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Control Theory: Mathematical Perspectives on Complex Networked Systems

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Abstract. Control theory is an interdisciplinary field that is located at the crossroads of pure and applied mathematics with systems engineering and the sciences. Its range of applicability and its techniques evolve rapidly with new developments in communication systems and electronic data processing. Thus, in recent years networked control systems emerged as a new fundamental topic, which combines complex communication structures with classical control methods and requires new mathematical methods. A substantial number of contributions to this workshop was devoted to the control of networks of systems. This was complemented by a series of lectures on other current topics like fundamentals of nonlinear control systems, model reduction and identification, algorithmic aspects in control, as well as open problems in control.

Mathematics Subject Classification (2000): 93xx, 37N35, 90Bxx.

Introduction by the Organisers

Control theory is now a classical field in mathematics which is permanently evolving due to new developments in the engineering sciences. The advent of new communication means like wireless signal transmission, or the internet has led to the development of *networked control systems*, which combine a possibly large number of classical control systems in a digital network. Control variables, measured variables and other signals are transmitted between the subsystems via communication channels. Properties of these channels like capacity and bandwidth, the protocol, or transmission delays and losses thus affect the possibility to control the system. On the other hand, wireless connections between distantly located parts of a system offer new strategies for control and monitoring. New mathematical questions which arise in this context are, for instance, related to the amount of information needed to control a system, the role of the toplogy of the connecting graph, the differences between event-driven and synchronized communication or centralized and decentralized control, as well as the statistical properties of the channel.

The field therefore covers a wide variety of topics, ranging from fundamental mathematical aspects and new control paradigms in the sciences to real world engineering applications of industrial relevance. In particular, it has deep connections to different branches of pure and applied mathematics, including e.g. ordinary and partial differential equations, operator theory, real and complex analysis, probability theory, numerical analysis, discrete mathematics, stochastics as well as algebraic and differential geometry.

The workshop *Control Theory: Mathematical Perspectives on Complex Networked Systems* brought together about 45 internationally active researchers from Austria, Belgium, France, Germany, Israel, Italy, The Netherlands, Sweden, Switzerland, the United Kingdom and the United States, with both a mathematical and systems engineering background. In order to address the new challenges posed by the new communication structures, a special focus of this workshop has been on networked control systems. This was complemented by challenging systems engineering topics. In all these talks, the interaction of mathematical methods from nonlinear dynamics and control with those from discrete mathematics (especially graph and information theory) played a crucial role. The program comprised 24 talks on the theory and applications of control theory. The lengths of the talks were different, between 30 and 45 minutes, where always enough time (at least about 10 minutes) was granted for the discussion. The lectures were organized into rather coherent sessions on the topics:

- Networks and Control
- Fundamentals of Nonlinear Control Systems
- Model reduction and Identification
- Algorithmic Aspects in Control
- Fundamental Control Problems

In addition to these lectures and the very active discussions throughout the workshop there was an informal open problem session on Tuesday evening, in which 10 participants presented open mathematical problems in control. Furthermore, as a new format, we implemented poster sessions on Wednesday and Thursday evening to have a more informal forum to discuss recent results. These sessions were accompanied by 'poster-teaser-sessions', where each presenter of a poster had about ten minutes to introduce the audience to the topic of the poster and to answer first questions. In particular the younger participants used this chance to present their work very actively. The extended abstracts of all lectures and posters are collected in this report.

The traditional Wednesday afternoon walk to St. Roman was replaced by a walk to Wolfach, where the participants enjoyed the exciting new MIMA-museum.

Workshop: Control Theory: Mathematical Perspectives on Complex Networked Systems

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Abstracts

Dynamics of Chemical Reaction Networks

David Angeli

Complex interactions of chemical species are a key ingredient of life when regarded at the cellular level. The goal of Chemical Reaction Networks Theory is to model such interactions by means of dynamical systems and to understand the qualitative features of their behaviour. We discuss some results in this area, with a focus on recent advances in the field of monotone chemical reaction networks and persistence analysis.
A Chemical Reaction Network can be formally defined as a list of reactions of the following type:

$$\sum_{i \in \mathcal{S}} \alpha_{ij} S_i \to \sum_{i \in \mathcal{S}} \beta_{ij} S_i \tag{1}$$

where i ranges on some finite set $\mathcal{S} = \{1, 2, \dots, n_s\}$, j ranges in $\mathcal{R} := \{1, 2, \dots, n_r\}$ and the S_i are symbols denoting the chemical species involved in the network. The α_{ij} and β_{ij} are non-negative integers called the stoichiometry coefficients of the network. Such coefficients define the structure of the network and are normally arranged in a matrix Γ, defined according to $[\Gamma]_{ij} = \beta_{ji} - \alpha_{ji}$ which is referred to as the stoichiometry matrix of the reaction.

In order to associate a dynamical description of the network to the list of reaction in (1) an expression for the individual rates of reaction is needed. In particular, this is a function $R : [0, +\infty)^{n_s} \to [0, +\infty)^{n_r}$ which maps species concentrations to rates of reactions. Once kinetics are specified, an ODE model of the network (1) is defined as follows:

$$\dot{x}(t) = \Gamma R(x)$$

where the state $x \in [0, +\infty)^{n_s}$ is the vector of species concentrations.

A natural way to represent chemical networks is by means of bipartite graphs. Nodes are associated to either chemical species and chemical reactions, an arc means that a particular species is either involved as a reactant and/or a product in a given reaction. It is an interesting question to understand the subtle links between the topological features of the graph associated to some network and its dynamics. We will discuss results related to monotonicity of chemical reaction networks (when regarded as dynamical systems on partially ordered state spaces, [1]) and with persistence (that is to the asymptotic availability of all species when initial conditions are selected in the interior of the positive orthant [2]).

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A stochastic approach to control of refrigerator appliances for frequency regulation

David Angeli
(joint work with A. Kountouriotis)

Dynamic demand management is a promising research direction for improving power system resilience. In a power network, the system frequency (mains frequency) can be interpreted as a measure of the balance between demand (load) and supply (generation), with perfect balance corresponding to the nominal value of 50Hz. In cases where demand exceeds the available supply, the frequency drops below 50Hz, while excess supply leads to frequency rising above 50Hz. As a result, system frequency continuously fluctuates around the nominal level, and the system operator ensures that the balance between demand and supply is continuously maintained, stabilizing the frequency within narrow bands around 50Hz, by regulating the available supply.

In order for such (supply) regulation to be possible, however, it is required that 'frequency response services', as well as sufficient reserves, are included in the system This is essential not only for instantaneous frequency balancing, but, more importantly, for the ability to respond to sudden power plant failures, which would otherwise lead to severe blackouts.

These 'support' services, however, significantly add to the cost of power generation, and any method which manages to reduce the magnitude of these services, without sacrificing system stability, is of significant importance. "Dynamic demand control" is a recent research direction, which focuses on the possibility of using frequency responsive loads, so as to reduce the amount of frequency response and reserve services that are required.

In this poster, we consider the problem of managing power demand by means of "smart" thermostatic control of domestic refrigerators. In this approach, the operating temperature of these appliances, and thus their energy consumption, is modified dynamically, within a safe range, in response to mains frequency fluctuations.

An individual refrigerator is represented as a hybrid automaton, capable of switching between 2 states (an ON state and an OFF states). Simple affine first order equations are assumed to describe the evolution of the temperature within the two states:

$$\dot{T} = -\alpha(T - T_{ON}) \qquad \dot{T} = -\alpha(T - T_{OFF})$$

where α is a coefficient which characterizes the thermal dispersion of the refrigerator, T_{OFF} is the ambient temperature, and T_{ON} is the temperature that the refrigerator would reach asymptotically if always ON.

In order to compensate for frequency fluctuations dynamically we define a stochastic and frequency dependent switching policy. In particular, individual refrigerators adopt a Markovian switching policy (that is they behave as stochastic hybrid automata), in which transition rates are frequency dependent. Suitable design of such transition rates λ_1 and λ_2 can provide desirable stability features for the closed-loop system as well as avoid potential synchronization issues arising from deterministic switching laws.

The results illustrated in the poster can be found in greater detail in [1].

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Model reduction of parametrized systems

Thanos Antoulas

Abstract. This talk was dedicated to the approximation of linear dynamical systems which depend on parameters. The main tool is the Loewner matrix framework which was recently extended to deal with parameters. This yields an all important tradeoff between accuracy of fit and model complexity.

Keywords: Model order reduction, parametrized dynamical systems, Loewner matrices, multivariate functions, interpolation, approximation, accuracy, model complexity.

Some details

Dynamical systems are a principal tool in modeling and control of physical processes in engineering, economics, the natural and social sciences. In many areas, direct numerical simulation (DNS) has become essential for studying the rich complexity of these phenomena and for the design process. Due to the increasing complexity of the underlying mathematical models, unmatched by the increase in computing power, model reduction has become an indispensable tool in order to facilitate or even enable simulation, control, and optimization of dynamical systems. Here, we focused on parametrized models where the preservation of parameters as symbolic quantities in the reduced-order model is essential. We pursued an approach which starts from empirical data and employs advanced interpolation techniques, overcoming limitations of standard projection methods. The empirical data may be provided by physical experimentation or by DNS. Our theme consisted in constructing reduced-order models satisfying interpolation conditions. This resulted in computationally efficient model reduction algorithms.

Many of the model reduction approaches which have been proposed can be interpreted as Petrov-Galerkin projections (see e.g. [1, 4]). The first class is that of SVD-based methods, and originates in systems and control. The second class, known as Krylov approximation methods, have their roots in numerical linear algebra. Krylov methods are moment matching methods and thus automatically provide Padé or Padé-type approximations. For an overview of these appraches to model reduction, see [1]. More recent developments are based on a further interpretation of Krylov methods as rational interpolation. Such methods can also be used for the conservation of structural properties [4].

In the use of mathematical models for applications, an important aspect is that such models may depend on parameters. This allows the use of the same model in simulations, for instance under changing material properties, geometric characteristics or varying boundary conditions. This is particularly important at the stage of design and optimization. Moreover, often parameters are affected by measurement noise and are thus uncertain. An uncertainty quantification thus also requires the possibility to vary the parameters in the model. Therefore, the preservation of parameters as symbolic variables in the reduced model becomes an important aspect for the use of model reduction in applications.

In this talk we proposed an approach to *model reduction of parameter dependent systems* based on measured data. In the case of systems not depending on parameters, it was shown in [2, 3], that the *Loewner matrix* is a powerful tool which allows the construction of models and reduced models from measured data, while providing a trade-off between accuracy of fit and model complexity. This approach was recently extended to the parameter dependent case (for details see [5]). The talk was built around this paper.

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Some Mathematical Formulations of Poblems in Network Control

Roger Brockett

1. In some cases the most important factor limiting the performance of a distributed control system is not the availability of computational power but rather the availability of communication channels of suitable capacity which can be used to establish feedback paths between the sensors, the control computer and the actuators. This may be modeled as follows. We assume that at each sampling instant there is a state x consisting of the state of all the individual physical devices together with the values of all past measurements currently in memory. There is also a set of linear functions, $\langle c_i(t), x(t)\rangle$, $i = 1, 2, ..., p$ defining those function of $x(t)$ that are available at time t for use by the control computer. We can think of

the $c_i(t)$ as describing the *availability* of sensed information. We also have a set of vectors $b_1(t), b_2(t), ..., b_m(t)$ which enter into the overall evolution equation

$$x(t+h) = Ax(t) + \sum_{i=1}^{m} b_i(t)u_i(t)$$

and serve to the possible control actions one can take at a given time. We can think of these as describing the extent to which the desired control policy is *deliverable*. We postulate that each communication event, such as the request to send the value of a variable associated with one of the sensors to one of the actuators takes a specific time, τ. We have in mind that τ is much larger than the sampling times associated with the individual sensors and actuators. Pick a periodic communication sequence of period p which specifies the sequence in which information is delivered. For a given linear system with an n-dimensional state vector, each choice of communication sequence of period $p\tau$, defines an affine subspace of the space of all real $pn \times pn$ matrices. Each matrix in this space corresponds to a specific choice of feedback gains; the mapping from the space of gains to the space of matrices takes the form

$$k \mapsto \mathcal{A}_0 + \sum k_i \mathcal{A}_i$$

The eigenvalues of the product of matrices in this subspace determine the rate of growth or decay of the solutions of the closed-loop system as described by Floquet theory. This leads to a novel eigenvalue placement problem.

2. The collective and individual behaviors of the elements of a set of identical units are investigated from the point of view of their response to a coordination signal broadcast by a leader. In terms of the model used here, we show that for such systems nonlinearity plays a critical role in making this type of control effective. A new method for establishing controllability of nonlinear, replicated systems is given and a flock stabilization problem, depending strongly on nonlinear effects, is solved. More details are to be found in reference [6] below.

3. For the third topic of the lecture we described problems in automatic control related to the Liouville equation. The mathematical problems can be interpreted either in terms of designing a feedback controller which effectively controls a particular system over repeated trials corresponding to different initial conditions or, alternatively, using a broadcast signal to simultaneously control many copies of a particular system. In many cases a certain continuum limit can be formulated and in this way we are led to problems involving the control of an associated Liouville equation.

Given an ordinary differential equation, $\dot{x}(t) = f(x(t))$ defined on a manifold X, and having the property that there exists a unique solution through each point, there is an associated partial differential equation which describes the evolution of an initial density of points. Let $\rho(0, \cdot)$ be the initial density, thought of as a probability density for $x(0)$. As such it is nonnegative and normalized. Under mild additional assumptions, the first oder equation

$$\frac{\partial \rho(t,x)}{\partial t} = -\left\langle \frac{\partial}{\partial x},\ f(x)\rho(t,x) \right\rangle$$

then describes the evolution of ρ. We can think of this as Cauchy problem. Obviously the properties of the Liouville equation reflect quite closely the properties of the underlying ordinary differential equation. However, the fact that in the Liouville formulation the control term $u(t, x)$ is operated on by derivative operators means that a feedback implementation, i.e., expressing u as a function of x, differ greatly from open loop implementation in which u is expressed as a function of t alone. This observation can serve as the starting point for the rationalization of some of the many benefits of feedback control.

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Hierarchical clustering of dynamical networks using a saddle-point analysis

Mathias Bürger

(joint work with Daniel Zelazo and Frank Allgöwer)

We analyze the phenomenon of cluster synchronization in dynamical networks. Cluster synchronization, or clustering, is the phenomenon that in a network of dynamical systems the network partitions into several groups and all systems within the same group agree upon a common state [1]. We consider a novel class of nonlinear dynamic networks that exhibit clustering in their asymptotic behavior. The distinguishing features in the model we adapt are (i) the uncoupled node dynamics have distinct equilibria, and (ii) the interaction rules between neighboring systems are bounded. We show that the network synchronizes for sufficiently large saturation bounds, but partitions into clusters otherwise.

A main contribution of our work is to establish a connection between the asymptotic behavior of the dynamic network and the solution of a static saddle-point problem. We show that the solution of the saddle-point problem, corresponding to the Lagrange-dual of a network optimization problem with additional constraints on the dual variables, exhibits a clustered structure. If the bounds on the dual variables are chosen as the saturation levels of the interaction functions of the dynamic network, then both clustering structures are equivalent, and the saddle-point problem can be used to predict the clustering structure of the dynamic network.

The relationship between the behavior of the dynamical network and the static optimization problem allows us to explain the connection between the clustering structure and the topology of the underlying graph, and builds a bridge between dynamic synchronization [2] and static community detection or graph partitioning [3]. We show that the network partitions according to a precise optimality criterion: it maximizes the ratio of the "power imbalance" between the partitions over the number of edges connecting them. Using these results, we can explain exactly how our setup is related to the combinatorial min s-t-cut problem and the "inhibiting bisection problem", which is used in the literature for power network analysis.

A simple variation of some network parameters leads to different clustering structures, which can be shown to be hierarchically ordered. In this way, a variation of the parameters can be used to identify the complete hierarchical clustering structure of a network. Our theoretical results are applied to detect the hierarchical structure of the IEEE 30-bus power system, a benchmark system widely used in power network analysis.

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Minimal Bit Rates and Entropy for Stabilization

Fritz Colonius

An approach is presented for the study of minimal bit rates for exponential stabilization of control systems in continuous time. Upper and lower bounds for the stabilization entropy are derived. In particular, for linear systems, a formula is given in terms of the real parts of eigenvalues. Furthermore, extensions to other control problems are discussed. In particular, entropy for the disturbance decoupling problem and controlled invariant subspaces is considered. The relations of the concepts of entropy for control problems introduced here is put into perspective by discussing the relation to topological entropy of flows.

Ensemble Controllability of Bilinear Systems

Gunther Dirr

During the last decade, the concept of *ensemble controllability* emerged and developed in the field of quantum control. Depending on the underlying applications, the corresponding models range from finite dimensional large-scale to infinite dimensional control systems. Either way, one is interested in *simultaneously* controlling an ensemble of identical systems via a single control signal. Thereby, the ensemble members are assumed to obey in principle the same dynamical law, yet model parameters may vary individually. Such control scenarios arise in many different areas such as

- control of spin ensembles (in NMR spectroscopy and imaging) [1, 2, 6],
- flock/formation control by a leader [3],
- "open loop" robust control for models with parameter uncertainties [5],
- control of bilinear systems with continuous spectrum [2].

In the talk, we focus on ensembles of bilinear systems evolving on semisimple (matrix) Lie groups. We present a complete characterization of simultaneous accessibility which for compact groups guarantees simultaneous controllability.

General setting. Consider a smooth manifold M and an ensemble of smooth control systems

$$(\Sigma_p) \qquad \dot{x} = f_p(x,u), \qquad u \in U \subset \mathbb{R}^m, \qquad p \in P,$$

evolving on M, where P can be in principle any set – finite, countable, etc.

Problem. What can be said about the controllability of the "joint system" under the assumption that all individual systems (Σ_p) are controllable?

