Arbeitsgemeinschaft mit aktuellem Thema:
Quantum ergodicity
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Introduction:

We first sketch the general framework of semiclassical analysis on $\mathbb{R}^n$ or on a manifold X, mostly referring to the lecture notes [EvZw09]. We then focus on the study of quantum eigenstates, especially for classically chaotic systems, which leads to the properties of Quantum ergodicity and Quantum unique ergodicity. The recent reviews [Zel09, Non10b] give a comprehensive and up-to-date account of these questions. A (shorter) review of progresses in the specific field of arithmetic chaotic systems can be found in [Sar11].

A minimal knowledge of semiclassical analysis and of the properties of chaotic dynamical systems is required for all participants, since the main concepts of the Arbeitsgemeinschaft cannot be understood without it. These concepts will be presented in the first few "basics" talks.

Semiclassical analysis

Quantum ergodicity is a subfield of *quantum chaos*, which can itself be considered a subfield of *semiclassical analysis*: the latter aims at describing quantum systems in the semiclassical limit. Originally, a quantum system is defined by a Schrödinger operator (the *quantum Hamiltonian*) $P(\hbar)$ acting on $L^2(\mathbb{R}^3)$, but this definition can be generalized in various ways, for instance, by considering Schrödinger, or Laplace-Beltrami operators on a Riemannian manifold (X, g). This Hamiltonian depends on Planck's constant $\hbar > 0$, which is assumed very small: mathematically, the *semiclassical limit* consists in studying the asymptotical regime $\hbar \downarrow 0$. In this limit, quantum mechanics "converges to" classical mechanics, which is described by a Hamiltonian dynamical system on the *phase space* $T^*\mathbb{R}^3$ (more generally T^*X). Yet this "convergence" is rather singular, which makes semiclassical analysis interesting and nontrivial. The Schrödinger operator is selfadjoint, so the dynamical evolution in time is governed by its spectrum (in case of discrete spectrum, the eigenvalues and eigenstates). For a "general" Schrödinger operator in dimension $d \geq 2$, this spectral problem does not admit any explicit, or even approximate solution, because one cannot separate variables to get back to a set of 1D equations. Still, by using the connection with classical mechanics, semiclassical analysis is able to collect some nontrivial, yet rather "rough" information on this spectrum.

Quantum chaos

Because we are interested in stationary properties of the quantum system, we first need a good understanding of the stationary, or long time, properties of the classical dynamical system. This information is embodied in the *ergodic theory* of these classical systems, which for instance describes the invariant probability measures on the phase space. Somewhat paradoxically, the ergodic theory is best understood for two antipodal types of Hamiltonian systems. On one side, the fully (Liouville-)integrable systems, for which one can (at least locally) construct a full set of action-angle coordinates for which the dynamics is simple (quasiperiodic). At the other extreme, the (fully) chaotic systems, where the only invariant of the motion is the energy, all trajectories are unstable, and the motion mixes up the whole energy shell. The aim of *quantum chaos* is to study the Schrödinger operators for which the limiting classical mechanics is of this type. Here we will mostly focus on systems for which the individual energy shells are compact, resulting in *discrete spectra* for the Schrödinger operators. One aim of quantum chaos is to describe, as precisely as possible, these discrete spectra $(E_{\hbar,n})$ in the limit $\hbar \to 0$.

One of the most studied system of this type is the Laplace-Beltrami operator on compact Riemannian manifolds of negative curvature (for which the limiting classical flow, namely the geodesic flow, is *uniformly hyperbolic*, the strongest possible form of chaos), or on certain Euclidean billiards (like the *stadium billiard*) for which the shape of the boundary also generates a chaotic geodesic flow. In this case, the semiclassical limit $\hbar \to 0$ is equivalent with the high frequency limit $\lambda \to \infty$, where λ^2 is the eigenvalue of the Laplacian.

The mother of all results in this domain is the celebrated *Weyl's law* which approximately counts the number of eigenvalues in a given interval, in terms of the classical phase space volume.

Quantum ergodicity

In particular, we would like to understand the spatial structure of the corresponding eigenfunctions $u_{\hbar,n} \in L^2$ (which we always assume to be L^2-normalized). In absence of any approximate expression for these states, one can only hope to gather collective, or weak information. These eigenfunctions strongly oscillate on a spatial scale $\sim \hbar$, so they are quite singular in the limit $\hbar \to 0$. Nevertheless, the *macroscopic structure* of these eigenfunctions lends itself to some analysis. Since $|u_{\hbar,n}(x)|^2$ represents the probability density of the quantum particle at the point x, it makes sense to consider the integral of this probability over a fixed domain $D \subset X$, which represents the probability to find the particle (in the stationary state $u_{\hbar,n}$) to be observed in this domain. One can jointly study localization in space and momentum variables using phase space quantum representations: it makes sense to measure the probability of presence $P_{\hbar,n}(\Omega)$ of the particle in a (macroscopic) phase space domain $\Omega \subset T^*X$. This probability does not depend much on the fluctuations of the density at the quantum scale, but rather on its macroscopic fluctuations. One can then study the semiclassical limits of the probabilities $P_{\hbar,n}(\cdot)$, which represent the asymptotic macroscopic phase space distribution of the eigenmodes. These limit distributions are called *semiclassical measures*, they are localized on single energy shells, and are necessarily invariant under the classical dynamics. The main question is:

> What are the possible semiclassical measures? Do they encompass all possible classically invariant measures?

If the limiting classical system is chaotic (in particular, if it is ergodic with respect to the Liouville measure), this question can be partially answered by a *quantum ergodicity* (QE) theorem. This theorem states that for *almost every eigenfunction $u_{\hbar,n}$*, the probability $P_{\hbar,n}(\Omega)$ is asymptotically given by the Liouville measure of Ω; thanks to the ergodicity assumption, this is also the asymptotic fraction of time *almost every* initial phase space point will visit Ω in the course of the classical evolution. One says that these eigenfunctions $u_{\hbar,n}$ are *macroscopically equidistributed* on the energy shell.

QE is a robust result: its proofs are rather elementary, once one has at its disposal a few basic semiclassical properties. QE holds for a wide variety of systems, like *quantized chaotic maps*, certain *quantum graphs*. Its extension to vector-valued systems leads to interesting questions on the intertwining between internal and external degrees of freedom. QE can also be proved for systems with singularities (e.g. chaotic billiards), as long as these singularities occupy a part of phase space of Liouville measure zero.

