}_{g,n}$ is the space of conformal structures on a topological surface Σ of genus g with n marked points or equivalently, via the uniformization theorem, the set of hyperbolic metrics on Σ up to conformal isometry. Moduli space can be seen as the quotient of Teichmüller space via the mapping class group Mod(Σ).

Teichmüller space can be endowed with a CAT(0) metric, called the Weil-Petersson metric, which is defined on the cotangent space at $X \in$ Teich(Σ). The definition itself is quite technical, but through the work of many authors, and in particular S. Wolpert, the metric can be studied through the geometry of hyperbolic surfaces. As an example: a theorem of Brock tells us that the rough scale geometry of WP can be studied via a combinatorial object related to the curve complex, called the pants graph and one of the main tools to show this is the use of *Bers' constants*. (Bers proved that there is a genus dependent constant which bounds the length of a shortest pants decomposition of all hyperbolic surfaces of a given genus; the optimal constants are Bers' constants.)

The Weil-Petersson metric descends to $\mathcal{M}_g$ but is non-complete. However the metric completion $\overline{\mathcal{M}}_g$ is (topologically) a well known object called the Deligne-Mumford compactification of moduli space by stable nodal curves. In terms of hyperbolic structures, this compactification is given by adjoining "strata" to moduli space whose points correspond to degenerate hyperbolic structures on Σ, which are appropriate limits of sequences of hyperbolic surfaces in which the lengths some collection of disjoint simple closed geodesics goes to zero. These strata are lower dimensional moduli spaces parameterizing families of cusped surfaces, and many geometric and topological properties of $\mathcal{M}_g$ can be understood inductively using properties of this stratified boundary. Since the completion of the Weil-Petersson metric on $\mathcal{M}_g$ is a compact space, the Weil-Petersson diameter of $\mathcal{M}_g$ is finite.

With Will Cavendish, we've shown the following [5]:

Theorem 9. *There exists a genus independent constant D such that*

$$\lim_{n\to\infty} \frac{\operatorname{diam}(\mathcal{M}_{g,n})}{\sqrt{n}} = D.$$

Theorem 10. *There exist a constant $C > 0$ such that for any $n \geq 0$ the Weil-Petersson diameter $\operatorname{diam}(\mathcal{M}_{g,n})$ satisfies*

$$\frac{1}{C} \leq \liminf_{g\to\infty} \frac{\operatorname{diam}(\mathcal{M}_{g,n})}{\sqrt{g}}, \ \limsup_{g\to\infty} \frac{\operatorname{diam}(\mathcal{M}_{g,n})}{\sqrt{g}\log(g)} \leq C.$$

The proofs use different tools, but in particular recent bounds on Bers' constants, which are related to some of the things in the first section.

3. The geometry of random surfaces

With Larry Guth and Robert Young [7], we've been interested in short pants decompositions where the length of a pants decomposition is defined as the sum of the lengths of the curves in the pants decomposition. The total pants length has not been studied as much as the usual pants length, but it also seems like a natural invariant. Since a pants decomposition has $3g-3$ curves in it, estimates of Buser and Seppälä imply that every genus g hyperbolic surface has total pants length at most Cg^2. This is the best known general upper bound. In the other direction, it is easy to construct hyperbolic surfaces with total pants length at least cg for every g by taking covers of a genus 2 surface. The only previous non-trivial estimate comes from Buser and Sarnak's examples mentioned in the first section where they proved that there exist families of surfaces, one in each genus g, with the property that every topologically non-trivial curve has length at least $\sim \log g$. Since each curve in a pants decomposition is non-trivial, the total pants length of these arithmetic hyperbolic surfaces is at least $\sim g \log g$.

We were able to show:

Theorem 11. *For any $\varepsilon > 0$, a "random" hyperbolic surface of genus g has total pants length at least $g^{7/6-\varepsilon}$ with probability tending to 1 as $g \to \infty$. In particular, for all sufficiently large g, there are hyperbolic surfaces with total pants length at least $g^{7/6-\varepsilon}$.*

(To define a "random" hyperbolic surface we need a probability measure on the moduli space of hyperbolic metrics. We use the renormalized Weil-Petersson volume form.)

Our lower bound is a lot stronger than the one coming from the Buser-Sarnak estimate. Instead of improving the trivial bound by a factor of $\log g$, we improve it by a polynomial factor $g^{1/6-\varepsilon}$. We obtain the same type of result for surfaces coming from random gluings of euclidean equilateral triangles with side length 1.

References

[1] Florent Balacheff, Hugo Parlier, and Stéphane Sabourau. Short loop decompositions of surfaces and the geometry of Jacobians. *Preprint*, 2011.

[2] Robert Brooks. Platonic surfaces. *Comment. Math. Helv.*, 74(1):156–170, 1999.

[3] Peter Buser. *Geometry and spectra of compact Riemann surfaces*, volume 106 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1992.

[4] Peter Buser and Peter Sarnak. On the period matrix of a Riemann surface of large genus. *Invent. Math.*, 117(1):27–56, 1994. With an appendix by J. H. Conway and N. J. A. Sloane.

[5] Will Cavendish and Hugo Parlier. Growth of the Weil-Petersson diameter of moduli space. *Preprint*, 2010.

[6] Mikhael Gromov. Systoles and intersystolic inequalities. In *Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992)*, volume 1 of *Sémin. Congr.*, pages 291–362. Soc. Math. France, Paris, 1996.

[7] Larry Guth, Hugo Parlier, and Robert Young. Pants decompositions of random surfaces. *Geom. Funct. Anal.*, to appear.

[8] Mikhail G. Katz, Mary Schaps, and Uzi Vishne. Logarithmic growth of systole of arithmetic Riemann surfaces along congruence subgroups. *J. Differential Geom.*, 76(3):399–422, 2007.

Discrete Zariski dense subgroups of $SL(n, \mathbb{R})$

T. N. Venkataramana

In this talk we consider ways to extend lattices in smaller (semi-)simple groups to Zariski dense discrete non-lattice subgroups of larger (semi-)simple groups and show that this can be achieved if the real rank of the ambient group is one. We exhibit examples where this fails and as a consequence obtain linear discrete groups which are super-rigid in their Zariski closures and show that in many cases the super-rigid groups are actually lattices.

Homomorphisms between mapping class groups

Juan Souto
(joint work with J. Aramayona)

I discussed a few results suggesting that whenever X, Y are surfaces of sufficiently large genus then any homomorphism $\phi : \mathrm{Map}(X) \to \mathrm{Map}(Y)$ between the corresponding mapping class groups is one of the "usual suspects". More concretely, if X has genus at least six and Y has genus less than twice that of X, then any nontrivial homomorphism $\phi : \mathrm{Map}(X) \to \mathrm{Map}(Y)$ is a combination of forgetting punctures, boundary components, and subsurface embeddings.

On classification of arithmetic hyperbolic reflection groups

Viacheslav V. Nikulin

In my short talk, I gave a review of known results about finiteness and classification of arithmetic hyperbolic reflection groups.

Constructing finite models for the classifying space of the proper action of lattices in semisimple Lie groups of $\mathbb{R}$-rank one

Hyosang Kang

1. Introduction

The main result is the following.

Theorem 12. (Kang [8]) *For any lattice* Γ *in a semisimple Lie group of* $\mathbb{R}$*-rank one, there exists a cofinite model for the proper classifying space of* Γ*.*

To prove the above theorem, we construct a partial compactification of symmetric space which generalize the Borel–Serre partial compactification.

Classifying spaces have been actively studied in algebraic K and L-theories. They appear in long-standing conjectures such as the *integral Novikov conjecture*, which is the injectivity of the assembly maps

$$\begin{aligned} &A : H_*(B\Gamma, \mathbb{K}(\mathbb{Z})) \to K_*(\mathbb{Z}\Gamma), \text{ and} \\ &A : H_*(B\Gamma, \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}\Gamma). \end{aligned} \tag{1}$$

The rational injectivity of A, i.e. the injectivity of $A \otimes \mathbb{Q}$, is called the *Novikov conjecture*, and it is equivalent to Novikov's original conjecture on the homotopy invariance of the higher signature of manifolds. For groups with torsion elements, the integral Novikov conjecture fails. Since many natural groups such as $SL(n, \mathbb{Z})$ are not torsion-free, it is important to generalize the integral Novikov conjecture. The *generalized integral Novikov conjecture* is the injectivity of the assembly maps

$$\begin{aligned} &A : H_*^\Gamma(\underline{E}\Gamma, \mathbb{K}(\mathbb{Z})) \to K_*(\mathbb{Z}\Gamma) \text{ and} \\ &A : H_*^\Gamma(\underline{E}\Gamma, \mathbb{L}(\mathbb{Z})) \to L_*(\mathbb{Z}\Gamma), \end{aligned} \tag{2}$$

where $\underline{E}\Gamma$ is the *proper classifying space* of Γ.

The existence of a cofinite (i.e. cocompact) model for the classifying space of Γ plays an important role in the proof of the (generalized) integral Novikov conjecture. In [19], Yu showed that the integral Novikov conjecture is true for any group Γ with finite asymptotic dimension which admits a cofinite classifying space of Γ.[1] In [3], Bartels and Rosenthal extended this result to the generalized integral

[1] This implies that the group Γ must be torsion-free.

Novikov conjecture. Using a similar idea, Ji [7] formulated geometric conditions on groups for which the generalized integral Novikov conjecture holds (c.f. Theorem 15). Using the Borel–Serre partial compactification as a cofinite proper classifying space for arithmetic groups, he proved that the generalized integral Novikov conjecture is true for any arithmetic groups. As an application of Theorem 12, the generalized integral Novikov conjecture is true for lattices in semisimple Lie groups also (c.f. Corollary 1).

2. Cofinite proper classifying spaces for arithmetic lattices

Let Γ denote a discrete group. The universal cover $E\Gamma$ of the classifying space $B\Gamma$ is a free Γ-space. The proper classifying space for Γ is a generalization of $E\Gamma$ where the free Γ-action is replaced by the proper Γ-action.

Definition 7. A Γ-CW-complex is called a model for the **proper classifying space for** Γ, denoted by $\underline{E}\Gamma$, if all isotropy groups are finite, and for every finite subgroup of Γ the fixed point set is non-empty and contractible. A proper classifying space is called **cofinite** if it consists of finitely many Γ-equivariant cells, i.e. cocompact.

If Γ is torsion-free, then $\underline{E}\Gamma$ is equal to $E\Gamma$. For any discrete subgroup Γ of a Lie group G of finitely many components, the homogenous space G/K (where K is a maximal compact subgroup of G) is the proper classifying space for Γ. For example, the upper half-plane $\mathbf{H}$ is the proper classifying space for a Fuchsian group of Möbius transformations.[2]

Let us further assume that Γ is an arithmetic subgroup of a semisimple Lie group G. The corresponding symmetric space X is a proper Γ-space, and the Borel–Serre partial compactification $\overline{X}^{BS}$ is a cofinite proper classifying space. This is first observed by Borel and Prasad (c.f. [1, Remark 5.8], [7, Theorem 3.2]).

Let P be a rational parabolic subgroup of G and $X = N_P \times A_P \times X_P$ be the horospherical decomposition with respect to the Langlands decomposition of P. The space $\overline{X}^{BS}$ is obtained by attaching a boundary component $e(P) = N_P \times X_P$ for every rational parabolic subgroup $P \subset G$:

$$\overline{X}^{BS} = X \cup \bigcup_{P:\ \text{rational}} e(P). \tag{3}$$

3. Cofinite model for a general lattice Γ

Although arithmetic subgroups of semisimple Lie groups are important examples of lattices, there are many non-arithmetic lattices in semisimple Lie groups (c.f. [5]). A natural question is to ask: is there an analogue of the Borel–Serre compactification for general lattices in semisimple Lie groups? A positive answer

[2]Other important classes of discrete groups that admit cofinite proper classifying space are the mapping class groups Mod_g [10], the groups $\mathrm{Out}(\mathbb{F}_n)$ of outer automorphisms of the free groups $\mathbb{F}_n$ [11, 18], the p-adic algebraic groups [12], and the hyperbolic groups [15].

is given in [8] for the case of rank-one semisimple Lie groups, and the complete answer is given in [9]. In the following sections, we explain these results in detail.

3.1. **Lattices in $\mathbb{R}$-rank one semisimple Lie groups.** We restate Theorem 12 as follows:

Theorem 13. [8] *Let Γ be a lattice in a semisimple Lie group G of $\mathbb{R}$-rank one and let X be the corresponding symmetric space. Then there exists a partial compactification $\overline{X}_\Gamma$ of X which is a cofinite proper classifying space of Γ.*

The idea of the construction of the space $\overline{X}_\Gamma$ is similar to the uniform construction of the Borel–Serre partial compactification in Equation (3).

Definition 8. Let $P \subset G$ be a real parabolic subgroup and N_P be the unipotent radical of P. P is called **Γ-rational** if $\Gamma \cap N_P$ is a cocompact lattice in N_P (c.f. [2, §3.5 p472]).

The space $\overline{X}_\Gamma$ is defined by

$$\overline{X}_\Gamma = X \cup \coprod_{P:\ \Gamma\text{-rational}} e(P). \tag{4}$$

Since the $\mathbb{R}$-rank of G is one, each boundary component $e(P)$ is equal to N_P. Thus, the space $\overline{X}_\Gamma$ is a manifold with boundary whose interior is the symmetric space X.[3]

If Γ is arithmetic, then a parabolic subgroup $P \subset G$ is rational if and only if it is Γ-rational [2, §3.5 p473]. This implies that the space $\overline{X}_\Gamma$ generalize the Borel–Serre partial compactification. To show the space $\overline{X}_\Gamma$ is cocompact, we use the reduction theory due to Garland and Raghunathan [6]. Roughly, their result says that the Dirichlet fundamental domain of a lattice in $\mathbb{R}$-rank one semisimple Lie group has finitely many cusp neighborhoods which are covered by Siegel sets. I showed that each cups corresponds to Γ-rational real parabolic subgroups, which gives a group theoretic way of choosing the boundary components.

3.2. **Lattices in higher $\mathbb{R}$-rank semisimple Lie groups.**

Theorem 14. [9] *Let Γ be a lattice in semisimple Lie group G and X be the corresponding symmetric space. There exists a partial compactification $\overline{X}_\Gamma$ of X which is a cofinite model for the proper classifying space $\underline{E}\Gamma$.*

Margulis's arithmeticity theorem [13] states that every irreducible lattice in semisimple Lie group of $\mathbb{R}$-rank greater than two is arithmetic. Thus we only need to consider reducible lattices. A lattice $\Gamma \subset G$ is *reducible* if there exist a subgroup

[3] Raghunathan [16] prove this result by using Morse theory without extended Γ-action on the boundary.

$\Gamma' \subset \Gamma$ of finite index and a splitting

$$\begin{array}{ccc} G & \xrightarrow{\cong} & G_1 \times G_2 \\ \uparrow & & \uparrow \\ \Gamma & \xrightarrow{\cong} & \Gamma_1 \times \Gamma_2. \end{array}$$

By induction on the $\mathbb{R}$-rank of the subgroups G_1 and G_2, one can assume that, for each $i = 1, 2$, either (1) the $\mathbb{R}$-rank of G_i is one, or (2) the lattice Γ_i is irreducible, equivalently, arithmetic. Thus, a cofinite model of $\underline{E}\Gamma'$ is given by the product of two partial compactifications:

$$\underline{E}\Gamma' = \overline{X}_1 \times \overline{X}_2. \tag{5}$$

Since every parabolic subgroup is Γ-rational if and only if it is Γ'-rational, the product is also a cofinite $\underline{E}\Gamma$.

4. Applications

In [7], Ji showed that

Theorem 15. *Let Γ be a discrete subgroup of finite asymptotic dimension in a Lie group with finitely many connected components. If Γ admits a cofinite model X of the proper classifying space of Γ such that for any pair of subgroups $H \subseteq I$ in Γ, the fixed point set X^H and the quotient $N_I(H)\backslash X^H$ are uniformly contractible and of bounded geometry, then the generalized integral Novikov conjecture holds for Γ.*

As a corollary, Ji proved that the generalized integral Novikov conjecture holds for arithmetic subgroups of semisimple Lie groups. Using the same theorem and the cofinite model for the proper classifying space in Theorem 14, we showed that

Corollary 1. [9] *The generalized integral Novikov conjecture holds for lattices in semisimple Lie groups.*

Theorem 14 generalizes to a general linear Lie groups. Mostow [14] showed that every linear Lie group $G \subset GL(N, \mathbb{R})$ decomposes into $G = L \ltimes U$ where L is Levi subgroup and U is unipotent radical, which is called the *Levi decomposition.* Moreover, any lattice $\Gamma \subset G$ decomposes into $\Lambda \ltimes \Gamma'$ with respect to the Levi decomposition.[4] Since every lattice in solvable Lie group is cocompact, a cofinite model for $\underline{E}\Gamma$ is obtained by using Theorem 14.

[4]Venkataramana told me that the existence of the lattices Λ and Γ' is due to Auslander and the proof can be found in [17]. Baues suggested a careful reading on this proof because there is a gap in it. He also said that Mostow first proved the splitting of such lattice.

References

[1] Alejandro Adem and Yongbin Ruan. Twisted orbifold K-theory. *Comm. Math. Phys.*, 237(3):533–556, 2003.

[2] W. L. Baily, Jr. and A. Borel. On the compactification of arithmetically defined quotients of bounded symmetric domains. *Bull. Amer. Math. Soc.*, 70:588–593, 1964.

[3] Arthur Bartels and David Rosenthal. On the K-theory of groups with finite asymptotic dimension. *J. Reine Angew. Math.*, 612:35–57, 2007.

[4] Martin Deraux, Elisha Falbel, and Julien Paupert. New constructions of fundamental polyhedra in complex hyperbolic space. *Acta Math.*, 194(2):155–201, 2005.

[5] M. Gromov and I. Piatetski-Shapiro. Nonarithmetic groups in Lobachevsky spaces. *Inst. Hautes Études Sci. Publ. Math.*, (66):93–103, 1988.

[6] H. Garland and M. S. Raghunathan. Fundamental domains for lattices in (R-)rank 1 semisimple Lie groups. *Ann. of Math. (2)*, 92:279–326, 1970.

[7] Lizhen Ji. Integral Novikov conjectures and arithmetic groups containing torsion elements. *Comm. Anal. Geom.*, 15(3):509–533, 2007.

[8] Hyosang Kang. Cofinite classifying spaces for lattices in $\mathbb{R}$-rank one semisimple Lie groups. *Thesis*, 2011.

[9] Hyosang Kang. Finite proper classifying spaces for lattices in semisimple lie groups. *preprint*, 2011.

[10] Steven P. Kerckhoff. The Nielsen realization problem. *Bull. Amer. Math. Soc. (N.S.)*, 2(3):452–454, 1980.

[11] Sava Krstić and Karen Vogtmann. Equivariant outer space and automorphisms of free-by-finite groups. *Comment. Math. Helv.*, 68(2):216–262, 1993.