For the finite case, i.e. for $P := \{1,\dots,s\}$ the "joint system" is simply defined on the product manifold $\widehat{M} := M \times \cdots \times M$ by

$$(\widehat{\Sigma}) \qquad \dot{\widehat{x}} := \begin{bmatrix}\dot{x}_1\\ \vdots \\ \dot{x}_s\end{bmatrix} = \widehat{f}(\widehat{x},u) := \begin{bmatrix}f_1(x_1,u)\\ \vdots \\ f_s(x_s,u)\end{bmatrix}, \qquad u \in U \subset \mathbb{R}^m.$$

Definition 1. (1) *An ensemble* $(\Sigma_1),\dots,(\Sigma_s)$ *is called* simultaneously accessible *if the joint system* $(\widehat{\Sigma})$ *is accessible.*

(2) *An ensemble* $(\Sigma_1),\dots,(\Sigma_s)$ *is called* simultaneously controllable *(or* ensemble controllable*) if the joint system* $(\widehat{\Sigma})$ *is controllable.*

Example. Let $G \subset GL_n(\mathbb{R})$ denote a matrix Lie group and $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ its Lie algebra, e.g. $G = SO_n$ and $\mathfrak{g} = \mathfrak{so}_n$. Consider an ensemble of bilinear systems

$$(\Sigma_p) \qquad \dot{X} = (A_p + uB_p)X, \qquad u \in U \subset \mathbb{R}, \qquad p = 1,\dots,s$$

with $X \in G$ and $A_p, B_p \in \mathfrak{g}$. Then the joint system on $G \times \cdots \times G$ can easily be pictured as block-diagonal system in $GL_{n\cdot s}(\mathbb{R})$ as follows

$$(\widehat{\Sigma}) \quad \begin{bmatrix} \dot{X}_1 & 0 & \cdots & 0 \\ 0 & \dot{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \dot{X}_s \end{bmatrix} = \left(\begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_s \end{bmatrix} + u \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & B_s \end{bmatrix} \right) \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & X_s \end{bmatrix}.$$

Note that the set of all block-diagonal matrices of the above type constitutes a semisimple Lie subalgebra of $\mathfrak{gl}_{n\cdot s}(\mathbb{R})$ whenever $\mathfrak{g}$ is semisimple.

Definition 2. *Given $A, B_1, \dots, B_m \in \mathfrak{g}$ and $A', B'_1, \dots, B'_m \in \mathfrak{g}'$, where $\mathfrak{g}$ and $\mathfrak{g}'$ are Lie algebras. We say that the tuples $(A, B_1, \dots, B_m)$ and $(A', B'_1, \dots, B'_m)$ are* Lie-related, *if there exists a Lie algebra isomorphism $\tau : \mathfrak{g} \to \mathfrak{g}'$ such that*

$$A' = \tau(A) \quad \text{and} \quad B'_k = \tau(B_k) \quad \text{for } k = 1, \dots, m.$$

Otherwise, $(A, B_1, \dots, B_m)$ and $(A', B'_1, \dots, B'_m)$ are called Lie-unrelated.

Theorem 3. *[D. 2012] Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ be a semisimple (matrix) Lie algebra, let G be the corresponding connected (matrix) Lie group, and let $A, B \in \mathfrak{g}$. Then the following statements are equivalent:*

(1) *The system*

$$(\Sigma) \qquad \dot{X} = \big(A + uB\big)X, \qquad u \in U \subset \mathbb{R}$$

is accessible on G.

(2) *For all $p \in \{1, \dots, s\}$ the Lie algebra generated by A_p, B_p coincides with $\mathfrak{g}_p$ and for $p \neq p'$ the pairs (A_p, B_p) and $(A_{p'}, B_{p'})$ are Lie-unrelated.*

Here, A_p and B_p denote the p-th component of A and B with respect to the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$.

Applying Theorem 1 to the semisimple Lie subalgebra of block-diagonal matrices described in the above example leads to the following result.

Corollary 4. *Let $\mathfrak{g}$ be a simple (matrix) Lie algebra, let G be the corresponding connected (matrix) Lie group, and let $A_p, B_p \in \mathfrak{g}$ for $p = 1, \dots, s$. Then the following statements are equivalent:*

(1) *The ensemble of bilinear systems*

$$(\Sigma_p) \qquad \dot{X}_p = (A_p + uB_p)X_p, \quad u \in \mathbb{R}, \quad p = 1, \dots s$$

is simultaneously accessible.

(2) *For all $p \in \{1, \dots, s\}$ the Lie algebra generated by A_p, B_p coincides with $\mathfrak{g}$ and for $p \neq p'$ the pairs (A_p, B_p) and $(A_{p'}, B_{p'})$ are Lie-unrelated.*

If G is compact the above results actually provide necessary and sufficient conditions for simultaneous controllability.

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On invariant subspaces and intertwining maps

Paul Fuhrmann

The main topic of the talk is is the representation of invariant subspaces of a linear transformation as kernels and images of maps commuting with it. This extends a result of Halmos [3]. This result has strong system theoretic connection. In particular, J.C. Willems' characterization of behaviors, given in Willems [5], is a result of this type.he context in which we work is that of polynomial models, introduced in Fuhrmann [1]. We treat also the embeddability of quotient modules of a polynomial model into the model, the relation between the invariant factors of a polynomial model and those of its submodules and quotient modules. We focus also on the study of how complementarity of invariant subspaces is related to the invertibility of linear maps. That such a connection exists is not surprising as both properties can be characterized in terms of coprimeness of polynomial matrices. This analysis connects to the concept of skew-primeness, introduced in Wolovich [6] as well as to a theorem of Roth [4]. Fuhrmann [2] contains an infinite dimensional generalization of skew-primeness. This opens up the possibility of establishing the analog of Halmos's theorem in the context of backward shift invariant subspaces.

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Nonlinear MPC with systematic handling of a class of constraints

KNUT GRAICHEN
(joint work with Bartosz Käpernick)

Model predictive control (MPC) is well suited for multiple input systems with constraints and nonlinear dynamics. However, the numerical solution of the underlying optimal control problem (OCP) is often numerically expensive, in particular if state constraints are involved.

In order to relax this problem, the talk presented an approach to systematically include a class of state and control constraints within a new unconstrained problem formulation that can be efficiently solved with unconstrained optimization methods. The OCP formulation for this approach is of the form

$$\text{(1)} \qquad \text{minimize} \quad V(x(T)) + \int_0^T l(x(\tau), u(\tau))\,\mathrm{d}\tau$$

$$\text{(2)} \qquad \text{subject to} \quad \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i\,, \quad x(0) = x_k$$

$$\text{(3)} \qquad c_i(x) \in [c_i^-, c_i^+]\,, \quad u_i \in [u_i^-, u_i^+]\,, \quad i = 1, \ldots, m$$

with the cost (1), the nonlinear input-affine system (2) with the state $x \in \mathbb{R}^n$, and the constraint set (3) consisting of m state and input constraints, where m corresponds to the number of controls $u \in \mathbb{R}^m$. The initial conditions in (2) are given by the current system state x_k at time instant $t_k = k\Delta t + t_0$, $k \in \mathbb{N}_0$ with the sampling time $0 < \Delta t \leq T$. The MPC horizon is denoted by T.

Under the assumption that the state constraint functions $c_i(x)$, $i = 1, \ldots, m$ possess a well-defined vector relative degree $\{r_1, \ldots, r_m\}$ [1], the system (2) can be transformed into a normal form representation with the state constraints appearing at the top of chains of integrators. The introduction of saturation functions and successive differentiation along the normal form cascades then leads to a state/input transformation [2] between (x, u) and new variables $(\tilde{x}, v)$ of the form

$$\text{(4)} \qquad \left.\begin{aligned} x &= h_x(\tilde{x}) \\ u &= h_u(\tilde{x}, v) \end{aligned}\right\} \quad \Longleftrightarrow \quad \left\{\begin{aligned} \tilde{x} &= h_x^{-1}(x) \\ v &= h_u^{-1}(h_x^{-1}(x), u)\,, \end{aligned}\right.$$

where the functions $h_x : \mathbb{R}^n \to \mathcal{X}$ and $h_u : \mathbb{R}^n \times \mathbb{R}^m \to \mathcal{U}$ are defined on the open intervals

$$\text{(5)} \qquad \begin{aligned} \mathcal{X} &= \{x \in \mathbb{R}^n : c_i^- < c_i(x) < c_i^+,\ i = 1, \ldots, m\} \\ \mathcal{U} &= \{u \in \mathbb{R}^m : u_i^- < u < u_i^+,\ i = 1, \ldots, m\}\,. \end{aligned}$$

Based on this transformation, the original OCP (1)-(3) can be replaced by a new unconstrained OCP with respect to the new control v:

$$\text{(6)} \qquad \text{minimize} \quad \tilde{V}(\tilde{x}(T)) + \int_0^T \tilde{l}(\tilde{x}(\tau), v(\tau)) + \varepsilon||v||^2\,\mathrm{d}\tau$$

$$\text{(7)} \qquad \text{subject to} \quad \dot{\tilde{x}} = \tilde{f}(\tilde{x}, v)\,, \quad \tilde{x}(0) = \tilde{x}_k = h_x^{-1}(x_k)$$

with the new system dynamics (7) and the cost functions $\tilde{V}(\tilde{x}) := V(h_x(\tilde{x}))$ and $\tilde{l}(\tilde{x}, v) := l(h_x(\tilde{x}), h_u(\tilde{x}, v))$. The additional regularization term $\varepsilon||v||^2$ in (6) is necessary to account for singular arcs that correspond to constrained arcs in the original OCP (1)-(3). Details on the regularizazion term and the convergence properties for $\varepsilon \to 0$ are given in [2]. For the MPC implementation, ε is kept constant at a small value. The derivation of the transformation (4) and functions in (6)-(7) can be automated with computer algebra software.

Due to the incorporation of the original constraints (3), the new OCP (6)-(7) can be solved with unconstrained optimization methods, e.g. by means of the gradient method in optimal control. The implementation of the gradient method in the context of input-constrained MPC is described in [3]. In the talk, the MPC-tailored gradient method was adapted to the new OCP (6)-(7) resulting from the constraint handling approach.

The overall concept was demonstrated for a nonlinear model of an overhead crane with state and input constraints and a sampling time of 3 ms. The unconstrained OCP (6)-(7) was computed with Mathematica and exported as C code. In addition, a C implementation of the gradient algorithm outlined in [3] with Matlab interface was used to numerically and experimentally demonstrate the efficiency and performance of the concept with a computation time of approximately 150 μs per MPC step.

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Economic MPC and the role of exponential turnpike properties

Lars Grüne

1. Problem formulation

We consider discrete time control systems with state $x \in X$ and control values $u \in U$, where X and U are normed spaces with norms denoted by $\|\cdot\|$. The control system under consideration is given by

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

with $f : X \times U \to X$. For a given control sequence $u = (u(0), \ldots, u(K-1)) \in U^K$ or $u = (u(0), u(1), \ldots) \in U^\infty$, by $x_u(k, x)$ we denote the solution of (1) with initial value $x = x_u(0, x) \in X$.

For given admissible sets of states $\mathbb{X} \subseteq X$ and control values $\mathbb{U} \subseteq U$ and an initial value $x \in \mathbb{X}$ we call the control sequences $u \in \mathbb{U}^K$ satisfying

$$x_u(k, x) \in \mathbb{X} \quad \text{for all } k = 0, \dots, K$$

admissible. The set of all admissible control sequences is denoted by $\mathbb{U}^K(x)$. Similarly, we define the set $\mathbb{U}^\infty(x)$ of admissible control sequences of infinite length. For simplicity of exposition we assume $\mathbb{U}^\infty(x) \neq \emptyset$ for all $x \in \mathbb{X}$, i.e., that for each initial value $x \in \mathbb{X}$ we can find a trajectory staying inside $\mathbb{X}$ for all future times.

Given a feedback map $\mu : X \to U$, we denote the solutions of the closed loop system

$$x(k+1) = f(x(k), \mu(x(k)))$$

by $x_\mu(k)$ or by $x_\mu(k, x)$ if we want to emphasize the dependence on the initial value $x = x_\mu(0)$. We say that a feedback law μ is admissible if it renders the admissible set $\mathbb{X}$ (forward) invariant, i.e., if $f(x, \mu(x)) \in \mathbb{X}$ holds for all $x \in \mathbb{X}$. Note that $\mathbb{U}^\infty(x) \neq \emptyset$ for all $x \in \mathbb{X}$ immediately implies that such a feedback law exists.

Our goal is now to find an admissible feedback controller which yields approximately optimal average performance. To this end, for a given stage cost $\ell : X \times U \to \mathbb{R}$ we define the averaged functionals and optimal value functions

$$\begin{aligned} J_N(x, u) &:= \frac{1}{N} \sum_{k=0}^{N-1} \ell(x_u(k, x), u(k)), & V_N(x) &:= \inf_{u \in \mathbb{U}^N(x)} J_N(x, u), \\ J_\infty(x, u) &:= \limsup_{N \to \infty} J_N(x, u) \quad \text{and} & V_\infty(x) &:= \inf_{u \in \mathbb{U}^\infty(x)} J_\infty(x, u). \end{aligned}$$

We assume that ℓ is bounded from below on $\mathbb{X}$, i.e., that $\ell_{\min} := \inf_{x \in \mathbb{X}, u \in \mathbb{U}} \ell(x, u)$ is finite. This assumption immediately yields $J_N(x, u) \geq \ell_{\min}$ and $J_\infty(x, u) \geq \ell_{\min}$ for all admissible control sequences. In order to simplify the exposition in what follows, we assume that (not necessarily unique) optimal control sequences for J_N exist which we denote by $u^\star_{N,x}$ or briefly by $u^\star$.

Similarly to the open loop functionals, we can define the average cost of the closed loop solution for any feedback law μ by

$$\begin{aligned} J_K(x, \mu) &= \frac{1}{K} \sum_{k=0}^{K-1} \ell(x_\mu(k, x), \mu(x_\mu(k, x))) \\ J_\infty(x, \mu) &= \limsup_{K \to \infty} J_K(x, \mu). \end{aligned}$$

In order to construct the desired feedback law, henceforth denoted by μ_N, we employ a model predictive control (MPC) approach: in each time instant k, we compute an optimal control $u^\star_{N,x_0}$ for the initial value $x_0 = x_{\mu_N}(k, x)$ and define the feedback value as $\mu_N(x_0) := u^\star_{N,x_0}$, i.e., as the first element of the finite horizon optimal control sequence.

2. Value and trajectory convergence results

The presented results hold for averaged optimal control problems exhibiting an optimal steady state, i.e., for which there exists a point $x^e \in \mathbb{X}$ and a control value

$u^e \in \mathbb{U}$ with

$$f(x^e, u^e) = x^e \quad \text{and} \quad V_\infty(x) \geq \ell(x^e, u^e)$$

for all $x \in \mathbb{X}$.

For such problems, it was shown in [1, 2, 4] that the receding horizon controller μ_N shows optimal infinite horizon averaged performance if the terminal constraint $x_u(N, x) = x^e$ is added as an additional condition to the finite horizon problem employed for computing μ_N.

Here, we consider the MPC formulation without such terminal constraints. Motivation for doing so is on the one hand that removing the terminal constraint also removes the need to compute x^e beforehand and on the other hand that not imposing terminal constraints increases the region of feasibility for the MPC problem.

The central result from [5] shows that under appropriate conditions the feedback μ_N indeed shows approximately optimal performance and that the gap to optimality, i.e., the difference $|J_\infty(x, \mu_N) - V_\infty(x)|$ decreases to 0 for $N \to \infty$. The assumptions for this result are

(i) Uniform continuity of V_N in a neighborhood of x^e for all sufficiently large N
(ii) A turnpike property, which describes the fact that the finite time optimal trajectory enters a neighborhood of the optimal equilibrium x^e which shrinks to 0 as $N \to \infty$

Both properties can, e.g., be ensured by suitable controllability and dissipativity properties involving both the dynamics and the stage cost, for details and a formal version of (i) see [5]. The turnpike property (ii) is formally expressed as follows:

There is $\sigma(N)$ such that any optimal trajectory $x_{u^\star}(k)$ with horizon N satisfies

$$\min_{k=0,\ldots,N} \|x_{u^\star}(k) - x^e\| \leq \sigma(N), \quad \text{with } \sigma(N) \to 0 \text{ as } N \to \infty.$$

In addition to the value convergence result, important additional results are proved in [5] under the condition that $\sigma(N)$ tends to 0 faster than $1/N$. More precisely, under this additional condition, convergence of the MPC closed loop trajectory to a neighborhood of x^e (shrinking to x^e as $N \to \infty$) and an approximate optimality condition during the transient phase can be shown.

Several numerical examples show that it is a reasonable condition to expect that $\sigma(N)$ tends to 0 faster than $1/N$. More precisely, in many examples $\sigma(N) \approx C\theta^N$ for constants $C > 0$ and $\theta \in (0, 1)$ can be observed, i.e., an exponential turnpike property.

3. Exponential turnpike properties

Since exponential turnpike properties play an important role in Economic MPC, it is of considerable importance to find conditions which ensure this property for a given example. The following condition for an exponential turnpike property will be presented and discussed in [3] (to which we also refer for the precise technical assumptions and the proof). For its formulation, for a modified stage cost $\tilde{\ell}$ defined in [5] we define

$$\widetilde{J}_N(x,u) := \frac{1}{N+1}\sum_{k=0}^{N} \tilde{\ell}(x_u(n), u(n)),$$

and the optimal value function of the terminal constrained problem

$$\widetilde{V}_N(x_0, x_N) := \inf_u \widetilde{J}_N(x,u), \quad \text{s.t. } x_u(0) = x_0,\ x_u(N) = x_N$$

Then, an exponential turnpike property holds if there exists $\gamma \geq 1$ and $\delta \geq 1$ such that for all $x_0, x_N \in \mathbb{X}$ and $N \in \mathbb{N}$ the inequality

$$V_N(x_0, x_N) \leq \frac{\gamma \min_u \tilde{\ell}(x_0) + \delta \min_u \tilde{\ell}(x_N)}{N+1}$$

holds.