Quantum unique ergodicity vs. exceptional eigenstates

Is this dominant behaviour "totalitarian", or does there exist exceptions to the rule? At the classical level, there exist systems (called *uniquely ergodic*) for which all initial points have the same asymptotic behaviour, but these systems are rather rare. In the textbook case of a compact manifold of negative sectional curvature, there exits many points (e.g. periodic points) with asymptotical behaviour different from the Liouville measure (equivalently, there exist many different classical invariant probability measures). For such chaotic systems, what happens at the quantum level:

> Do there exist *exceptional eigenmodes*, leading to semiclassical measures different from Liouville?

The negative answer to this question is called the *quantum unique ergodicity* property (QUE). It states that quantum mechanics selects the Liouville measure as the only possible macroscopic behaviour. It has been conjectured to hold for the Laplace-Beltrami operator on manifolds of negative curvature. So far it has been proven only for very specific manifolds, namely surfaces of constant curvature enjoying a rich *arithmetic structure*, embodied by a commutative algebra of Hecke operators commuting with the Laplacian. It is then natural to consider only joint eigenbases of these commuting operators; these modes were proved to be all asymptotically equidistributed, a property sometimes denoted as *Arithmetic QUE*.

On the opposite, without these arithmetic symmetries, the possibility of exceptional eigenmodes remains open. Numerical computations of Laplacian eigenmodes on 2D chaotic billiards have shown the possibility of strong enhancements of the probability density in the neighbourhood of certain periodic orbits. It remains unclear whether these enhancements (*scars*) persist in the high frequency limit, and if they are strong enough to modify the macroscopic distribution. In particular, the possible existence of *strong scars*, that is families of eigenstates asymptotically concentrating (in the L^2 sense) along one or several periodic orbits, remained open until recently.

Results on this question were first obtained in the framework of certain quantum chaotic maps, like the *quantum "cat" maps* (hyperbolic automorphisms of the 2D torus), for which the algebraic structure allows some explicit computations. On the one hand, these maps are equipped with arithmetic symmetries ("Hecke" operators), so one can also restrict oneself to joint eigenstates. The latter were shown to be all asymptotically equidistributed. On the other hand, due to the possibility of very large spectral degeneracies, one can construct sequences of exceptional eigenstates, with half of the probability concentrated along some periodic orbit, the other half being equidistributed on the torus. Such constructions are very specific to these linear automorphisms, but they show that the QUE conjecture does not hold for all quantized hyperbolic systems.

A counterexample to QUE was also obtained for the (much more physical) stadium billiard: numerics had observed eigenstates strongly concentrating along the 1D family of *bouncing ball orbits*, which are not hyperbolic. It was recently proved that, indeed, some eigenstates must (at least partially) concentrate along these orbits, thereby disproving QUE for such billiards.

A recent approach has been developed in the case of hyperbolic chaotic systems, to show that not all invariant measures can be obtained as semiclassical measures; in particular, eigenstates cannot fully concentrate near periodic orbits (no strong scars). The argument is based on *hyperbolic dispersion estimates*, which reflect both the minimal delocalization due to Heisenberg's uncertainty principle, and the classical hyperbolicity. The second ingredient is the *entropy* of an invariant measure, which measures its complexity (in the information theoretic sense) but also gives information on its localization. Putting the two ingredients together one obtains nontrivial lower bounds on the entropy of a semiclassical measure, which, roughly speaking, show that the semiclassical measure must be *at least half delocalized*.

Schematic overview

A Background

1. Semiclassical Analysis [EvZw09]
   (i) Quantization and Pseudodifferential calculus, Wigner functions, quantum limits
   (ii) quantum-classical correspondence: Egorov theorem
   (iii) (global and local) Weyl's law
2. Concepts from the theory of dynamical systems [BriStu02]
   (i) ergodicity, weak mixing, mixing
   (ii) hyperbolicity, Anosov systems, manifolds of negative curvature as example
   (iii) statistical measures of complexity, entropy

B Quantum ergodicity

1. original proof: bound the quantum variance using time evolution [Sun97]. Variations: non-diagonal matrix elements and inverse results [Zel96-1]
2. alternative proof: ergodicity = extremal measures [GerLei93, CKST08]
3. Rate of quantum ergodicity: upper and lower bounds for Anosov systems.
   (i) estimates on the quantum variance [Zel94, Schu06, Schu08].
   (ii) large deviation estimates [AnRiv10]
4. Physics background: numerics for billiards [McDK79], random wave model [Ber77], heuristics for the quantum variance [FeinPer86].

C Quantum unique ergodicity vs. strong scarring

1. Quantum "cat" map
   (i) strongly scarred states and upper bounds on scarring [FNDB03, FN04]
   (ii) Hecke bases, QUE [KurRud00]
   (iii) failure of Arithmetic QUE on higher dimensional tori [Kelm10]
2. other quantum maps [CKST08]

3. bouncing-ball modes for stadium-like billiards [Hass10]

D Arithmetic QUE for hyperbolic surfaces

1. basics on Hecke correspondences
2. lower bound on the pointwise entropies of semiclassical measures
3. measure rigidity and QUE
4. estimates on the quantum variance

E Lower bounds on eigenstate delocalization

1. hyperbolic dispersive estimates
2. lower bounds on the entropy (topological/metric)
3. eigenmodes of large discrete graphs

F other developments

1. other systems: graphs, more on billiards
2. restriction to submanifolds, boundary functions of graphs
3. complex extensions of eigenfunctions
4. Systems of equations: Dirac, Maxwell, Vector bundles
5. Quantum ergodicity in the time domain, equidistribution of waves.

Talks:

The purpose of the next 4 talks is to introduce some of the basic concepts from microlocal analysis and ergodic theory. We expect that the participants have studied already at least part of the material beforehand in preparation of their talks, so the purpose of these first talks is to give a guided tour through the background needed.

1. **$\hbar$-pseudodifferential calculus on $\mathbb{R}^d$ or a compact manifold. Egorov's theorem**
   [EvZw09, Chap.4,App.E] [DimSj99] This talk should cover the material in Sections 4.1-4.5, Appendix E and finally Egorov's Theorem in Section 9.2 of the Lecture notes [EvZw09]. In particular the relation between

operators and symbols, the definition of symbol classes, the product formula and conditions for L^2 boundedness and (essential) selfadjointness (see [DimSj99, Prop 8.5]) should be covered. The adaptation to the case of a (say, compact) manifold should be explained. The quantum-classical correspondence (Egorov's Theorem) and its proof should be discussed in detail, including the special case that the symbol of the Hamilton operator is quadratic, in which case the correspondence is exact.