[12] Wolfgang Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.

[13] G. A. Margulis. Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than 1. *Invent. Math.*, 76(1):93–120, 1984.

[14] G. D. Mostow. Fully reducible subgroups of algebraic groups. *Amer. J. Math.*, 78:200–221, 1956.

[15] David Meintrup and Thomas Schick. A model for the universal space for proper actions of a hyperbolic group. *New York J. Math.*, 8:1–7 (electronic), 2002.

[16] M. S. Raghunathan. A note on quotients of real algebraic groups by arithmetic subgroups. *Invent. Math.*, 4:318–335, 1967/1968.

[17] M. S. Raghunathan. *Discrete subgroups of Lie groups*. Springer-Verlag, New York, 1972. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.

[18] Tad White. Fixed points of finite groups of free group automorphisms. *Proc. Amer. Math. Soc.*, 118(3):681–688, 1993.

[19] Guoliang Yu. The Novikov conjecture for groups with finite asymptotic dimension. *Ann. of Math. (2)*, 147(2):325–355, 1998.

Spectral geometry for the Riemann moduli space

Rafe Mazzeo

(joint work with Lizhen Ji, Werner Müller and Andras Vasy)

There are many similarities between the structure of the Riemann moduli spaces $\mathcal{M}_g$, $g > 1$, and of locally symmetric spaces, though of course these spaces have many different features as well. Developing analogies between these types of spaces has become an interesting field of research, and we point immediately to the paper of Ji [4] for a thorough exploration of the current knowledge on this topic. By locally symmetric spaces we mean Riemannian locally symmetric spaces $M = \Gamma \setminus G/K$, where Γ is a discrete group of isometries of X, endowed with a locally symmetric metric. The Riemann moduli space, on the other hand, is the quotient of Teichmüller space $\mathcal{T}_g$ by the mapping class group $\mathtt{Map_g}$. This space carries several different natural metrics, some Riemannian and others Finsler. We focus here primarily on the Weil-Petersson metric g_{WP}, which is the natural L^2 metric for the realization of $\mathcal{M}_g$ as the moduli space of hyperbolic metrics, and on the Ricci metric g_{Ric}, which is the negative of the Ricci curvature of g_{WP}, or at least on some smoothings of this metric. The Weil-Petersson metric is incomplete; its metric completion is called the Deligne-Mumford compactification, denoted $\overline{\mathcal{M}_g}$; by contrast, g_{Ric} is complete. Increasingly refined information is being discovered about the geometry of these spaces, particularly for the Weil-Petersson metric. We refer to Wolpert's recent survey [8] which gives a good summary of these geometric and topological developments, and includes an extensive bibliography.

Since the study of the spectral and scattering theory for locally symmetric spaces is highly developed, it is natural to enquire about the spectral and scattering properties of the Laplacians, and other natural elliptic operators, associated to these two metrics on $\mathcal{M}_g$. Surprisingly, this question does not seem to have been considered seriously before. The purpose of this report is to mention a few current and ongoing projects, in collaboration with L. Ji, W. Müller and A. Vasy, on these questions. One origin for this new work is the series of papers by the author and Vasy, see [7] in particular, which employs some tools of geometric scattering theory to study the resolvent of the Laplacian on an arbitrary Riemannian symmetric space (X, g). A natural outgrowth of that work would be to extend those methods to a similar analysis of the resolvent of the Laplacian for arbitrary Riemannian locally symmetric spaces. This is closely related to the meromorphic continuation of Eisenstein series. For various (good) historical reasons, the resolvent of the Laplacian had rarely been studied in this setting except in rank one. During the work on these extensions (by the author, Müller and Vasy), we became aware that it should not only be possible to prove similar results for the Laplacian associated to (a mollified version of) g_{Ric}, but should be even easier than for locally symmetric spaces. This led to the current collaboration and to the work described below on the spectral theory of g_{WP}. While the techniques needed to study this incomplete metric are rather different, the overall theme and motivations are similar.

We report here on our analysis of the spectral properties of $(\overline{\mathcal{M}_g}, g_{\mathrm{WP}})$. There are now many different approaches to studying elliptic operators on various classes of stratified spaces. Much of this work is aimed at developing general techniques for studying various types of stratified and metric singularities, and this has now reached a very refined state for metrics with conic, simple edge, and iterated edge singularities. We refer to [2], [6], [3], [1] for some examples of all of this.

The types of metric singularities exhibited by the Weil-Petersson metric are of a slightly more complicated nature. The Deligne-Mumford compactification $\overline{\mathcal{M}_g}$ is a smooth complex space which is singular along the union of a collection

$D_0, \ldots, D_{[g/2]}$ of immersed divisors with simple normal crossings. The approximate structure of g_{WP} near the singular divisors was first obtained by Masur in the 1970's [5]; this was later substantially sharpened by Yamada [9], with further results on its structure by Wolpert [8]. The upshot of these papers is that if p is a point in the singular set, then there is a local set of holomorphic coordinates $(z_1, \ldots, z_n$, $n = 3g - 3$, in some punctured neighbourhood, such that if we write $z_j = r_j e^{i\theta_j}$, then

$$g_{\mathrm{WP}} = \sum_{j=1}^{k} (dr_j^2 + r^6 d\theta_j^2)(1 + \mathcal{O}(r^3)) + g_D + k;$$

here $r = |(r_1, \ldots, r_k)|$ and g_D is some induced metric on the singular stratum where $z_1 = \ldots = z_k = 0$; the final term k is a higher order remainder term which is irrelevant for our purposes below. Before carrying out more refined spectral geometric analysis of this space, it will be necessary to establish higher order asymptotics of g_{WP}, but these are not yet known.

We announce some basic results about the scalar Laplacian Δ for the Weil-Petersson metric.

Theorem 16. *The operator Δ acting on $\mathcal{C}_0^\infty(\mathcal{M}_g)$ has a unique self-adjoint extension to an unbounded operator on $L^2(\overline{\mathcal{M}_g})$, which we continue to denote by Δ. Furthermore, this operator has discrete spectrum $\{\lambda_j\}$, and the spectral counting function $N(\lambda) = \#\{j : \lambda_j \leq \lambda\}$ satisfies the Weyl law*

$$N(\lambda) = \frac{\omega_{6g-6}}{(2\pi)^{6g-6}} \mathrm{Vol}(\mathcal{M}_g) \lambda^{n/2} + o(\lambda^{n/2}).$$

Here ω_ℓ is the volume of the Euclidean ball B^ℓ.

This sets the stage for further investigations: one expects interesting connections between $\mathrm{spec}(\Delta)$ and other aspects of the geometry of this space.

Essential self-adjointness is obtained by showing that there are no L^2 solutions to $(\Delta \pm i)u = 0$. These can be ruled out if we prove the standard identity $\langle -\Delta u, u \rangle = ||\nabla u||^2$ without additional boundary terms. In other words, it suffices to prove that if u and Δu both lie in L^2, then we can control the growth of u near the singular set enough to carry out this integration by parts. This is done using an elaboration of the Hardy inequality. The discreteness of the spectrum is then proved by showing that the (now unique) domain $\mathrm{Dom}(\Delta)$ is compactly contained in L^2. Finally, to obtain the Weyl law in this relatively crude form (i.e. with no analysis of the remainder), we can use standard comparison techniques (Dirichlet-Neumann bracketing), once we have verified that an arbitrarily small neighbourhood of the singular set contributes a lower order term. Amongst these three arguments, the first one is more difficult than the others.

There are many further directions. Current work of Gell-Redman proves a full asymptotic expansion for the heat kernel associated to metrics with the same 'crossing cubic cuspidal' structure as g_{WP}, but assuming that these metrics themselves have full asymptotic regularity. The asymptotics of the heat trace involve some new and potentially interesting terms. One significant goal is to obtain a signature formula for $(\overline{\mathcal{M}_g}, g_{\mathrm{WP}})$, and this heat kernel analysis should provide a crucial tool for this. However, it remains to show that g_{WP} itself does indeed have full asymptotic regularity or at least that what is known about its asymptotic structure suffices to understand enough of the expansion of the heat trace to obtain such index formulas.

References

[1] Albin, P., Leichtnam, E., Mazzeo, R., Piazza, P. "The signature package on Witt spaces". To appear, Ann. Sci. de l'Ecole Norm. Sup.

[2] Cheeger, J. "Spectral geometry of singular Riemannian spaces" Jour. Diff. Geom. 18 (1983) No. 4, 575-657.

[3] Gil, J., Krainer, T., Mendoza, G. "On the closure of elliptic wedge operators" arXiv:1007.2397

[4] Ji, L. "A tale of two groups: arithmetic groups and mapping class groups" In preparation.

[5] Masur, H. "Extension of the Weil-Petersson metric to the boundary of Teichmüller space" Duke Math. J. 43 (3) (1976), 623-635.

[6] Mazzeo, R. "Elliptic theory of differential edge operators, I" Comm. Par. Diff. Eqns. 16 (1991) No. 10, 1615-1664.

[7] Mazzeo, R., Vasy, A. "Analytic continuation of the resolvent of the Laplacian on symmetric spaces of noncompact type" Jour. Func. Anal. 228 (2005) No. 2, 311-368.

[8] Wolpert, S. "The Weil-Petersson metric geometry" in Handbook of Teichmüller theory Vol. II, 47-64. IRMA Lect. Math. Theor. Phys. 13, Eur. Math. Soc., Zürich (2009).

[9] Yamada, S. "On the geometry of Weil-Petersson completion of Teichmüller spaces" Math. Res. Lett. 10 (2-3) (2003), 391-400.

Rank and divergence for lattices and mapping class groups

Cornelia Drutu

(joint work with J. Behrstock)

This talk pursues the comparison between lattices in semisimple Lie groups and mapping class groups of surfaces via another aspect: the notion of rank (in its various forms).

There exist several notions of rank, all of them introduced first in the setting of Hadamard manifolds (i.e. complete, simply connected, non-positively curved manifolds). Three will be discussed here.

1. The flat rank

In a Hadamard manifold, by *flat* is always meant an isometrically embedded copy of an Euclidean space $\mathbb{R}^k$. The *flat rank* is usually defined as the maximal dimension of a flat.

Correspondingly, a group G acting cocompactly on a Hadamard manifold M contains quasi-flats (i.e. quasi-isometric embeddings of $\mathbb{R}^k$) of the same dimension as the rank of M. Thus, for a group (especially one acting on a $CAT(0)$-space) one can define the quasi-flat rank as the maximal dimension of a quasi-flat contained in a Cayley graph of that group.

When X is a symmetric space, this yields the standard notion of rank, which coincides also with the quasi-flat rank of a lattice of isometries.

Consider now a surface S of genus g and with p boundary components, and consider its complexity $\xi(S) = 3g + p - 3$. The mapping class group of the surface $Mod(S)$ contains copies of $\mathbb{Z}^{\xi(S)}$ coming from groups generated by Dehn twists around maximal families of disjoint curves. It was proved by Farb, Lubotzky and Minsky that these are quasi-flats, i.e. they are undistorted subgroups of $Mod(S)$. Brock and Farb asked whether $\xi(S)$ is the quasi-flat rank of $Mod(S)$. This was answered in the affirmative independently by Behrstock-Minsky and Hamenstädt.

2. The isoperimetric rank

Again in the setting of a Hadamard manifold X, one can define *k-dimensional spheres* and *k-dimensional balls* as Lipschitz maps of Euclidean spheres and balls of the appropriate dimension into X. Since by Rademacher theorem such Lipschitz maps are differentiable almost everywhere, one can define the volume of such spheres and balls. A $(k+1)$-dimensional ball is said to *fill* a k-dimensional sphere if the corresponding Lipschitz map defining the ball extends that of the sphere.

The *filling volume of a sphere* is the infimum of the volumes of all the balls filing it (it is ∞ if no filling ball exists). The *k-th isoperimetric function* $Iso_k(x)$ is defined as the supremum of all the filling volumes of all the k-dimensional spheres of volume at most Ax^k. Here the constant A is considered large enough, and assumed fixed for the given space. We nevertheless do not define it explicitly because, among other things, we want to be able to say that Iso_k is a quasi-isometry invariant.

The *isoperimetric rank* can be defined either as the minimal dimension k such that $Iso_k(x) \asymp x^k$, or as the maximal k such that $Iso_k(x) \asymp x^{k+1}$ plus one.

For a group, under certain conditions, one can define a notion of k-sphere and $(k+1)$-filling ball and define the functions Iso_k as above.

For symmetric spaces and uniform lattices this notion of rank coincides to that of flat rank. For mapping class groups we have proved that likewise the quasi-flat rank coincides with the isoperimetric rank, because the following holds: $Iso_k(x) \asymp x^{k+1}$ for $k \leq \xi(S) - 1$ and $Iso_k(x) = o(x^{k+1})$ for $k \geq \xi(S)$.

3. The divergence rank

The divergence functions measure the isoperimetry when both the spheres to fill and the balls filling them are pushed towards infinity. In other words, we fix a basepoint p (since all the spaces that we consider are homogeneous the choice of p is irrelevant) and for each $x > 0$ we only consider k-dimensional spheres $\mathcal{S}$ of volume $\leq Ax^k$ that are disjoint of the ball centred at p and of radius x. We then fix $\delta \in (0, 1)$ and for every such sphere $\mathcal{S}$ we only consider balls filling it and disjoint of $B(p, \delta x)$. The minimal volume of such a ball defines the *divergence of the sphere* $\mathcal{S}$.

The *k-th divergence function* $Div_k(x)$ is defined as the supremum over all divergences of spheres $\mathcal{S}$ of volume $\leq Ax^k$ and disjoint of the ball $B(p, x)$.

Brady and Farb proved that for products of k rank one symmetric spaces (which are therefore of flat rank k) the $(k-1)$-divergence is exponential. They asked whether the rank of symmetric spaces can be detected through divergence functions. This was answered in the affirmative later on: Leuzinger proved that in every symmetric space of rank k the$(k-1)$-divergence is exponential, and Hindawi showed that for $r \geq k$, the divergence $Div_r(x) \preceq x^{r+1}$.

We prove that for mapping class groups, if $k \leq \xi(S) - 1$ then the divergence is $Div_k(x) \succeq x^{k+2}$, while if $k \geq \xi(S)$ the divergence is $o(x^{k+1})$. In particular the rank of the mapping class groups too can be detected through divergence.

Handlebody groups and mapping class groups

Ursula Hamenstädt
(joint work with Sebastian Hensel)

A handlebody H of genus $g \geq 2$ is a compact 3-manifold whose boundary is a closed surface of genus g. It can be realized as a standard neighborhood of a bouquet of g circles embedded in $\mathbb{R}^3$. The *handlebody group* $\mathrm{Map}(H)$ is the group of isotopy classes of orientation preserving diffeomorphisms of H. By a result of Laudenbach, the handlebody group is a subgroup of the *mapping class group* $\mathrm{Mod}(\partial H)$ of ∂H consisting of all mapping classes which can be realized by diffeomorphisms extending to H. Here the mapping class group of ∂H is the group of isotopy classes of orientation preserving diffeomorphisms of ∂H.

Each diffeomorphism of H acts as as automorphism on the fundamental group $\pi_1(H) = F_g$ of H, and isotopy classes of diffeomorphisms act as outer automorphisms on $\pi_1(H)$. Thus there is a natural homomorphism of $\mathrm{Map}(H)$ into the outer automorphism group $\mathrm{Out}(F_g)$ of the free group with g generators.. This homomorphism is surjective, but its kernel, the so-called *twist group*, is infinitely generated.

As for $\mathrm{Mod}(\partial H)$ and $\mathrm{Out}(F_g)$, the handlebody group $\mathrm{Map}(H)$ is finitely presented. In particular, every finite set of generators defines a word norm $\| \, \|_H$ and hence a distance on $\mathrm{Map}(H)$ which is unique up to quasi-isometry. On the other hand, the restriction of a word norm on $\mathrm{Mod}(\partial H)$ also defines a word norm $\| \, \|_{\partial H}$. The group $\mathrm{Map}(H)$ is *undistorted* in $\mathrm{Mod}(\partial H)$ if there is a number $L > 0$ so that

$$\|g\|_H \leq L\|g\|_{\partial H} \text{ for all } g \in \mathrm{Map}(H).$$

It is *exponentially distorted* if there are numbers $\ell \leq L$ so that $\|g\|_H \leq e^{L\|g\|_{\partial H}}$ for all g and if moreover there is a sequence $g_i \subset \mathrm{Map}(H)$ with $\|g_i\|_H \to \infty$ and $\|g_i\|_H \geq e^{\ell\|g_i\|_{\partial H}}$.

In [1] the following is shown.

Theorem: The handlebody group is exponentially distorted in the mapping class group.

As a consequence, the understanding of the geometry of the mapping class group does not yield an understanding of the geometry of the handlebody group.

Instead one can try to develop tools for studying the handlebody group which resemble the tools developed for the mapping class group.

A very important such tool for Mod(∂H) is the *curve graph* $\mathcal{C}(S)$. It is the locally infinite graph whose vertices are isotopy classes of simple closed curves on ∂H and where two such curves are connected by an edge of length one if they can be realized disjointly. Masur and Minsky showed that the curve graph is a hyperbolic geometric metric space.

There are various analogs of a curve graph for a handlebody. Namely, an essential *disk* in H is a properly embedded disk whose boundary is not contractible in ∂H. There are three Map(H)-graphs whose vertices are isotopy classes of essential disks.

(1) The *disk graph* is the graph whose vertices are disks and where two such disks are connected by an edge of length one if and only if they can be realized disjointly.
(2) The *electrified disk graph* is obtained from the disk graph by adding an edge between any two disks which a not disjoint but which are disjoint from a common essential simple closed curve in ∂H.
(3) The *superconducting disk graph* is the graph which is obtained from the electrified disk graph by adding an edge between any two disks which intersect the same *diskbusting I-bundle* in exactly two points.

Here a diskbusting I-bundle is a simple closed curve in ∂H which can be realized as the boundary circle of a bordered surface F with connected boundary and such that the oriented I-bundle over this surface F is homeomorphic to H.

Masur and Schleimer showed [3]

Theorem: The disk graph is hyperbolic.

We complemented this result in [2] by showing

Theorem:

(1) The superconducting disk graph is quasi-isometrically embedded in the curve graph.
(2) The electrified disk graph and the disk graph are hyperbolic.

This result can be used to show that the handlebody group for a handlebody of genus 2 is semi-hyperbolic. The case of higher genus is open.

References

[1] U. Hamenstädt, Sebastian Hensel *Geometry of the handlebody groups I: Distortion*, arXiv:1101.1838.
[2] U. Hamenstädt, *Geometry of graphs of discs in a handlebody*, arXiv:1101.1843.
[3] H. Masur, S. Schleimer, *The geometry of the disk complex*, arXiv:1010.3174.