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Kernel Methods for Model Reduction of Parameterized Nonlinear Systems

Bernard Haasdonk

(joint work with Daniel Wirtz)

This work is concerned with model order reduction for for parameterized, nonlinear kernel-based systems. The dynamical systems under consideration consist of a nonlinear, time- and parameter-dependent kernel expansion representing the system's inner dynamics as well as time- and parameter-affine inputs, initial conditions and outputs. The class of dynamical systems we consider is given by

$$x'(t) = f(x(t), t, \mu) + B(t,\mu)u(t), \tag{1}$$

$$x(0) = x_0(\mu), \qquad y(t) = C(t,\mu)x(t) \tag{2}$$

with $x(t) \in \mathbb{R}^d$ denoting the system state, x_0 initial condition, B, C input/output matrix, input/control $u(t)$ and parameters $\mu \in P \subseteq \mathbb{R}^p$. Further, f is a *kernel expansion* $f(x,t,\mu) = \sum_{i=1}^{N} c_i \Phi_s(x, x_i)\Phi_t(t, t_i)\Phi_P(\mu, \mu_i)$, having scalar state, time and parameter *kernels* Φ_s, Φ_t, Φ_P, expansion centers $x_i \in \mathbb{R}^d,\ t_i \in [0,T],\ \mu_i \in P,$

and coefficient vectors $c_i \in \mathbb{R}^d, i = 1 \ldots N$. The components B, C and x_0 are time- and parameter-affine, e.g. $B(t, \mu) = \sum_{i=1}^{Q_B} \theta_i^B(t, \mu) B_i$, with $Q_B \in \mathbb{N}$ small, constant matrices $B_i \in \mathbb{R}^{d \times m}$ and low-complexity coefficient functions $\theta_i^B : [0, T] \times P \to \mathbb{R}$. The reduction technique we use was originally proposed in [1] which we extend here to the full parametric and time-dependent case. The system above is reduced applying a Galerkin projection with biorthogonal matrices $V, W \in \mathbb{R}^{r \times d}, V^t W = I_r$. For more general settings, the evaluation of the projected nonlinear term $W^t f(V \cdot, t, \mu)$ would involve d-dimensional computations. Therefore, our key ingredient is to use translation & rotation-invariant kernels $\Phi(x, x') = \phi(\|x - x'\|)$ induced by a so called *bell function* ϕ. Key features for efficient error estimation [2] are to use local Lipschitz constants and an iterative scheme to balance computation costs against estimation sharpness. Together with the time- and parameter-affine system components a full offline/online decomposition for both the reduction process and the error estimators is possible. Some experimental results for synthetic systems illustrate the efficient evaluation of the derived error estimators for different parameters. The figure shows improving estimation results for a synthetic system using global (square), local (star) and iterated local (star/triangle) Lipschitz constants. The right image shows a parameter sweep with system output and a-posteriori error bounds (transparent red).

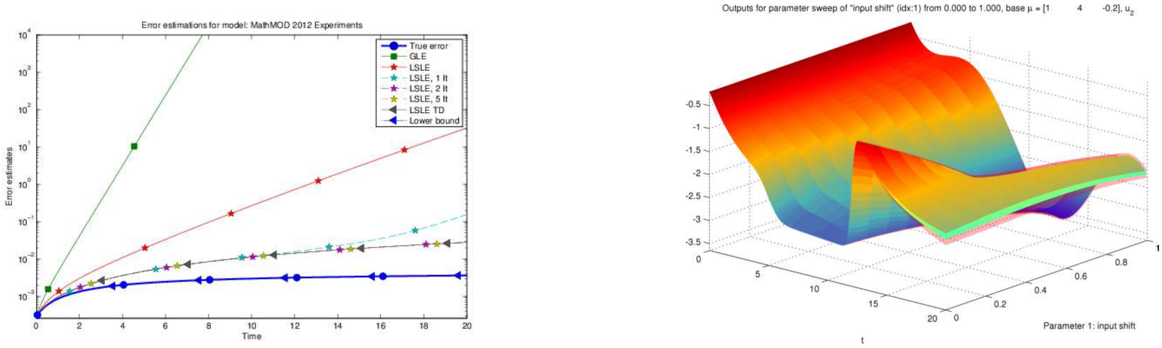

FIGURE 1. Left: Estimation results using different (local) Lipschitz constants. Right: Parameter sweep with system output and error bounds

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Optimal Event-based Control - The Extended Linear Quadratic Problem

Sandra Hirche
(joint work with Adam Molin)

Summary. We investigate the structure of joint optimal control and scheduling policies for event-based feedback control systems. The problem setting is an extension of the stochastic linear quadratic system framework, where the joint design of the control law and the event-triggering law minimizing a common objective is considered. We study three differing variants that reflect the resource constraints: a penalty term to acquire the resource, a limitation on the number of resource acquisitions, and a constraint on the average number of resource acquisitions. By reformulating the underlying optimization problem, a characterization of the optimal control law is possible. This characterization shows that the certainty equivalence controller is optimal for all three optimization problems.

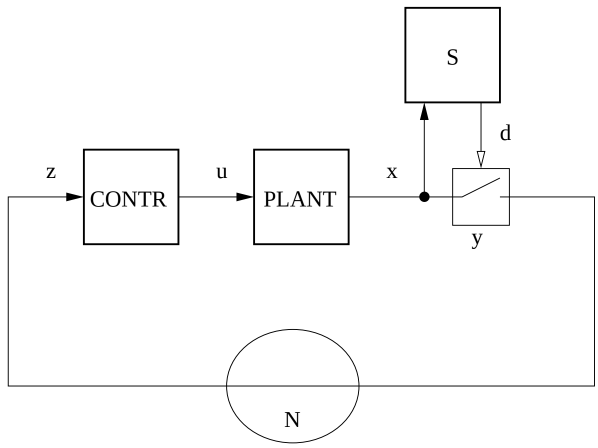

Figure 1. System model of the resource-constrained control system with process $\mathcal{P}$, event-trigger $\mathcal{E}$, controller $\mathcal{C}$, and communication channel $\mathcal{N}$.

The Extended Linear Quadratic Problem. The system under consideration is illustrated in Figure 1 and can be viewed as a resource-constrained control system. For the sake of illustration, the constraint is represented by a resource-constrained communication channel $\mathcal{N}$. The other part of the system consist of a process $\mathcal{P}$, an event-trigger $\mathcal{E}$ and a controller $\mathcal{C}$. The stochastic discrete-time process $\mathcal{P}$ to be controlled is described by the following time-invariant difference equation

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times d}$. The variables x_k and u_k denote the state and the control input and are taking values in $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^d$, respectively. The initial state x_0 is a random variable with finite mean and covariance. The system noise proces $\{w_k\}$ is i.i.d. (independent identically distributed); w_k takes values in

$\mathbb{R}^n$ and is zero-mean and has finite covariance. The random variables x_0 and w_k are statistically independent for each k. The statistics of the process $\mathcal{P}$ are known a-priori to both, the event-trigger $\mathcal{E}$ and the controller $\mathcal{C}$. The event-trigger $\mathcal{E}$ situated at the sensor station has access to the complete state information and decides, whether the controller $\mathcal{C}$ should receive an update over the feedback channel $\mathcal{N}$. The controller calculates inputs u_k to regulate the process $\mathcal{P}$. Concerning the amount of information available at the control station at each time step k we assume the following: The output signal of the event-trigger, δ_k, takes values in $\{0,1\}$ deciding whether information is transmitted at time k, i.e.,

$$\delta_k = \begin{cases} 1, & \text{measurement } x_k \text{ sent}, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the signal y_k is defined as

$$y_k = \begin{cases} x_k, & \delta_k = 1, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{2}$$

We allow the control input and the event-triggering output to depend on their complete past history. Let the event-triggering law $f = \{f_0, f_1, ..., f_{N-1}\}$ and the control law $\gamma = \{\gamma_0, \gamma_1, ..., \gamma_{N-1}\}$ denote admissible policies for the finite horizon N with

$$\delta_k = f_k(\mathcal{I}_k^{\mathcal{E}}), \quad u_k = \gamma_k(\mathcal{I}_k^{\mathcal{C}}),$$

where $\mathcal{I}_k^{\mathcal{E}}$ and $\mathcal{I}_k^{\mathcal{C}}$ represent the available information at the event-trigger and the controller, respectively. The communication channel $\mathcal{N}$ takes the role of restricting or penalizing transmissions in the feedback loop. This will be reflected in the optimization problem. Let $J_{\mathcal{C}}$ be the control objective defined as

$$J_{\mathcal{C}} = x_N^{\mathsf{T}} Q_N x_N + \sum_{k=0}^{N-1} x_k^{\mathsf{T}} Q x_k + u_k^{\mathsf{T}} R u_k$$

and let $J_{\mathcal{E}}$ be the communication cost given by the number of transmissions, i.e.,

$$J_{\mathcal{E}} = \sum_{k=0}^{N-1} \delta_k$$

We consider three different optimization problems, which represent extensions to the classical linear quadratic setting.
Problem A: Let $\lambda \geq 0$. Find the optimal f^* and γ^* that

$$\inf_{f,\gamma} \mathsf{E}[J_{\mathcal{C}} + \lambda J_{\mathcal{E}}]$$

Problem B: Let m be a non-negative integer. Find the optimal f^* and γ^* that

$$\inf_{f,\gamma} \mathsf{E}[J_{\mathcal{C}}], \quad \text{s.t. } J_{\mathcal{E}} \leq m$$

Problem C: Let $\bar{m} \geq 0$. Find the optimal f^* and γ^* that

$$\inf_{f,\gamma} \mathsf{E}[J_{\mathcal{C}}], \quad \text{s.t. } \mathsf{E}[J_{\mathcal{E}}] \leq \bar{m}$$

The Optimal Control Policy - Preliminary Result. Finding the joint optimal policies of the event-triggered controller is in general difficult for all three problem settings. The controller and event-trigger have different information available, and it is well known that such problems are usually very hard to solve [1]. Stochastic control problems with non-classical information pattern generally do not allow to apply concepts like dynamic programming directly. Nevertheless, it is possible to obtain structural results of the optimal solution. As a preliminary result concerning the optimal control policy we can state that certainty equivalence controller is optimal for all three problem settings A-C. the following.

Proposition Let the system be given by (1) and (2). The class of policies $\mathcal{U}_{CE} \subset \mathcal{U}$ defined by

$$\mathcal{U}_{CE} = \{(f, \gamma^*) | \gamma^* = -L_k \mathsf{E}[x_k | \mathcal{I}_k^{\mathcal{C}}]$$

with

$$L_k = (R + B^\mathsf{T} P_{k+1} B)^{-1} B^\mathsf{T} P_{k+1} A$$

$$P_k = A^\mathsf{T} P_{k+1} A + Q - A^\mathsf{T} P_{k+1} B (R + B^\mathsf{T} P_{k+1} B)^{-1} B^\mathsf{T} P_{k+1} A \text{ and } P_N = Q_N,$$

is a dominating class of policies for the problem settings A-C.
Details on the proof can be found in [2]. Further extensions towards the joint optimal control and transmission scheme design are concerned with a separation principle for the optimal control and event-triggering law [4], and optimal decentralized control laws and transmission schemes for multiple control loops coupled via resource constraints [3].

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Post-processing internal models for robust nonlinear output regulation

Alberto Isidori

(joint work with Lorenzo Marconi)

The problem of robust output regulation for linear multivariable control systems has been thoroughly studied and fully solved in the works [1], [2]. In particular, these works demonstrate that, in a general multivariable setting, robust regulation is achieved *if and only if* the controller possesses a realization in which *the regulated variable drives an internal model*, consisting of as many identical copies of the exosystem (to be precise, of the largest cyclic component of the exosystem) as the number of components of output to be regulated. The realization in question, therefore, can be seen as embedding an internal model that *directly post-processes* the regulated output, cascaded with stabilizer that, driven by the state of the internal model as well as by any other variable available for measurement, produces the appropriate control input.

The problem of output regulation for nonlinear systems, beginning with the work [3], has been addressed by several authors and a rather satisfactory corpus of results has been developed, the majority of which address the problem in question for single-input single-output systems (see for instance [4]) and references therein). For the class of systems in question, (robust) regulation is typically achieved by means of a controller consisting of *an internal model that provides a control input*, to the purpose of forcing the existence of a "steady-state (invariant) manifold" on which the regulated variable vanishes, complemented by a stabilizer that makes the manifold in question attractive for the cascade of two such subsystems. Thus, the approach is somewhat complementary to the one derived in linear multivariable systems, because this structure can be seen as embedding an internal model that *directly pre-processes* the control input.

Of course, in the case of linear single-input single-output systems, the two structures are fully equivalent, because internal model and stabilizer can be swapped. But the structures are not equivalent in the case of multivariable linear systems. In fact, it is not fully understood, in a general multivariable setting, how to design a robust controller in which an internal model directly preprocesses the control input. If the system has q outputs, only $p < q$ of which need to be regulated, but *all* of which are necessary for detectability, it is not immediately clear how to handle these extra $q - p$ outputs in the context of a controller structure embedding a pre-processing internal model. Likewise, if the system has $m > p$ controls, only p of which are needed for regulation purposes, but *all* of which might be necessary for stabilization, it is not immediately clear how to identify a "robust selection" of the inputs that are to be driven by the internal model. All such problems disappear in the case of the control structure embedding an "internal model" that directly post-processes the regulated output and this is why the structure in question appears as the most "natural" one in a multivariable setting.

In view of a (systematic) extension of the theory of nonlinear output regulation to general multivariable systems, it seems appropriate to investigate to what extent controllers in which the internal model post-processes the regulated output are feasible. This is this a problem that, to the best of our knowledge, has not been addressed so far. Consider a nonlinear plant modeled as

$$\begin{aligned} \dot{w} &= s(w) \\ \dot{x} &= f(w,x,u) \\ e &= h_{\mathrm{e}}(w,x) \\ y_{\mathrm{r}} &= h_{\mathrm{r}}(w,x)\,, \end{aligned} \tag{1}$$

in which $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $e \in \mathbb{R}^p$, $y_{\mathrm{r}} \in \mathbb{R}^{q-p}$. The state w of the exosystem evolves on a compact invariant set W. All maps are assumed to be smooth. Take a *post-processing* nonlinear internal model

$$\begin{array}{rcl} \dot\eta &=& \Phi(\eta) + Ge \\ \bar e &=& \gamma(\eta) + e \end{array} \tag{2}$$

cascaded with a system having the following structure

$$\begin{array}{rcl} \dot\xi &=& \varphi(\xi, e, y_{\rm r}) + M_{\rm s}\bar e \\ u &=& \vartheta(\xi, e, y_{\rm r}) + N_{\rm s}\bar e\,. \end{array} \tag{3}$$

Then, the following claim is trivially true.

The controller (2)-(3) solves the problem of output regulation if

- *the equations*

$$\begin{array}{rcl} \left[\begin{array}{c} \frac{\partial\pi}{\partial w} \\ \frac{\partial\pi_{\rm s}}{\partial w} \end{array}\right] s(w) &=& \left[\begin{array}{c} f(w,\pi(w),\vartheta(\pi_{\rm s}(w),0,h_{\rm r}(w,\pi(w)) + N_{\rm s}\psi(w)) \\ \varphi(\pi_{\rm s}(w),0,h_{\rm r}(w,\pi(w)) + M_{\rm s}\psi(w) \end{array}\right] \\ 0 &=& h_{\rm e}(w,\pi(w)) \end{array} \tag{4}$$

  have a solution $\pi(w), \pi_{\rm s}(w), \psi(w)$*,*
- *there exists* $\sigma(w)$ *satisfying*

$$\begin{array}{rcl} \frac{\partial\sigma}{\partial w}s(w) &=& \Phi(\sigma(w)) \\ \psi(w) &=& \gamma(\sigma(w))\,, \end{array} \tag{5}$$

- *the manifold* $x = \pi(w), \eta = \sigma(w), \xi = \pi_{\rm s}(w)$*, that by construction is invariant in the closed-loop system, attracts all its trajectories.*

Note that, in view of the general results of [4], given any $\psi(w)$ satisfying (4) for some $\pi(w), \pi_{\rm s}(w)$, there always exist $\Phi(\eta)$ and $\gamma(\eta)$ such that (5) holds for some $\sigma(w)$. Hence, the intermediate condition in the previous Proposition is immaterial. As a matter of fact, it is known that, if d is large enough, there always exists a $d\times d$ Hurwitz matrix F, a $d\times p$ matrix G such that (F,G) is controllable, a smooth map $\sigma(w)$ and a continuous map $\gamma : \mathbb{R}^d \to \mathbb{R}^p$ such that

$$\begin{array}{rcl} \frac{\partial\sigma}{\partial w}s(w) &=& F\sigma(w) + G\psi(w) \\ \psi(w) &=& \gamma(\sigma(w))\,. \end{array} \tag{6}$$

Thus, the choice $\Phi(\eta) = F\eta + G\gamma(\eta)$ makes (5) fulfilled. From this viewpoint, the previous result can (trivially again) be restated as follows.