A summary of the main concepts can be found in [Schu01, Chap 2]

2. **Phase space representations. Wigner / Bargmann-Husimi functions. Semiclassical measures. Weyl's law: local and integrated version**
Wigner and Bargmann representations: [Fol88] Microlocal measures: [EvZw09, Chap.5,Chap.8] [Bur97] Weyl's law: [EvZw09, Chap.6,App.E]

   In the first talk the relation between operators, which act on functions on some manifold M, and symbols, which are functions on phase space T^*X, was introduced. If one takes a state $\psi \in L^2(X)$ and considers the orthogonal projection onto the one-dimensional subspace of $L^2(X)$ spanned by ψ, then the symbol of this operator is a phase space representation of the function ψ. In case that $M = R^d$ this defines the Wigner function of the state ψ.

   To a semiclassical sequence of (normalized) states $(\psi_\hbar \in L^2)_{\hbar \to 0}$ one can associate the corresponding sequence of phase space representations. Any semiclassical limit (in a weak topology) of these representations defines a probability measure on phase space, which has been given various names (semiclassical measure, quantum limit, defect measure, microlocal lift). If the states $\psi_\hbar$ are eigenstates of a Hamiltonian $P(\hbar)$, then by Egorov's theorem any quantum limit is invariant under the classical flow, which results in a natural question: which invariant measures can occur as quantum limits? Quantum ergodicity will provide a partial answer if the classical flow is ergodic.

   This talk should guide through the main properties of such phase space representations and quantum limits. [EvZw09, Sec. 5.1,5.2] cover the basic properties of quantum limits. Complementary material on Wigner and Husimi functions, and on Anti-Wick quantisation, can be found in [Schu01, Chap 4].

In the second part the functional calculus for pseudodifferential operators, following [DimSj99, Chap 8], should be developed and used to prove the local and integrated versions of Weyl's law, see [EvZw09, Sec 9.3]. The functional calculus is furthermore very useful because it allows to localise in energy and therefore we can, in the semiclassical setting, always assume that our operators are bounded.

3. **Concepts from ergodic theory**

   This talk should introduce/recall basic notions from dynamical systems and ergodic theory.

   Invariant measures, Birkoff theorem, ergodicity (definitions in terms of sets, time averages, and extremal elements in the convex set of invariant measures), unique ergodicity, weak mixing, mixing. [KatHas95, Chap. 4] [BriStu02, Chap. 4] [Wal82, Chap 1,6]

   Examples: Hamiltonian flows. [KatHas95, Chap. 4] Symplectic maps on the torus: skew translations, "cat" map (hyperbolic automorphisms) [KatHas95, Chap. 1] [BriStu02, Chap. 1]

   Complexity theory of dynamical systems: topological entropy of a closed invariant set, metric entropy of an invariant measure. [KatHas95, Chap. 3,4] [BriStu02, Chap. 2,9] [Wal82, Chap 4,7.8]

4. **Hyperbolic systems, chaotic billiards**
   The talk should be specifically oriented to the description of hyperbolic (unstable) dynamical systems, and their ergodic properties.

   Hyperbolicity, construction of stable/unstable manifolds. [KatHas95, Chap 6] [BriStu02, Chap 5] Uniform hyperbolicity; Anosov map or flow. [KatHas95, Chap 17] [BriStu02, Chap 5.10]

   Main examples of symplectic Anosov systems: "cat map" and its perturbations on the torus [BriStu02, Chap 5.11] [KatHas95, Chap 18.6]. Compact Riemannian manifold of negative curvature [KatHas95, Chap 17.5-6].

   Periodic orbits and equilibrium states of Anosov maps [KatHas95, Chap 18.5, 20]. Ergodicity of the Liouville measure for symplectic Anosov maps [BriStu02, Chap 6.3] [KatHas95, Chap 20.4].

Chaotic billiards: dispersive (Sinai's) and nondispersive (Bunimovich's) planar billiards. Expository book: [CherMar03, Chap.4] More technical treatment: [CherMar06, Chap.8,9]

5. **QE for smooth Hamiltonian flows**
(see [EvZw09, Chap.9] and the reviews [Sun97, Zel09]).
The concept of Quantum Ergodicity appeared in the study of the eigenfunctions of the Laplace-Beltrami operator on compact Riemannian manifolds of negative curvature. This property, which describes the eigenfunctions in the high-frequency limit, states that as a consequence of ergodicity of the geosisc flow "almost all" eigenfunctions become equidistributed over the manifold.

The property was sketched by Schnirelman in [Schn74], which is why QE is also called "Schnirelman's theorem". However, the first full proof was given by Zelditch in the case of constant negative curvature [Zel87], using a specific way to lift the spatial density $|u_n(x)|^2$ into a measure on S^*X (this "microlocal lift" was using the group theory underlying X).

The proof in the case of general smooth manifolds with ergodic geodesic flow was given by Colin de Verdière [CdV85]. It uses a local Weyl's law, and Egorov's theorem (the fact that evolution and quantization almost commute in the semiclassical limit) in order to estimate the "quantum variance", that is, the variance of the distribution of matrix elements $\langle u_n, \mathrm{Op}(a)u_n\rangle$, for a an observable on T^*X.

The ideas were generalized to the semiclassical setting in [HMR87]. One considers a (classical) Hamiltonian $p(x,\xi)$ on the phase space $T^*\mathbb{R}^d$, such that for some energy $E \in \mathbb{R}$ the energy shell $p^{-1}(E)$ is compact, and the Hamiltonian flow on $p^{-1}(E)$ is nonsingular and ergodic (w.r.to the Liouville measure).

The quantum problem is obtained by $\hbar$-quantizing the Hamiltonian into a family of selfadjoint operators ($P_\hbar = \mathrm{Op}_\hbar(p)$, $\hbar \to 0$). The spectrum near E is then discrete. One is interested in the eigenstates of $P_\hbar$ in small energy intervals near E, in the limit $\hbar \to 0$. The proof of QE then proceeds similarly as in [CdV85], namely by bounding the quantum variance associated with the states in this interval, using the semiclassical version of Egorov's theorem. The only new element that one needs is a sharper version of the local Weyl law [PetRob85].