Veech groups of translation surfaces

Frank Herrlich

In this talk, a *translation surface* (X, μ) is a surface X together with an atlas μ such that all transition maps in μ are translations. More specifically, we require X to be of the form $\overline{X} - S$, where $\overline{X}$ is a compact oriented surface and S is a finite subset. Each translation surface carries a flat metric, namely the pullback of the Euclidean metric on $\mathbb{R}^2$. The points in S are cone singularities for this metric.

Example. The basic example of a translation surface is obtained as follows: Let $\overline{X}$ be a compact Riemann surface, $\omega \in \Omega^1(\overline{X})$ a nonzero holomorphic 1-form and S the set of zeroes of ω. Outside S, in a simply connected neighbourhood of a point P_0, the map $P \mapsto \int_{P_0}^{P} \omega$ is a chart map. Clearly, these charts endow $X = \overline{X} - S$ with a translation structure.

In fact, each translation surface in the above sense can be obtained by this construction: any translation structure is in particular a complex structure on X, and the pullback of the differential dz on $\mathbb{C} = \mathbb{R}^2$ is a holomorphic differential on X (with a zero of order $k-1$ in $s \in S$, if s is a singularity of total angle $2\pi k$).

Definition 9. For a translation surface (X, μ), let $\mathit{Aff}(X, \mu)$ be the set of all orientation preserving diffeomorphisms of X, that are affine w.r.t. μ.

Here "affine" means that locally in the charts of μ, f is of the form $z \mapsto Az + b$ for some $A \in \mathrm{GL}_2^+(\mathbb{R})$ and some $b \in \mathbb{R}^2$. Note that, since the transition maps are translations, the matrix part A is the same on all charts. Therefore, we have a well defined homomorphism

$$D : \mathrm{Aff}(X, \mu) \to \mathrm{SL}_2(\mathbb{R}).$$

Note that the determinant of the matrix has to be 1, since f must preserve the area of X.

Definition 10. $\Gamma(X, \mu) = D(\mathit{Aff}(X, \mu))$ is called the *Veech group* of (X, μ).

Remark. Slightly more general than translation surfaces is the notion of a *flat surface*: in addition to translations, also rotations by an angle of π are allowed as transition maps. They are obtained from holomorphic quadratic differentials on a Riemann surface as in the above example. The matrix part of an affine map is only defined up to sign. Therefore flat surfaces have a well defined *projective Veech group* in $\mathrm{PSL}_2(\mathbb{R})$.

Translation surfaces always come in families: For a translation surface (X, μ) and a matrix $A \in \mathrm{SL}_2(\mathbb{R})$, denote by $(X, A\mu)$ the translation surface with chart maps $A \circ \varphi : U \to \mathbb{R}^2$ for all charts $(U, \varphi) \in \mu$. If we consider (X, μ) as a (reference) point in the Teichmüller space $T_{g,n}$, where $g = g(X)$ is the genus of X and $n = |S|$, then $(X, A\mu)$ is another point in the same Teichmüller space (using the identity map on X as marking). Since rotations do not change the complex structure, $A \mapsto (X, A\mu)$ induces a map

$$\iota : \mathbb{H} = \mathrm{SO}(2)\backslash\mathrm{SL}_2(\mathbb{R}) \to T_{g,n}.$$

Proposition 1. *ι is holomorphic and isometric w.r.t. the hyperbolic metric on $\mathbb{H}$ and the Teichmüller metric on $T_{g,n}$.*

The image $\Delta(X,\mu) = \iota(\mathbb{H})$ of this embedding is called a *Teichmüller disk*.

Proposition 2. $\mathrm{Aff}(X,\mu)$ *is equal to the stabilizer of $\Delta(X,\mu)$ in the mapping class group $\Gamma_{g,n}$, and the Veech group $\Gamma(X,\mu)$ is isomorphic to the quotient of this stabilizer by the pointwise stabilizer.*

Denote by $C(X,\mu)$ the image of $\Delta(X,\mu)$ in the moduli space $M_{g,n} = T_{g,n}/\Gamma_{g,n}$, where $\Gamma_{g,n}$ is the mapping class group that acts on $T_{g,n}$. Then we have

Proposition 3. *$C(X,\mu)$ is an (affine) algebraic curve if and only if $\Gamma(X,\mu)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. In this case, the induced map $q : \mathbb{H}/\Gamma(X,\mu) \to C(X,\mu)$ is birational, and $C(X,\mu)$ is called a* Teichmüller curve.

Examples of Veech groups.
1. The Veech group of the flat torus E is equal to $\mathrm{SL}_2(\mathbb{Z})$.
2. Origamis: Let $p : \overline{X} \to E$ be a finite covering, ramified over at most one point $\infty \in E$ and endow $X = \overline{X} - p^{-1}(\infty)$ with the translation structure pulled back from E. The resulting translation surface O is called an *origami* or *square-tiled surface*. It can be shown that the Veech group of an origami is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index.
3. Veech's double n-gon: Reflect a regular n-gon over one of its sides and then glue parallel sides (by translations). Veech proved that the Veech group of the resulting translation surface is the Hecke triangle group $\Delta(2,n,\infty)$.
4. Triangle groups: It has been shown by Bouw and Möller [1] and with a different approach by Hooper [5] that all triangle groups $\Delta(m,n,\infty)$ (with $m,n \geq 2, mn \geq 6$) arise as Veech groups of translation surfaces. For a very readable account of Hooper's construction see [6].

In general, for a given subgroup of $\mathrm{SL}_2(\mathbb{R})$, it is impossible to decide whether it is the Veech group of some translation surface or not. However, there are two necessary properties, as was noted already by Veech [8]:

Proposition 4. *The Veech group $\Gamma = \Gamma(X,\mu)$ of a translation surface is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, and $\mathbb{H}/\Gamma$ is not compact.*

The only class of Fuchsian groups for which there is a good (although by far not complete) knowledge are the finite index subgroups of $\mathrm{SL}_2(\mathbb{Z})$, i. e. the Veech groups of origamis. The basis for this is the following result of G. Schmithüsen:

Theorem 17 ([7])**.** *Let $O = (p : \overline{X} \to E)$ be an origami and $H(O) = \pi_1(X)$ the subgroup of $F_2 = \pi_1(E - \{\infty\})$ that corresponds to the unramified covering $p|X$. Then $\Gamma(O)$ is the image of $\mathrm{Stab}(O)$ in $\mathrm{Out}^+(F_2) = \mathrm{SL}_2(\mathbb{Z})$, where $\mathrm{Stab}(O)$ denotes the stabilizer of $H(O)$ in $\mathrm{Aut}^+(F_2)$.*

Using this theorem, Schmithüsen proved that almost all congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are Veech groups, s. [7] for a precise statement. An even more general result, also relying on the characterization of Veech groups in Theorem 1, was obtained by Ellenberg and McReynolds:

Theorem 18 ([2]). *Every finite index subgroup of the principle congruence subgroup* $\Gamma(2) \subset \mathrm{SL}_2(\mathbb{Z})$ *that contains* $-I$ *is the Veech group of a translation surface.*

It is still an open question, whether every finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is a Veech group.

In the other direction, one might ask how much information about a translation surface can be recovered from its Veech group. We show by one prominent example that there can be very different translation surfaces having the same Veech group. The following proposition is an immediate corollary to Theorem 1:

Proposition 5. *Let* O *be an origami such that* $H(O)$ *is a characteristic subgroup of* F_2. *Then* $\Gamma(O) = \mathrm{SL}_2(\mathbb{Z})$.

F_2 contains many characteristic subgroups of finite index. Such groups and thus origamis with Veech group $\mathrm{SL}_2(\mathbb{Z})$ can be constructed explicitly:

Proposition 6 ([3]). *Let* G *be a finite group and* Φ *the set of surjective homomorphisms* $F_2 \to G$*; suppose* Φ *is nonempty. Let* $h_1, \dots, h_s$ *be representatives of* $\mathrm{Aut}(G)\backslash\Phi$ *and* $h = (h_1, \dots, h_s) : F_2 \to G^s$ *the diagonal map.*
Then $K = \ker(h)$ *is a characteristic subgroup of* F_2.

Examples.
1. For the origami $L_{2,2}$ with 3 squares in L-shape, the construction of Proposition 6 leads to an origami with 108 squares. The group G is the subgroup

$$\{(\sigma_1, \sigma_2, \sigma_3) \in S_3 \times S_3 \times S_3 : \mathrm{sign}(\sigma_1) \cdot \mathrm{sign}(\sigma_2) \cdot \mathrm{sign}(\sigma_3) = 1\}.$$

The genus of the corresponding Riemann surface is 37.
2. The "Wollmilchsau": For the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, the kernel of the homomorphism $F_2 \to Q_8$, $x \mapsto i, y \mapsto j$, is a characteristic subgroup. The corresponding origami (with 8 squares) is of genus 3 and has many surprising properties, see [4].

References

[1] I. Bouw, M. Möller, *Teichmüller curves, triangle groups, and Lyapunov exponents*, Ann. of Math. **172** (2010), 139-185.

[2] J. Ellenberg, D. McReynolds, *Every curve is a Teichmüller curve*, Preprint 2009.

[3] F. Herrlich, *Teichmüller curves defined by characteristic origamis*, Contemp. Math. **397** (2006), 133-144.

[4] F. Herrlich, G.Schmithüsen, *An extraordinary origami curve*, Math. Nachrichten **281** (2008), 219-237.

[5] P. Hooper, *Grid graphs and lattice surfaces*, Preprint 2009.

[6] A. Randecker, *Dreiecksgruppen, die als Veechgruppen auftreten*, Diploma thesis, Karlsruhe 2011.

[7] G. Schmithüsen, *Veech groups of origamis*, PhD thesis, Karlsruhe 2005.

[8] W. Veech, *Teichmüller curves in moduli space, Eisenstein series and an application to triangular billards*, Invent. math. **97** (1989), 553-583.

Cohomology of Locally Symmetric Spaces and the Moduli Space of Curves

Leslie Saper

Let Γ be a group, E a Γ-module, and consider the group cohomology $H^{\cdot}(\Gamma; E)$. If X is a contractible space on which Γ acts properly one may represent this cohomology topologically as $H^{\cdot}(\Gamma\backslash X; \mathbb{E})$ for a certain sheaf $\mathbb{E}$. (In the case Γ acts freely then $\mathbb{E}$ is the local system corresponding to the representation of $\pi_1(\Gamma\backslash X) = \Gamma$.) Our primary interest is when Γ is arithmetic and X is a symmetric space or when Γ is a mapping class group and X is Teichmüller space. When $M = \Gamma\backslash X$ is a compact Riemannian manifold it is profitable to use the Hodge-de Rahm theory and represent cohomology by harmonic forms; from this one may deduce Poincaré duality. When M is non-compact, Poincaré duality no longer holds for the ordinary cohomology but the same reasoning applies instead to the L^2-cohomology $H^{\cdot}_{(2)}(M; \mathbb{E})$, provided it is finite-dimensional and M is complete. The L^2-cohomology is an invariant of the quasi-isometry class of the metric.

We consider three examples to indicate that $H^{\cdot}_{(2)}(M; \mathbb{E})$ can represent a topological invariant: (1) Cheeger's analysis [2] of the L^2-cohomology of horn metrics on triangulated pseudomanifolds; (2) Saper's proof [9] that the L^2-cohomology of the Weil-Petersson metric on the moduli space of curves $\mathcal{M}_g$ is the cohomology of the Deligne-Mumford compactification $\overline{\mathcal{M}_g}$; and (3) Zucker's conjecture [12] (proved by Saper-Stern [11] and Looijenga [7]) that the L^2-cohomology of a Hermitian locally symmetric space $\Gamma\backslash X$ is the middle perversity intersection cohomology of the Baily-Borel Satake compactification $\Gamma\backslash X^*$ [1]. We conclude this section with a heuristic for Zucker's conjecture.

Example (2) above answered a question of Hain and Looijenga, perhaps motivated in analogy with Zucker's conjecture. A better analogy suggests one consider the Siegel metric on $\mathcal{M}_g$, the pull-back of the locally symmetric metric under the Torelli embedding $\tau\colon \mathcal{M}_g \to \mathcal{A}_g$, and the Satake compactification $\mathcal{M}_g^*$, the closure of $\tau(\mathcal{M}_g)$ in the Baily-Borel Satake compactification $\mathcal{A}_g^*$. We conjectured in 1993 that the analogue of Zucker's conjecture holds in this setting. Although no progress has been made on this conjecture for $g > 3$, more recent work on Rapoport's conjecture suggests a possible approach.

Rapoport's conjecture [8] (made independently by Goresky and MacPherson [6]) asserts that for a Hermitian symmetric space X, either middle perversity intersection cohomology [4, 5] of the reductive Borel-Serre compactification $\Gamma\backslash\overline{X}^{RBS}$ [12] (see also [3]) is isomorphic to the middle perversity intersection cohomology of $\Gamma\backslash X^*$. The conjecture was motivated by Langlands's program—the point is that $\Gamma\backslash\overline{X}^{RBS}$ is a far less singular compactification making local calculations easier. Saper proved the conjecture (actually a generalization to equal-rank spaces) in 2001 [10] by introducing the theory of $\mathcal{L}$-modules, a combinatorial model of sheaves on $\Gamma\backslash\overline{X}^{RBS}$.

In current work, $\mathcal{L}$-modules are being used to study $H^{\cdot}(\Gamma; E)$ itself for Γ arithmetic. We now suggest that an analogue of $\mathcal{L}$-modules can be applied to address the 1993 conjecture on the moduli space of curves. Namely $\overline{\mathcal{M}_g}$ could play the role of the reductive Borel-Serre compactification. What is needed is to understand the Siegel metric locally on $\overline{\mathcal{M}_g}$ (as opposed to locally on $\mathcal{M}_g^*$), to understand the fibers of the extended Torelli map $\overline{\mathcal{M}_g} \to \mathcal{M}_g^*$, and to prove a vanishing theorem on these fibers. Progress in some simple concrete examples has been made.

References

[1] W. Baily and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. **84** (1966), 442–528.

[2] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Geometry of the Laplace Operator, 91–146, Proc. Sympos. Pure Math. **36**, Amer. Math. Soc., Providence, 1980.

[3] M. Goresky, G. Harder, and R. MacPherson, *Weighted cohomology*, Invent. Math. **116** (1994), 139–213.

[4] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135162.

[5] ———, *Intersection homology II*, Invent. Math. **72** (1983), 77–129.

[6] ———, *Weighted cohomology of Satake compactifications*, Centre de recherches mathématiques, preprint #1593, 1988.

[7] E. Looijenga, *L^2-cohomology of locally symmetric varieties*, Compositio Math. **67** (1988), 3–20.

[8] M. Rapoport, *On the shape of the contribution of a fixed point on the boundary: The case of $\mathbb{Q}$-rank 1 (with an appendix by L. Saper and M. Stern)*, 479–488, The Zeta functions of Picard modular surfaces (R. P. Langlands and D. Ramakrishnan, eds.), Les Publications CRM, Montréal, 1992.

[9] L. Saper, *L^2-cohomology of the Weil-Petersson metric*, Mapping Class Groups and Moduli Spaces of Riemann Surfaces (C.-F. Bödigheimer and R. Hain, eds.), 345–360, Contemporary Mathematics, vol. 150, Amer. Math. Soc., Providence, 1993.

[10] ———, *$\mathcal{L}$-modules and micro-support*, Ann. of Math., accepted, `math.RT/0112251`, 2001.

[11] L. Saper and M. Stern, *L^2-cohomology of arithmetic varieties*, Annals of Math. **132** (1990), 1–69.

[12] S. Zucker, *L_2 cohomology of warped products and arithmetic groups*, Invent. Math. **70** (1982), 169–218.

Euler characteristic, simplicial volume and the Schläfli volume formula

Michelle Bucher

The simplicial volume $\|M\| \in \mathbb{R}_{\geq 0}$ of a closed oriented n-dimensional manifold M is defined as

$$\|M\| = \inf \{ \Sigma |a_\sigma| \mid [\Sigma a_\sigma \sigma] = [M] \in H_n(M, \mathbb{R}) \} .$$

It was introduced by Gromov in his seminal paper [2] to give a topological definition of the volume of certain families of Riemannian manifolds. In turn, it led Gromov to a new elegant proof of Mostow Rigidity for closed hyperbolic manifolds.

The positivity of the simplicial volume has many consequences for the geometry of the manifold, such as degree theorems or positivity of the minimal volume. While the simplicial volume vanishes for some manifolds, for example those with amenable fundamental group, it is now known to be positive for many families of manifolds, typically such exhibiting some nonpositive curvature features. One of the first result in this direction is due to Gromov:

Theorem 19. *(Gromov, [2]) Let M be a closed oriented locally symmetric space of noncompact type. If $\chi(M) \neq 0$, then $\|M\| > 0$.*

Answering a question of Gromov, Lafont and Schmidt later proved that the simplicial volume of any closed oriented locally symmetric space of noncompact type is positive, relying on previous computations of Connell and Farb that do not apply in the case when M has a local $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ or $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$, in which case the positivity of the simplicial volume follows from estimates by Thurston, respectively Savage and myself.

There is another way to generalize Theorem 19 by asking if $\chi(M) \neq 0$ implies $\|M\| > 0$ for any aspherical manifolds, which is also a question of Gromov:

Question 2. [3, Section 8.A_4] Let M be an aspherical manifold. Is it true that $\|M\| = 0$ implies $\chi(M) = 0$?

Or an even stronger version of the question:

Question 3. [3, Section 8.A_4] Does there exist a constant $C(n)$ depending only on the dimension n such that

$$|\chi(M)| \leq C(n) \cdot \|M\|,$$

for any aspherical manifold M?

Note that Question 2 is related to another question of Gromov asking if the vanishing of the simplicial volume of aspherical manifolds implies the vanishing of their ℓ^2-Betti numbers. Since the Euler characteristic is equal to the alternating sum of ℓ^2-Betti numbers, the latter question clearly implies Question 2. Note that in fact, the two questions would be equivalent if the Singer Conjecture, predicting the vanishing of all but the middle ℓ^2-Betti number, were true.

For a somehow modified version of the simplicial volume, I can answer Question 3: Define the *immersive simplicial volume* of a closed oriented manifold M as

$$\|M\|^{\mathrm{imm}} = \inf \left\{ \Sigma |a_\sigma| \mid [\Sigma a_\sigma \sigma] = [M] \in H_n(M, \mathbb{R}),\ \sigma : \Delta^n \to M \text{ immersive} \right\}.$$

where a singular simplex $\sigma : \Delta^n \to M$ is said to be immersive if it is the restriction of an immersive map defined on an open neighborhood of Δ^n. One has the trivial inequality

$$\|M\| \leq \|M\|^{\mathrm{imm}},$$

which is in general not an equality since for example, it is not difficult to show that $\|S^2\|^{\mathrm{imm}} = 2$, while $\|S^2\| = 0$. For this modified simplicial volume, I can answer Question 3 with constant $C(n) = 1$, and this for any manifold [1]:

Theorem 20. *Let M be any smooth closed oriented manifold. Then*

$$|\chi(M)| \leq \|M\|^{\mathrm{imm}}.$$

In view of Question 3, it is now natural to ask:

Question 4. Let M be an aspherical manifold. Is it true that

$$\|M\| = \|M\|^{\text{imm}}?$$

This is true for locally symmetric spaces modeled on $\mathbb{H}^n$ or $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ or their products, since these spaces admit an immersive straightening. Unfortunately in these cases the answer to Question 3 is already known.