A controller of the form

$$\begin{array}{rcl} \dot\eta &=& F\eta + G[\gamma(\eta) + e] \\ \dot\xi &=& \varphi(\xi, e, y_{\rm r}) + M_{\rm s}[\gamma(\eta) + e] \\ u &=& \vartheta(\xi, e, y_{\rm r}) + N_{\rm s}[\gamma(\eta) + e]\,. \end{array} \tag{7}$$

solves the problem of output regulation if :

- *the equations (4) have a solution* $\pi(w), \pi_{\rm s}(w), \psi(w)$*,*
- $\gamma(\eta)$ *is chosen as to fulfill (6), as it is always possible,*
- *the manifold* $x = \pi(w), \eta = \sigma(w), \xi = \pi_{\rm s}(w)$*, that by construction is invariant in the closed-loop system, attracts all its trajectories.*

Of course, the previous results are *all but constructive*. The problem remains to determine how to find a system of the form (3) such that the indicated properties hold. As a matter of fact, it is possible to show that this is always possible for single-input single-output nonlinear systems possessing a well-defined relative degree r, a normal form and an asymptotically stable zero dynamics. In these cases, the controller in question is a controller of the form

$$\begin{array}{rcl}\dot{\eta} & = & F\eta + G[\gamma(\eta) + e] \\ \dot{\hat{y}} & = & A\hat{y} + Be \\ u & = & -k(\mathrm{sat}_L(H\hat{y}) + g^{r-1}c_0[\gamma(\eta) + e])\end{array} \tag{8}$$

in which $\mathrm{sat}_L(\cdot)$ is a saturation function

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On the Role of Plant Model Information in Large-scale Control Systems

Karl H. Johansson

(joint work with Farhad Farokhi and Cédric Langbort)

Introduction. Many complex networked control systems, such as aircraft formations, vehicle platoons, and power grids, consist of several subsystems coupled through their dynamics, controls, or performance objectives. When regulating these systems, we often adopt a distributed control architecture, in which the overall controller is composed of several local subcontrollers that only access local state measurements. A common, but often implicit, assumption is that the design is performed with the full knowledge of the plant model. However, in practice, this assumption is quite seldom being warranted due to the following reasons:

- *Maintenance*: Requirements from control systems maintenance and tuning impose that each subcontroller is a function only of local subsystem parameters, so that the resulting subcontroller does not need to be modified even if the model parameters of a particular subsystem change over time.

- *Availability*: The lack of a complete plant model at the time of the design restricts the control designer to only use local model information in the computation of each subcontroller.
- *Privacy*: Privacy constraints caused by financial or security incentives limit the amount of the model information available in the control design for each subsystem.

Removing the assumption of full plant model knowledge from the control design procedure generates a new class of problems, namely, limited model information control design problems. Some instances of these problems are discussed in this talk.

Main Results. The main contribution is on the achievable performance for large-scale networked control design under limited plant model information. The considered limited model information setup was introduced in [1] and is here extended to a network setting with plants, controllers, and design information constraints represented as graphs. First, we consider limited model information control design for interconnections of fully-actuated discrete-time linear time-invariant subsystems with a quadratic separable cost function [2, 3]. We investigate the best closed-loop performance achievable by structured static state-feedback controllers based on limited model information design strategies. To do so, we introduce control design strategies as mappings from the set of plants of interest to the set of eligible controllers. These control design strategies are compared using the competitive ratio as a performance metric and the domination as a partial order on the set of limited model information control design strategies. We define the competitive ratio as the worst case ratio of the cost of a control design strategy to the cost of the optimal control design with full model information. We show that the competitive ratio depends crucially on how the subsystems are interconnected and what state measurements that are available. We prove that the deadbeat control design strategy is the best limited model information design strategy when there is no subsystem that cannot affect any other subsystems and each subcontroller has access to at least the state measurements of those subsystems that affect it. However, the deadbeat control design strategy is dominated when there is a subsystem that cannot affect any other subsystem. We find an undominated limited model information control design strategy that achieves a better closed-loop performance in average while having the same competitive ratio. We also characterize the amount of model information needed to achieve a better competitive ratio than the deadbeat control design strategy. We generalize these results to structured dynamic state-feedback controllers when the closed-loop performance criterion is H_2-norm of the closed-loop transfer function [4]. Surprisingly, the optimal limited model information control design strategy is static. This is the case even though the optimal decentralized state-feedback controller with full model information is dynamic. We partially relax the assumption that all the subsystems are fully-actuated and generalize the result for a class of under-actuated systems where the sinks (in the plant graph) are not necessarily fully-actuated. Later, we also discuss the design of dynamic controllers for disturbance accommodation [5, 6]. This problem is of special interest, because the best limited model information control design is dynamic in this case. The dynamic control design strategy can be divided into two parts: a static feedback law and a dynamic observer. For constant disturbances, it is shown that this structure corresponds to proportional-integral control.

Example. Vehicle platooning is used to illustrate the results and the applicability of the approach. We consider the problem of regulating the distance between trucks in a platoon. The characteristics of each truck (e.g., mass, tire quality, break capability) influence its model parameters. The designer of the controller of each truck may want its controller to only be a function of its own truck parameters due to several reasons. One reason could be that the designer wants its controller to be fixed because of safety constraints since changing a truck's subcontroller may result in an unpredictable behavior. Another reason is simply that the models of the other trucks in the platoon are not available at the time of design. Privacy constraints might also limit the amount of information available in the design procedure since different trucks might belong to different companies and these companies may wish to disclose information about the performance of their trucks. Independent of these design constraints, the designers want to guarantee some reasonable bounds on the closed-loop performance of the platoon in terms of reduction of the fuel consumption. This problem is hence a viable candidate for optimal control design with limited model information.

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Coordination Control of Linear Systems (poster)

Pia L. Kempker
(joint work with André C.M. Ran, Jan H. van Schuppen)

Coordinated linear systems are a special class of hierarchical systems, with a top-to-bottom information structure. Our interest in coordinated linear systems is motivated by the need for coordination in many engineering systems, with several subsystems collaborating to achieve a common goal.

A coordinated linear system consists of three subsystems, of which the coordinator subsystem influences the local subsystems but is not influenced by them, and when disregarding the influence of the coordinator, the local subsystems are independent. This structure allows for a partially decentralized approach to control synthesis: given the closed-loop coordinator, each local subsystem can be treated independently. Some global control objectives can be approached in a fully decentralized manner: For example, global stabilizability reduces to local stabilizability of all subsystems.

The theory developed in [1], [2] and [3] is based on the concepts of conditional independence of linear subspaces and of invariant spaces with respect to linear maps. Coordinated linear systems have a state space representation of the form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_c\end{bmatrix} = \begin{bmatrix}A_{11} & 0 & A_{1c}\\ 0 & A_{22} & A_{2c}\\ 0 & 0 & A_{cc}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_c\end{bmatrix} + \begin{bmatrix}B_{11} & 0 & B_{1c}\\ 0 & B_{22} & B_{2c}\\ 0 & 0 & B_{cc}\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_c\end{bmatrix},$$
$$\begin{bmatrix}y_1\\ y_2\\ y_c\end{bmatrix} = \begin{bmatrix}C_{11} & 0 & C_{1c}\\ 0 & C_{22} & C_{2c}\\ 0 & 0 & C_{cc}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_c\end{bmatrix},$$

where subscripts 1 and 2 correspond to the local subsystems, and c corresponds to the coordinator. The set of matrices of this type forms an algebraic ring. The structure can easily be extended to hierarchical systems with more subsystems and several coordination layers.

Coordinated linear systems can be constructed from arbitrary interconnected systems, by moving those parts of each component which require interaction with the rest of the system to the coordinator, and moving those parts of each component which have no influence on the rest of the system to one of the local subsystems ([2]).

For coordinated linear systems, the concepts of controllability and observability are refined ([3]), taking into account which part of the system is controllable via *which input*, and observable from *which output*: The local subsystems may be controllable via their local inputs, or via the coordinator input or (part of) both, and similarly the subsystem observations may involve their local state, the coordinator state, or (part of) both. This distinction leads to refined controllability and observability decompositions, and to new concepts of controllability and observability.

The corresponding LQ optimal control problem separates into independent LQ problems at the lower level, which can be solved locally, and a more involved control problem at the coordinator level. For the latter problem, possible approaches include using numerical optimization ([5]), and approximating the centralized optimum by means of event-based bottom-to-top feedback ([4]).

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Linear time invariant minimax filtering

Arthur J. Krener
(joint work with Wei Kang)

The problem of filtering a linear time invariant system with white Gaussian observation noise and unknown but bounded driving noise is considered. This models the possibility that the driving noise is under the control of an intelligence adversary who is trying to corrupt the filter or that the driving noise is a stochastic process about which little is known except that its sample paths are bounded.

We review the minimax filter of Johansen and Berkovitz-Pollard for the double integrator. While their solution is very elegant, the optimal filter is infinite dimensional. We show that there is a two dimensional filter that is within 2.5% of optimal and a four dimensional filter that is within 0.7% of optimal. The best Kalman filter is within 2.6% of optimal.

We show that similar results hold for other linear time invariant systems.

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Necessary and sufficient dissipativity conditions for stability of interconnected systems

Mircea Lazar, Rob H. Gielen

In what follows, necessary and sufficient conditions based on dissipativity theory are established for global exponential stability of interconnected dynamical systems. The non-conservative nature of these conditions is due to a relaxation of local dissipation inequalities inspired by the asymptotic stability criterion of [1] for time-variant dynamical systems. A simple example of two scalar linear interconnected systems, for which the original dissipativity conditions of [2] are not feasible, demonstrates the non-conservatism of the proposed framework for stability analysis of interconnected systems.

Consider a set of $N \in \mathbb{Z}_{\geq 2}$ interconnected systems

$$x_i(k+1) = g_i(x_1(k), \ldots, x_N(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

with $x_i \in \mathbb{R}^{n_i}$ and $g_i : \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_N} \to \mathbb{R}^{n_i}$ for all $i \in \mathbb{Z}_{[1,N]}$. For notational convenience, throughout this abstract we use x_i to denote both an arbitrary vector in $\mathbb{R}^{n_i}$ and a solution of system (1), i.e., $x_i(k)$, $k \in \mathbb{Z}_+$. Also, for arbitrary sets $\mathcal{S}, \mathcal{P} \subseteq \mathbb{R}$, we use the notation $\mathcal{S}_{\mathcal{P}} := \mathcal{S} \cap \mathcal{P}$. To describe the overall system, let $x = \operatorname{col}(\{x_i\}_{i \in \mathbb{Z}_{[1,N]}}) := \left[x_1^\top \ldots x_N^\top \right]^\top$ and let $n = \sum_{i=1}^{N} n_i$, which yields

$$x(k+1) = G(x(k)), \quad k \in \mathbb{Z}_+, \tag{2}$$

where $G(x) = \operatorname{col}(\{g_i(x_1, \ldots, x_N)\}_{i \in \mathbb{Z}_{[1,N]}})$.

Definition 1. *The interconnected system* (2) *is* globally exponentially stable *(GES) if for all* $(x(0), k) \in \mathbb{R}^n \times \mathbb{Z}_+$ *it holds that* $\|x(k)\| \leq c\mu^k \|x(0)\|$ *for some* $(c, \mu) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{[0,1)}$.

Since the fundamental work [2], the following result has been heavily employed in establishing GES of interconnected systems of the form (2).

Theorem 2. *Suppose that there exists a set of storage and supply functions* $\{W_i, S_{i,j}\}_{(i,j) \in \mathbb{Z}^2_{[1,N]}}$, *with* $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ *and* $S_{i,j} : \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}$, *such that the following conditions hold:* (i) *for all* $i \in \mathbb{Z}_{[1,N]}$

$$c_1 \|x_i\|^\lambda \leq W_i(x_i) \leq c_2 \|x_i\|^\lambda, \quad \forall x_i \in \mathbb{R}^{n_i},$$

for some $(c_1, c_2, \lambda) \in \mathbb{R}^2_{>0} \times \mathbb{Z}_{\geq 1}$; (ii) *for all* $i \in \mathbb{Z}_{[1,N]}$

$$W_i(g_i(x_1, \ldots, x_N)) \leq \rho W_i(x_i) + \textstyle\sum_{j=1}^{N} S_{i,j}(x_i, x_j),$$

for all $x \in \mathbb{R}^n$ *and some* $\rho \in \mathbb{R}_{[0,1)}$; *and* (iii) $S_{i,j}(x_i, x_j) + S_{j,i}(x_j, x_i) \leq 0$ *for all* $(i,j) \in \mathbb{Z}^2_{[1,N]}$ *and all* $x \in \mathbb{R}^n$. *Then, the overall interconnected system* (2) *is GES.*

The following example demonstrates that the hypothesis of Theorem 2 is conservative even for very simple linear interconnected systems.

Example 3 (Part I). *Consider the interconnected scalar systems*

$$\begin{array}{l} x_1(k+1) = x_1(k) - 0.5x_2(k) \\ x_2(k+1) = x_1(k) \end{array}, \quad k \in \mathbb{Z}_+. \tag{3}$$

It can be concluded from standard Lyapunov arguments that the interconnected system is GES with $\mu = 0.7071$ and $c = 2.62$. Next, suppose that there exists a set of storage and supply functions, i.e., $(W_1, W_2, S_{1,2}, S_{2,1})$, that satisfies the hypothesis of Theorem 2. Then, for any $\rho \in \mathbb{R}_{[0,1)}$, it follows from property (ii), *by choosing $x = col(1,0)$, that*

$$W_1(1) \leq \rho W_1(1) + S_{1,2}(1,0),$$
$$W_2(1) \leq \rho W_2(0) + S_{2,1}(0,1).$$

The above inequalities together with property (i) *imply that $0 < S_{1,2}(1,0) + S_{2,1}(0,1)$, which contradicts property* (iii) *of Theorem 2. Therefore, the interconnected system* (3)*, although it is GES, it does not admit a set of storage and supply functions that satisfy the hypothesis of Theorem 2.* □

In what follows, the relaxation of Lyapunov inequalities proposed in [1] for time-variant systems is exploited to obtain non-conservative dissipativity-like conditions for GES of time-invariant interconnected systems.

Theorem 4. *Suppose that the map $G : \mathbb{R}^n \to \mathbb{R}^n$ corresponding to the dynamics* (2) *is Lipschitz continuous. The interconnected system* (2) *is GES* if and only if *there exists a finite $M \in \mathbb{Z}_{\geq 1}$ (e.g., any integer $M \geq \log_\mu(\frac{\rho}{Nc})$ will do) and a set of storage and supply functions $\{W_i, S_{i,j}\}_{(i,j)\in\mathbb{Z}^2_{[1,N]}}$ where $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ and $S_{i,j} : \mathbb{R}^{n_i} \times \mathbb{R}^{n_j} \to \mathbb{R}$ such that the following conditions hold:* (i) *for all $i \in \mathbb{Z}_{[1,N]}$*

$$c_1\|x_i\|^\lambda \leq W_i(x_i) \leq c_2\|x_i\|^\lambda, \quad \forall x_i \in \mathbb{R}^{n_i},$$

for some $(c_1, c_2, \lambda) \in \mathbb{R}^2_{>0} \times \mathbb{Z}_{\geq 1}$; (ii) *for all $i \in \mathbb{Z}_{[1,N]}$*

$$W_i(x_i(M)) \leq \rho W_i(x_i) + \textstyle\sum_{j=1}^N S_{i,j}(x_i, x_j),$$

for all $x \in \mathbb{R}^n$ and some $\rho \in \mathbb{R}_{[0,1)}$ (above $x_i(k+1) := g_i(x_1(k), \ldots, x_N(k))$ and $x_i(0) := x_i$ for all $(k,i) \in \mathbb{Z}_+ \times \mathbb{Z}_{[1,N]}$); and (iii) *$S_{i,j}(x_i, x_j) + S_{j,i}(x_j, x_i) \leq 0$ for all $(i,j) \in \mathbb{Z}^2_{[1,N]}$ and all $x \in \mathbb{R}^n$.*

Remark 5. *Theorem 4 holds in terms of global asymptotic stability (GAS) as well, with a suitable change in condition* (i)*. Moreover, the Lipschitz continuity assumption can be relaxed to Hölder continuity or even $\mathcal{K}$-continuity. However, then it is not clear how to obtain an estimate of the value of M.*

Clearly, the proposed non-conservative dissipativity conditions recover the original dissipativity conditions of Theorem 2 for $M = 1$. Furthermore, they are also equivalent to a linear matrix inequality for linear interconnected systems with quadratic storage and supply functions.

Example 3 is revisited next to demonstrate the non-conservativeness of the hypothesis of Theorem 4.

Example 3 (Part II)**.** *Consider again the interconnected scalar systems* (3)*. In Example 3, Part I it was shown that there does not exist a set of storage and supply functions that satisfy the hypothesis of Theorem 2. However, the functions*

$$W_1(x_1) = x_1^2, \quad W_2(x_2) = x_2^2,$$
$$S_{1,2}(x_1, x_2) = -S_{2,1}(x_2, x_1) = -0.51x_1^2 + 0.063x_2^2,$$

satisfy the hypothesis of Theorem 4 for any $\rho \in \mathbb{R}_{[0.51,1)}$ *and* $M = 3$*. Interestingly, the lower bound on* M *indicated in Theorem 4 yields that for any* $M \geq 5$ *the hypothesis of Theorem 4 can be satisfied with* $\rho \in \mathbb{R}_{[0.95,1)}$*.*

The lower bound obtained for the above example indicates that further relaxations of condition *(ii)* of Theorem 4 may be possible, i.e., by reducing N to the number $N_i \in \mathbb{Z}_{\geq 1}$ of systems that have a direct interconnection with system i and by allowing for a different $M_i \in \mathbb{Z}_{\geq 1}$ for each system i. Clearly, the situation $M_i = N_i = 1$ for all $i \in \mathbb{Z}_{[1,N]}$ would correspond to a standard fully decentralized stability condition.

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ARMA Identification of Graphical Models

Anders Lindquist

(joint work with Enrico Avventi and Bo Wahlberg)

Consider a Gaussian stationary stochastic vector process with the property that designated pairs of components are conditionally independent given the rest of the components. Such processes can be represented on a graph where the components are nodes and the lack of a connecting link between two nodes signifies conditional independence. This leads to a sparsity pattern in the inverse of the matrix-valued spectral density. Such graphical models find applications in speech, bioinformatics, image processing, econometrics and many other fields, where the problem to fit an autoregressive (AR) model to such a process has been considered. In this paper we take this problem one step further, namely to fit an autoregressive moving-average (ARMA) model to the same data. We develop a theoretical framework and an optimization procedure which also spreads further light on previous approaches and results. This procedure is then applied to the identification problem of estimating the ARMA parameters as well as the topology of the graph from statistical data.