6. **QE for chaotic billiards**
Chaotic billiards form a *physically relevant* family of chaotic systems. The classical ray dynamics is fully determined by the shape of the billiard. There exist simple examples of 2D Euclidean billiards which were proved to be chaotic, e.g. the "dispersive" billiards studied by Sinai, or Bunimovich's "stadium" billiard [Buni79]. At the quantum level, understanding the eigenmodes of the Laplacian (say, with Dirichlet b.c.) is a relevant question in various physical contexts, and these billiards are the quantum chaotic systems most studied numerically [McDK79, Hel84].

   Compared with the case of smooth flows, the difficulty of applying semiclassical methods to billiards lies in the singularities of the flow at the boundary. First, the flow in a billiard is always discontinuous at the boundary; second, the boundary of an ergodic billiard cannot be smooth (C^∞).

   However, Zelditch and Zworski showed that, provided the set of singular rays has negligible Liouville measure, the proof of QE on smooth manifolds can be adapted rather easily [ZelZwo96]. The proof of QE for (convex) chaotic billiards was first given in [GerLei93] through a different method: from the eigenstates they define positive measures on the reduced phase space associated with the *boundary* of the billiard. Instead of estimating the quantum variance through time propagation, the authors use the fact that ergodic invariant measures are extremal in the (convex) set of invariant probability measures.

7. **QE for quantized chaotic maps**
A useful toy model for chaotic Hamiltonian flows are chaotic symplectic maps acting on some compact phase space (that is, a compact symplectic manifold). The main example of phase space where strongly chaotic (uniformly hyperbolic) maps can be constructed is the 2-dimensional torus. This toy model can be adaped to the quantum framework. One first defines a "ladder" of finite dimensional Hilbert spaces $\mathcal{H}_N$ of quantum states "living" on the torus. These spaces have arbitrary large dimensions $N \sim \hbar^{-1}$. One may quantize observables into selfadjoint operators (matrices) on $\mathcal{H}_N$. One also wants to quantize symplectic maps κ into unitary propagators (quantum maps) $U_N(\kappa)$; these operators have similar properties as the Fourier Integral Operators associated

with certain symplectic transformations on $T^*\mathbb{R}^d$ [EvZw09, Sect. 10]. For a given map κ there is no canonical quantization procedure, but rather quantization "recipes" (see e.g. [Zel97] for a geometric quantization procedure, or the more "hands-on" approach of [KMR99]). Specific chaotic maps on the 2D torus have received a particular attention: the hyperbolic torus automorphisms ("cat" maps), which were quantized e.g. in [HanBer80, DE93, BDB96] which enjoy rich algebraic and arithmetic structures. Another popular chaotic map: the baker's map, quantized in [BalVor98].

In the case where κ is chaotic, the objective is to analyze the eigenstates $(u_{N,n})_{n=1,\dots,N}$ of the propagators $U_N(\kappa)$, in the semiclassical limit $N \to \infty$. For this aim one may use the same phase space representations as on $T^*\mathbb{R}^d$, e.g. semiclassical measures associated with sequences of eigenstates. Quantum ergodicity holds for quantized (smooth) ergodic maps [BDB96, Thm 1.1].

Ref. [MOK05, Sec 2] introduces the minimal "quantization axioms" necessary to prove a form of Weyl's law, as well as QE, for quantum maps [MOK05, Thm 8.1]. In particular, one can allow the map κ to have singularities, as long as those have negligible Liouville measure. The authors consider a family of linked twist maps. The particular example of the quantum baker's map (which is discontinuous) is treated in [DENW06].

In the talk the definition and properties of quantum maps on the torus should be presented, with specific description of the "cat" map and its perturbations, and the baker's map. The proof of QE for smooth maps should be given in detail, and possibly also the modifications necessary in presence of discontinuities.

8. **Reverse QE, Quantum Mixing**
(see [Sun97], [Zel09, Sec.4])
We have seen that the implication (classical ergodicity implies QE) holds for a vast set of systems. The reverse implication (QE implies classical ergodicity) does not hold in general (see e.g. the counterexample found by Gutkin [Gut09]). However, one can show that ergodicity is equivalent to a quantum property slightly stronger than QE, involving transition amplitudes between different eigenstates [Zel90, Sun97].

A chaotic property stronger than ergodicity is the *mixing* property, or decay of time correlations. In [Zel96-1] Zelditch shows that a slightly weaker property, namely weak mixing, implies a property of the quantum transition amplitudes, which he baptized *quantum weak mixing*.

The results mentioned above should be presented in the talk, as well as the proof for quantum weak mixing.

9. **QE for quantum graphs**
Quantum graphs form a family of simplified quantum models which can be easily investigated, both numerically and analytically. The idea is to replace the Laplacian on a manifold by a form of "triangulation", namely a graph (network) composed of 1D segments (edges, links, bonds) with their ends connected at vertices. One can then construct various self-adjoint Laplacians on the graph, with $\Delta = -\partial^2/\partial x^2$ on the edges, plus specific boundary conditions at the vertices.

If the graph is finite, the spectrum of this Laplacian is discrete, and can be obtained by solving a finite dimensional generalized eigenvalue problem. One can then investigate the high-frequency behaviour of its eigenvalues or eigenfunctions [KotSmil97]. It appears that sticking to a fixed graph is not very interesting: one should rather consider a family of graphs $(\mathcal{G}_N)_{N\to\infty}$ sharing some features, and investigate both limits $N \to \infty$ and $\lambda \to \infty$.

QE-like problem: exhibit such a family of graphs, for which (most of) the eigenfunctions become equidistributed. Notice that the classical limit of such a graph is not a deterministic system, but rather a Markov process: at each vertex the particle has a choice between several future directions. So the notion of ergodicity is not really inappropriate in this context.

Eigenmodes of various graphs were investiged numerically in [Kap01]. It is proved in [BKW05] that the simple family of *star graphs* does not satisfy QE. On the opposite, in [BKS07] a family of graphs, derived from 1D maps on the interval, are shown to satisfy QE.

At a nonrigorous level, in [GKP08, GKP10] an effective field theoretic model is used to obtain criterion on the structure of the graphs G_N, leading to the equidistribution of the high frequency eigenmodes. This criterion depends on the spectrum of the Markov matrices representing the "classical limits" of the quantum graphs.