To prove Theorem 20, one needs to exhibit a singular cocycle representing the Euler class of the tangent bundle of M, whose value on immersive singular simplices is bounded, in absolute value, by 1. This is done by a generalization of a classical formula of Schläfli.

References

[1] M. Bucher, *Euler characteristic, immersive simplicial volume and generalized Schlffli Formulas*, Preprint in preparation 2011.

[2] M. Gromov, *Volume and Bounded cohomology* Inst. Hautes E?tudes Sci. Publ. Math. No. 56, (1982), 599.

[3] M. Gromov, *Geometric Group Theory, Vol. 2: Asymptotic invariants of infinite groups*, edited by A. Niblo and Martin A. Roller, London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993.

On convex subsets of spherical buildings

Bernhard Leeb

(joint work with Carlos Ramos-Cuevas)

I report on work of Carlos Ramos-Cuevas the initial part of which is joint work.

In the unit sphere one observes that convex subsets are either small in the sense that they have circumradius $\leq \frac{\pi}{2}$, or they have special geometry, i.e. are geodesic subspheres. It is natural to ask whether a similar dichotomy holds in spherical buildings since these are CAT(1) spaces whose geometry is rigidified by the presence of "plenty" of embedded top-dimensional unit spheres. Are sufficiently large (radius $> \frac{\pi}{2}$) convex subsets of spherical buildings rigid, i.e. are they subbuildings? A consequence would be a fixed point property for isometric group actions on spherical buildings, namely that the existence of an invariant convex subset which is not a subbuilding implies the existence of a fixed point.

These questions seem considerably easier when one restricts to convex subcomplexes with respect to the natural polyhedral structure of the spherical building. The fixed point question in the case of convex subcomplexes has been asked by Tits already in the 50s and is referred to as his Center Conjecture. For spherical buildings of the classical types (A_n, B_n and D_n) it has been proven by Mühlherr and Tits [MT]. Regarding the exceptional types, the F_4 case has been announced by Parker and Tent (2008), written proofs for the F_4 and E_6 cases have been given by Ramos-Cuevas and myself [LR], and the (much harder) E_7 and E_8 cases have been proven by Ramos-Cuevas [R]. His results, together with the earlier ones and general results by Balser and Lytchak in dimension 2, imply the Center Conjecture for all thick spherical buildings and more generally for all spherical buildings without factors of type H_4.

References

[LR] B. Leeb, C. Ramos-Cuevas: *The center conjecture for spherical buildings of types F4 and E6*, Geometric and Functional Analysis 21 no. 3 (2011), 525-559.

[MT] B. Mühlherr, J. Tits, *The center conjecture for non-exceptional buildings*, J. Algebra 300, No. 2, 687-706 (2006).

[R] C. Ramos-Cuevas, *The center conjecture for thick spherical buildings*, Preprint, September 2009. arXiv:0909.2761v1.

Groups, hyperbolic spaces and Teichmüller spaces

Christopher J. Leininger

(joint work with Matt Clay, Richard P. Kent IV, Johanna Mangahas, Saul Schleimer)

There are a number of analogies between hyperbolic spaces and Teichmüller spaces of finite type surfaces, and the groups which naturally act on them. For example, they are both complete unique geodesic metric spaces, both are homeomorphic to a ball and admit isometry group–invariant compactifications by a "sphere at infinity". The action of the mapping class group, and in particular the actions of its subgroups, on Teichmüller space has some similarities with Kleinian groups acting on hyperbolic space. For example, the groups act properly discontinuously, the elements of the mapping class group admit a classification into 3 types similar to the classification into elliptic, parabolic and hyperbolic isometries of hyperbolic space, and for subgroups of the mapping class group, a dynamical decomposition of the action on the sphere at infinity is possible analogous to that studied for Kleinian groups. See the survey article [2] for a complete list of references.

In joint work with Richard Kent in [3, 4] we extended the notion of convex cocompactness discovered by Farb and Mosher and proved the equivalence of several formulations analogous to those used in the setting of Kleinian groups. There are subtleties involved, and in [5] for example, Kent and I construct a group using an analogue of the "Schottky construction" which is nonetheless not convex cocompact.

The importance of convex cocompactness in the world of Kleinian groups is that they are the most well behaved types of groups. For example, they form the *stable* class of groups in the sense that a small perturbation of a convex cocompact group is still convex cocompact. In the mapping class group, the importance of convex cocompactness comes from work of Farb–Mosher and Hamenstädt who prove that convex cocompactness for a subgroup of the mapping class group is equivalent to Gromov-hyperbolicity of the associated surface group extension. This group can be thought of as the fundamental group of a surface bundle with injective monodromy onto given by the subgroup.

There are a number of open questions regarding convex cocompactness. A first question is whether or not every finitely generated, purely pseudo-Anosov subgroup is convex cocompact. If this is true, then this gives a positive answer to Gromov's coarse hyperbolization question for fundamental groups of surface bundles. If it is false for a sufficiently nice subgroup, then it gives a negative answer to Gromov's question in the form of an example. In joint work with Kent and Saul Schleimer [6], we provide a positive answer to this question for a certain class of subgroups.

Another question is whether there are any non-virtually free convex cocompact groups. Because of the discussion above, this is equivalent to asking whether there are surface bundles over spaces B, which have Gromov hyperbolic fundamental group, when B is essentially more complicated than a graph. A particular instance of this question is whether or not there are surface bundles over surfaces with Gromov hyperbolic fundamental group, or surface bundles over closed hyperbolic n–manifolds with Gromov hyperbolic fundamental group. These latter questions can be rephrased as questions about homomorphisms of surface groups and hyperbolic n–manifold groups into the mapping class group, and equivariant maps of hyperbolic n–space into Teichmüller space with certain geometric properties. Specifically, we want these to be quasi-isometric embedding, quasi-convex into the thick part.

In work with Matt Clay and Johanna Mangahas [1], we constructed injective homomorphisms from surface groups into the mapping class group and equivariant maps of the hyperbolic plane into Teichmüller space which are quasi-isometric embeddings. Unfortunately, these fail to have quasi-convex image. In joint work with Schleimer [7], we were able to construct quasi-isometric embedding of any real hyperbolic n–space into some Teichmüller space which is quasi-convex and lies in the thick part. However, there is no group equivariance. In particular, the existence of surface bundles over closed hyperbolic n–manifolds with Gromov hyperbolic fundamental group when $n \geq 2$ is still open.

References

[1] Matt Clay and Christopher J. Leininger and Johanna Mangahas, The geometry of right angled Artin subgroups of mapping class groups, To appear in Groups Geom. Dyn..

[2] Richard P. Kent IV and Christopher J. Leininger, Subgroups of the mapping class group from the geometrical viewpoint a survey with R. P. Kent IV, In the tradition of Ahlfors–Bers IV, Contemp. Math., 432, Amer. Math. Soc. Providence, RI, 2007.

[3] Richard P. Kent IV and Christopher J. Leininger, Shadows of mapping class groups: capturing convex cocompactness, with R. P. Kent IV, Geom. Funct. Anal. 18 (2008), 1270–1325.

[4] Richard P. Kent IV and Christopher J. Leininger, Uniform convergence in the mapping class group with R. P. Kent IV, Ergodic Theory and Dynamical Systems (2008), 28, 1177–1195.

[5] Richard P. Kent IV and Christopher J. Leininger, A fake Schottky group in Mod(S) with R. P. Kent IV, In the tradition of Ahlfors-Bers. V, 185–196, Contemp. Math., 510, Amer. Math. Soc., Providence, RI, 2010.

[6] Richard P. Kent IV and Christopher J. Leininger and Saul Schleimer, Trees and mapping class groups with R. P. Kent IV and S. Schleimer, J. Reine Angew Math. 637 (2009), 1–21.

[7] Christopher J. Leininger and Saul Schleimer, Hyperbolic spaces in Teichmüller spaces, in preparation.

Generating Torelli groups

Dan Margalit
(joint work with Allen Hatcher)

The mapping class group $\mathrm{Mod}(S)$ of a surface S is the group of isotopy classes of orientation-preserving homeomorphisms of S. The mapping class group is a central object in low-dimensional topology, relating to the theory of surface bundles, moduli space, and 3-manifolds. The Torelli group $\mathcal{I}(S)$ is the subgroup of $\mathrm{Mod}(S)$ consisting of all elements that act trivially on $H_1(S_g;\mathbb{Z})$. As the action of $\mathrm{Mod}(S)$ on $H_1(S;\mathbb{Z})$ is linear, we can think of $\mathcal{I}(S)$ as embodying the more mysterious aspects of $\mathrm{Mod}(S)$.

A classical theorem of Dehn states that, when S is a closed, orientable surface, $\mathrm{Mod}(S)$ is generated by Dehn twists [3]. These are elements of $\mathrm{Mod}(S)$ supported on an annulus. The modern proof of Dehn's theorem is to consider the action of $\mathrm{Mod}(S)$ on the complex of curves, an abstract simplicial complex whose vertices are in bijection with the isotopy classes of simple closed curves in S. Birman and Powell proved an analogous theorem for $\mathcal{I}(S)$: it is generated by bounding pair maps, which are each supported on a pair of disjoint annuli [1, 5]. Birman and Powell proved this result via combinatorial group theory. Recently, Putman give a proof using group actions on complexes [6].

My work with Allen Hatcher gives a proof of the Birman–Powell result that is completely analogous to the curve complex proof that $\mathrm{Mod}(S)$ is generated by Dehn twists [4]. Instead of the complex of curves, we consider the complex of homologous curves, which is the subcomplex of the complex of curves consisting of all curves lying in a given homology complex. The key step is to show that the complex of homologous curves is connected.

Tara Brendle and I have shown that the hyperelliptic Torelli group (the subgroup of $\mathcal{I}(S)$ consisting of all elements that commute with some fixed hyperelliptic involution in $\mathrm{Mod}(S)$) has the property that every reducible element is a product of Dehn twists [2]. We would like to show that the hyperelliptic Torelli group is itself generated by Dehn twists, since this would give very concrete information about the topology of the branch locus of the period mapping from Torelli space to the Siegel upper half plane. We hope that the new proof of the Birman–Powell result will shed light on this problem.

References

[1] Joan S. Birman. On Siegel's modular group. *Math. Ann.*, 191:59–68, 1971.
[2] Tara Brendle and Dan Margalit. Point pushing, homology, and the hyperelliptic involution. *In preparation.*
[3] M. Dehn. Die Gruppe der Abbildungsklassen. *Acta Math.*, 69(1):135–206, 1938. Das arithmetische Feld auf Flächen.
[4] Allen Hatcher and Dan Margalit. Generating the Torelli group. *In preparation.*
[5] Jerome Powell. Two theorems on the mapping class group of a surface. *Proc. Amer. Math. Soc.*, 68(3):347–350, 1978.
[6] Andrew Putman. Cutting and pasting in the Torelli group. *Geom. Topol.*, 11:829–865, 2007.

On Weil–Petersson symmetries of Teichmüller spaces

Sumio Yamada

Let Σ_g be a closed topological surface of genus larger than one. We assume that Σ_g is equipped with some hyperbolic metric. In [6], we have studied geometry of the Weil-Petersson completed Teichmüller space $\overline{\mathcal{T}}$ of the surface Σ. It was shown that the metric completion $\overline{\mathcal{T}}$ is a CAT(0) space, and the $\overline{\mathcal{T}}$ has a stratification by its boundary sets. In [7], we have further introduced the Weil-Petersson-Coxeter space $D(\overline{\mathcal{T}}, \iota)$ of the surface Σ, also a CAT(0) space, which is geodesically complete. In this talk, we would like to demonstrate several features of these CAT(0) spaces which have some parallels to the subject of symmetric spaces of noncompact type.

We recall (see [5]) that the Teichmüller space of the surface Σ_g is the space of the hyperbolic metrics defined on Σ up to the diffeormorphism equivalence via the pull-back action $\mathcal{T}(\Sigma) = \mathcal{M}_{-1}/\mathrm{Diff}_0\Sigma$ where $\mathrm{Diff}_0\Sigma$ is the identity component of the full (orientation preserving) diffeomorphism group $\mathrm{Diff}\Sigma$. The Weil–Petersson metric on the Teichmüller space is the L^2 metric on the surface Σ for deformation tensors of the hyperbolic metric G; $\langle h_1, h_2\rangle_{\mathrm{WP}} = \int_\Sigma \langle h_1(x), h_2(x)\rangle_{G(x)}\ d\mu_G(x)$ where the tangency condition for the tensors h_1, h_2 are traceless and divergence-free with respect to G, which preserves the constant curvature condition as well as the perpendicularity to the diffeomorphism fibers. We denote by $d(x, y)$ the Weil–Petersson distance between the points x and y.

The Weil–Petersson completion $\overline{\mathcal{T}}$, a space of Cauchy sequences in $(\mathcal{T}, d)$, consists of the original Teichmüller space $\mathcal{T}$ as well as the bordification points of $\mathcal{T}$ so that Σ is allowed to have nodes, which are geometrically interpreted as simple closed geodesics of zero hyperbolic length. The completed space $\overline{\mathcal{T}}$ (also identified as augmented Teichmüller space by Bers and Abikoff) has the stratification

$$\overline{\mathcal{T}} = \cup_{\sigma\in C(\mathcal{S})}\mathcal{T}_\sigma$$

where the original Teichmüller space $\mathcal{T}$ is expressed as $\mathcal{T}_\emptyset$, and where $C(\mathcal{S})$ is the complex of curves. A k-complex σ in $C(\mathcal{S})$ consists of $k+1$ homotopy classes of mutually disjoint simple closed curves. And each strata $\mathcal{T}_\sigma$ is the Teichmüller space of the nodal surface Σ_σ. Here an important fact is that the set of nodal surfaces exactly corresponds to the set of admissible degenerations of conformal structures along Weil–Petersson geodesics where the surfaces are uniformized by hyperbolic metrics.

We showed in [6] that this stratification is very much compatible with the Weil–Petersson geometry. Namely for each collection $\sigma \in C(\mathcal{S})$, each boundary Teichmüller space $\mathcal{T}_\sigma$ is a Weil–Petersson geodesically convex subset of $\overline{\mathcal{T}}$. Here geodesic convexity means that given a pair of points in $\mathcal{T}_\sigma$, there is a distance-realizing Weil–Petersson geodesic segment connecting them lying entirely in $\mathcal{T}_\sigma$. Note that the Weil–Petersson convexity of the individual stratum says that each stratum is totally geodesic, namely each set is neither convex or concave with respect to other strata. The non-positive curvature implies the uniqueness of the geodesics.

In considering the Weil–Petersson symmetries, we first recall the Thurston classification theorem of elements of the mapping class group $\mathrm{Map}(\Sigma)$. An element γ of $\mathrm{Map}(\Sigma)$ is classified as one of the three types; 1) finite order, 2) reducible, 3) pseudo-Anosov (also called irreducible). The classification is relevant to the Weil–Petersson geometry in the sense that the pseudo-Anosov elements are hyperbolic (Wolpert[3], G.Daskalopoulos- R.Wentworth[2].) namely the infimum of the translation distance $d(x, \gamma x)$ is achieved on a pseudo-Anosov axis in $\mathcal{T}$, and the reducible (by σ) elements are loxodromic in the respective stratum $\mathcal{T}_\sigma$.

In order to introduce a new set of symmetries, we first construct an auxiliary space. Jacques Tits has introduced abstract "Coxeter group" (W, S) generated by a set of reflections S, and a collection of relations among the reflections $\{(ss')^{m(s,s')}\}$. Here $m(s, s')$ denotes the order of ss' and the relations range over all unordered pairs $s, s' \in S$ with $m(s, s') \neq \infty$. In other words, $m(s, s') = \infty$ means no relation between s and s'. The linchpin connecting the Weil–Petersson geometry of $\overline{\mathcal{T}}$ and the Coxeter theory is the following theorem of Wolpert's[4].

Theorem *Given a point p in $\mathcal{T}_\sigma \subset \overline{\mathcal{T}}$, representing a nodal surface Σ_σ, the Alexandrov tangent cone with respect to the Weil–Petersson distance function is isometric to $\mathbf{R}^{|\sigma|}_{\geq 0} \times T_p\mathcal{T}_\sigma$, where $\mathbf{R}^{|\sigma|}_{\geq 0}$ is the orthant in $\mathbf{R}^{|\sigma|}$ with the standard metric.*

The significance of this result in our context is that it describes the geometry around the vertices, given as the Weil–Petersson tangent cone angles, when $\overline{\mathcal{T}}$ is seen as a convex polygon. This picture specifies a particular choice of the Coxeter matrix. Namely for each σ with $|\sigma| = 1$, one can *reflect* $\overline{\mathcal{T}}$ across the totally geodesic stratum $\overline{\mathcal{T}}_\sigma$. Now for $\tau = \sigma \cup \sigma'$ with σ and σ' representing a pair of disjoint simple closed geodesics, the relation $m(s_\sigma, s_{\sigma'}) = 2$ has a geometric meaning where four copies of $\overline{\mathcal{T}}$ can be glued together around a point $q \in \mathcal{T}_\tau$ to form a space whose tangent cone at q is a union of four copies of $\mathbf{R}^2_{\geq 0} \times T_p\mathcal{T}_\tau$ (each $\mathbf{R}^2_{\geq 0}$ is regarded as a quadrant in the plane) isometric to $\mathbf{R}^2 \times T_p\mathcal{T}_\tau$ on which the reflections $s_\sigma, s_{\sigma'}$ act as reversing of the orientations of the x, y axes for $\mathbf{R}^2$. Hence we define Coxeter group (W, S) by letting the generating set S be the elements of $\mathcal{S}$, and the relations among the generating set are specified by the Coxeter matrix whose components satisfying i) $m_{ss} = 1$, ii) if $s \neq s'$, and if there is some simplex σ in $C(\mathcal{S})$ containing s and s', then define $m_{ss'} = 2$, and iii) if $s \neq s'$, and if the geodesics representing s and s' intersect on Σ_0 then $m_{ss'} = \infty$. This group has a *geometric realization* $D(\overline{\mathcal{T}}, \iota)$ defined as the set which is the quotient of $W \times \overline{\mathcal{T}}$ by the following equivalence relation

$$(g, y) \sim (g', y') \Longleftrightarrow y = y' \text{ and } g^{-1}g' \in W_{\sigma(y)}$$

where $\overline{\mathcal{T}}_{\sigma(y)}$ denotes the smallest stratum containing y, and the subgroup $W_{\sigma(y)}$ fixes the stratum $\overline{\mathcal{T}}_{\sigma(y)}$ pointwise. We write $[g, y]$ to denote the equivalence class of (g, y). Furthermore ι denotes a simple morphsm of groups, which specifies a system of subgroups $W_\sigma \subset W$, compatible with the poset structure of the complex of curves $\mathcal{C}(\mathcal{S})$ (see [7] for details.)