Event-based control: A state-feedback approach

Jan Lunze

Event-based control is a means to reduce the communication between the sensors, the controller and the actuators in a control loop by invoking a communication among these components only after an event has indicated that the control error exceeds a tolerable bound. This working principle differs fundamentally from that of the usual feedback loop, in which data are communicated from the sensor to the controller and from the controller to the actuator continuously or at every sampling instance given by a clock. Hence, in the control schemes currently used a communication takes place independently of the size of the control error and, in particular, also in case of small control errors when no information feedback is necessary to satisfy the performance requirements on the plant. In these situations, the communication resources are used unnecessarily.

This paper considers the event-based control loop, which consists of

- a plant with state $x(t)$ and input $u(t)$,
- an event generator,
- a control input generator and
- a digital communication network that connects the event generator with the control input generator.

In this scheme, the control law is implemented in the event generator and the control input generator.

The event generator determines the time instants t_k, $(k = 0, 1, ...)$ at which the next communication from the event generator towards the control input generator is invoked. The control input generator determines the input $u(t)$ for the time interval $t \in [t_k, t_{k+1})$ in dependence upon the information $x(t_k)$ obtained at time t_k. The information link from the event generator towards the control input generator is only used after an event has been generated.

The aim of this paper is to propose a new scheme of event-based control, where the communication link is only used if the disturbance $d(t)$ has caused an intolerable effect on the loop performance. As the main result, algorithms for the event generation and the control input generation are described for which the event-based control loop mimics the continuous state-feedback loop with adjustable accuracy.

In comparison with the numerous recent publications on event-based control, the approach described in this talk has three novelties. First, the control input generator is no longer a zero-order hold, but uses a model of the continuous closed-loop system to adapt the input continuously to the plant state. Second, the event generator evaluates the current plant state in comparison with the state that a continuous state-feedback system has. Hence, an event does not indicate a large control error but a large deviation of the event-based control loop from the continuous loop. Third, both the event generator and the control input generator include a disturbance estimator.

The talk also reports on the experimental evaluation of event-based control for a thermofluid process. The experiments show that event-based control leads to a considerable reduction of the communication within the control loop in comparison with sampled-data control. Furthermore, the experiments demonstrate a considerable robustness of the closed-loop system with respect to model uncertainties.

The paper is based on the following references:

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Hybrid Linear Regulation

Lorenzo Marconi
(joint work with Andrew Teel)

In this work we focus on the problem of output regulation for a class of hybrid linear systems and exosystems governed by the flow dynamics

$$\dot{\tau} = 1\,, \qquad \dot{w} = Sw\,, \qquad \dot{x} = Ax + Bu + Pw$$

whenever $((\tau, w), x, u) \in W \times \mathbb{R}^n \times \mathbb{R}$ and subject to jumps according to the rules

$$\tau^+ = 0\,, \qquad w^+ = Jw\,, \qquad x^+ = Nw + Mx$$

whenever $((\tau, w), x, u) \in (W \cap (\{\tau_{\max}\} \times \mathbb{R}^s)) \times \mathbb{R}^n \times \mathbb{R}$, where W is a compact set that is invariant for the hybrid dynamics with state (τ, w). In the equations above, $\tau_{\max}$ is a positive known constant that imposes a dwell-time constraint between two consecutive jumps. The dynamics of the exogenous variable is what, in the literature on output regulation, is usually called the "exosystem". Associated to the previous system there is a *regulation error* given by $e = Cx + Qw$, $e \in \mathbb{R}$ which jumps, whenever the state jumps, as $e^+ = (CN + QJ)w + CMx$. In this context, the problem we are interested in is to develop a *continuous* hybrid regulator, processing the error e and the clock τ, of the form

$$\begin{array}{rcll} \dot{\xi} & = & \Phi(\tau)\xi + \Lambda(\tau)e & (\tau, \xi, e) \in [0, \tau_{\max}] \times \mathbb{R}^m \times \mathbb{R} \\ \xi^+ & = & \Sigma\xi + \Delta e & (\tau, \xi, e) \in \{\tau_{\max}\} \times \mathbb{R}^m \times \mathbb{R} \\ u & = & \Gamma(\tau)\xi + K(\tau)e & \end{array}$$

so that the resulting hybrid closed-loop system has bounded trajectories and the regulation error converges to zero uniformly over compact sets of initial conditions.

The objective of this work is to lay the foundation for an output regulation theory for linear hybrid systems described in the flow-jump hybrid formalism. A fundamental tool that is developed in the work is a notion of steady-state response for hybrid cascade systems. This tool will form the basis to generalize the notion of regulator equations and of the internal model principle ([1]) to the considered case of hybrid linear regulation and, in turn, to present necessary and sufficient conditions for the regulator design. We also present constructive design procedures for the regulator by emphasizing how to achieve robustness to possible system uncertainties. We precisely characterize scenarios in which the hybrid regulator can be taken to be independent of time, while showing how robust asymptotic regulation can be sometimes achieved by necessarily adopting regulators embedding time-varying internal models. Am extended version of the work is presented in [2].

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Synchronization in networks with time-delay coupling: "the sympathy of pendulum clocks and beyond"

Henk Nijmeijer, Jonatan Peña-Ramírez

1. Introduction

In the 17-th century, the Dutch scientist Christiaan Huygens observed that two pendulum clocks hanging from a common support (a wooden bar supported by two chairs) kept in pace relative to each other such that the two pendulums always swung together (in opposite motion) and never varied. Huygens called this "the sympathy of the two clocks" [1]. Since then, several attempts have been made to better understand the true mechanism behind the sympathy of pendulum clocks described by Huygens. For instance, in [2] and [3], an experimental study related to the original Huygens' experiment is presented. In such works, the pendulum clocks have been replaced by arbitrary nonlinear oscillators and instead of the flexible wooden bar, a one degree of freedom rigid bar is considered. Then, it is shown that the oscillators may exhibit in-phase and anti-phase synchronization. These results in fact suggest that the synchronized motion observed by Huygens extends beyond pendulum clocks.

The purpose of the present contribution is to pursue further the nature of synchronization of arbitrary oscillators and see to what extent such sympathy is also meaningful in understanding synchronization in a network of identical systems.

2. Framework

There exist many different types of synchronization, in nature often perceived as surprising or fairly difficult to understand. In particular, the mechanism underlying synchronization effects in a swarm of fireflies blinking simultaneously, or a flock of birds flying in a v-shape, or, the reader may start to generate numerous other examples, is often intriguing and extremely challenging for scientific study, see for instance the stimulating book of Strogatz [4]. On the other hand, the famous example by Christiaan Huygens of two pendulum clocks exhibiting anti-phase or in-phase synchronized motion as brought forward in his notebook features exactly the crucial point: despite the lack of good modeling tools, Huygens did realize that there is a "medium" responsible for the synchronized motion, namely the bar to which both pendula are attached! Despite this sharp observation, even today a complete rigorous mathematical derivation using proper models for pendula and flexible beam showing the occurrence of synchronized motion is still lacking. Obviously, understanding of synchronized motion of a larger set of coupled identical systems is in general even more difficult and therefore a fitting framework for this type of study seems mandatory. The ingredients required in the framework that we use in our work are semi-passive systems, convergence, and network topology (or rather coupling topology).

First, we briefly describe semi-passive systems, but before doing this, it is necessary to mention two (related) properties: dissipativity and passivity. We say that a system is dissipative if it does not generate energy and dissipates (or at most conserves) the energy supplied to it. A system is called passive if it is dissipative and its supply rate is given by the bilinear product of the input(s) and output(s). On the other hand, a semi-passive system is an input-output system with as many inputs as outputs, and which is passive outside some ball in the state space. Basically, this amounts to the fact that without external forcing the system trajectories are confined to that bounded region. It should also be noted that in contrast with passive systems, semi-passive systems may generate a finite amount of energy itself. Moreover, semi-passive systems might feature much richer dynamical behaviour (in comparison with passive systems) like for instance self-sustained oscillatory behaviour.

One of the key properties of semi-passive systems is that when two or more semi-passive systems are linearly interconnected, they posses bounded solutions. This property can actually be exploited in the analysis of synchronization of networks of semi-passive systems, cf. [5, 6]

The second key element in our framework is the property of convergence. An input-output system is said to be convergent if any solution of the system starting in a certain region "forgets" its initial condition such that, after transients, the solutions converge to a steady-state solution which only depends on the input signal applied to the system. Such input can be either a disturbance or a feed-forward control signal. In fact, the property of convergence is associated with the output regulation problem in control theory, where an internal stability condition for the closed-loop system is required. Namely, that every solution of the closed-loop system must be independent of its initial condition and it should converge to a unique solution, which is determined only by the input. In the case of asymptotically stable linear systems, convergence is a natural property, whereas nonlinear systems do not have this property in general. For more formal and general definitions on the convergence property, the reader is referred to [7].

In the study of synchronization of interconnected systems, the property of convergence can be useful in determining internal stability properties of the (interconnected) systems. For instance, the notion of convergence can be used in order to find conditions such that it is ensured that the (interconnected) systems are minimum-phase.

The final ingredient is the so called network or coupling topology. This is illustrated by means of an example. Suppose that one has a collection of dynamical systems and it is desired to connect them in such way that they synchronize their states either partially (clustering) or completely. Then, a natural question would be what kind of network structure and coupling functions will lead to (partial/complete) synchronization of the systems? Moreover, what happens if either the number of couplings and /or systems increases? This kind of questions can be addressed by, for instance, the use of graph theory. Under this formalism, each dynamical system is considered as a node and the coupling between two systems, which normally it is assumed to be either unidirectional or bidirectional, is taken to be an edge of the graph.

Existing results suggest that when all members in the network are assumed to be identical, the existence of invariant (partial/complete) synchronization manifolds is likely to follow from the specific topology of the network. In particular, in [8] it is shown how the structure of the network can be chosen such that partial synchronization occurs. Further results related to the topology of diffusive networks and synchronization in time-varying network topologies are presented in [9] and [10], respectively.

3. Networks with time-delay coupling

So far, the case where the interaction between the dynamical systems within the network is instantaneous has been considered, i.e. it has been assumed that the communication delay between two or more systems is negligible. However, when a signal is traveling through a complex network, the time that it takes to the signal to go from one system to another (time delay) is generally not negligible. The time delay in the network can be caused, for instance, by speed transmission and/or traffic congestion. Consider for example the interaction between two distant neurons; due to the finite propagation speed of the membrane potential through the neuron's axon, a neuron “feels” the change of membrane potential of the other neuron to which it is connected only after some time has elapsed. Another familiar example can be found in communication systems, like in computer networks, where the transmission of a message from one client to another takes certain amount of time, i.e. is subject to a delay; in this case the time delay can be caused either by excess of traffic through the network and/or because of processing time required by the servers in order to process the information.

In practical situations, time delays caused by signal transmission may affect the behavior of coupled systems. It is therefore necessary to study the effect that time delays have in existing synchronization schemes. Actually, at this point the natural question would be, can systems in networks show synchronous behaviour even in the presence of time delays?

By using the framework above described in combination with some basic theory about retarded functional differential equations (like the well known Lyapunov-Razumikhin Theorem), it is possible to prove under some mild assumptions, that identical strictly semi-passive systems, whose internal dynamics are stable, always will synchronize given that the coupling between the systems is sufficiently strong and a possible constant time delay is sufficiently small [11].

4. Conclusions

A theoretical framework than can be used to study synchronization in networks of certain dynamical systems has been presented. The ingredients required in the proposed framework are semi-passive systems, the convergence property and the network (or coupling) topology. Furthermore, the proposed framework is valid even in the case where the communication between the members/systems of the network is subject to a time delay.

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An Optimal Controller Architecture for Poset-Causal Systems

Pablo A. Parrilo
(joint work with Parikshit Shah)

Summary: We propose a novel and natural architecture for decentralized control that is applicable whenever the underlying system has the structure of a partially ordered set (poset). This controller architecture is based on the concept of Möbius inversion for posets, and enjoys simple and appealing separation properties, since the closed-loop dynamics can be analyzed in terms of decoupled subsystems. The controller structure provides rich and interesting connections between concepts from order theory such as Möbius inversion and control-theoretic concepts such as state prediction, correction, and separability. In addition, using our earlier results on $\mathcal{H}_2$-optimal decentralized control for arbitrary posets, we prove that the $\mathcal{H}_2$-optimal controller in fact possesses the proposed structure, thereby establishing the optimality of the new controller architecture. A complete version of these results can be found in [2].

In this work we are concerned with the following questions: "What is a sensible architecture of controllers for poset-causal systems? What should be the role of controller states, and what computations should be involved in the controller?"

Our main contributions are:

- We propose a controller architecture that involves natural concepts from order theory and control theory as building blocks.
- We show that a natural coordinate transformation of the state variables yields a novel *separation principle.*
- We show that the optimal state-feedback $\mathcal{H}_2$ controller that we studied earlier in [3] has precisely the proposed controller structure.
- We establish novel connections that tie together three well-known concepts: (a) Youla parameterization in control, (b) the concept of purified output feedback in robust optimization and (c) Möbius inversion on posets.

Poset-causal systems. We consider state-space systems in continuous time:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + w(t) + Bu(t) \\ z(t) &= Cx(t) + Du(t) \\ y(t) &= x(t). \end{aligned} \tag{1}$$

The system matrices (A, B, C, D) are partitioned into blocks in the following natural way. Let $\mathcal{P} = (P, \preceq)$ be a finite poset with $P = \{1, \ldots, s\}$. We think of system (1) as being divided into s subsystems, with subsystem i having states $x_i(t) \in \mathbb{R}^{n_i}$. The control inputs at the subsystems are $u_i(t) \in \mathbb{R}^{m_i}$ for $i \in \{1, \ldots, s\}$. The external output is $z(t) \in \mathbb{R}^p$. The signal $w(t)$ is a disturbance signal. The states and inputs are partitioned in the natural way such that the subsystems correspond to elements of the poset $\mathcal{P}$ with $x(t) = [x_1(t)\,|x_2(t)\,|\ldots|x_s(t)\ ]^T$, and $u(t) = [u_1(t)\,|u_2(t)\,|\ldots|u_s(t)\ ]^T$. This naturally partitions the matrices A, B, C, D

into appropriate blocks so that $A = [A_{ij}]_{i,j \in P}$, $B = [B_{ij}]_{i,j \in P}$, $C = [C_j]_{j \in P}$ (partitioned into columns), $D = [D_j]_{j \in P}$.

We call such systems *poset-causal* due to the following causality-like property among the subsystems. If an input is applied to subsystem i via u_i at some time t, the effect of the input is seen by the downstream states x_j for all subsystems $j \in \downarrow i$ (at or after time t). Thus $\downarrow i$ may be seen as the cone of influence of input i. We refer to this causality-like property as *poset-causality.* This notion of causality enforces (in addition to causality with respect to time), a causality relation between the subsystems with respect to a poset.

Information Constraints on Controller. We require the controller K to also be poset-causal, with respect to the same (block) incidence algebra as the system. We want to compute the best structure-preserving controller, that minimizes a measure of performance given by the standard $\mathcal{H}_2$ norm of the feedback interconnection.

Controller architecture and properties: The proposed controller structure is as follows. At each subsystem, the partial ordering allows a decomposition of the global state into "upstream" states (i.e. states that are available), "downstream" (these are unavailable) and "off-stream" states (corresponding to uncomparable elements of the poset). The downstream and off-stream states are (partially) predicted using available upstream information using a "simulator" (see Figure 1), this prediction is the role of the controller states. The best available information of the global state at each subsystem is then described using a matrix X; each column of X corresponds to the best local guess or estimate of the overall state at a particular subsystem.

Having computed these local partial estimates, the controller then performs certain natural local operations on X that preserve the structure of the poset. These local operations are the well-known ζ and μ operations in Möbius inversion. These operations, which are intimately related to the inclusion-exclusion formula and its generalizations, have a rich and interesting theory, and appear in a variety of mathematical contexts [1]. The control inputs are of the form $U = \zeta(\mathbf{F} \circ \mu(X))$. The operators μ and ζ can be interpreted as generalized notions of differentiation and integration on the poset so that $\mu(X)$ may be interpreted as the differential improvement in the prediction of the local state. Here $\mathbf{F} = \{F(1), \ldots, F(s)\}$ are feedback gain matrices corresponding to the different subsystems. The quantity $\mathbf{F} \circ \mu(X)$ may therefore be interpreted as a local "differential contribution" to the overall control signal. The overall control law then aggregates all these local contributions by "integration" along the poset using ζ. This architecture is been shown diagrammatically in Figure 1.

The key property of this architecture is the *decoupling* or *separation* of the dynamics of the independent subsystems. Each subsystem can simulate the global dynamics using local states and inputs. This can be compactly written as

$$\dot{X}(t) = AX(t) + BU(t). \tag{2}$$

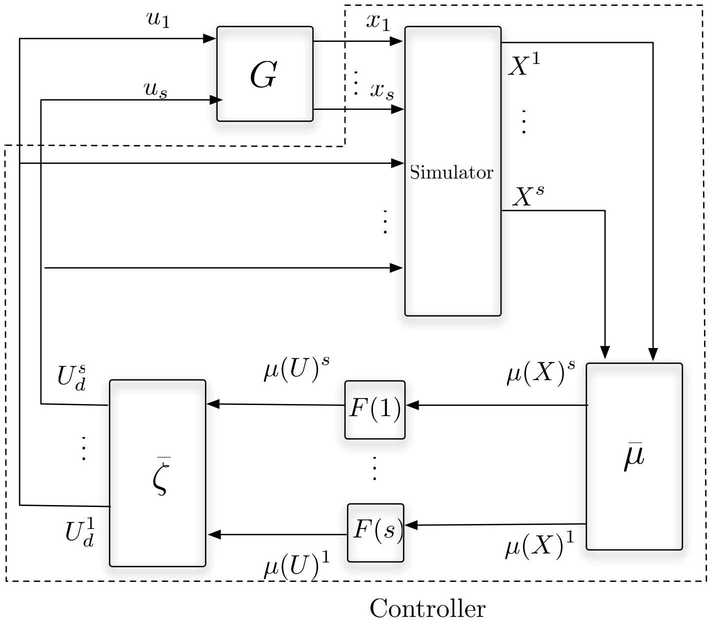

FIGURE 1. Block diagram representation of the control architecture. The simulator predicts the unknown states at each subsystem (predictions at subsystem k are denoted by X^k). The controller then computes a differential improvement on the prediction using μ, acts on it with local gains $F(k)$ and then "integrates" them along the poset using ζ to produce the control inputs.