A different point of view is to consider the eigenstates of the *discrete Laplacian* on a family of large graphs. In [Elon08] Elon studies the value distribution of eigenstates of random regular graphs, and checks that this distribution is asymptotically Gaussian. Brooks and Lindenstrauss focus on regular graph with large "girth" (such graphs have locally the structure of a regular tree, which allows to use discrete harmonic analysis on the regular tree). In [BroLin10, Thm 1] they prove a discrete version of *hyperbolic dispersive estimate* (see the last talk), which implies some delocalization for the discrete eigenstates.

The talk should present the formalism of quantum graphs and the definition of QE in this context. The (numerical, heuristic or rigorous) results on the graph eigenstates mentioned above should be presented. Time permitting, the results on the eigenstates of the discrete Laplacian should be mentioned as well.

10. **QE for systems**

In interesting quantum systems the dynamics is generated by a Dirac- or Laplace type operator which typically acts on sections of a vector bundle. Examples from global analysis are the Hodge Laplacians on differential forms in the Riemannian or Kähler case, or Dirac operators and their twisted forms. In physically relevant cases one needs to go from the scalar case to systems, resp. matrix valued operators, in order to include the spin of particles or gauge-fields with non-abelian gauge symmetries. In these cases the algebra of classical observables is typically a non-commutative algebra of endomorphism valued symbols. QE for such systems has been investigated in two slightly different cases:

(a) The case of the dynamics generated by Laplace type operator has been considered in [JakStro07, JSZ08], [BuOl04]. As a typical feature the principal symbol of the Hamiltonian is scalar and QE is considered in the high energy regime.

(b) In [BolGla00, BolGla02, BolGla04] QE is studied in the semiclassical limit. The most interesting new features arise from the non-scalar principal symbol of the Hamiltonian which leads to reconsider the definition of the correct algebra of observables.

The conditions for QE in both cases have to take the mixing not only on the underlying phase space, but also for the internal finitely many degrees of freedom into account.

This talk should concentrate on one of these two cases.

(a) In this case one should follow [JakStro07] and explain the notion of a frame flow and conditions for its ergodicity. Further one should show the versions of Egorov's theorem [JakStro07, Prop 3.3, Prop. 4.1]. The main results to be presented should be the statements for QE for the Hodge Laplacians on forms and the spin Dirac operator in terms, [JakStro07, Sec. 4,5]. If time permits, the indicate in the Kähler case following [JSZ08], how one incorporates internal symmetries in the conditions for QE.

(b) In this case one should follow [BolGla04] and explain the new features due to the non-scalar principal symbol of the Hamiltonian. Explain the semiclassical resolution of the identity [BolGla04, Sec 2], identify and characterize the relevant algebra of observables [BolGla04, Thm 3.2, Prop. 3.4] and explain the Stratonovich-Weyl symbol [BolGla04, Def. 4.4]. The main theorems to be discussed are the Egorov-type theorem on the flow of principal symbols [BolGla04, Prop. 4.6], the Szegö-type formula [BolGla04, Prop 5.1], and the QE theorem [BolGla04, Thm. 6.1]. Put the emphasis on the explanation of the new features compared with the scalar case.

11. **Estimating the rate of QE**

The proof of QE is based on the fact that the *quantum variance*, namely the variance of the distribution of matrix elements $\langle \psi_n, \mathrm{Op}(f)\psi_n \rangle$, with ψ_n eigenmodes in a frequency window around λ, decays to zero in the limit $\lambda \to \infty$. One is interested in finer informations on this distribution. Heuristics suggest that this distribution is approximately a Gaussian centered at the ergodic average $\bar{f}$, with variance $\sim C(f)/\lambda^{d-1}$, where $C(f)$ is an explicit "classical" variance [FeinPer86, Ekh+95]. Logarithmic upper bounds for the quantum variance and higher moments were obtained in [Zel94] for geodesic flows and [Schu06] for Hamiltonian flows. In [Schu08] it is shown that this logarithmic bound can be sharp for certain quantum maps.

Using a different method (large deviation estimates), it is shown in [AnRiv10, Thm 2.2] that the number of eigenstates showing large deviations from equidistribution satisfies an algebraic upper bound.

Precise estimates for the quantum variance were obtained for some chaotic systems with arithmetic symmetries: the Hecke-Maass eigenfunctions on the modular surface [Zhao10], and the Hecke eigenstates of the quantum cat map [KurRud05]. The methods are completely different from the previous ones: they use explicit formulas for the matrix elements, arising from analytic number theory.

The talk should explain the heuristics for the quantum variance, the dynamical proofs of [Schu06] for the upper bounds, and mention the specific lower bounds in [Schu08]. The result of [AnRiv10] should also be explained. Mention should be made (without too many details) of the arithmetic results in [Zhao10, KurRud05].

12. **Holomorphic Q(U)E and equidistribution of zero sets**
QE and QUE address the localization properties of spatial or phase space densities which represent the probability of presence of the quantum particle. For systems defined on the phase space $T^*\mathbb{R}$, the Bargmann-Husimi representation of a state $u \in L^2(\mathbb{R})$ is an entire function $\mathbb{B}u(z)$ (with complex phase space parameter $z = x - i\xi$). This formalism applies as well to the phase space $\mathbb{T}^2 = (T^*\mathbb{R})/\mathbb{Z}^2$: a state $u_N \in \mathcal{H}_N$ is represented by an entire, quasiperiodic function $\mathbb{B}u_N(z)$, or equivalently by a section of a certain holomorphic line bundle over $\mathbb{T}^2$. This section has exactly N zeroes on the torus, and it can be reconstructed from these zeroes by an explicit Weierstraßproduct formula. The data of these zeroes therefore represents a faithful, minimal description of the state u_N, which has been named the *stellar representation* [TuaVor95]. Lebœuf and Voros proposed [LebVor90] to characterize the difference between chaotic and "regular" eigenstates by studying this stellar representation, and performed numerical experiments. In the case of a Liouville integrable system, the zeroes sit on certain 1D curves of the torus (called anti-Stokes lines); on the opposite, for chaotic states the zeroes seem distributed over the full torus. It was conjectured that the distribution is then asymptotically uniform, as for the Husimi function itself. Notice that the relation between the measure carried by the zeroes and the Husimi function is very nonlinear.