The remarkable phenomena here is that despite of the fact that the generating set is infinite, we have a geometric realization of the Coxeter group (W, S)

action (which is very far from linear, but still Weil–Petersson isometric) on a space modeled on a finite dimensional space $\mathcal{T}$, albeit the partial bordification $\overline{\mathcal{T}}$ encodes non-locally compact geometry due to the singular behavior of the Weil–Petersson metric tensor. The space obtained by the action of the Coxeter group on $\overline{\mathcal{T}}$, which we will call development $D(\overline{\mathcal{T}}, \iota)$, is then shown to be CAT(0) via the Cartan-Hadamard theorem for metric spaces, and also to be geodesically complete. Furthermore, this construction is used to show that the development $D(\overline{\mathcal{T}}, \iota)$, and it covex subset $\overline{\mathcal{T}}$, are of finite rank, implying existence of upper bounds for the dimensions of flats dependent of genus g.

As the Coxeter group W is generated by $\mathcal{S}$, and the group W is completely determined by the Coxeter matrix $[m_{st}]_{s,t\in\mathcal{S}}$, it follows that each element γ in $\widehat{\mathrm{Map}_\Sigma}$ induces an automorphism of W. Such an automorphism of W is called *diagram automorphism* [1]. As for the new set of Weil–Petersson symmetries, the formalism laid out in [1] gives us a natural action (Proposition 9.1.7) of the semi-direct product $G := W \rtimes \widehat{\mathrm{Map}_\Sigma}$ on the development $D(\overline{\mathcal{T}}, \iota)$ as follows: given $u = (g, \gamma) \in G$ and $[g', y] \in D(\overline{\mathcal{T}}, \iota)$,

$$u \cdot [g', y] := [g\gamma(g'), \gamma y]$$

where $\gamma(g')$ is the image of g' by the automorphism of W induced by $\gamma : C(\mathcal{S}) \to C(\mathcal{S})$.

Another point which justifies the analogy with the classical Coxeter theory is that Wolpert([4]) has shown that the Weil–Petersson completion $\overline{\mathcal{T}}$ is the closed convex hull of the vertex set given by the *zero dimensional* Teichmüller spaces of the maximally degenerate surfaces $\{\mathcal{T}_\sigma : |\sigma| = 3g - 3\}$. Each face/stratum $\overline{\mathcal{T}}_\sigma$ of $\overline{\mathcal{T}}$ is then a simplex spanned by the subset of those vertex sets, which lies as a complete convex subset in $\overline{\mathcal{T}}$; a picture analogous to the standard simplicial complex theory. This situation encourages us to treat Teichmüller spaces from the viewpoint of convex geometry within the "ambient space" $D(\overline{\mathcal{T}}, \iota)$, a direction currently under investigation [8].

References

[1] M.Davis. *The Geometry and Topology of Coxeter Groups.* Princeton Univ.Press, 2007.

[2] G. Daskalopoulos and R. Wentworth. Classification of Weil-Petersson isometries American J. of Math., **125**,(2003), 941-975

[3] S. Wolpert. Geometry of the Weil-Petersson completion of Teichmueller space. Surveys in Differential Geometry VIII: Papers in Honor of Calabi, Lawson, Siu and Uhlenbeck, editor S. T. Yau, International Press, (2003).

[4] S. Wolpert. Behavior of geodesic-length functions on Teichmüller space, J. Diff. Geom. **79** (2008),277–334.

[5] S.Yamada. Weil-Petersson convexity of the energy functional on classical and universal Teichmüller spaces. J.Diff.Geom. **51** (1999) 35-96.

[6] S.Yamada. On the geometry of Weil-Petersson completion of Teichmüller spaces. Math.Res.Let. **11** (2004) 327-344.

[7] S.Yamada. Weil–Petersson geometry of Teichmüller-Coxeter complex and its finite rank property. Geometria Dedicata, **145** (2010) 43-63.

[8] S.Yamada. Funk-Hilbert geometry on Teichmüller spaces. In preparation.

The Congruence Subgroup Problem and the Fundamental Group of Reductive Borel-Serre Compactifications

John Scherk

(joint work with Lizhen Ji, V. Kumar Murty, Les Saper)

Let k be a number field and let S be a finite set of places of k which includes the infinite places. Suppose that $\mathbf{G}$ is an absolutely almost simple, simply connected algebraic group over k. Let $C(S, G)$ be the congruence subgroup kernel of $\mathbf{G}$. Then the following has been established by Raghunathan, Prasad, Gille and others:

Theorem 21. *If S-rank $\mathbf{G} \geq \mathbf{2}$ and if k_v-rank $\mathbf{G} \geq \mathbf{1}$ for all finite places $v \in S$, then $C(S, G)$ is finite. Furthermore, $C(S, G)$ is a quotient of $\mu(k)$, the roots of unity of k.*

Now let $\mathcal{O}$ be the ring of S-integers of k. For any ideal $\mathfrak{a} \in \mathcal{O}$ let $\Gamma(\mathfrak{a}) \subset \mathbf{G}(\mathcal{O})$ be the congruence subgroup defined by $\mathfrak{a}$ (defined with respect to some faithful representation of $\mathbf{G}$ over k). For any S-arithmetic subgroup Γ of $\mathbf{G}$, denote by $E\Gamma$ the normal subgroup of unipotent elements in $\mathbf{G}(k)$. Then Ragunathan and Venkataramana show that

Theorem 22. *If S-rank $\mathbf{G} \geq \mathbf{2}$, then*

$$C(S, G) \cong \varprojlim_{\mathfrak{a}} \Gamma(\mathfrak{a})/E\Gamma(\mathfrak{a})$$

We noticed that in some examples, the computations of the fundamental group of the Baily-Borel compactification of Hermitian locally symmetric spaces also involved the subgroup of unipotents $E\Gamma$. This lead us to a general result which relates fundamental groups of compactifications of locally symmetric spaces to $C(S, G)$ via subgroups of unipotents.

Let X_∞ be the product of the symmetric spaces associated with $\mathbf{G}(k_v)$ where v is an infinite place, and let X_v be the Bruhat-Tits building associated with $\mathbf{G}(k_v)$ where $v \in S_f$, the set of finite places in S. Set

$$X = X_\infty \times \prod_{v \in S_f} X_v$$

The group $G_S = G_\infty \times \prod_{v \in S_f} \mathbf{G}(k_v)$ acts properly on X. We can embed $\mathbf{G}(k)$ diagonally in this group. This also defines an action of Γ on X. Now let $\bar{X}_\infty^{RBS}$ denote the reductive Borel-Serre partial compactification of X_∞. Then define

$$\bar{X}^{RBS} = \bar{X}_S^{RBS} = \bar{X}_\infty^{RBS} \times \prod_{v \in S_f} X_v$$

The action of Γ extends to a discontinuous action on this space, and the quotient space $\Gamma \backslash \bar{X}^{RBS}$ is a compact Hausdorff space.

In general, suppose that Γ is a group acting continuously on a topological space Y. For each point $y \in Y$, let $\Gamma_y = \{ g \in \Gamma \mid gy = y \}$ be the stabilizer subgroup of y in Γ. Then the fixed subgroup Γ_f is defined to be the subgroup generated by the stabilizer subgroups Γ_y for all $y \in Y$.

Proposition 7. *We have:* $E\Gamma \subseteq \Gamma_{f,RBS}$. *Furthermore, if* Γ *is neat then equality holds.*

Our result is:

Theorem 23. *For any* S*-arithemtic group* Γ, $\pi_1(\Gamma\backslash\bar{X}^{RBS}) \cong \Gamma/\Gamma_f$. *If* Γ *is neat then* $\pi_1(\Gamma\backslash\bar{X}^{RBS}) \cong \Gamma/E\Gamma$.

There is an analogous result for Satake compactifications. The two theorems above imply that under the given conditions these fundamental groups are finite.

The proof is based on the following general result due to Grosche and Armstrong:

Proposition 8. *Let* Y *be a simply connected topological space and* Γ *a discrete group acting on* Y*. Assume that either*

(1) *the* Γ*-action is discontinuous and admissible, or that*
(2) *the* Γ*-action is proper and* Y *is a locally compact metric space.*

Then the natural morphism $\Gamma \to \pi_1(\Gamma\backslash Y)$ *induces an isomorphism* $\Gamma/\Gamma_f \cong \pi_1(\Gamma\backslash Y)$.

A continuous surjection $p\colon Y \to X$ of topological spaces is *admissible* if for any path ω in X with initial point x_0 and final point x_1 and for any $y_0 \in p^{-1}(x_0)$, there exists a path $\tilde{\omega}$ in Y starting at y_0 and ending at $y_1 \in p^{-1}(x_1)$ such that $p \circ \tilde{\omega}$ is homotopic to ω relative to the endpoints. An action of a group Γ on a topological space Y is *admissible* if the quotient map $Y \to \Gamma\backslash Y$ is admissible.

References

[1] L. Ji, V.K. Murty, J. Scherk, L. Saper, *The Congruence Subgroup Problem and the Fundamental Group of the Reductive Borel-Serre Compactification*, preprint: http://arxiv.org/pdf/1106.4810v1 .

Quantum representations of mapping class groups and Zariski density

Louis Funar

Our first aim is to explain a largeness result for images of quantum representations of mapping class groups in genus at least 3. The main motivation is the construction of large families of finite quotients of (central extensions of the) mapping class groups. Some results in this direction are already known. In [6] we proved that the images are infinite and non-abelian (for all but finitely many explicit cases) using earlier results of Jones who proved in [12] that the same holds true for the braid group representations factorizing through the Temperley-Lieb algebra at roots of unity. Masbaum then found in [14] explicit elements of infinite order in the image. General arguments concerning Lie groups actually show that the image should contain a free non-abelian group. Furthermore, Larsen and Wang showed (see [13]) that the image of the quantum representations of the mapping class groups at roots of unity of the form $\pm\exp\left(\frac{2(r+1)\pi i}{4r}\right)$, for prime $r \geq 5$, is dense in the projective unitary group.

In [7] the authors proved that although the images are large in the sense that they contain (explicit) free non-abelian groups, from another viewpoint these images are small because they are of infinite index in the group of unitary matrices with cyclotomic integers entries. The latter group can be embedded as an irreducible higher rank lattice in a semi-simple Lie group $\mathbb{G}_p$ (depending on the genus and the order of the roots of unity) obtained by restriction of scalars. In this talk we will strengthen the largeness property above by showing that, in general, the image of a quantum representation is Zariski dense in the non-compact group $\mathbb{G}_p$.

In order to be precise we have to specify the quantum representations we are considering. Recall that in [1] the authors defined the TQFT functor $\mathcal{V}_p$, for every $p \geq 3$ and a primitive root of unity A of order $2p$. These TQFT should correspond to the so-called $SU(2)$-TQFT, for even p and to the $SO(3)$-TQFT, for odd p (see also [13] for another $SO(3)$-TQFT).

Definition 11. Let $p \in \mathbb{Z}_+$, $p \geq 3$ and A be a primitive $2p$-th root of unity. The quantum representation $\rho_{p,A}$ is the projective representation of the mapping class group associated to the TQFT $\mathcal{V}_p$ at the root of unity A. We denote therefore by $\widetilde{\rho}$ the linear representation of the central extension $\widetilde{M_g}$ of the mapping class groups M_g (of the genus g closed orientable surface) which resolves the projective ambiguity of $\rho_{p,A}$ (see [10, 16]). Furthermore $N(g,p)$ will denote the dimension of the space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the closed orientable surface of genus g.

Remark 1. *The unitary TQFTs arising usually correspond to the following choices of the root of unity:*

$$A_p = \begin{cases} -\exp\left(\frac{2\pi i}{2p}\right), & \text{if } p \equiv 0 (\operatorname{mod} 2); \\ -\exp\left(\frac{(p+1)\pi i}{p}\right), & \text{if } p \equiv 1 (\operatorname{mod} 2). \end{cases}$$

Gilmer and Masbaum proved in [11] that the mapping class group preserves a certain free lattice within the space of conformal blocks associated to the $SO(3)$-TQFT. Let us introduce the following notation. For $p \geq 5$ an odd prime we denote by $\mathcal{O}_p$ the ring of integers in the cyclotomic field $\mathbb{Q}(\zeta_p)$, where ζ_p is a primitive p-th root of unity. Thus $\mathcal{O}_p = \mathbb{Z}[\zeta_p]$, if $p \equiv -1(\operatorname{mod} 4)$ and $\mathcal{O}_p = \mathbb{Z}[\zeta_{4p}]$, if $p \equiv 1(\operatorname{mod} 4)$.

The main result of [11] states that, for every odd prime $p \geq 5$, there exists a free $\mathcal{O}_p$-lattice $S_{g,p}$ in the $\mathbb{C}$-vector space of conformal blocks associated by the TQFT $\mathcal{V}_p$ to the genus g closed orientable surface and a non-degenerate Hermitian $\mathcal{O}_p$-valued form on $S_{g,p}$ such that (a central extension of) the mapping class group preserves $S_{g,p}$ and keeps invariant the Hermitian form. Therefore the image of the mapping class group consists of unitary matrices (with respect to the Hermitian form) with entries in $\mathcal{O}_p$. Let $P\mathbb{U}(\mathcal{O}_p)$ be the group of all such matrices, up to scalar multiplication.

A natural question is to compare the image $\rho_p(M_g)$ and the discrete group $P\mathbb{U}(\mathcal{O}_p)$. The main result of [7] shows that the image is small with respect to the whole group:

Theorem 24 ([7]). *Suppose that $g \geq 4$ and $p \notin \{3, 4, 5, 8, 12, 16, 24, 40\}$. Suppose moreover that in the case $p = 8k$ with k odd there exists a proper divisor of k which is greater than or equal to 7. Then the group $\rho_{p,A}(M_g)$ is not an irreducible lattice in a higher rank semi-simple Lie group. In particular, if $p \geq 7$ is an odd prime, then $\rho_{p,A}(M_g)$ is of infinite index in $P\mathbb{U}(\mathcal{O}_p)$.*

It is known that $P\mathbb{U}(\mathcal{O}_p)$ is an irreducible lattice in a semi-simple Lie group $P\mathbb{G}_p$ obtained by the so-called restriction of scalars construction from the totally real cyclotomic field $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$, ζ_p being a p-th root of unity, to $\mathbb{Q}$. Specifically, let us denote by $\mathbb{G}_p$ the product $\prod_{\sigma \in S(p)} S\mathbb{U}^{\sigma}$. Here $S(p)$ stands for a set of representatives for the classes of complex valuations σ of $\mathcal{O}_p$ modulo complex conjugacy. The factor $S\mathbb{U}^{\sigma}$ is the special unitary group associated to the Hermitian form conjugated by σ, thus corresponding to some Galois conjugate root of unity. Denote also by $\widetilde{\rho}_p$ and ρ_p the representations $\prod_{\sigma \in S(p)} \widetilde{\rho}_{p,\sigma(A_p)}$ and $\prod_{\sigma \in S(p)} \rho_{p,\sigma(A_p)}$, respectively. When p is an odd prime $p \geq 5$ and $g \geq 3$ then it is known that $\widetilde{\rho}_{p,A_p}$ takes values in $S\mathbb{U}$ (see [2]).
Notice that the real Lie group $\mathbb{G}_p$ is a semi-simple algebraic group defined over $\mathbb{Q}$. Eventually, the main density result from [6] can be stated now as follows:

Theorem 25. *Suppose that $g \geq 3$ and $p \geq 5$ is an odd prime. Then $\widetilde{\rho}_p(M_g)$ is a discrete Zariski dense subgroup of $\mathbb{G}_p$.*

Remark 2. *A similar result holds for the $SU(2)$-TQFT. Specifically let $p = 2r$ where $r \geq 5$ is prime. According to ([1], 1.5) there is an isomorphism of TQFTs between $\mathcal{V}_{2r}$ and $\mathcal{V}'_2 \otimes \mathcal{V}_r$, and hence the projection on the second factor gives us a homomorphism $\pi : \rho_{2r}(M_g) \to \mathbb{G}_r$. Furthermore the image of the TQFT representation associated to $\mathcal{V}'_2$ is finite. Therefore the $\pi \circ \rho_{2r}(M_g)$ is a discrete Zariski dense subgroup of $\mathbb{G}_r$. Notice that the result holds also for $g = 2$ and prime $p \geq 5$ using the modifications from [7] in the constructions of free non-abelian subgroups in the image. We will skip the details.*

We will now consider the Johnson filtration by the subgroups $I_g(k)$ of the mapping class group M_g of the closed orientable surface of genus g, consisting of those elements having a trivial outer action on the k-th nilpotent quotient of the fundamental group of the surface, for some $k \in \mathbb{Z}_+$.

The main application of our main density result is the following consequence of the Nori-Weisfeiler strong approximation theorem (see [18]):

Theorem 26. *For every $g \geq 3$, prime $p \geq 5$ and $k \geq 1$ there exists some homomorphism $\widetilde{M_g} \to \mathbb{G}_p(\mathbb{Z}/q^k\mathbb{Z})$, whose restriction to $I_g(3)$ is surjective for all large enough primes q. In particular, the surjectivity holds also for $\widetilde{M_g}$, the Torelli group $I_g(1)$ and the Johnson kernel $I_g(2)$, respectively.*

Corollary 2. *For any prime p, $g \geq 4$ and $k \geq 1$ there is a homomorphism $M_g \to \mathbb{PG}_p(\mathbb{Z}/q^k\mathbb{Z})$, which is surjective for all large enough primes $q \geq 5$. Here $\mathbb{PG}_p$ is the product of the projective unitary groups whose associated special unitary groups occur as factors of $\mathbb{G}_p$.*

Remark 3. (1) *The first construction of finite quotients of mapping class group by this method was given in* [17].

(2) *A stronger result, namely that every finite group embeds in some finite quotient of the genus $g \geq 3$ mapping class group, was obtained independently by Masbaum and Reid (see* [15]*). This answers a question of U. Hamenstaedt.*

(3) *The set of finite quotients of a particular M_g (with $g \geq 4$) provided by Theorem 26 is rather large. Indeed, among the factors of the semi-simple groups $P\mathbb{G}_p$, p running over the primes, we can find indefinite unitary groups $PU(m,n)$, with arbitrarily large m. In particular, the alternate group on m elements is contained into $PU(m,n)(\mathbb{Z}/p\mathbb{Z})$ and hence into some finite quotient of M_g. This gives an alternate proof of Hamenstaedt's conjecture above, first proved by Masbaum and Reid in* [15].

