OBERWOLFACH-SEMINAR ON "AFFINE ALGEBRAIC GEOMETRY AND GROUP ACTIONS"

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Introduction. Over the last 10 years there was a very interesting development in affine algebraic geometry, in particular in questions related to group actions.

- Koras and Russell showed that every $\mathbb{C}^*$-action on complex affine 3-space $\mathbb{A}^3$ is *linearizable*, i.e. conjugate to a linear action within the group $G\mathbb{A}_3$ of polynomial automorphisms of $\mathbb{A}^3$ [KR97, KR99].
- Kalimann proved that a fixed point free action of the additive group $\mathbb{C}^+$ on affine 3-space $\mathbb{A}^3$ is *conjugate to a translation*; i.e. to an action $(s, x) \mapsto x + s \cdot v$ with a fixed $v \in \mathbb{C}^3$ [KS04, Kal04].
- Makar-Limanov developed the *ML-invariant* of an affine variety X as a important tool to distinguish X from affine n-space $\mathbb{A}^n$ [KKMLR97, KML97, MLvRSY04].
- Kraft and Russell showed that every faithful action of a non-finite reductive group on affine 3-space $\mathbb{A}^3$ is *linearizable* [KR11].
- There is an extensive work of Dubouloz on Danielewski surfaces, in relation with the *cancellation problem* and the *ML-invariant* ([Dub06, Dub07, Dub09, Dub04, Dub05, DP09]).
- Flenner, Kalimann and Zaidenberg studied and classified $\mathbb{C}^*$-surfaces, from several different point of view.
- Moser-Jauslin and Freudenburg constructed new examples of *non-linearizable* actions of reductive and finite groups on affine n-space [FMJ02, FMJ04].
- Arzhantsev, Flenner, Kaliman et al have introduced and studied the *special automorphism group* of a variety X which generated by the unipotent one-parameter subgroups, and proved a very interesting connection between the infinite transitivity of this group and the *flexibility* of the variety ([AFKKZ10]).

Since the Bourbaki survey from 1996 [Kra96] a lot has happened. We believe it is now a good time to set up a seminar for young mathematicians introducing them to this challenging field and presenting some recent development.

The aim of the Seminar. With this seminar we would like to reach two groups of young researchers. First of all we assume that the participants have some basic background in algebraic geometry and in group theory (see below). On one hand we have in mind those who would like to get some insight, to learn new subjects and to get an idea what is presently going on in this field. On the other there are quite a number of young researchers who are already working in some specific areas of affine algebraic geometry and group actions and who would like to learn what is happening in the neighboring fields. It will be our challenge to serve both groups!

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Prerequisites. We assume that the participants have some basic knowledge in the following two areas.

- Algebraic Geometry. We recommend the notes *Basics from algebraic geometry* (by H. Kraft, freely available on www.math.unibas.ch/kraft), the first chapter of Hartshorne's book [Har77], or the first chapters from Shafarevich's book [Sha94]
- Group actions and representations of groups, as found in many books on group theory.

Additional knowledge in projective geometry (see [Sha94]) and algebraic group theory (see [Bor91], [Hum75], [Spr09], [Pro07] or the notes on www.math.unibas.ch/kraft) would be helpful. However, we will recall some basic facts in the first lectures.

Content of the lectures. We plan to set up three series of lectures in the morning, (5 times 45 minutes each), combined with special short lectures, discussions and exercises in the afternoon. For the lectures we have the following in mind.

Adrien Dubouloz: *Elements of Classification of Surfaces.*

I) General facts on projective surfaces
  - Line bundles and the Riemann-Roch Theorem
  - Birational geometry, minimal models
  - Differential forms and Kodaira dimension
  - Overview of the classification of projective surfaces

II) Overview of the classification of affine surfaces
  - Projective completions of quasi-projective surfaces
  - Logarithmic forms and logarithmic Kodaira dimension
  - Structure of affine surfaces with non negative logarithmic Kodarai dimension

III) Birationally ruled quasi-projective surfaces
  - Ruled projective surfaces and Castelnuovo criterion for rationality
  - Affine-ruled quasi-projective surfaces and Miyanishi criterion for affine-ruledness
  - Structure of degenerate fibers of affine rulings on normal affine surfaces
  - Application to the classification of affine surfaces with additive group actions

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Hubert Flenner: *Torus actions on affine varieties*

I) General facts on torus actions.
  - Torus actions on affine varieties and gradings

- Example: Affine toric varieties in dimension 2 and quotients of $\mathbb{C}^2$
- Affine toric varieties of any dimension
- Toric varieties and fans: an overview.

II) $\mathbb{C}^*$-actions on surfaces
- The DPD description of Dolgachev, Pinkham and Demazure
- When does a $\mathbb{C}^*$-surface admit a $\mathbb{C}_+$-action?
- Quasihomogeneous surfaces: the classification of Gizatullin
- Completions of affine surfaces: the theorem of Gizatullin

III) Higher dimensions: the case of complexity 1
- Affine varieties and polyhedral divisors (after Altmann Hausen)
- When do affine varieties with a torus action admit a $\mathbb{C}_+$-action?

IV) Automorphism groups of affine varieties
- Algebraically generated groups acting on affine varieties
- Gizatullin surfaces
- Infinite transitivity and applications
- Problems

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Hanspeter Kraft: *Algebraic groups actions on varieties.*

I) Introduction to algebraic transformation groups
- Algebraic group actions
- Representations
- Quotients and classification problems

II) $\mathbb{C}^+$-actions and bundles
- $\mathbb{C}^+$-actions and locally nilpotent derivations
- Free actions and local triviality
- $\mathbb{C}^+$-actions on affine spaces (Theorems of Gutwirth and Kaliman, Examples of Winkelmann et al)

III) The Linearlization Problem
- The affine Cremona group $G\mathbb{A}_n$
- The structure of $G\mathbb{A}_2$
- Summary of results and some problems

Final remarks. The program above might be too ambitious. Depending on the audience, we are ready to stick to more elementary material and to spend more time for basic notions and fundamental material. It is important for us that the participants are able to follow the main parts of lectures and to understand the basic ides.

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