It was then proved in [NonVor98, Thm 1], and in a more general framework in [ShifZel99, Lemma 1.4], that if the Husimi densities associated with a sequence ($u_N \in \mathcal{H}_N$) become equidistributed, then the corresponding zero sets equidistribute as well. Hence, QE (resp. QUE) implies that for almost all (resp. all) sequences of eigenstates, the zero sets equidistribute. The proof of this property uses some basic potential theory, in particular properties of subharmonic functions.

This property (which uses the compactness of the torus) was extended to the case of holomorphic cusp forms on the modular surface by Rudnick [Rud05, Thm 2]: in this case the Planck's parameter N is replaced by the *level* $k \in 2\mathbb{N}$ of the modular form. The proof is very similar with the ones in [NonVor98, ShifZel99], the cusp bringing only a minor difficulty.

The motivation in [Rud05] was to study the *Hecke eigencuspforms*, that is the holomorphic cusp forms which are eigenfunctions of the family of Hecke operators. Indeed, QE had just been proved for these eigenforms [LuoSar03], and QUE had been established under the Generalized Riemann Hypothesis. Note that in this problem there is no underlying classical dynamics or phase space, so the term Q(U)E is rather inappropriate; one should rather speak of *asymptotic spatial equidistribution*. The semiclassical limit is replaced by the high level limit $k \to \infty$.

QUE for Hecke eigenforms was proved only recently. Holowinsky [Hol10, Cor. 1.3] and Soundararajan [Sound10-1, Cor. 1] obtained two different upper bounds for the diagonal matrix elements of Hecke eigenforms; each bound implies that the possible exceptions to equidistribution are rare (namely a form of QE). Yet, it appeared that the two set of exceptional eigenforms obtained by both methods can be sufficiently characterized to show that they have no intersection: this proves QUE [HolSou10, Thm 1]. Using Rudnick's result, one automatically gets the equidistribution of the zero sets of these eigenforms [HolSou10, Rem. 2].

The talk should present the stellar representation formalism, as well as the proof of the implication between mass equidistribution and equidistribution of the zero set, say on $\mathbb{T}^2$. The developments results on the Hecke eigencuspforms should be also mentioned as well, without giving too many details of the proofs.

13. **Arithmetic Quantum unique ergodicity**
The QE property describes the asymptotic equidistribution *almost all* the high frequency eigenmodes. It is natural to ask [CdV85] whether *all* eigenstates actually follow this behaviour, a property called Quantum Unique Ergodicity (QUE), or on the opposite some exceptional eigenstates show a different asymptotic distribution on phase space, e.g. have a positive mass in the vicinity of an unstable periodic orbit (a phenomenon referred to as a *strong scar*).

Rudnick and Sarnak conjectured in [RudSar94] that QUE holds true for any compact manifold of negative curvature. They then focussed on *arithmetic* surfaces (quotients $\Gamma\backslash\mathbb{H}$ by co-compact arithmetic subgroups $\Gamma < SL(2,\mathbb{R})$), for which it is natural to consider joint eigenstates of the Laplacian and of all Hecke operators; in this framework, one speaks of arithmetic QUE. Using the full set of Hecke symmetries, they show that Hecke eigenstates cannot be too concentrated near a closed geodesic, thereby ruling out the possibility of a "strong scar" component in the semiclassical measure [RudSar94, Thm 1.1].

The proof of arithmetic QUE on such arithmetic surfaces was given by Lindenstrauss [Lin06, Thm 1.4]. One first proves that semiclassical measures cannot be too concentrated along geodesics [BourLin03, Thm 2.1], by refining the method used in [RudSar94]; this nonconcentration has the consequence that the entropies of the ergodic components of the semiclassical measure are bounded below. The second step relies on a *measure rigidity* result for the joint action of the geodesic flow and one Hecke correspondence [Lin06, Thm 1.1]. The semiclassical measure of a joint Laplace-Hecke eigenstate is invariant under the geodesic flow, but is also proved to be *recurrent* w.r.to the Hecke correspondence (recurrence is a weaker property than invariance). Using the positive entropy property, Lindenstrauss proves that such a measure must be the Haar measure on S^*X.

A more recent approach avoids the sieve argument of [BourLin03], and proves that every ergodic component of a semiclassical measure associated with joint eigenstates of the Laplacian and a *single* Hecke operator has positive entropy [BroLin10-1, Thm 1]. The authors use harmonic analysis on the Hecke tree to show that eigenfunctions must be "spread enough" across the tree (see also [BroLin10]). Once this is established, the proof of QUE proceeds as in [Lin06].

The talk should provide a proof of the relative delocalization of Hecke eigenfunctions (following either [BourLin03] or [BroLin10-1]), and then give the ideas of the main steps of Lindenstrauss's proof in [Lin06]. The ideas of the proof are also explained in the proceedings [Lin05]

14. **QUE for the modular surface**
There exists a natural class of surfaces of negative curvature which are not compact, but have finite volume: they present one or several *cusps* towards infinity, leading to a continuous spectrum on $[1/4, \infty)$ spanned by the so-called Eisensteing series.

However, because this cusp is "thin", such a surface can still accomodate L^2 eigenstates. Zelditch [Zel91] proved a version of QE for finite volume hyperbolic surfaces, which groups together the L^2 eigenstates and the Eisenstein series.

In the case of the modular surface $X = PSL(2, \mathbb{Z})\backslash\mathbb{H}$, the discrete spectrum is infinite, and "dominates" the continuous one. This surface is arithmetic, so it is normal to consider the joint Hecke eigenstates (called Hecke-Maass waveforms). The equidistribution of the Eisenstein series (spanning the continuous spectrum) was proved by Luo and Sarnak [LuoSar95] and lifted to S^*X by Jakobson [Jak94].

The proof of QUE by Lindenstrauss [Lin06, Thm 1.4] applies as well to the Hecke eigenstates on the modular surface, with the proviso that the probability could partially "leak" into the cusp. This possibility was later ruled out in [Sound10] using purely number theoretic methods. See a survey of these results in [Sar11].

The talk should introduce the spectral decomposition of the modular surface (continuous spectrum made of Eisenstein series, discrete embedded spectrum of Hecke-Maass eigenfunctions), an explain Zelditch's proof of QE [Zel91]. The Hecke operators should be made explicit. The ideas of the proofs of QUE for both Eisenstein series [LuoSar95] and Hecke eigenfunctions [Lin06, Sound10] should be given.