(4) *In* [8] *we already obtained results showing that a given mapping class group has many more finite quotients than the family of all symplectic groups, as it can be measured by their 2-homology groups.*

(5) *The number $n(p)$ of the non-compact factors in $\mathbb{G}_p$ goes to infinity with p.*

Another consequence of the main density theorem is the following description of the normalizer $N_{\mathbb{G}_p}(\widetilde{\rho}_p(\widetilde{M_g}))$ of $\widetilde{\rho}_p(\widetilde{M_g})$ within $\mathbb{G}_p$. This will be a consequence of a deep result of Eskin and Margulis from [3], valid for arbitrary Zariski dense subgroups of higher rank lattices, at it was pointed out to us by Yves Benoist. Specifically, we proved in [9] the following:

Theorem 27. *Let $g \geq 4$ and $p \geq 7$ prime. Then the normalizer $N_{\mathbb{G}_p}(\widetilde{\rho}_p(\widetilde{M_g}))$ contains the lattice normalizer $N_{\mathbb{G}_p(\mathbb{Z})}(\widetilde{\rho}_p(\widetilde{M_g})) \subset \mathbb{G}_p(\mathbb{Z})$ as finite index subgroup and therefore is a discrete subgroup of $\mathbb{G}_p$ of infinite covolume.*

It seems that the arithmetic properties of the groups $\widetilde{\rho}_p(\widetilde{M_g})$ are worth to further study. We formulate the following questions:

(**1**) For given g and p find the bad primes q in the sense of Nori-Weisfeiler theorem, namely for which the homomorphism $\widetilde{\rho}_p(\widetilde{M_g}) \to \mathbb{G}_p(\mathbb{Z}/q\mathbb{Z})$ is not surjective. The analogous question for the Apollonian group was recently solved by E.Fuchs in [5], but the methods used there do not extend to more general Zariski dense subgroups of Lie groups.

(**2**) Determine the kernel of the map between the pro-finite completions $\widehat{\widetilde{\rho}_p(\widetilde{M_g})} \to \widehat{\mathbb{G}_p(\mathbb{Z})}$.

(**3**) Prove that the inclusion $\widetilde{\rho}_p(\widetilde{M_g}) \hookrightarrow \mathbb{G}_p$ is locally rigid.

References

[1] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology 34 (1995), 883–927.

[2] N. Dunfield and H. Wong, *Quantum invariants of random 3-manifolds*, arXiv:1009.1653.

[3] A. Eskin and G. Margulis, *Recurrence properties of random walks on finite volume homogeneous manifolds*, Random walks and geometry, (V.A. Kaimanovich, K. Schmidt and W. Woess, Editors), 431–444, Walter de Gruyter 2004.

[4] M. H. Freedman, K. Walker and Z. Wang, *Quantum* SU(2) *faithfully detects mapping class groups modulo center*, Geom. Topol. 6 (2002), 523–539.
[5] E. Fuchs, *Strong approximation in the Apollonian group*, preprint 2010.
[6] L. Funar, *On the TQFT representations of the mapping class groups*, Pacific J. Math. 188 (2)(1999), 251–274.
[7] L. Funar and T. Kohno, *On images of quantum representations of mapping class groups*, arXiv:0907.0568.
[8] L. Funar and W. Pitsch, *Finite quotients of symplectic groups vs mapping class groups*, arxiv:1103.1855.
[9] L. Funar, *Zariski density and finite quotients of mapping class groups*, arxiv:1106.4165.
[10] S. Gervais, *Presentation and central extensions of mapping class groups*, Trans. Amer. Math. Soc. 348 (1996), 3097–3132.
[11] P. Gilmer and G. Masbaum, *Integral lattices in TQFT*, Ann. Sci. Ecole Norm. Sup. (4) 40 (2007), 815–844.
[12] V. F. R. Jones, *Braid groups, Hecke algebras and type II_1 factors*, in Geometric methods in operator algebras, Proc. US-Japan Seminar, (H. Araki, E. G. Effros Eds.), 1986, 242–271. (2011), 11–40.
[13] M. Larsen and Z. Wang, *Density of the SO(3) TQFT representation of mapping class groups*, Comm. Math. Phys. 260 (2005), 641–658.
[14] G. Masbaum, *An element of infinite order in TQFT-representations of mapping class groups*, Low-dimensional topology (Funchal, 1998), 137–139, Contemp. Math., 233, Amer. Math. Soc., Providence, RI, 1999.
[15] G. Masbaum and A. Reid, *All finite groups are involved in the mapping class groups*, arXiv:1106.4261.
[16] G. Masbaum and J. Roberts, *On central extensions of mapping class groups*, Math. Ann. 302 (1995), 131–150.
[17] G. Masbaum, *On representations of mapping class groups in integral TQFT*, Oberwolfach Reports, Vol. 5, issue 2, 2008, pp. 1202-1205.
[18] M. Nori, *On subgroups of $GL_n(\mathbb{F}_p)$*, Inventiones Math. 88 (1987), 257–275.

Thurston compactifications of spaces of marked lattices and of the Torelli space

Thomas Haettel

The content of this report comes from the preprint [7].

1. Thurston compactification of spaces of marked lattices

Let $g \geq 1$ be an integer, fix b the standard symplectic form on $\mathbb{R}^{2G}$, and the standard euclidian norm $\|\cdot\|$. If $\Lambda \subset \mathbb{R}^{2g}$ is a lattice, it is said to be b-self-dual, or symplectic, if

$$\Lambda^* := \{x \in \mathbb{R}^{2g} \ : \ \forall y \in \Lambda, b(x,y) \in \mathbb{Z}\} = \Lambda.$$

Consider the space of isometry classes of marked unimodular symplectic lattices of $\mathbb{R}^{2g}$:

$$\mathcal{E} = \{f : \mathbb{Z}^{2g} \to \mathbb{R}^{2g} \ : \ f(\mathbb{Z}^{2g}) \text{ is a covolume 1 symplectic lattice of } \mathbb{R}^{2g}\}/\text{isometry}.$$

It is the symmetric space $\mathcal{E} = U(g)\backslash Sp_{2g}(\mathbb{R})$ of $Sp_{2g}(\mathbb{R})$, also known as the Siegel upper half-plane, and comes with a natural action of $Sp_{2g}(\mathbb{Z})$.

Let us copy the definition of the Thurston compactification of the Teichmller space (see [1] for instance) in this setting, by considering the $Sp_{2g}(\mathbb{Z})$-equivariant embedding

$$\begin{aligned} \phi : \mathcal{E}^b &\to \mathbb{P}({\mathbb{R}_+}^{\mathbb{Z}^{2g}}) \\ [f] &\mapsto [u \in \mathbb{Z}^{2g} \mapsto \|f(u)\|]. \end{aligned} \tag{1}$$

Define the closure of its image $\overline{\mathcal{E}}^T = \overline{\phi(\mathcal{E})}$ to be the Thurston compactification of the symmetric space $\mathcal{E}$.

Let us compare it with the Satake compactification associated with the natural representation of $Sp_{2g}(\mathbb{R})$ on $\mathbb{R}^{2g}$, namely the closure $\overline{\mathcal{E}}^S$ of the embedding

$$\begin{aligned} \mathcal{E} = U(g)\backslash Sp_{2g}(\mathbb{R}) &\to \mathbb{P}(\mathrm{Sym}_{2g}(\mathbb{R})) \\ U(g)h &\mapsto [{}^t hh], \end{aligned} \tag{2}$$

where $\mathbb{P}(\mathrm{Sym}_{2g}(\mathbb{R})$ denotes the projective space of the symmetric matrices of size $2g$.

Theorem 28. *The Thurston compactification $\overline{\mathcal{E}}^T$ and the Satake compactification $\overline{\mathcal{E}}^S$ of the symmetric space $\mathcal{E}$ are $Sp_{2g}(\mathbb{Z})$-isomorphic.*

This isomorphism extends at least to all classical symmetric spaces of non-compact type, and to the symmetric space of non-compact of the exceptional Lie group $E_{6(-26)}$.

2. Thurston compactification of the Torelli space

Let S be a closed surface of genus $g \geq 1$. The Torelli space $Tor(S)$ of S is the quotient of the Teichmller space of S by the Torelli group, which is the subgroup of the mapping class group of S which has trivial action in homology. It is the kernel of the surjective map from the mapping class group to $Sp_{2g}(\mathbb{Z})$, so the Torelli space comes with a natural action of $Sp_{2g}(\mathbb{Z})$. The Torelli groups are still not completely understood, see [2] for instance.

Fix a marked hyperbolic surface $h : S \to X$, and let us define a euclidian norm on $H_1(X, \mathbb{R})$. The Hodge theorem identifies the vector space of harmonic 1-forms with the first cohomology group $H^1(X, \mathbb{R})$, so the L^2 product of harmonic forms defines an inner product on $H^1(X, \mathbb{R})$, and hence an inner product on $H_1(X, \mathbb{R})$ (see [3] for a comparison between this euclidian norm and the stable norm).

Consider the mapping

$$\begin{aligned} \psi : Tor(S) &\to P({\mathbb{R}_+}^{H_1(S,\mathbb{Z})}) \\ [X, h] &\mapsto [u \in H_1(S, \mathbb{Z}) \mapsto \|h_*(u)\|]. \end{aligned} \tag{3}$$

Let us define the closure of this map to be the Thurston compactification $\overline{Tor(S)}^T$ of the Torelli space.

We will compare it to a Satake compactification of the Siegel upper half-plane. Consider the mapping

$$\begin{aligned} p : Tor(S) &\to \mathcal{E} \\ [X, h] &\mapsto [h_* : H_1(S, \mathbb{Z}) \to H_1(X, \mathbb{R})], \end{aligned} \tag{4}$$

it is well-defined since the lattice $h_*(H_1(S,\mathbb{Z}))$ is symplectic with respect to the intersection form on $H_1(X,\mathbb{R})$.

Theorem 29. *The map p is the classical period map (see* [2], [4], [5], [6]*).*

We can define the Satake compactification of the Torelli space by $\overline{Tor(S)}^S = \overline{p(Tor(S))} \subset \overline{\mathcal{E}}^S$. Since $\psi = \phi \circ p$, we get the following.

Theorem 30. *The Thurston compactification $\overline{Tor(S)}^T$ and the Satake compactification $\overline{Tor(S)}^S$ of the Torelli space $Tor(S)$ are $Sp_{2g}(\mathbb{Z})$-isomorphic.*

3. Partial stratification of the boundary

We will now describe a subset of the boundary of the compactification, namely the closure of the image of the map

$$\begin{aligned} \widetilde{\psi} : Tor(S) &\to {\mathbb{R}_+}^{H_1(S,\mathbb{Z})} \\ [X,h] &\mapsto (u \in H_1(S,\mathbb{Z}) \mapsto \|h_*(u)\|). \end{aligned} \tag{5}$$

Let K^{sep} denote the complex of separating simple closed curves, and let $\sigma = \{\gamma_1, \ldots, \gamma_k\}$ be a $(k-1)$-simplex. Topologically, $S \cup \sigma$ is the disjoint union of $k+1$ surface with punctures, $(S_i \cup P_i)_{0 \le i \le k}$. Consider the application

$$\begin{aligned} \widetilde{\psi_\sigma} : Tor_\sigma(S) = \textstyle\prod_{i=0}^k Tor(S_i \cup P_i) &\to {\mathbb{R}_+}^{H_1(S,\mathbb{Z})} \\ ([X_j,h_j])_{j\in[0,k]} &\mapsto \left\{ u \mapsto \sqrt{\textstyle\sum_{j=0}^k \|(h_j)_*(u_j)\|^2} \right\}, \end{aligned} \tag{6}$$

where $u = \sum_{j=0}^k u_j \in H_1(S,\mathbb{Z}) = \oplus_{j=0}^k H_1(S_j,\mathbb{Z})$.

Theorem 31. *We have the following stratification*

$$\overline{\widetilde{\psi}(Tor(S))} = \bigsqcup_{\sigma \in K^{sep}} \widetilde{\psi_\sigma}(Tor_\sigma(S)).$$

One would like to understand the full boundary of this compactification, and the relationship with the Thurston compactification of the Teichmller space.

References

[1] A. Fathi, F. Laudenbach and V. Poenaru – *Travaux de Thurston sur les surfaces* , Astrisque **66-67**, Soc. Math. France, 1979.

[2] B. Farb (ed.) – *Problems on mapping class groups and related topics* , Proc. of Symp. in Pure Math. **74**, Amer. Math. Soc., 2006.

[3] D. Massart – Normes stables des surfaces, PhD Thesis, ENS Lyon, 1996, HAL : tel-00589624, version 1.

[4] P. A. Griffiths – Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping., *Publ. Math. IHÉS* **38** (170), p. 125–180.

[5] Debarre, Olivier – *Periods and moduli* , Current Developments in Algebraic Geometry, MSRI Publications, 2010.

[6] G. Mess – The Torelli groups for genus 2 and 3 surfaces, *Topology* **31** (1992), p. 775–790.

[7] ———, Compactification de Thurston d'espaces de rseaux marqus et de l'espace de Torelli, *Prpub. Univ. Orsay, arXiv:1104.3417v2* (2011).

Arithmetic quotients at infinity

Gregory Margulis

The title above is too general and actually I talked only about a certain class of proper functions on the space of lattices.

Let $\Omega_n \simeq SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ denote the space of unimodular lattices in $\mathbb{R}^n, n > 1$. This space is not compact but has a finite Haar measure. If f is an integrable function on $\mathbb{R}^n$ then we can define a function $\tilde{f}$ on Ω_n by

$$\tilde{f}(\Delta) = \Sigma_{v\in\Delta, v\neq 0} f(v), \Delta \in \Omega_n. \tag{1}$$

According to a theorem of C.L. Siegel,

$$\int_{\Omega_n} \tilde{f} d\mu = \int_{\mathbb{R}^n} f dm, \tag{2}$$

where μ is the probability $SL(n,\mathbb{R})$-invariant measure on Ω_n and m is the Lebesque measure on $\mathbb{R}^n$. As an application of this theorem, Siegel gave a proof of the Minkowski–Hlawka theorem about the existence of a unimodular lattice in $\mathbb{R}^n$ which does not contain a non-zero vector in a ball of volume less than 1.

If f is a positive continuous non-zero function on $\mathbb{R}^n$ then $\tilde{f}$ is unbounded even if f is compactly supported. But it is well know that

$$\tilde{f} < c(f)\alpha,$$

where α is a function on Ω_n defined below.

Let $\Delta \in \Omega_n$ be a unimodular lattice in $\mathbb{R}^n$. We say that a (linear) subspace L of $\mathbb{R}^n$ is Δ-rational if $L/L\cap\Delta$ is compact. For any Δ-rational subspace L we define

$$d(L) = \mathrm{vol}(L/L\cap\Delta) = ||v_1 \wedge v_2 \wedge \ldots \wedge v_i||, \tag{3}$$

where $i = \dim L$ and $(v_1, v_2, \ldots, v_i)$ is a basis of the sublattice $L\cap\Delta$. We put

$$\alpha_i(\Delta) = \max\{1/d(L) : L \text{ is a } \Delta\text{-rational subspace of dimension } i\}, \tag{4}$$

$1 \le i \le n$. We have $\alpha_n(\Delta) = 1$, and we define α_0 by $\alpha_0(\Delta) = 1$. For $1 \le i \le n-1$, the functions α_i are proper (i.e., $\alpha_i(\Delta) \to \infty$ as Δ tends to infinity on Ω_n). Finally we define

$$\alpha(\Delta) = \max_{0\le i\le n} \alpha_i(\Delta). \tag{5}$$

Roughly speaking, $\alpha(\Delta)$ characterizes the number of points in the intersection of Δ with the ball B of radius 1 centered at 0. More precisely, there exists $c > 0$ such that

$$c < \frac{\alpha(\Delta)}{|\Delta\cap B|} < c^{-1}. \tag{6}$$

Now let σ be a compactly supported probability measure on $SL(n,\mathbb{R})$. Let us define an operator A_σ on the space of continuous functions on Ω_n by

$$(A_\sigma h)(x) = \int_G h(gx)d\sigma(g), x \in \Omega_n. \tag{7}$$

Let $1 \le i \le n-1$ and define $\bar{i} = \min\{i, n-i\}$. It easily follows from the Hadamard inequality that

$$d(L)d(M) \ge d(L \cap M)d(L+M) \tag{8}$$

for any two Δ-rational subspaces L and M. From this one can easily get that for any $s > 0$

$$A_\sigma \alpha_i^s \le c_i(s,\sigma)\alpha_i^s + \omega^{2s} \max_{0<j\le\bar{i}} \sqrt{\alpha_{i+j}^s \alpha_{i-j}^s}, \tag{9}$$

where

$$c_i(s,\sigma) = \sup_{v\in F(i), ||v||=1} \int_G \frac{d\sigma(g)}{||gv||^s}, \tag{10}$$

$$F(i) = \{v_1 \wedge v_2 \wedge \ldots \wedge v_i : v_1, v_2, \ldots, v_i \in \mathbb{R}^n\} \subset \Lambda^i(\mathbb{R}^n), \tag{11}$$

and

$$\omega = \sup\{||\Lambda^j(g)|| : 0 < j < n, g \in \text{supp}\,\sigma \cup (\text{supp}\,\sigma)^{-1}\}. \tag{12}$$

Let us denote $q(i) = i(n-i)$. Then by direct computations $2q(i) - q(i+j) - q(i-j) = 2j^2$. Therefore we get from (9) that for any positive $\epsilon < 1$

$$A_\sigma(\epsilon^{q(i)}\alpha_i^s) \le c_i(s,\sigma)\epsilon^{q(i)}\alpha_i^s + \epsilon\omega^{2s} \max_{o<j\le\bar{i}} \sqrt{\epsilon^{q(i+j)}\alpha_{i+j}^s \epsilon^{q(i-j)}\alpha_{i-j}^s} \tag{13}$$

Consider the linear combination

$$\alpha_{\epsilon,s} = \sum_{0\le i\le n} \epsilon^{q(i)}\alpha_i^s. \tag{14}$$

Since $\epsilon^{q(i)}\alpha_i^s < \alpha_{\epsilon,s}, \alpha_0 = 1$, and $\alpha_1 = 1$, the inequalities in (13) imply the following inequality

$$A_\sigma \alpha_{\epsilon,s} < 2 + (c + n\epsilon\omega^{2s})\alpha_{\epsilon,s}, \tag{15}$$

where $c = \max_{0<i<n} c_i(s,\sigma)$. The inequality (15) plays an important role in various applications such as a quantatative version of the Oppenheim conjecture (distribution of values of irrational indefinite quadratic forms at integral points), recurrence properties of random walks on homogeneous spaces, etc. In these applications, the detailed analysis of constants $c_i(s,\sigma)$ is involved and sometimes the functions α_i are replaced by their modifications.