Applying μ to these equations we obtain the following closed-loop dynamics in the new variables $\mu(X)$:

$$\mu(\dot{X})(t) = A\mu(X)(t) + B\mu(U)(t).$$

Define $\mathbf{A} + \mathbf{BF} = \left\{ A + B\hat{F}(1), \ldots, A + B\hat{F}(s) \right\}$. From the fact that $\mu(\zeta(\mathbf{F} \circ \mu(X))) = \mathbf{F} \circ \mu(X)$ the modified closed-loop dynamics are then decoupled:

$$\mu(\dot{X})(t) = (\mathbf{A} + \mathbf{BF}) \circ \mu(X)(t).$$

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Generalized Factor Analysis Models

GIORGIO PICCI
(joint work with Giulio Bottegal)

An interesting generalization of factor analysis models has been proposed by Chamberlain and Rothschild in 1983 [2] and recently extended to the dynamic setting by Forni, Lippi and collaborators [3].
These models, called *generalized factor analysis (GFA) models* describe observations of infinite cross-sectional dimension. Restricting to the static case, let $\mathbf{y}$ be an infinite string of zero-mean finite variance random variables, which we write as a column vector; then a GFA is a infinite dimensional linear model of the following type

$$\mathbf{y} = \sum_{i=1}^{q} f_i \mathbf{x}_i + \tilde{\mathbf{y}}$$

where:
$f_i \in \mathbb{R}^\infty$, $i = 1, \dots, q$ i-th **factor loading** vector,
$\mathbf{x}_i = i$-th **latent factor**, w.l.g. normalized to unit variance $\mathbb{E}\,\mathbf{x}\mathbf{x}^\top = I_q$
$\tilde{\mathbf{y}}$ is the **"idiosyncratic" noise** vector with $\mathbf{x} \perp \tilde{\mathbf{y}}$.

The idiosyncratic term is no longer required to have uncorrelated components as in the classical factor analysis model, but to satisfy instead a *zero-average* condition. This condition implies that the covariance of any two variables $\tilde{\mathbf{y}}(k)$ and $\tilde{\mathbf{y}}(j)$, say $\tilde{\sigma}(k, j)$ tends to zero when $|k - j| \to \infty$.

It can be shown that with this new definition the inherent non-uniqueness of classical factor analysis models does not occur. Moreover in this generalized context the dimension q of the latent factors vector can be characterized as the number of "infinite eigenvalues" of the covariance matrix of $\mathbf{y}$.

We attempt to use this class of models to do modeling of large interconnected systems. We show that the overall covariance of the observed process can be decomposed in the sum of two contributions.

- A *long range* correlation structure which describes the component of $\mathbf{y}$ driven by the latent vector. The *long range* property means that the covariance of two varible $\hat{\mathbf{y}}(k)$ and $\hat{\mathbf{y}}(j)$, say $\hat{\sigma}(k, j)$ does not go to zero when $|k - j| \to \infty$.
- A *short range* correlation structure which corresponds to the idiosyncratic component $\tilde{\mathbf{y}}$. The *short range* property means that the covarinace of two varible $\mathbf{y}(k)$ and $\mathbf{y}(j)$, say $\tilde{\sigma}(k, j) \to 0$ when $|k - j| \to \infty$.

There is a natural interpretation of generalized factor analysis models in terms of Wold decomposition of stationary sequences. A stationary sequence admits a (unique) generalized factor analysis decomposition if and only if two rather natural conditions are satisfied.

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Distributed Control of Positive Systems

Anders Rantzer

Classical methods for multi-variable control, such as LQG and H_∞, suffer from a lack of scalability that make them hard to use for large-scale systems. The difficulties come from both computational complexity and from the absence of distributed structure in the resulting controllers. The complexity can be traced back to the fact that even stability verification of a linear system with n states generally requires a Lyapunov function involving n^2 quadratic terms. This is true even if the system matrices are sparse. However, the situation improves drastically if we restrict our attention to system matrices with nonnegative off-diagonal entries. Then stability and performance can be verified using a Lyapunov function with only n linear terms. Sparsity can be exploited in performance verification and even synthesis of distributed controllers can be done with a complexity that grows linearly with the number of nonzero entries in the system matrices. These basic observations have far-reaching implications:

(1) The essential mathematical mechanism extends beyond system matrices with nonnegative off-diagonal entries. A sufficient assumption is that the transfer functions involved are "positively dominated".
(2) The desired structure appears naturally in many important application areas, such as mechanical systems, economics, transportation networks, power systems and biology.
(3) In control applications, the condition on positive dominance need not apply to the open loop process. Instead, a large-scale control system can often be structured into local control loops that give positive dominance, thereby enabling scalable methods for optimization of the global performance.

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Flatness of driftless systems equivalent to the canonical Cartan distributions

Witold Respondek
(joint work with Shun-Jie Li)

Our talk is devoted to flatness of control systems and based on two recent publications [6],[7] of Shun-Jie Li and the author. The notion of flatness was introduced in control theory around 1990 by Fliess, Lévine, Martin and Rouchon [1],[2] (see also [3],[8],[12]). Roughly speaking, a control system of the form

$$\Xi \ : \ \dot{x} = f(x,u),$$

$x \in M$, a smooth manifold (state space), and $u \in \mathbb{R}^m$, is flat if there exist m functions (of the state, controls, and their time derivatives) that allow to represent the whole state and all controls via differentiation only. More precisely:

Definition *A smooth control system* $\Xi : \ \dot{x} = f(x,u)$, *with* m *controls, is locally flat if there exit* m *functions* $h_i(x, u, \ldots, u^{(r)})$, *called flat outputs, such that*

$$x = \phi(h, \ldots, h^{(s)})$$
$$u = \psi(h, \ldots, h^{(s)}),$$

where $h = (h_1, \ldots, h_m)$, *holds locally for some smooth maps* ϕ *and* ψ. *If* $h_i = h_i(x)$, *then we say that the system is* x*-flat.*

In other words, a system is flat if it is locally linearizable via a (dynamic) endogenous invertible feedback, that is, it has, locally, the same trajectories as a linear controllable system.

In the first part of the talk, we deal with two-input driftless (equivalently, control-linear) systems

$$\Sigma : \quad \dot{x} = f_1(x)u_1 + f_2(x)u_2,$$

on an $(n+2)$-dimensional state-space M and $u = (u_1, u_2)^\top \in \mathbb{R}^2$. To the system Σ, we associate the distribution $\mathcal{D}$ spanned by the vector fields f_1, f_2, i.e.,

$$\mathcal{D} = \operatorname{span}\{f_1, f_2\}.$$

The *derived flag* of $\mathcal{D}$ is the sequence of distributions defined inductively by

$$\mathcal{D}^{(0)} = \mathcal{D} \quad \text{and} \quad \mathcal{D}^{(i+1)} = \mathcal{D}^{(i)} + [\mathcal{D}^{(i)}, \mathcal{D}^{(i)}], \quad \text{for} \quad i \geq 0.$$

The characteristic subdistribution $\mathcal{C}$ of $\mathcal{D}$ is $\mathcal{C} = \{f \in \mathcal{D} \ : \ [f, \mathcal{D}] \subset \mathcal{D}\}$ and is always involutive.

For two-input driftless sytems the three following properties are closely related: flatness, the condition $\operatorname{rank} \mathcal{D}^{(i)} = i+2$, for $0 \leq i \leq n$, and feedback equivalence to the chained form

$$\dot{z}_0 = v_0, \quad \dot{z}^j = z^{j+1} v_0, \quad \dot{z}^n = v_1$$

where $(z_0, z^0, \ldots, z^n)$ are coordinates on $\mathbb{R}^{n+2}$ and $0 \leq j \leq n-1$.

It is easy to see that the chained form is x-flat with flat outputs chosen as $h = (h_1, h_2) = (z_0, z^0)$ and provided that the control $v_0 \neq 0$. The converse implication that flatness implies the chained form was proved by Martin and Rouchon [9] to hold almost everywhere on M. It is is a classical result, going back to von Weber (1898), Cartan (1914), and Goursat (1923), that $\operatorname{rank}\mathcal{D}^{(i)} = i+2, \quad 0 \le i \le n$, implies the chained form, called also Goursat normal form or Cartan distribution for curves in $\mathbb{R}$. Finally, Shun-Jie Li and the author proved [7] that x-flatness and equivalence to the chained from coincide around any point of M (and not only generically).

Knowing that only 2-input driftless systems equivalent to the chained form are flat, we answer the following natural question: when does a pair of two functions on M form flat outputs?

Theorem *Consider a 2-input driftless control system Σ feedback equivalent to the chained form everywhere on M. A pair of smooth functions (φ_1, φ_2) is an x-flat output of Σ at x_0 if and only if the following conditions hold:*

(i) $\mathrm{d}\varphi_1$ *and* $\mathrm{d}\varphi_2$ *are independent at* x_0;
(ii) $\mathcal{L} = (\operatorname{span}\{\mathrm{d}\varphi_1, \mathrm{d}\varphi_2\})^{\perp} \subset \mathcal{D}^{\mathrm{n}-1}$ in $\mathcal{M}$;
(iii) $\mathcal{D}(x_0)$ is not contained in $\mathcal{L}(x_0)$.

The conditions of the theorem are verifiable, i.e., we can easily verify whether for a pair of functions (φ_1, φ_2) on M forms an x-flat output of a given system and verification involves derivations and algebraic operations only (without solving PDE's or bringing the system into a normal form).

In the second part of the talk, we propose a kinematic model of a system moving in an $(m+1)$-dimensional Euclidean space and consisting of n rigid bars attached successively to each other and subject to the nonholonomic constraints that the instantaneous velocity of the source point of each bar is parallel to that bar [6]. The n-bar system is a natural generalization of the n-trailer system [4],[5]. We prove that the associated control system is controllable and feedback equivalent to the m-chained form around any *regular configuration*, that is, provided that the angles of any two consecutive bars are not $\pm\pi/2$. The m-chain form is the following control system

$$\dot{z}_0 = v_0, \quad \dot{z}_i^j = z_i^{j+1} u_0, \quad \dot{z}_i^n = v_i,$$

where $1 \le i \le m$, $0 \le j \le n-1$ and (z_0, z_i^j) are coordinates on $\mathbb{R}^{m(n+1)+1}$. As a consequence, we deduce that the n-bar system is flat at any regular configuration and show that the Cartesian position of the source point of the last (from the top) bar is a flat output.

The m-chained form is a natural generalization of the chained form (in which we have replaced one chain of length $n+1$ by m such chains) and the geometry of both classes exhibits many similarities. Indeed, the inclusions $\mathcal{D}^{(i)} \subset \mathcal{D}^{(i+1)}$ are of corank m for the m-chain form and of corank one for the chained form while the inclusions $\mathcal{C}_i \subset \mathcal{D}^{(i-1)}$ are of corank one for both classes (where $\mathcal{C}_i$ is the charactersitic subdistribution of $\mathcal{D}^{(i)}$):

$$\begin{array}{ccccccccccc} \mathcal{D}^{(0)} & \subset & \mathcal{D}^{(1)} & \subset & \cdots & \subset & \mathcal{D}^{(n-2)} & \subset & \mathcal{D}^{(n-1)} & \subset & \mathcal{D}^{(n)} = TM \\ \cup & & \cup & & & & \cup & & \cup & & \\ \mathcal{C}_1 & \subset & \mathcal{C}_2 & \subset & \cdots & \subset & \mathcal{C}_{n-1} & \subset & \mathcal{L} & & \end{array}$$

On the other hand, there is a substantial difference concerning the involutive subdistribution $\mathcal{L} \subset \mathcal{D}^{(n-1)}$ of corank one. For the m-chained form, $m \geq 2$, it is unique and thus it determines minimal flat outputs uniquely (up to a diffeomorphism), see [6] for a detailed analysis. For the chained form, $\mathcal{D}^{(n-1)}$ contains many involutive subdistributions $\mathcal{L}$ (although none of them is canonical) and any involutive subdistribution $\mathcal{L}$ of $\mathcal{D}^{(n-1)}$ gives a pair of flat outputs, as described by the theorem above.

For the chained and m-chained forms, we provide also systems of partial differential equations to be solved in order to find all flat outputs.

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Stability and stabilization of piezoeletric beams

Jacquelien M. A. Scherpen
(joint work with Thomas Voß)

1. Introduction

We analyze the finite dimensional dynamics of a piezoelectric beam in the port-Hamiltonian (pH) framework. These dynamics have been previously derived using a structure preserving spatial discretization scheme [2], see [7, 8]. We show that although the method proposed in [2] yields a finite dimensional pH system, it is not guaranteed that this model can then be used for the design of a controller because the system may not satisfy a necessary condition for stabilization. It turns out that the system is not stabilizable due to modeling simplifications made for the infinite dimensional system. Two different type of simplifications are analyzed, as well as the model without these simplifications, see [10]. .

The pH framework was originally developed for modeling finite dimensional systems, but was later on extended to the case of infinite dimensional systems as shown in [3, 4]. For more details we refer the interested reader to [1]. Here we consider smooth spatially discretized pH systems with neglectable dissipation that takes the following form,

$$
\begin{aligned}
(1) \qquad \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + B_{int}u_{int} + B_{ext}u_{ext} \\
y_{int} &= B_{int}^{\top}\frac{\partial H}{\partial x}(x) + D_{int}u_{int}, \quad y_{ext} = B_{ext}^{\top}\frac{\partial H}{\partial x}(x) + D_{ext}u_{ext}
\end{aligned}
$$

where $x = (x_1, \ldots, x_n)$ are the local coordinates of an n-dimensional state space manifold $\mathcal{X}$, $u_{int} \in \mathbb{R}^m$ and $y_{int} \in \mathbb{R}^m$ are the inputs and outputs corresponding to the boundary ports, and $u_{ext} \in \mathbb{R}^p$ and $y_{ext} \in \mathbb{R}^p$ are the distributed inputs and outputs corresponding to the distributed ports. $J(x) : \mathcal{X} \to \mathbb{R}^{n\times n}$ is the smooth skew-symmetric interconnection matrix. $B_{int}(x) : \mathcal{X} \to \mathbb{R}^{n\times m}$ and $B_{ext}(x) : \mathcal{X} \to \mathbb{R}^{n\times p}$ are the smooth input matrices, and $H(x) : \mathcal{X} \to \mathbb{R}$ with $H(x) > c > -\infty \, \forall x \in \mathcal{X}$ is the smooth so called Hamiltonian of the system, $H(x)$ represents the stored energy in the system. Interconnection of two finite dimensional pH systems yields again a finite dimensional pH system. This property can be exploited for finite dimensional control design.

The model we use was derived in [7, 8] We first modeled the infinite dimensional dynamics of the piezoelectric beam in the pH framework [8] and then we used the method proposed in [2] to spatially discretize an infinite dimensional nonlinear piezoelectric Timoshenko beam while preserving the pH structure [7]. Different to the very simple system in [2] the model in [7] consists of 8 states and has a non constant interconnection structure.

We consider a piezoelectric composite beam which consists of a base layer to which a piezoelectric layer is bonded. We assume that the base layer has a constant thickness and a constant height, while its length is L. Each side of the piezoelectric layer in the z_1z_2 plane is covered by an electrode to which a homogeneous voltage distribution is applied. The voltage distribution will generate an electrical field between the electrodes. Hence, due to the piezoelectric properties, the material will deform. This electrical field can be controlled and thus we can also control the shape of the piezoelectric beam.

Following the spatial discretization scheme proposed in [2] we first divide a patch of the beam with one piezoelectric element, which is described in the interval $Z = [0, L]$, into n subintervals. On each of these n subintervals, e.g., $Z_{ab} = [a, b]$ with $0 \leq a < b \leq L$, we spatially discretize the dynamics, resulting in a finite dimensional approximation for the inifinite dimensional dynamics of our piezoelectric composite on the interval Z_{ab}, for more details see [7]. Then, these n finite dimensional pH models are interconnected in a physical way via the boundary ports u_{int}, y_{int}. The interconnected model then approximates the dynamics of the total piezoelectric beam on the interval Z.

2. Discretized models and stabilizability

First we consider the model of the piezoelectric Timoshenko beam with a quasi-static electrical field, a standard assumption in engineering, see [5], i.e., the magnetic coupling between the mechanical and the electrical domain is neglected.

Important physical quantities are $[p_u, p_w, p_\phi, u', w', \phi, \phi', E]$ on the interval Z_{ab}. Here ϕ is the angle of the deformation, u the horizontal displacement, and w the vertical displacement. The p_i, $i \in \{u, w, \phi\}$ are the momenta in the u, w, and ϕ direction, u', w', ϕ, and ϕ' are strain parameters ε, where the prime operator stands for $x' = \frac{\partial x}{\partial z}$, where z is the relevant spatial coordinate. Finally, E is the electrical field generated between the two electrodes. The energy function H is given by $H = \frac{1}{2} p^\top M^{-1} p + \frac{1}{2} \varepsilon^\top \mathbf{C}(\varepsilon) \varepsilon + \frac{1}{2} \epsilon^e E^2$, where where $\mathbf{M}$ is the matrix of the beam, $\mathbf{C}$ is a nonlinear smooth positive definite matrix which relates the stresses and the strains in the system and ϵ^e is the permittivity of the piezoelectric material.