15. **Q(U)E for higher rank symmetric spaces**
The surfaces $\Gamma\backslash\mathbb{H}$ are symmetric spaces of rank $r = 1$. On locally symmetric spaces of the form $\Gamma\backslash G/K$ of rank $r > 1$ (G a connected semisimple Lie group, K a maximal compact subgroup, $\Gamma < G$ a lattice) there exist r independent Hamiltonians in involution (meaning that

the flows they generate commute with one another). Hence, any flow generated by some combination of these Hamiltonians leaves invariant the *joint energy shells*. The regular energy shells admit an algebraic description: they are isomorphic to $\Gamma\backslash G$.

At the quantum level, there exist a corresponding commutative algebra G-invariant differential operators with r generators; it is natural to study the joint eigenstates of this algebra in the limit where at least one of the eigenvalues goes to $+\infty$. In particular, in this case QE or QUE are defined by the equidistribution of the eigenstates on the *joint energy shell*.

The first example is made of manifolds of the type $X = \Gamma\backslash\mathbb{H} \times \mathbb{H}$, with $\Gamma < SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ a cocompact irreducible lattice. At the quantum level, one considers the joint eigenstates of the two partial Laplacians on X, each associated with one copy of $\mathbb{H}$, in the limit where at least one of the eigenvalues goes to $+\infty$. Lindenstrauss [Lin06, Thm 1.6] proved QUE for these joint eigenfunctions, under the condition that almost all ergodic components of the semiclassical measure have positive entropy; this condition can be proved to hold [BroLin10-1] without any need for Hecke symmetries.

Meanwhile, Kelmer [Kelm08-1, Thm 1] has proved a form of QE for the quotients $X = \Gamma\backslash(\mathbb{H} \times \mathbb{H} \cdots \times \mathbb{H})$ (the rank equals the number of factors $\mathbb{H}$).

The more general setting of locally symmetric spaces $X = \Gamma\backslash G/K$ was studied by Silberman and Venkatesh. Generalizing Zelditch's method, in [SilVen07] they construct equivariant microlocal lifts for the joint eigenstates, which naturally live on the joint energy shell. In a second work, they focus on certain arithmetic quotients of $G = PGL(d, \mathbb{R})$ (d prime), which allows to consider the joint eigenstates of the differential algebra and of infinitely many Hecke operators. In this setting they prove Arithmetic QUE, namely the fact that the lifts of these joint eigenstates converge to the Haar measure [SilVen11, Thm 1.1]. Like in [Lin06], they use the Hecke operators to prove that all the ergodic components of the semiclassical measure have positive entropies. Then, they invoke a measure rigidity result due to Einsiedler and Katok to conclude.

In the case of symmetric spaces of nonpositive curvature, Anantharaman and Silberman [AnSil10, Thms 1.8,1.9,1.10] proved some delocalization properties for the microlocal lifts of the joint eigenfunctions. Their arguments are purely dynamical and semiclassical, in particular they do not use any arithmetic structure. The delocalization is expressed through lower bounds on the entropy of the semiclassical measure [AnSil10, Thm 1.6] (see the corresponding talk below).

The talk should present the specific features of higher rank symmetric spaces (in particular the joint energy shells and the algebra of equivariant differential operators).

16. **QUE for the quantized torus automorphisms**
Considering a given torus automorphism $A \in SL(2,\mathbb{Z})$ and its quantizations $U_N(A)$, Kurlberg and Rudnick introduce a commutative algebra of Hecke operators commuting with $U_N(A)$ [KurRud00, Cor. 6], which allows to consider the joint Hecke eigenbasis. They then prove that all the Hecke eigenstates semiclassically equidistribute [KurRud00, Thm 1]. In [KurRud01, Thm 1] they show that QUE holds for all eigenstates (Hecke or not) along a subsequence of Planck's parameter $(N_k)_{k\geq 1}$ of of density one (namely the values for which the spectral multiplicities are not too large).

In [KurRud05, Thms 2,3] the authors compute the variance and fourth moment of the distribution of diagonal matrix elements of a given observable over the Hecke eigenstates; they present a conjecture for the explicit form of the distribution, which slightly differs from the Gaussian distribution conjectured to hold for generic chaotic systems.

The talk should present the ideas of the proofs of the above results, in particular the link between QUE with certain arithmetic counting problems.

17. **Scars of periodic orbits**
This talk should present some of the ideas from physics about eigenfunctions of chaotic systems and show lots of pictures of (numerically computed) eigenfunctions, which started with a rough study of the stadium billiard in [McDK79]. Berry conjectured in [Ber77] that eigenfunctions of a chaotic billiard should statistically behave like random superposition of simple (plane) waves: this is the *random wave conjecture*. This conjecture would imply QUE, but much more: it predicts a

Gaussian value distribution for the eigenfunction $u_\hbar(x)$, and a growth $\sim \sqrt{\log \hbar^{-1}}$ of their L^∞ norms. Numerical test of some of these predictions can be found in [ABST99].

In [FeinPer86] a relation between the quantum and the classical variance was given, which leads to a prediction for the rate of quantum ergodicity. The behaviour of the quantum variance was studied numerically in detail in [AuTag98, BSS98] for surfaces of negative curvature and planar billiards, respectively, and more recently in [Bar06].

Seemingly in stark contrast to the random wave picture, Heller observed [Hel84] the presence of *scars* of periodic orbits on some billiard eigenfunctions; these scars appear as *enhanced* amplitudes of the eigenfunctions along certain (unstable) periodic orbits. Although originally observed on the spatial density $|u_\hbar(x)|^2$, scars are easier to detect in phase space representations, especially in the case of quantum maps on the 2D torus (in that case a periodic orbit consists in finitely many points) [Kap99].

Attempts were made to explain this phenomenon by averaging over some energy window (that is, effectively treating quasimodes rather than individual eigenstates) [Bogo88, Ber89]. Part of the problem is to find a precise, quantitative definition of what a scarred wavefunction is. Nowadays one usually distinguishes between weak and strong scars: (weak) scars are enhanced amplitudes along periodic orbits, but which do not necessarily lead to a violation of QUE. Still, they violate the random wave picture, but are not strong enough to show up in the quantum limit. Indeed, scars were defined in [Kap99] by a statistical deviation of the value distribution from the random wave predictions.

Strong scars, on the other hand side, survive the semiclassical limit and lead to a violation of QUE, i.e., quantum limits which are different from the Lebesgue measure. Explicit examples are constructed in a the next talk.