Functorial semi-norms in homology and mapping degrees

Clara Löh
(joint work with Roman Sauer, Dieter Kotschick, Diarmuid Crowley)

Functorial semi-norms on singular homology add metric information to homology classes that is compatible with continuous maps. Functorial semi-norms give rise to degree theorems for certain classes of manifolds; conversely, knowledge about mapping degrees can help to construct functorial semi-norms with interesting properties. In this context, hyperbolicity, locally symmetric spaces, and groups such as mapping class groups of surfaces or outer automorphism groups of free groups provide interesting examples. The talk consisted of an introduction into the subject and gave a survey of recent developments.

1. Functorial semi-norms $\longrightarrow$ mapping degrees

Definition 12 (Functorial semi-norm [3, Section 5.34])**.** A *functorial semi-norm* on singular homology in degree d consists of a choice of a (possibly infinite) semi-norm $|\cdot|$ on $H_d(X;\mathbb{R})$ for all topological spaces X such that

$$\bigl|H_d(f;\mathbb{R})(\alpha)\bigr| \leq |\alpha|$$

holds for all continuous maps $f\colon X \longrightarrow Y$ and all $\alpha \in H_d(X;\mathbb{R})$.

Functorial semi-norms on singular homology provide a systematic approach to degree theorems:

Remark 4 (Degree theorems out of functorial semi-norms)**.** *If $|\cdot|$ is a functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$ and $f\colon M \longrightarrow N$ is a continuous map between oriented closed connected d-manifolds, then $|\deg f|\cdot|[N]_\mathbb{R}| \leq |[M]_\mathbb{R}|$; in particular, if $|[N]_\mathbb{R}| \neq 0$, then*

$$|\deg f| \leq \frac{|[M]_\mathbb{R}|}{|[N]_\mathbb{R}|}.$$

A key example of a (finite) functorial semi-norm is the ℓ^1-semi-norm [4]; roughly speaking, the ℓ^1-semi-norm measures how many simplices are needed to represent a singular homology class with real coefficients. Evaluating the ℓ^1-semi-norm on fundamental classes of manifolds leads to the simplicial volume [4, 9]. In view of Remark 4, non-vanishing results for the simplicial volume give rise to degree theorems.

- If M is an oriented closed connected hyperbolic n-manifold, then

$$\|M\| = \frac{\operatorname{vol} M}{v_n} > 0,$$

  where v_n is the supremal volume of all geodesic n-simplices in hyperbolic n-space [4, 12], and where $\|M\|$ denotes the simplicial volume of M.
- The simplicial volume of all oriented closed connected locally symmetric spaces of non-compact type is non-zero [8].

- The Lipschitz simplicial volume of all oriented connected locally symmetric spaces of non-compact type with finite volume is finite and non-zero [11]. The corresponding degree theorems complement the results by Connell and Farb on degree theorems for locally symmetric spaces [1].

  However, the locally finite simplicial volume of all oriented connected locally symmetric spaces of non-compact type with finite volume and $\mathbb{Q}$-rank at least 3 is zero [11].

  Further vanishing results for the locally finite simplicial volume in the non-compact case have been established by Ji [5], e.g., for most moduli spaces of surfaces.
- Notice that the simplicial volume of all oriented closed connected simply connected manifolds is zero [4], and so the simplicial volume cannot contribute to interesting degree theorems in the simply connected case.

2. Mapping degrees ⟶ functorial semi-norms

Conversely, we can use knowledge about mapping degrees to obtain interesting functorial semi-norms:

Theorem 32 (Generating functorial semi-norms [2])**.** *Let $d \in \mathbb{N}$, let S be a class of oriented closed connected d-manifolds, and let $v\colon S \longrightarrow [0,\infty]$ be a functorial semi-norm on S, i.e., for all continuous maps $f\colon M \longrightarrow N$ with M, $N \in S$ we have $|\deg f| \cdot v(N) \leq v(M)$. If X is a topological space and $\alpha \in H_d(X;\mathbb{R})$, then we define*

$$|\alpha|_v := \inf\biggl\{\sum_{j=1}^{k} |a_j| \cdot v(M_j) \biggm| k \in \mathbb{N},\ a_1,\dots,a_k \in \mathbb{R}\setminus\{0\},\ M_1,\dots,M_k \in S,\ f_1\colon M_1 \to X,\dots,f_k\colon M_k \to X \text{ continuous with } \sum_{j=1}^{k} a_j \cdot H_d(f_j;\mathbb{R})[M_j]_\mathbb{R} = \alpha\biggr\}.$$

Then $|\cdot|_v$ is a functorial semi-norm on $H_d(\,\cdot\,;\mathbb{R})$ and for all $M \in S$ we have

$$|[M]_\mathbb{R}| = v(M).$$

For example, the functorial semi-norm generated by the simplicial volume coincides with the ℓ^1-semi-norm [2], except possibly in dimension 3.

Question 5. Does the functorial semi-norm on $H_3(\,\cdot\,;\mathbb{R})$ associated to the simplicial volume coincide with the ℓ^1-semi-norm?

Another example can be obtained by looking at products of surfaces: The Euler characteristic of products of surfaces of genus at least 1 is functorial [3, p. 303]. Gromov suggested that the corresponding functorial semi-norm should be infinite for most "interesting" manifolds [3, 5.36 on p. 303f]. We confirmed this by showing that oriented closed connected manifolds M

- whose fundamental group knows enough about the topology of M (e.g., M is aspherical or $\mathbb{Q}$-essential),
- and where all "generic" elements of $\pi_1(M)$ have "small" centralisers

cannot be dominated by any non-trivial product of manifolds [6]; in particular, the products of surfaces semi-norms of such manifolds is infinite. Concrete examples include the following [6, 7]:

- closed Riemannian manifolds of negative sectional curvature,
- irreducible closed locally symmetric spaces of non-compact type,
- $\mathbb{Q}$-essential manifolds whose fundamental group is non-elementary hyperbolic, or a mapping class group of a surface of genus at least 1, or the outer automorphism group of a free group of rank at least 1.

Gromov [3, Remark (b) in 5.35] raised the question whether any functorial semi-norm on singular homology is trivial on all simply connected spaces. For this type of question, the fundamental group, the cohomology ring, and the ℓ^1-semi-norm cannot serve as an obstruction. Using methods from rational homotopy theory, we show that there exist simply connected "inflexible" manifolds and hence that there indeed exist functorial semi-norms on singular homology that take finite non-zero values on certain classes of simply-connected spaces [2].

However, in general, not much is known about functorial semi-norms: Because the volume of hyperbolic manifolds is proportional to the simplicial volume (and hence functorial), one can consider the corresponding functorial semi-norm:

Question 6. Is the functorial semi-norm on singular homology associated with the volume of oriented closed connected hyperbolic manifolds finite?

References

[1] C. Connell, B. Farb, *The degree theorem in higher rank*, J. Differential Geom. **65.1** (2003), 19–59.

[2] D. Crowley, C. Löh, *Functorial semi-norms on singular homology and (in)flexible manifolds*, preprint, arXiv: 1103.4139 [math.GT] (2011).

[3] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces* with appendices by M. Katz, P. Pansu and S. Semmes, translated from the French by Sean Michael Bates, Birkhäuser (1999).

[4] M. Gromov, *Volume and bounded cohomology*, Publ. Math. IHES **56** (1982), 5–99.

[5] L. Ji, *Simplicial volume of moduli spaces of Riemann surfaces*, preprint (2011).

[6] D. Kotschick, C. Löh, *Fundamental classes not representable by products*, J. London Math. Soc. **79.3** (2009), 545–561.

[7] D. Kotschick, C. Löh, *Groups not presentable by products*, preprint, arXiv: 1003.3853 [math.GR] (2010).

[8] J.-F. Lafont, B. Schmidt, *Simplicial volume of closed locally symmetric spaces of non-compact type*, Acta Math. **197.1** (2006), 129–143.

[9] C. Löh, *Simplicial volume*, survey article, Bull. Man. Atl. (2011), 7–18.

[10] C. Löh, R. Sauer, *Simplicial volume of Hilbert modular varieties*, omment. Math. Helv. **84** (2009), 457–470.

[11] C. Löh, R. Sauer, *Degree theorems and Lipschitz simplicial volume for non-positively curved manifolds of finite volume*, Journal of Topology **2** (2009), 193–225.

[12] W.P. Thurston, *Geometry and Topology of 3-Manifolds*, lecture notes, Princeton (1978). http://www.msri.org/publications/books/gt3m.

Survey of invariant orders on arithmetic groups (and mapping class groups)

Dave Witte Morris

Assumption 1. *Γ is always assumed to be a finitely generated, infinite group.*

The basic question to be discussed is:

Does there exist an invariant order relation $\prec$ on Γ?

Four different versions of the question are considered, because

- the order relation may be assumed to be either *bi-invariant* or only *left-invariant*, and
- the order relation may be assumed to be either *total* or only *partial.*

We mainly discuss the case where Γ is an arithmetic group, but there are also some comments on mapping class groups.

We start with a trivial observation:

Proposition 9. *Γ has a left-invariant partial order (unless it is a torsion group).*

Proof. Fix $g \in \Gamma$ (of infinite order), and let $P = \{ g^n \mid n > 0 \}$. Define

$$x \prec y \iff x^{-1}y \in P.$$

This is a partial order, because it is:

- transitive: if $x \prec y$ and $y \prec z$, then $x^{-1}z = (x^{-1}y)(y^{-1}z) \in P$
- irreflexive: $x^{-1}x = e \notin P$, so $x \not\prec x$

It is also left-invariant:

$$x \prec y \implies (ax)^{-1}(ay) = x^{-1}y \in P. \quad \square$$

Remark 5. *The same proof works if P is any semigroup in Γ, such that $e \notin P$.*

Recall that a *quasimorphism* on Γ is a map $\rho\colon \Gamma \to \mathbb{R}$, such that

$$\{ \rho(xy) - \rho(x) - \rho(y) \mid x, y \in \Gamma \}$$

is a bounded set.

Proposition 10. *If Γ has an unbounded quasimorphism, then Γ has a nontrivial, bi-invariant partial order.*

Corollary 3. *The following groups have nontrivial bi-invariant partial orders, because they have unbounded quasimorphisms:*

(1) *arithmetic subgroups of semisimple Lie groups of real rank one,*
(2) *mapping class groups,*
(3) *Gromov-hyperbolic groups.*

I believe that irreducible arithmetic subgroups of semisimple Lie groups of higher rank do not have bi-invariant partial orders, but this has not even been proved for $SL(3, \mathbb{Z})$. The nonexistence of such an order can be restated in other ways:

Proposition 11. *The following are equivalent:*

(1) Γ *has no nontrivial, bi-invariant partial order.*
(2) *Every conjugation-invariant subsemigroup of* Γ *is a subgroup.*
(3) *If* g *is any element of* Γ*, then the identity element of* Γ *is a product of conjugates of* Γ*.*

Here is a classical result:

Theorem 33. *If* Γ *has a bi-invariant total order, then the abelianization of* Γ *is infinite.*

From work of D. Kazhdan and others, it is known that lattices in most semisimple Lie groups have finite abelianization. This has the following consequence:

Corollary 4. *Suppose* Γ *is an irreducible arithmetic subgroup of a connected, semisimple Lie group* G *with no compact factors. If* Γ *has a bi-invariant total order, then* G *is locally isomorphic to either* $SO(1,n)$ *or* $SU(1,n)$*.*

Remark 6. (1) *Most mapping class groups also have trivial abelianization, and therefore do not admit bi-invariant total orders.*
(2) *There does exist a bi-invariant total order on any group that is residually torsion-free nilpotent. Thus, for example, every free group has a bi-invariant total order.*

Conjecture 1. *Suppose* Γ *is an irreducible arithmetic subgroup of a connected, semisimple Lie group* G*, such that* $\text{rank}_{\mathbb{R}} G \geq 2$*. Then* Γ *does not have a left-invariant total order.*

There has been some progress in the non-cocompact case:

Theorem 34 (Chernousov-Lifschitz-Morris)**.** *If the conjecture is true for all of the non-cocompact arithmetic subgroups of* $SL(3,\mathbb{R})$ *and* $SL(3,\mathbb{C})$*, then it is true for all non-cocompact arithmetic subgroups.*

In contrast, there has been essentially no progress in the cocompact case:

Open problem 1. *Show there is a cocompact arithmetic subgroup* Γ *of some semisimple Lie group* G*, such that no finite-index subgroup of* Γ *has a left-invariant total order.*

Remark 7. *If* S *is a surface with nonempty boundary, then it is known that the mapping class group of* S *has a faithful action on the real line, and therefore also has a left-invariant total order.*

Acknowledgment 1. I am grateful to the audience of the lecture for informing me of many relevant results on mapping class groups and related groups.

References

Left-invariant total orders on arithmetic groups:

[1] D. W. Morris and Lucy Lifschitz: Bounded generation and lattices that cannot act on the line, Pure and Applied Mathematics Quarterly 4 (2008) 99–126. http://arxiv.org/abs/0604612

[2] D. W. Morris: Can lattices in $SL(n,\mathbb{R})$ act on the circle?, in B. Farb and D. Fisher, eds., *Geometry, Rigidity, and Group Actions,* U. of Chicago Press, Chicago, 2011, pp. 158–207. http://arxiv.org/abs/0811.0051
[3] S. Boyer, D. Rolfsen, and B. Wiest: Orderable 3-manifold groups, Ann. Inst. Fourier, Grenoble 55 (2005) 243–288. http://arxiv.org/abs/0211110

Bi-invariant partial orders:
[4] D. Witte: Products of similar matrices. Proc. Amer. Math. Soc. 126 (1998) 1005–1015.

Arithmetic groups with infinite abelianization:
[5] A. Lubotzky: Eigenvalues of the Laplacian, the first Betti number and the congruence subgroup problem. Ann. of Math. (2) 144 (1996), no. 2, 441–452. MR1418904 (98h:22013)

Orders on general groups:
[6] A. M. W. Glass: Partially Ordered Groups, World Scientific, River Edge, NJ, 1999. ISBN 981-02-3493-7. MR1791008 (2001g:06002)
[7] V. M. Kopytov and N. Ya. Medvedev: Right-ordered groups. Consultants Bureau, New York, 1996. ISBN: 0-306-11060-1 MR1393199 (97h:06024a)
[8] R. B. Mura and A. Rhemtulla: Orderable groups. Marcel Dekker, Inc., New York-Basel, 1977. MR0491396 (58 #10652)

Weil-Petersson Riemannian and symplectic geometry: geodesic-lengths, Fenchel-Nielsen twists and Thurston shears

Scott A. Wolpert

We discuss the correspondence between Weil-Petersson (WP) geometry on Teichmüller space $\mathcal{T}$ and the hyperbolic geometry of surfaces, the unions of thrice punctured spheres. A theme is that the mapping class group (MCG) is the symmetry group of geometries of $\mathcal{T}$.

Theorem 35. (W) $\overline{\mathcal{T}}$ - a right infinite polyhedron. *The augmented Teichmüller space $\overline{\mathcal{T}}$ is the closed convex hull of the countable set of maximally noded hyperbolic structures. At each maximally noded structure, the Alexandrov tangent space is isometric to a Euclidean orthant of dimension $\frac{1}{2}\dim\mathcal{T}$.*

The description combines with results of Korkmaz, Ivanov and Luo on automorphisms of the curve complex to give a proof of the Masur-Wolf Theorem that the MCG is the WP isometry group - the symmetry group of a geometry.

The symplectic geometry of $\mathcal{T}$ begins with the geodesic-length functions ℓ_α and Fenchel-Nielsen (FN) infinitesimal twist deformations t_α. A twist deformation is given by cutting a surface along a simple closed geodesic α, and reassembling the boundaries with a relative rotation. The basic formulas are

$$2t_\alpha = i\,\mathrm{grad}\,\ell_\alpha, \quad \text{and} \quad 2\omega_{WP}(\ ,t_\alpha) = d\ell_\alpha.$$

The formulas provide the elements of a symplectic geometry - ω_{WP} is invariant under FN twist flows and geodesic-lengths are Hamiltonian potentials for FN twist vector fields. Hermitian products of length gradients are given by the combined Riera-Wolpert formula for the WP Riemannian and symplectic pairing

$$\langle \operatorname{grad} \ell_\alpha, \operatorname{grad} \ell_\beta \rangle = 4\langle t_\alpha, t_\beta \rangle = \frac{2}{\pi}\delta_{\alpha\beta}\ell_\alpha + \sum_{D\in\langle A\rangle\backslash\Gamma \mathrm{SL}ash\langle B\rangle} \mathcal{R}_D,$$

where for the geodesics α and β, the corresponding axes $\operatorname{axis}(A), \operatorname{axis}(DBD^{-1})$ disjoint in $\mathbb{H}$, then

$$\mathcal{R}_D = \frac{2}{\pi}R(\cosh d(\operatorname{axis}(A), \operatorname{axis}(DBD^{-1})))$$

and for the axes intersecting with angle θ_D, then

$$\mathcal{R}_D = \frac{2}{\pi}R(\cos\theta_D) - 2i\cos\theta_D,$$

and

$$R(u) = u\log\left|\frac{u+1}{u-1}\right| - 2,$$

and where $\delta_{\alpha\beta}$ is the Kronecker delta for the geodesic pair. The formula incorporates the original twist-length-cosine formula. For disjoint simple geodesics α and β, all summands are positive and the pairing of gradients is positive - length gradients of simple geodesics never vanish and the length gradients of a pair of disjoint simple geodesics lie in a common half space. Symplectic duality and the rescaling $T_\alpha = 4\sinh \ell_\alpha\, t_\alpha$ lead to a further element of the geometry - the Goldman-Wolpert Lie bracket evaluation

$$[T_\alpha, T_\beta] = \sum_{p\in\alpha\cap\beta} T_{A_pB^{-1}} - T_{A_pB},$$

at each intersection of the geodesics the two elementary surgeries are performed to form new curves and geodesics. For suitable geodesics on surfaces with cusps or boundaries, Moira Chas has shown that there is no cancellation in the bracket evaluation.
Conjecture. The MCG is the automorphism group of the Lie algebra.

We also discuss generalizing the above considerations to Thurston's shear deformations.

Theorem 36. (Thurston) *The space of measured geodesic laminations $\mathcal{MGL}$ is a PL manifold with a non degenerate PL symplectic structure.* (Bonahon) *Tangent vectors of $\mathcal{MGL}$ are given as transverse cocycles - finitely additive 'measures' on underlying geodesic laminations.*

Weights on branches of train tracks provide finite linear models for the affine flats in $\mathcal{MGL}$.

Theorem 37. (Bonahon, Penner after Thurston) *Teichmüller space real analytically embeds into an explicit cone in the space of transverse cocycles for a maximal geodesic lamination.* (Bonahon) *Elements of the cone are characterized by positive pairing with every measure on the geodesic lamination.* (Bonahon-Sözen)

The embedding tangent space mapping identifies the WP and Thurston symplectic forms.