Using the above described physical quantities as states, we have a constant interconnection (Dirac) structure, and hence, we can apply the discretization method described above. However, it turns out that the resulting finite dimensional port-Hamiltonian system with constant interconnection structure and 8 states on Z_{ab} does not fulfill the necessary condition for stabilizability (see e.g., Proposition 4.2.14 from [6]) by the distributed inputs. This necessary condition is easily checkable by checking the rank of $J(x)$ and B_{ext}, and stems from Brockett's necessary condition. The reason for that may be the dependency of two state variables, namely ϕ and ϕ'.

In fact, the strains of an infinite dimensional nonlinear Timoshenko beam are given by $\varepsilon_{11} = u_0' + \frac{1}{2} (w')^2 - z\phi'$, $\varepsilon_{13} = \frac{1}{2} (w - \phi')$. The strain is parametrized to define the states of the dynamical system. In order to obtain a constant interconnection structure, we we have chosen a linear strain parametrization $\varepsilon := [u_0', w', \phi, \phi']^\top$ which results in a constant infinite dimensional interconnection structure, but yields a finite dimensional system that cannot be stabilized. Another way to parametrize the strains is to define the following nonlinear strain states

$\varepsilon := \left[\varepsilon_{11}^0, \varepsilon_{11}^1, \varepsilon_{13}\right]^\top$ where $\varepsilon_{11}^0 := \left(u_0' + \frac{1}{2}(w')^2\right), \quad \varepsilon_{11}^1 := \phi', \quad \varepsilon_{13} := \frac{1}{2}(w - \phi')$. resulting in a non-constant interconnection structure. To translate this to the finite dimensional case, we define a coordinate change for the 8th order finite dimensional model, resulting in a 7th order finite dimensional model with a non-constant interconnection structure on Z_{ab}. It is readily checked that the system on Z_{ab} does fulfill the necessary condition for stabilizability. However, when the n subsystems are interconnected through the internal ports, so that we obtain a model for $Z = [0, L]$, and taking into account that the input from u_{ext} is the same for all n subsystems, it turns out that the system in Z does *not* fulfill the necessary condition for stabilizability anymore.

A reason for these results could be that the magnetic coupling between the piezoelectric element and the mechanical beam is important for the control, i.e., the coupling may be very small, but on the other hand it may be crucial for the stabilizability Inclusion of the magnetic field in the model is rather straightforward. The mechanical states are the same as described above. For the electrical states we take the flux and charge distribution of the piezoelectrical element. The structure of the interconnection matrix is now such that the rank is 8, different from the previous cases, resulting in an interconnection matrix of the n interconnected systems on Z with rank $8n$, and thus the necessary condition for stabilizability is fulfilled. In [9] a energy shaping and damping injection shape controller is developed.

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A novel Lyapunov function for kick-synchronization models

Rodolphe Sepulchre
(joint work with Alexandre Mauroy)

We consider a continuum of phase oscillators on the circle interacting through an impulsive instantaneous coupling. In contrast with previous studies on related pulse-coupled models, the stability results obtained in the continuum limit are global. For the nonlinear transport equation governing the evolution of the oscillators, we propose (under technical assumptions) a global Lyapunov function which is induced by a total variation distance between quantile densities. The monotone time evolution of the Lyapunov function completely characterizes the dichotomic behavior of the oscillators: either the oscillators converge in finite time to a synchronous state or they asymptotically converge to an asynchronous state uniformly spread on the circle. The results of the present paper apply to popular phase oscillators models (e.g. the well-known leaky integrate-and-fire model) and draw a strong parallel between the analysis of finite and infinite populations. In addition, they provide a novel approach for the (global) analysis of pulse-coupled oscillators.

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Circulant and Pseudo-circulant Control Systems

Frederike Rüppel
(joint work with Uwe Helmke)

Linear dynamical systems with circulant interconnection structures have been investigated quite often in the control literature; mainly in connections with consensus algorithms; see e.g. [3]. More general interconnection structures such as rings, chains or trees were also investigated by [2], with applications to e.g. multiagent systems and cellular chemistry. These structures are often of a Toeplitz-form. Before analyzing the controllability properties of Toeplitz formations we consider the much easier case of circulant formations. Since circulant systems are never controllable, this motivates to consider more general classes of systems, defined by pseudo-circulants or Toeplitz matrices.
Consider now the task of characterizing the reachable sets of bilinear control systems on $\mathbb{C}^n$

$$\dot{x} = \left(\sum_{j=1}^{d} u_j C_j\right) x, \tag{1}$$

defined by circulant matrices $C_1, \ldots, C_d \in \mathbb{C}^{n\times n}$, $d \leq n$. Without loss of generality we can assume that the circulant matrices $C_j := \mathrm{Circ}(c_0^{(j)}, ..., c_{n-1}^{(j)})$ are linearly

independent. Let $p_j(z) = \sum_{k=0}^{n-1} c_k^{(j)} z^k$ denote the corresponding generating polynomials, $j = 1, \dots, d$, and let V_d denote the vector space of complex polynomials spanned by $p_1, \dots, p_d$. Identify a vector of coefficients $x = (x_0, \dots, x_{n-1})^\top \in \mathbb{C}^n$ with the associated polynomial $\pi_x(z) := \sum_{k=0}^{n-1} x_k z^k$.
Any element in the system Lie algebra of (1) is of the form $\Phi p(\Omega)\Phi^*$, where p runs through the elements of V_d and Ω is a diagonal matrix. Therefore, the elements of the system Lie group are exactly $\Phi e^{p(\Omega)}\Phi^*$, where $p \in V_d$. Thus, for $x \in \mathbb{C}^n$, the reachable sets of (1) are

$$\mathcal{R}(x) = \{\Phi e^{p(\Omega)}\Phi^* x \mid p \in V_d\}.$$

We conclude

Theorem 1. (1) *The circulant control system* (1) *is never controllable on* $\mathbb{C}^n$.
(2) *For $d = n$ there is a unique reachable set $\mathcal{R}(x)$ that is dense in $\mathbb{C}^n$. $\mathcal{R}(x)$ is characterized by $\pi_x(\omega^j) \neq 0$ for $j = 0, \dots, n-1$.*
(3) *For $d < n$ all reachable sets have empty interior.*

Before presenting our main results, we begin with a reformulation of general controllability results by Gauthier and Bornard [1] and Silva-Leite and Crouch [4] for arbitrary bilinear control systems on the Lie group of complex invertible $n \times n$ matrices $GL_n(\mathbb{C})$

$$\dot{X} = \left(\sum_{j=1}^{d} u_j A_j\right) X, \quad X(0) = I_n.$$

Theorem 2 (Gauthier, Bonnard; Silva-Leite, Crouch)**.** *Suppose that complex matrices $A_1, \dots, A_d$ (or complex skew-Hermitian matrices $A_1, \dots, A_d$, respectively,) satisfy*

(1) *There exist $u_1, \dots, u_d \in \mathbb{C}$ (or $u_1, \dots, u_d \in \mathbb{R}$, respectively,) such that $\sum_{j=1}^{d} u_j A_j$ is strongly regular.*
(2) *$A_1, \dots, A_d$ possess no non-trivial common invariant subspace $V \subset \mathbb{C}^n$.*

Then the system Lie algebra $\mathcal{L}(A_1, \dots, A_d)$ is either equal to $sl_n(\mathbb{C})$ or equal to $gl_n(\mathbb{C})$ (or equal to $su_n(\mathbb{C})$ or $u_n(\mathbb{C})$, respectively).

One can show that there exists $\lambda \in \mathbb{C}^n$ such that C_λ is strongly regular if and only if the first row vector of C_λ lies in $\mathbb{C}^n - W$. Here W denotes the subspace $W := \bigcup_{ijkl} \ker\left(I_n - \Omega(\omega^{j-i}) + \Omega(\omega^{l-i}) - \Omega(\omega^{k-i})\right)$ of $\mathbb{C}^n$ with $(i,j) \neq (k,l), i < j, k \neq l$. Hence one can use Theorem 2 for proving the following theorem:

Theorem 3. *Let $c \in \mathbb{C}^n - W$, $c_0 \neq 0$. Then the bilinear control system on $GL_n(\mathbb{C})$*

$$\dot{X} = \mathrm{Circ}_{u(t)}(c_0, \dots, c_{n-1}) X$$

is accessible.

By Theorem 3, the Lie algebra generated by pseudo-circulant matrices is equal to the full matrix Lie algebra $\mathbb{C}^{n \times n}$. Therefore, the products of exponentials of pseudo-circulant matrices generate $GL_n(\mathbb{C})$. As every matrix exponential of a pseudo-circulant is an invertible pseudo-circulant we obtain:

Corollary 4. *$GL_n(\mathbb{C})$ is the smallest Lie group that contains all invertible pseudo-circulant matrices. In particular, the discrete-time pseudo-circulant control system*

$$X_{t+1} = \mathrm{Circ}_{u_n(t)}(u_0(t), \ldots, u_{n-1}(t))X_t, \quad X_0 = I_n,$$

is controllable on $GL_n(\mathbb{C})$. Here $u_0(t), \ldots, u_n(t)$ denote arbitrary control sequences such that $\sum_{j=0}^{n-1} u_j z^j \neq 0$ on all $n-th$ roots of $u_n \neq 0$.

Since the elements of the reachable sets of a symmetric bilinear control system on a Lie group are always reached in finite time, this has an interesting consequence in Linear Algebra.

Corollary 5. *Any complex invertible matrix is a finite product of invertible pseudo-circulant matrices.*

Since pseudo-circulant matrices being special Toeplitz matrices, we obtain similar results for invertible Toeplitz matrices and Toeplitz copntrol systems:

Corollary 6. *Any complex invertible matrix is a finite product of invertible Toeplitz matrices.*

Theorem 7. *The Lie algebra generated by $n \times n$ complex Toeplitz matrices is equal to the full matrix Lie algebra $\mathbb{C}^{n \times n}$. In particularly, the bilinear Toeplitz control system*

$$\dot{X} = \left(\sum_{k=-(n-1)}^{n-1} u_k T_k \right) X$$

is controllable on $GL_n(\mathbb{C})$.

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Synchronization in networks of linear parameter-varying systems

Georg S. Seyboth

(joint work with Frank Allgöwer)

We study synchronization problems in heterogeneous networks of linear parameter-varying (LPV) dynamical systems. The heterogeneity is described by local time-varying parameters in the subsystems. The agents in the network are given as

$$\begin{aligned} \dot{x}_k &= A(\gamma(t), \lambda_k(t))x_k + B(\gamma(t), \lambda_k(t))u_k \\ y_k &= Cx_k, \end{aligned} \tag{1}$$

where $x_k(t) \in \mathbb{R}^n$ is the state, $y_k(t) \in \mathbb{R}^p$ is the output and $u_k(t) \in \mathbb{R}^q$ the input of agent k, $k = 1, ..., N$. The time-varying parameters $\gamma(t)$ and $\lambda_k(t)$ are assumed to be available as real-time measurements. In particular, agent k has access to the global parameter $\gamma(t)$ and its own local parameter $\lambda_k(t)$.

Synchronization problem: For a given set of N agents (1), find dynamic LPV controllers such that the outputs y_k and ζ_k synchronize, i.e., that for all $k, j = 1, ..., N$, $(y_k - y_j) \to 0$ and $(\zeta_k - \zeta_j) \to 0$ exponentially as $t \to \infty$.

The situation is illustrated in Fig. 1. As indicated, each agent has a local

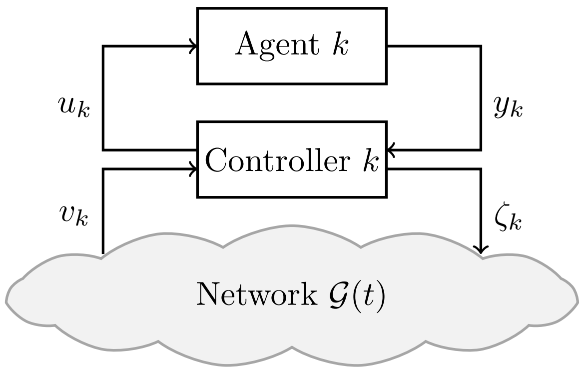

FIGURE 1. Network of agents (1) and local controllers, interconnected with graph $\mathcal{G}(t)$.

controller which has access to y_k and may communicate with neighbors in the network according to a time-varying communication graph $\mathcal{G}(t)$.

We propose a solution for the synchronization problem and show that concepts from synchronization in heterogeneous networks of linear time-invariant systems [1] can be adapted to the more general framework of LPV systems.

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Classical Network Synthesis

Malcolm C. Smith

The talk consists of four parts. In the first part, the motivation is explained for revisiting certain questions in circuit theory, in particular, the synthesis theory of "one-port" (i.e. with a pair of external driving terminals) RLC networks and the questions of minimality associated with the Bott-Duffin procedure. The motivation relates to the synthesis of passive mechanical impedances and the need for a new ideal modelling element the "inerter" [7]. The development of the inerter from a mathematical concept and ideal modelling element through to its adoption as a standard component in Formula One racing and beyond is described [2].

In the second part, classical results from electrical circuit synthesis are reviewed including the procedures of Foster, Cauer, Brune, Darlington, Bott and Duffin. The reactance theorem of Foster [3] for lossless networks and the Bott-Duffin construction [1] for arbitrary positive-real functions are highlighted.

In the third part, the concept of regular positive-real functions is described. A positive-real function $Z(s)$ is defined to be *regular* if the smallest value of $\mathrm{Re}\,(Z(j\omega))$ or $\mathrm{Re}\,\big(Z^{-1}(j\omega)\big)$ occurs at $\omega = 0$ or $\omega = \infty$. It is shown how the concept can aid the classification of low-complexity networks. A generating set for the two-reactive/three-resistive-element networks is described [4].

The fourth part presents a reworking and amplification [5] of the proof of the theorem of Reichert [6] which states that any driving-point impedance which may be realised using two reactive elements and an arbitrary number of resistors can be realised using two reactive elements and three resistors.

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Universal regularity results for open-loop optimal controls

Héctor J. Sussmann

A "universal regularity result" for a class of problems is a theorem that says that for all problems in the class, whenever a solution exists, then a solution having certain regularity properties exists. Here we consider optimal control problems of the Lagrange type, with a given state space which is a real analytic manifold, a control set which is a compact subanalytic subset of some other real analytic manifold, a dynamics which is real analytic in the state and the control, a Lagrangian that is also real analytic in the state and the control, and fixed initial and terminal states. For such problems, the universal regularity theorem says that whenever an optimal open-loop control exists, then there exists one which is real analytic on an open dense subset of its interval of definition. The proof is a construction by "induction" on the dimension of the control set, where the *dimension* of a compact subanalytic set U is defined as follows: let (V, f) be a "desingularization" of U. (This means that V is a compact manifold which is a finite union of tori, possibly of different dimensions, and $f : V \mapsto U$ is a real analytic surjective map. Such a pair (V, f) exists by Hironaka's desingularization theorem.) Let $\mathbf{n} = (n_0, n_1, n_2, \ldots)$ be the sequence such that n_k is the number of k-dimensional connected components of V. Then the *dimension* of U is the minimum—with respect to the lexicographical ordering—of all sequences $\mathbf{n}$ arising from desingularizations of U. Induction is carried out with respect to the lexicographical ordering. The inductive step consists, roughly, of using the Pontryagin Maximum Principle to conclude that every optimal trajectory is actually an optimal trajectory of a problem with a smaller U. In the lecture, the much simpler case of a Lagrangian equal to 1 (minimum time control), a control set U equal to the interval $[-1, 1]$, and a dynamics which is affine linear in the control, was considered. In that case, the regularity theorem can be proved by successively differentiating the switching function.

From Gossip to Voting

Patrick Thiran
(joint work with Florence Bénézit, Martin Vetterli)

An increasingly larger number of applications require networks to perform decentralized computations over distributed data. A representative problem of these "in-network processing" tasks is the distributed computation of the average of values present at nodes of a network, known as gossip algorithms. They have received recently significant attention across different communities (networking, algorithms, signal processing, control) because they constitute simple and robust methods for distributed information processing over networks.

The first part of the talk is a survey some recent results on these stochastic, linear, discrete-time dynamical systems. The conditions for convergence are well established, but the convergence time or cost (number of message exchanges) is more challenging to estimate. Classical nearest-neighbor gossip algorithms are slow, but a variation of these algorithms can be proven to order optimal (cost of $O(n)$ messages for a network of n nodes) for some random geometric graphs. The reader is referred to [1, 2, 3] for more details on "analog" gossip algorithms.

The second part of the talk is devoted to quantized gossip on arbitrary connected networks. By their nature, quantized algorithms cannot produce a real, analog average, but they can (almost surely) reach consensus on the quantized interval that contains the average, in finite time.

The *interval consensus* problem [4, 7] can be described as follows. At time $t = 0$, n nodes measure some quantized values $(x_1[0], x_2[0], \ldots, x_n[0])$, where $\mathbb{R}$ has been uniformly quantized with step δ. We denote by x_{ave} the average of the n measurements:

$$x_{ave} = \frac{1}{n} \sum_{i=1}^{n} x_i[0].$$

The nodes can communicate through a connected network $\mathcal{G}$ and we are given an ordered subset of quantization levels $\Theta_1 < \Theta_2 < \ldots < \Theta_r$ called thresholds.

The goal is to design a quantized distributed algorithm such that nodes can tell whether the average x_{ave} is smaller than Θ_1, or between Θ_1 and Θ_2, ..., or larger than Θ_r. At each step of the algorithm, nodes can store a limited number of bits as their current state, and neighboring nodes can exchange their states. Based on their final state, all the nodes should reach a consensus on the interval $[\Theta_i, \Theta_{i+1}]$ which contains x_{ave}. To simplify, we assume that x_{ave} cannot be threshold level ($x_{ave} \neq \Theta$). A particular instance of this problem, where states are quantized on two values, is the voting problem: nodes initially vote for $x = 1$ or $x = 0$, and they want to know the majority opinion. The interval consensus parameters are therefore $\delta = 0.5$, $\Theta = 0.5$, and 3 quantization levels: $\{0, 0.5, 1\}$, n odd.