18. **Counterexamples to QUE for quantum maps**
Here we present an array of quantum chaotic maps which, in a certain sense, are "solvable": one has explicitly construct the eigenstates, some of which can be shown to be "exceptional". One has a good description of the corresponding semiclassical measures. These examples are counterexamples of QUE for chaotic maps.

The first class of maps to be considered are the quantized hyperbolic automorphism of $\mathbb{T}^2$ ("quantum cat maps"), namely the sequence of propagators $(U_N(A))_{N\geq 1}$ quantizing the action of a hyperbolic matrix $A \in SL(2,\mathbb{Z})$. Due to the algebraic properties of this construction, the quantum maps $U_N(A)$ are periodic (up to a phase), i.e, there exist $T(N) \in \mathbb{N}$ such that $U_N(A)^{T(N)} = e^{i\varphi_N} I$. For a rare, yet infinite sequence $(N_k)_{k\geq 1}$, the periods are short, $T(N) \sim C \log N$, which implies that the spectrum of U_{N_k} is very degenerate, and leaves a lot of freedom to construct "weird" eigenstates. It is shown in [FNDB03, Thm 1] that one can indeed construct eigenstates u_{N_k} of $U_{N_k}(A)$ which partically concentrate near any given periodic orbit. The description of these eigenstates is very explicit.

In [FN04, Thm 1.1] a restriction on semiclassical measures is obtained: the "pure point" component of a semiclassical measure, supported on a family of periodic orbits, implies the existence of a Liouville component of greater or equal mass.

A similar construction can be set up for a nonstandard (Walsh) quantization of the baker's map on $\mathbb{T}^2$ [AnaNo07-1, Thm 1.3, Sec. 4.1] (this quantum map is also periodic). In this case, one can also exhibit semiclassical measures supported on proper fractal subsets of $\mathbb{T}^2$ [AnaNo07-1, Sec. 4.2].

In the case of (certain) hyperbolic automorphisms on higher dimensional tori $\mathbb{T}^{2d}$, Kelmer constructed *Hecke* eigenstates completely localized on certain invariant co-isotropic subtori [Kelm10, Thm 1], thereby disproving arithmetic QUE in this context (these states were baptized "superscars"). He then showed that the construction generalizes to certain nonlinear perturbations of those automorphisms, and that this phenomenon is not due to spectral degeneracies [Kelm08].

In [CKST08] a specific quantization of certain cutting-and-stacking maps on the unit interval (lifted to $\mathbb{T}^2$) lead to a a full characterization of the set semiclassical measures, including cases of ergodic maps which violate QUE.

The talk should recall the definition of the quantum cat map, and explain the construction of the half-scarred states of [FNDB03]. The (very different) construction of the "superscars" in higher dimension [Kelm10, Kelm08] should also be explained. The results pertaining to the quantum maps in [CKST08] should also be explained.

19. **Bouncing ball modes on ergodic billiards**
On certain 2D billiards (or manifolds) for which the geodesic flow is ergodic, there may exist a 1-parameter family of periodic orbits which are marginally stable. This is the case if the billiard contains a rectangle, like in the case of the "stadium" billiard proved to be chaotic by Bunimovich [Buni79]. One then speaks of *bouncing ball orbits*. Donnelly [Donn03] also constructed a smooth surface containing a "flat tube" which carries a family of marginally stable trajectories.

    At the quantum level, numerical studies of the Laplacian on such billiards have revealed the presence of many eigenstates which are stronly localized along this family of trajectories [Hel84, HelOCon88]. Several questions arose: do such states persist in the high frequency limit? How strongly are they localized?

    It is easy to construct *quasimodes* localized on these trajectories [BSS97, Donn03, Zel04]. But the proof that actual eigenmodes have this property, or at least partially concentrate on these orbits, has taken more time [Hass10, Thm.4]. The main enemy when going from quasimodes to eigenvalues is the possibility that many eigenvalues cluster in a small energy range. This possibility can be ruled out (at least for "almost all" aspect ratio of the rectangle) by carefully following the dynamics of the eigenvalues when deforming the billiard.

    This result is the only known counterexample to QUE in the framework of chaotic geodesic flows.

    The talk should clearly explain the specific problem of bouncing ball modes in billiards, and explain Hassell's proof.

20. **Entropy bounds for semiclassical measures**
Investigating QUE for Anosov manifolds without any arithmeticity condition, Anantharaman has initiated a new approach to QUE by obtaining nontrivial lower bounds for the Kolmogorov-Sinai entropy of semiclassical measures. Her arguments uses a *hyperbolic dispersion estimate*: due to the classical hyperbolicity, any intially localized state will necessarily disperse after evolving for a time $\sim \log \hbar^{-1}$. One uses this (local) dispersion to prove a lower bound on a quantum dynamical

entropy associated with logarithmic times. One then invokes a subadditivity property of the entropy to bring this lower bound on entropies for short times, and finally on the entropy of the semiclassical measure.

In [Ana08, Thm 1.1.1] she shows that such a semiclassical measure cannot be supported on a "small" invariant set, namely a set on which the geodesic flow has a small topological entropy. In [AnaNo07-2, Thm 1.2] a more explicit lower bound is obtained for the metric (Kolmogorov-Sinai) entropy of any semiclassical measure. The proof uses an *entropic uncertainty principle* to derive the lower bound on the quantum entropy. This lower bound has been improved in [AKN09, Thm 4], and an "optimal" lower bound has been obtained in [Riv10-1, Thm 1.2] for hyperbolic surfaces, and generalized in [Riv10-2, Thm 1.1] for surfaces of nonpositive curvature. Roughly speaking, the lower bound amounts to half the maximial entropy.

The method applies as well to the case of quantized Anosov maps [Broo10, Non10a], as well as a nonstandard (Walsh-like) quantization of the baker's map [AnaNo07-1, Gut10].

These entropy bounds were refined in [Broo10] in the specific cases of the hyperbolic torus automorphisms and the Walsh-quantized baker's map, to show that high-entropy components (that is, delocalized components) of the semiclassical measure must outweight low-entropy components (that is, more localized ones).

The talk should present the proof of the lower bound on the Kolmogorov-Sinai entropy, from the quantum partition of unity, hyperbolic dispersive estimate, entropy uncertainty principle, and subadditivity argument. The improvements due to Brooks and Rivière should be mentioned.

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Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`ulrich.bunke@mathematik.uni-regensburg.de`

by **August 15, 2011** at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.