A transverse cocycle ρ has a total length $L(\rho)$, given by integrating the local product of hyperbolic length along leaves and the transverse cocycle measure. A transverse cocycle also determines a finite and then infinitesimal shear deformation $\sigma(\rho)$ given as a combination of left and right earthquakes along the leaves of the geodesic lamination, according to the mass of the transverse cocycle.

We consider the special case of a weighted finite sum of ideal geodesics on a surface with cusps. The type preserving condition for the deformation is that the sum of weights vanishes for the geodesic rays entering a cusp region.

Theorem 38. (W) *A weighted finite sum of ideal geodesics shear $\sigma(\rho)$ is a symplectic vector field with Hamiltonian potential $L(\rho)/2$. There is a Poisson bracket for the total length functions L. There is an elementary formula for the Thurston symplectic form as a sum of weights at each cusp.*

We generalize the Riera-Wolpert formula to give a formula for the pairing product $\langle \operatorname{grad} \ell_\alpha, \operatorname{grad} \ell_\beta \rangle$. The formula gives an infinite sum relation for the distances between ultraparallels in the $SL(2;\mathbb{Z})$ tessellation.

References

[Bon96] Francis Bonahon. Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):233–297, 1996.

[Wol10] Scott A. Wolpert. *Families of Riemann surfaces and Weil-Petersson Geometry*, volume 113 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2010.

Higher order invariants

Anton Deitmar

For a group Γ acting linearly on a $\mathbb{C}$-vector space V we define the invariants of order $q \geq 1$ as the set $H^0_q(\Gamma, V)$ of all elements which are annihilated by J^q, where J is the augmentation ideal in the group ring $A = \mathbb{C}\Gamma$.

Example. Let $\Gamma = \pi_1(X, \bar{x}_0)$, where X is a Riemann surface and let ω be a holomorphic 1-form on X. Let $\tilde{X}$ be the universal cover and x_0 a lift of $\bar{x}_0$. Let $f(x) = \int_{x_0}^x \omega$ for $x \in \tilde{X}$. Then $f(\gamma x) - f(x) = \int_x^{\gamma x} \omega$ does not depend on x but only on $\gamma \in \Gamma$. Therefore $f \in H^0_2(\Gamma, \mathcal{O}(\tilde{X}))$.

Iterated integrals. Let X be a smooth manifold, $p : [0,1] \to X$ a path, and let $\omega_1, \dots, \omega_s$ be 1-forms on X. Define the iterated integral given by these data as

$$\int_p \omega_1 \dots \omega_s = \int_0^1 \int_0^{t_s} \dots \int_0^{t_2} p^*\omega_1(t_1) p^*\omega_2(t_2) \dots p^*\omega_s(t_s).$$

Let $B_s^{\mathrm{hom}}(X)$ be the set of all homotopy stable iterated integrals, then a theorem of Ivan Horozov and the author says that

- the map $\omega \mapsto f(x) = \int_{x_0}^{x} \omega$ is an injection

$$B_s^{\mathrm{hom}}(X)_{x_0} \hookrightarrow H^0_{s+1}(\Gamma, C^\infty(\tilde{X})),$$

- after tensoring with $C^\infty(X)$, this map is surjective and the kernel is explicitely computable.

So one might say that all higher order invariants are generated by iterated integrals.

Automorphic forms. Let X be a locally symmetric space, so $X = \Gamma\backslash G/K$ for a semisimple Lie group G and its maximal compact subgroup K. In the theory of automorphic forms one is interested in the spectral decomposition of

$$L^2(X) = L^2(\Gamma\backslash G/K) = L^2(\Gamma\backslash G)^K.$$

As this embeds into $L^2(\Gamma\backslash G)$ it is the more general task to understand the latter space as G-representation. In the case when X is compact, one has

$$L^2(\Gamma\backslash G) = L^2_{\mathrm{loc}}(G)^\Gamma,$$

where L^2_{loc} is the space of locally square integrable functions. One introduces

$$L^2_q(\Gamma\backslash G) = H^0_q(\Gamma, L^2_{\mathrm{loc}}(G)).$$

Chosing a fundamental domain F, one embeds $L^2_q(\Gamma\backslash G)$ into the L^2-space of a sufficiently large number of translates of F to get a unitary structure, which is not unique, but the induced topology is unique. One then finds

$$L^2_q(\Gamma\backslash G) \cong \bigoplus_{\pi\in\hat{G}} \dim(A/J^q) N_\Gamma(\pi) \begin{pmatrix} \pi & * & \\ & \ddots & * \\ & & \pi \end{pmatrix},$$

where $N_\Gamma(\pi)$ is the multiplicity of π in $L^2(\Gamma\backslash G)$ and the matrix is $q \times q$.

Hecke algebra. Suppose that (G, Γ) is a Hecke pair, then for a given $\mathbb{C}G$-module V, the group G acts on the inductive limit of all $H^0_q(\Sigma, V)$, where Σ runs through all congruence subgroups of Γ. In the example case $G = \mathrm{SL}_2(\mathbb{Q})$ and $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ one gets a module V for the group $\mathrm{SL}_2(\mathbb{A})$, where $\mathbb{A}$ is the adele ring, which injects into

$$(S_2 \oplus \overline{S_2}) \otimes L^2(\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A})).$$

It is conjectured that V is actually isomorphic to this tensor product.

References

[1] Anton Deitmasr, *Higher order invariants, cohomology, and automorphic forms*, http://arxiv.org/abs/0811.1088.

Homomorphisms to mapping class groups: Parametrizing surface bundles

Daniel Groves

Let S be an oriented surface of finite type with genus g and p punctures, and suppose that $3g + p > 3$. Let $\mathrm{Mod}(S)$ be the mapping class group of S. If G is a group, let $\mathcal{X}_S(G) = \mathrm{Hom}(G, \mathrm{Mod}(S))/\sim$, where $\sim$ denotes $\mathrm{Mod}(S)$-conjugacy. If B is a connected CW-complex, then there is a one-to-one correspondence between isomorphism classes of oriented S-bundles over B and $\mathcal{X}_S(\pi_1(B))$.

This talk was about the structure of the set $\mathcal{X}_S(G)$, where G is an arbitrary finitely generated group.

Theorem 39. *Suppose that G is finitely generated and $\mathcal{X}_S(G)$ is infinite. Then a finite-index subgroup G_0 of G acts without global fixed point on a simplicial tree. (So G_0 admits a nontrivial graph of groups decomposition.)*

(For G finitely presented, this theorem was previously proved by Berhstock, Druţu and Sapir [1].)

In order to delve deeper into the structure of the set $\mathcal{X}_S(G)$, one needs to understand the edge stabilisers of the graph of groups decomposition of G_0. This leads to a more technical, but more useful, version of Theorem 39. It also allows a proof of the following result (first proved by Bowditch [2] and by Dahmani-Fujiwara [3] in case G is finitely presented):

Theorem 40. *Let G be finitely generated and one-ended. Then there are only finitely many elements of $\mathcal{X}_S(G)$ which are represented by injective homomorphisms for which every nontrivial element of G is mapped to a pseudo-Anosov mapping class in* $\mathrm{Mod}(S)$*.*

It is worth noting that there are no known homomorphisms satisfying the requirements of Theorem 40.

The following theorem (which can be rephrased as a 'Compact Core Theorem' for S-bundles) is basic to the understanding of the structure of the set $\mathcal{X}_S(G)$:

Theorem 41. *Suppose that G is a finitely generated group. There is a finitely presented group $\hat{G}$, together with an epimorphism $\eta : \hat{G} \twoheadrightarrow G$ so that precomposition with η induces a bijection:*

$$\eta^* : \mathcal{X}_S(G) \to \mathcal{X}_S(\hat{G}).$$

This theorem says that $\mathrm{Mod}(S)$ is *equationally noetherian.*

The methods of proof involve considering the action of G on the curve complex $\mathcal{C}(S)$ and marking complex $\mathcal{M}(S)$ of S. Masur and Minsky [5] proved that the $\mathcal{C}(S)$ is δ-hyperbolic. However, it is not locally finite, and the action of $\mathrm{Mod}(S)$ on it is not proper. If $\{f_i : G \to \mathrm{Mod}(S)\}$ leads to a sequence of *divergent* actions on $\mathcal{C}(S)$, then taking an equivariant Gromov-Hausdorff limit of rescaled copies of $\mathcal{C}(S)$ leads to an action of G on an $\mathbb{R}$-tree. However, in general, a sequence will not lead to a divergent sequence of actions. In this case, we use the *hierarchies* of Masur and Minsky [6], which give quasi-geodesics in $\mathcal{M}(S)$, and are assembled from geodesics not just in $\mathcal{C}(S)$ but also in the curve complexes of sub-surfaces of S. Since the action of $\mathrm{Mod}(S)$ on $\mathcal{M}(S)$ *is* proper, there is some sub-surface Y of S (considered up to homeomorphism of the pair (S,Y)) for which the lengths in $\mathcal{C}(Y)$ go to infinity.

This allows us to immerse a complex into a presentation 2-complex of G recording the interaction with $\mathcal{C}(Y)$. It is an immersion rather than an embedding because of sub-surfaces disjoint from Y. This immersed complex can be embedded in a finite cover, due to a separability result of Leininger and McReynolds [4].

Taking a limit gives a proof of Theorem 39 in the finitely presented case, and carefully taking a limit of finite presentation 2-complexes gives a proof in the finitely generated case. Theorem 40 follows almost immediately from an understanding of the edge stabilizers in the splitting of G_0 obtained from Theorem 39.

To prove Theorem 41, we consider all of the ways in which splittings arise, and prove the existence of an appropriate JSJ decomposition. After that, the proof is similar to the analogous fact for torsion-free hyperbolic groups, proved by Sela [7] In fact, it is fair to say that the methods throughout this project are strongly influenced by the work of Sela from [7] (and his earlier work on the Tarski problem).

References

[1] J. Behrstock, C. Druţu and M. Sapir, Homomorphisms into mapping class groups. An addendum, *Proc. LMS*, to appear.

[2] B. Bowditch, One-ended subgroups of mapping class groups, preprint.

[3] F. Dahmani and K. Fujiwara, Copies of one-ended groups in mapping class groups, *Groups, Geom. Dyn.* **3** (2009), 359–377.

[4] C. Leininger and D. McReynolds, Separable subgroups of mapping class groups, *Topol. Appl.* **154** (2007), 1–10.

[5] H. Masur and Y. Minsky, Geometry of the complex of curves. I. Hyperbolicity, *Invent. Math.* **138** (1999), 103–149.

[6] H. Masur and Y. Minsky, Geometry of the complex of curves. II. Hierarchical structure, *GAFA* **10** (2000), 902–974.

[7] Z. Sela, Diophantine geometry over groups. VII. The elementary theory of a hyperbolic group, *Proc. LMS* **99** (2009), 217–273.

Stationary measures on homogeneous spaces

Yves Benoist

(joint work with Jean-François Quint)

Let G be a real Lie group, Λ be a lattice of G, and Γ be a subgroup of G such that the Zariski closure of the adjoint group $Ad\Gamma$ is semisimple connected with no compact factors.

We prove in [1] that for any point x in the quotient space $X = G/\Lambda$ the closure F of the orbit Γx is homogeneous, i.e. that the stabilizer of F in G acts transitively on F.

For that, when Γ is compactly generated, we introduce a probability measure μ on Γ whose support is compact and generates Γ. We prove in [1] that any μ-ergodic

μ-stationary probability measure on X is homogeneous, i.e. that the stabilizer of ν in G acts transitively on the support of ν.

During this talk, I have presented a sketch of the proof for the special case where X is the 2-dimensional torus, G is the group of affine automorphisms of this torus, Λ is the group of linear automorphisms of this torus and Γ is a non elementary subgroup of Λ generated by two elements.

References

[1] Yves Benoist, Jean-François Quint, *Fermés invariants et probabilités stationnaires des espaces homogènes II*, CRAS to appear.

Reporter: Patrick Boland

Participants

HDoz. Dr. Oliver Baues
Institut für Algebra u. Geometrie
Fakultät für Mathematik
KIT
Kaiserstr. 89-93
76133 Karlsruhe

Prof. Dr. Yves Benoist
Laboratoire de Mathematiques
Universite Paris Sud (Paris XI)
Batiment 425
F-91405 Orsay Cedex

Dr. Patrick Boland
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor , MI 48109-1043
USA

Prof. Dr. Nathan Broaddus
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus , OH 43210-1174
USA

Prof. Dr. Jeffrey F. Brock
Department of Mathematics
Brown University
Box 1917
Providence , RI 02912
USA

Prof. Dr. Michelle Bucher-Karlsson
Universite de Geneve
Section de Mathematiques
2-4, rue du Lievre
Case Postale 64
CH-1211 Geneve 4

Prof. Dr. Kai-Uwe Bux
Fakultät für Mathematik
Abteilung II
Universität Bielefeld
Postfach 100131
33501 Bielefeld

Prof. Dr. Anton Deitmar
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen

Valentina Disarlo
Institut de Recherche
Mathematique Avancee
Universite de Strasbourg et CNRS
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Prof. Dr. Cornelia Drutu Badea
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Louis Funar
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Prof. Dr. Daniel Groves
Dept. of Mathematics, Statistics
and Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago , IL 60607-7045
USA

Dr. Thomas Haettel
Universite de Paris-Sud
Laboratoire de Mathematiques
Batiment 430
F-91405 Orsay Cedex

Prof. Dr. Ursula Hamenstädt
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Frank Herrlich
Institut für Algebra u. Geometrie
Karlsruher Institut für Technologie
Kaiserstr. 89-93
76128 Karlsruhe

Prof. Dr. Lizhen Ji
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor , MI 48109-1043
USA

Hyosang Kang
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor , MI 48109-1043
USA

Dr. Fanny Kassel
Universite des Sciences et
Technologies de Lille (Lille I)
Laboratoire Paul Painleve
F-59655 Villeneuve d'Ascq Cedex

Prof. Dr. Ralf Köhl
Mathematisches Institut
Justus-Liebig-Universität Gießen
Arndtstr. 2
35392 Gießen

Dr. Jean-Francois Lafont
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus , OH 43210-1174
USA

Prof. Dr. Bernhard Leeb
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstr. 39
80333 München

Prof. Dr. Christopher J. Leininger
Dept. of Mathematics, University of
Illinois at Urbana-Champaign
273 Altgeld Hall MC-382
1409 West Green Street
Urbana , IL 61801-2975
USA

Prof. Dr. Enrico Leuzinger
Institut für Algebra u. Geometrie
Fakultät für Mathematik
KIT
Kaiserstr. 89-93
76133 Karlsruhe

Prof. Dr. Seonhee Lim
Department of Mathematical Sciences
Seoul National University
Seoul 151-747
KOREA

Dr. Gabriele Link
Institut für Algebra u. Geometrie
Fakultät für Mathematik
KIT
Kaiserstr. 89-93
76133 Karlsruhe

Prof. Dr. Clara Löh
Naturwissenschaftliche Fakultät I
Mathematik
Universität Regensburg
93040 Regensburg

Dr. Ousama Malouf
Institut de Recherche
Mathematique Avancee
Universite de Strasbourg et CNRS
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Prof. Dr. Jason Fox Manning
Department of Mathematics
State University of New York at
Buffalo
244 Math. Bldg.
Buffalo NY 14260-2900
USA

Prof. Dr. Dan Margalit
School of Mathematics
Georgia Institute of Technology
686 Cherry Street
Atlanta , GA 30332-0160
USA

Prof. Dr. Grigorii A. Margulis
Department of Mathematics
Yale University
Box 208 283
New Haven , CT 06520
USA

Dr. Gregor Masbaum
Institut de Mathematiques de Jussieu
Case 247
Universite de Paris VI
4, Place Jussieu
F-75252 Paris Cedex 05

Prof. Dr. Rafe Mazzeo
Department of Mathematics
Stanford University
Stanford , CA 94305-2125
USA

Prof. Dr. Mahan Mj
School of Mathematical Sciences
RKM Vivekananda University
P.O.Belur Math
Dt. Howrah , WB-711202
INDIA

Prof. Dr. Dave Witte Morris
Department of Mathematics and
Computer Science
University of Lethbridge
4401 University Drive
Lethbridge AB T1K 3M4
CANADA

Prof. Dr. Werner Müller
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn

Prof. Dr. Kumar Murty
Department of Mathematics
University of Toronto
40 St George Street
Toronto , Ont. M5S 2E4
CANADA

Prof. Dr. Viacheslav V. Nikulin
Dept. of Pure Mathematics
University of Liverpool
GB-Liverpool L69 3BX

Prof. Dr. Martin Olbrich
Universite du Luxembourg
Unite de Recherche en Mathematiques
6, Rue Richard Coudenhove-Kalergi
L-1359 Luxembourg

Dr. Hugo Parlier
Departement de Mathematiques
Universite de Fribourg
Perolles
Chemin du Musee 23
CH-1700 Fribourg

Prof. Dr. Piotr Przytycki
Institute of Mathematics of the
Polish Academy of Sciences
P.O. Box 21
ul. Sniadeckich 8
00-956 Warszawa
POLAND

Prof. Dr. Andy Putman
Department of Mathematics
Rice University
6100 Main Street
Houston , TX 77005-1892
USA

Prof. Dr. Kasra Rafi
Department of Mathematics
University of Oklahoma
601 Elm Avenue
Norman , OK 73019-0315
USA

Ahmad Said
Institut de Mathematiques
Universite de Strasbourg
7, rue Rene Descartes
F-67084 Strasbourg Cedex

Andrew Sale
Mathematical Institute
Oxford University
24-29 St. Giles
GB-Oxford OX1 3LB

Prof. Dr. Leslie D. Saper
Department of Mathematics
Duke University
P.O.Box 90320
Durham , NC 27708-0320
USA

Prof. Dr. John Scherk
Department of Mathematics
University of Toronto
40 St George Street
Toronto , Ont. M5S 2E4
CANADA

Prof. Dr. Vlad Sergiescu
Laboratoire de Mathematiques
Universite de Grenoble I
Institut Fourier
B.P. 74
F-38402 Saint-Martin-d'Heres Cedex

Prof. Dr. Juan Souto
Department of Mathematics
University of Michigan
2074 East Hall
530 Church Street
Ann Arbor , MI 48109-1043
USA

Prof. Dr. Tyakal N. Venkataramana
Tata Institute of Fundamental
Research
School of Mathematics
Homi Bhabha Road, Colaba
400 005 Mumbai
INDIA

Prof. Dr. Michael Wolf
Department of Mathematics
Rice University
P.O. Box 1892
Houston , TX 77005-1892
USA

Prof. Dr. Scott A. Wolpert
Department of Mathematics
University of Maryland
College Park , MD 20742-4015
USA

Dr. Kevin Wortman
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City , UT 84112-0090
USA

Prof. Dr. Sumio Yamada
Mathematical Institute
Faculty of Science
Tohoku University
6-3 Aoba
Sendai 980-8578
JAPAN