Just as in [5], our algorithm is a quantized version of the pairwise gossip algorithm. Similarly to gossip, at the beginning of every round of our algorithm, an undirected edge of the communication network is randomly selected and its two end-nodes exchange their states. We denote by p_e the probability that edge e is chosen. The most common way of selecting edges is to assign a random exponential clock to each node. When their clock activates, nodes wake up and choose a neighbor uniformly at random among their neighbors. In that setting, if i and j are neighbors, edge $e = (i, j)$ is chosen with positive probability $p_e = (1/nd_i)+(1/nd_j)$, where d_i and d_j are the degrees of nodes i and j.

In pairwise gossip, nodes update their states to the average of the two states. By iterating this update rule over the successively chosen edges, all the states progressively converge to the average of the initial states. Our goal is to modify this simple averaging rule so that the states are quantized and so that the nodes reach an interval consensus.

In order to achieve our goal, we assign *two states* to each threshold level Θ while all the other quantization levels are represented by one state only. An ordinary state will be denoted by its quantization value. The two states with threshold value Θ are distinguished by Θ^- and Θ^+. We order the set of states: if the quantization level of state x is smaller than the quantization level of state y, we write $x \prec y$. Also, for any threshold level Θ, we adopt the following convention: $\Theta^- \prec \Theta^+$. As a result, for example, the voting problem functions with 4 states, coded by the two bits we announced. The four states are ordered: $0 \prec 0.5^- \prec 0.5^+ \prec 1$. We adopt all the natural ordering vocabulary, which we adapt to $\prec$: min, max, $=$, $\preceq$, $\succ$, $\succeq$. In particular the notion of *consecutive states* is crucial. In previous example, 0 and 0.5^- are consecutive states. So are 0.5^- and 0.5^+. But 0 and 0.5^+ are *not* consecutive states.

A quantized gossip algorithm has converged if and only if all the nodes have either equal or consecutive states. In other words, an algorithm has converged when there are two consecutive states x and y such that every state in the network is equal to x or y. Suppose that we have run a converging algorithm that *preserves average*, and that the average x_{ave} is in $[\Theta_1, \Theta_2]$, then necessarily, the two converging states are Θ_1^+, Θ_2^- or quantized levels between these two. Individually, each node knows in which interval x_{ave} is. Even a node with state Θ_1^+ makes the correct decision, because the + sign tells it that $x_{ave} \geq \Theta_1$. Thanks to the threshold state splitting, we are able to locally decide common intervals. To simplify the wording, we say that state x has color C, if a node that has state s decides that interval C contains x_{ave}. For example in the voting problem, states 0 and 0.5^- have color 0 (they lead to the conclusion that 0 wins), and states 1 and 0.5^+ have color 1 (they lead to the conclusion that 1 wins).

The following properties of the interval consensus algorithm are sufficient for the algorithm to converge, but not necessary. If nodes i and j are activated at time t:

- **Conservation property** The average should be preserved: $x_i[t+1] + x_j[t+1] = x_i[t] + x_j[t]$.
- **Contraction property** $x_i[t+1]$ and $x_j[t+1]$ should be either equal or consecutive states. Furthermore, if $x_i[t] = x_j[t]$ then $x_i[t+1] = x_j[t+1] = x_i[t] = x_j[t]$.
- **Mixing property** If $x_i[t] \preceq x_j[t]$, then $x_i[t+1] \succeq x_j[t+1]$. In particular, if $x_i[t]$ and $x_j[t]$ are consecutive states, then states are swapped: $x_i[t+1] = x_j[t]$ and $x_j[t+1] = x_i[t]$.

The algorithm in [5] has similar properties, which we have adapted to our setting.

We can now explicit the interval consensus algorithm. To simplify, we consider that quantization levels are centered at 0, i.e. they can be written $k\delta$, with k integer. At time t, an edge is randomly chosen. We denote by i the activated node with smaller state and by j the other activated node: $x_i[t] \preceq x_j[t]$. Nodes i and j update their states according to the following rules:

$$\begin{aligned} x_i[t+1] &= \left\lceil \frac{x_i[t] + x_j[t]}{2\delta} \right\rceil \delta \\ x_j[t+1] &= \left\lfloor \frac{x_i[t] + x_j[t]}{2\delta} \right\rfloor \delta. \end{aligned}$$

When $x_i[t+1]$ or $x_j[t+1]$ is equal to a threshold value Θ, we need to specify whether they are equal to Θ^+ or Θ^-. There are 4 cases:

- If $x_i[t] = x_j[t]$, then $x_i[t+1] = x_j[t+1] = x_i[t]$.
- If $x_i[t] \neq x_j[t]$ and $x_i[t+1] = x_j[t+1] = \Theta$, then $x_i[t+1] = \Theta^+$ and $x_j[t+1] = \Theta^-$.
- If only $x_i[t+1] = \Theta$, then $x_i[t+1] = \Theta^-$.
- If only $x_j[t+1] = \Theta$, then $x_j[t+1] = \Theta^+$.

It is easy to check that this algorithm conserves averages through the iterations, contracts and mixes states. By the conservation property, if the algorithm converges, then the network reaches consensus on the interval containing x_{ave}. In addition, we can prove that our algorithm converges in finite time with probability 1 [4, 7]:

Theorem 1. *Let T be the first time the algorithm has converged. If the updating rules have the conservation, contraction and mixing properties, $\mathbb{P}[T < \infty] = 1$.*

An upper bound on $\mathbb{E}[T]$ has been computed in [6].

The interval consensus problem can be extended to multiple voting: a finite set $\mathcal{V}$ of agents is connected in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which is possibly time-varying. Each agent initially votes for a "color" among a finite set of colors. By means of local communications only, and using constant, identical and simple updating rules, the agents want to distributively reach a state of consensus indicating the initial majority color. We have defined pairwise asynchronous graph automata (PAGA), for binary (which is given above), ternary and quaternary voting (which are described in [7]).

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Switched differential algebraic equations

Stephan Trenn

Mathematical modeling of physical (or other) systems naturally leads to a combination of differential equations and algebraic constraints. For analysis purposes the algebraic equations are often resolved and plugged into the differential equations to obtain an explicit ordinary differential equation (ODE). However, in the presence of switches or sudden component faults the algebraic equations may depend on the mode of the system and the corresponding ODEs for each mode might not be compatible anymore to obtain an overall system description. For this reason it is necessary (and in many other situations possibly advantageous) to study switched systems where each mode is described by a differential algebraic equation (DAE):

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \\ y(t) &= C_{\sigma(t)}x(t). \end{aligned} \tag{1}$$

In contrast to switched ODEs, solutions of (1) may contain intrinsic jumps or even Dirac impulses. This introduces some mathematical difficulties to interpret (1) because the product of a piecewise-constant function with a Dirac impulse (and its derivatives) has to be defined. It is not possible to define such a multiplication consistently for general distributions; however, if one restricts oneself to the smaller space of *piecewise-smooth distributions* [1] one can utilize the *Fuchssteiner multiplication* [2, 3] in order to interpret (1) as an equation in the space of piecewise-smooth distributions. Within this solution framework typical system theoretical questions can be studied. There are already some promising results like characterization of existence and uniqueness of solutions [4], stability analysis via Lyapunov functions [5, 6, 7], impulse detection [8], and observability [9]. However, there are still many open questions like characterization of controllability, observer and controller design as well as possible extensions to the nonlinear case.

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Robustness and adaptation of biological networks under kinetic perturbations

Steffen Waldherr

(joint work with Frank Allgöwer, Elling W. Jacobsen, Stefan Streif)

We study biochemical reaction networks described by the ODE $\dot{x} = Nv(x)$ where $x \in \mathbb{R}^n$ is the concentration vector, $N \in \mathbb{R}^{n \times m}$ the stoichiometric matrix, and $v(x) \in \mathbb{R}^m$ the reaction rate vector. The network is assumed to have a steady state $\bar{x}$ with $Nv(\bar{x}) = 0$.

A kinetic perturbation is defined as a change from $v(x)$ to $\tilde{v}(x)$ such that $\tilde{v}(\bar{x}) = v(\bar{x})$. Defining a suitable perturbation matrix Δ, the change in the reaction rate

Jacobian at steady state is given by

$$\frac{\partial \bar{v}}{\partial x}(\bar{x}) - \frac{\partial v}{\partial x}(\bar{x}) = \operatorname{diag} v(\bar{x})\Delta(\operatorname{diag} \bar{x})^{-1}. \tag{1}$$

Considering the linear approximation of the network at steady state, we have the Jacobian

$$\tilde{A}(\Delta) = N\frac{\partial v}{\partial x}(\bar{x}) + \operatorname{diag} v(\bar{x})\Delta(\operatorname{diag} \bar{x})^{-1}. \tag{2}$$

In my talk, I discuss first the robustness problem: find Δ such that $\tilde{A}(\Delta)$ has eigenvalues on the imaginary axis.

In the adaptation problem, we study a network given by $\dot{x} = Nv(x, u)$ and $y = Cx$, where u and y are a scalar input and output, respectively, added to the previous network. Let $\bar{u}$ be a stationary input and define $B = N\frac{\partial v}{\partial u}(\bar{x}, \bar{u})$. The adaptation problem is to find Δ such that

$$\det \begin{pmatrix} \tilde{A}(\Delta) & B \\ C & 0 \end{pmatrix} = 0,$$

and $\tilde{A}(\Delta)$ is Hurwitz.

Both the robustness and adaptation problem are solved by robust control techniques.

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Open Stochastic Systems

Jan C. Willems

In this presentation we propose a definition of stochastic system, of a linear stochastic system, of interconnection of stochastic systems, and of constrained probability.

Our aim is the notion of 'open' stochastic system, that is, a model that incorporates the environment as an unmodeled feature. A basic requirement imposed on this concept is that it should accommodate interconnection as an elementary operation on stochastic systems. Interconnection is the central feature of systems thinking. It a plays a crucial role in modeling, in analysis, and in synthesis.

The larger picture underlying this presentation it to put forward a framework that does for stochastic systems what [1] does for deterministic systems.

Definition 1. *A* stochastic system *is a triple* $(\mathbb{W}, \mathfrak{E}, P)$ *with*

- $\mathbb{W}$ *a non-empty set, the* outcome space*, with elements called* outcomes*,*
- $\mathfrak{E}$ *a* σ*-algebra of subsets of* $\mathbb{W}$ *with elements are called* events*,*

- P *a* probability *measure on* $\mathfrak{E}$.

Classical stochastic systems use for $\mathfrak{E}$ the Borel σ-algebra induced by the topology on $\mathbb{W}$. No probabilistic modeling then enters into the specification of the events. In Definition 1 on the other hand, the event space $\mathfrak{E}$ is very much a part of the stochastic model. One of the aims of the presentation is to demonstrate that notions as linearity and interconnection of stochastic systems require coarse σ-algebras and therefore the full generality of Definition 1.

Definition 2. *The stochastic system* $(\mathbb{R}^n, \mathfrak{E}, P)$ *is said to be* linear *if there exists a linear subspace* $\mathbb{L}$ *of* $\mathbb{R}^n$ *such that the events are the Borel subsets of the quotient space* $\mathbb{R}^n/\mathbb{L}$, *and the probability is a Borel probability on* $\mathbb{R}^n/\mathbb{L}$. *Note that* $\mathbb{R}^n/\mathbb{L}$ *is a finite dimensional real vector space, with therefore well-defined Borel sets.* $\mathbb{L}$ *is called the* fiber *and* `dimension`$(\mathbb{L})$ the number of degrees of freedom *of the linear stochastic system* $(\mathbb{R}^n, \mathfrak{E}, P)$.

The stochastic system $(\mathbb{R}^n, \mathfrak{E}, P)$ *is said to be* gaussian *if it is linear and if the Borel probability on* $\mathbb{R}^n/\mathbb{L}$ *is gaussian. We consider a probability measure that is concentrated on a singleton to be gaussian. More generally, a gaussian probability measure may be concentrated on a linear variety.*

In order to deal with system interconnection, we first need to discuss complementarity. Two σ*-algebras* $\mathfrak{E}_1$ *and* $\mathfrak{E}_2$ *on a set* $\mathbb{W}$ *are said to be* complementary *if for all nonempty sets* $E_1, E_1' \in \mathfrak{E}_1, E_2, E_2' \in \mathfrak{E}_2$ *there holds*

$$[E_1 \cap E_2 = E_1' \cap E_2'] \Rightarrow [E_1 = E_1' \text{ and } E_2 = E_2'].$$

The stochastic systems $\Sigma_1 = (\mathbb{W}, \mathfrak{E}_1, P_1)$ *and* $\Sigma_2 = (\mathbb{W}, \mathfrak{E}_2, P_2)$ *are said to be* complementary *if for all* $E_1, E_1' \in \mathfrak{E}_1$ *and* $E_2, E_2' \in \mathfrak{E}_2$ *there holds*

$$[E_1 \cap E_2 = E_1' \cap E_2'] \Rightarrow [P_1(E_1)P_2(E_2) = P_1(E_1')P_2(E_2')].$$

Complementarity of two stochastic systems is implied by complementarity of the associated σ-algebras. It is easy to construct examples involving zero probability events that show that complementarity of two stochastic systems does not imply complementarity of the associated σ-algebras. Complementarity of the event σ-algebras is a more primitive condition that is convenient for proving complementarity of stochastic systems.

Definition 3. *Let* $\Sigma_1 = (\mathbb{W}, \mathfrak{E}_1, P_1)$ *and* $\Sigma_2 = (\mathbb{W}, \mathfrak{E}_2, P_2)$ *be complementary stochastic systems (assumed stochastically independent). Then the* interconnection *of* Σ_1 *and* Σ_2 *is defined as the stochastic system* $(\mathbb{W}, \mathfrak{E}, P)$ *with* $\mathfrak{E}$ *the* σ*-algebra generated by* $\mathfrak{E}_1 \cup \mathfrak{E}_2$, *and the probability* P *defined through 'rectangles'* $\{E_1 \cap E_2 \mid E_1 \in \mathfrak{E}_1, E_2 \in \mathfrak{E}_2\}$ *by*

$$P(E_1 \cap E_2) := P_1(E_1)P_2(E_2).$$

The definition of the probability P for rectangles uses complementarity in an essential way and $\mathfrak{E}$ is in fact the σ-algebra generated by these rectangles. It is readily seen that the class of subsets of $\mathbb{W}$ that consist of the union of a finite number of disjoint rectangles forms an algebra of sets, that is, a class of subsets of $\mathbb{W}$ that is closed under taking the complement, intersection, and union. The probability of rectangles defines the probability on the subsets of $\mathbb{W}$ that consist of a union of a finite number of disjoint rectangles. By the Hahn-Kolmogorov extension theorem, this leads to a unique probability measure P on $\mathfrak{E}$, the σ-algebra generated by the rectangles. This construction of the probability measure P is completely analogous to the construction of a product measure.

Consider the stochastic system $\Sigma = (\mathbb{W}, \mathfrak{E}, P)$. Let $\mathbb{S}$ be a nonempty subset of $\mathbb{W}$ (with typically $\mathbb{S} \notin \mathfrak{E}$). We discuss the meaning of *the stochastic system induced by* Σ *with outcomes constrained to be in* $\mathbb{S}$ by considering the interconnection of Σ with the (deterministic) system $\Sigma' = (\mathbb{W}, \{\emptyset, \mathbb{S}, \mathbb{S}^{\text{complement}}, \mathbb{W}\}, P')$, $P'(\mathbb{S}) = 1$. Complementarity imposes the regularity condition

$$[E_1, E_2 \in \mathfrak{E} \text{ and } E_1 \cap \mathbb{S} = E_2 \cap \mathbb{S}] \Rightarrow [P(E_1) = P(E_2)].$$

Definition 4. *Let* $\Sigma = (\mathbb{W}, \mathcal{E}, P)$ *be a stochastic system and* $\mathbb{S} \subseteq \mathbb{W}$. *Assume that the regularity condition holds. Then the stochastic system*

$$\Sigma|_{\mathbb{S}} := (\mathbb{S}, \mathfrak{E}|_{\mathbb{S}}, P|_{\mathbb{S}})$$

with

$$\mathfrak{E}|_{\mathbb{S}} := \{E' \subseteq \mathbb{S} \mid E' = E \cap \mathbb{S} \textit{ for some } E \in \mathfrak{E}\},$$

and

$$P|_{\mathbb{S}}(E') := P(E) \textit{ with } E \in \mathfrak{E} \textit{ such that } E' = E \cap \mathbb{S},$$

is called the stochastic system Σ with outcomes constrained to be in $\mathbb{S}$*.*

The notion of *the stochastic system* Σ *with outcomes constrained to be in* $\mathbb{S}$, while reminiscent of the notion of *the stochastic system* Σ *conditioned on outcomes in* $\mathbb{S}$, is quite different from it. The former basically requires $\mathbb{S} \notin \mathfrak{E}$, while the latter requires $\mathbb{S} \in \mathfrak{E}$. Secondly, constraining associates with the event $E \in \mathfrak{E}$ of Σ, the event $E \cap \mathbb{S}$ of $\Sigma|_{\mathbb{S}}$ with probability $P(E)$, while conditioning associates with the event $E \in \mathfrak{E}$ of Σ the event $E \cap \mathbb{S}$, also in $\mathfrak{E}$, with probability $P(E \cap \mathbb{S})/P(\mathbb{S})$. So, constraining pulls the probability of E 'globally' into $E \cap \mathbb{S}$, while conditioning associates with E 'locally' the probability of $E \cap \mathbb{S}$, renormalized by dividing by $P(\mathbb{S})$.

More details on the matters discussed in this extended abstract may be found in [2].

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Performance and Design of Cycles in Consensus Networks

Daniel Zelazo
(joint work with Simone Schuler and Frank Allgöwer)

Consensus networks have emerged as a canonical model for studying networked systems. The linear consensus protocol, typically expressed as a collection of single integrators interacting over a communication graph, reveals a deep connection between its dynamic behavior and the underlying properties of the graph. Beyond its analytic elegance, it has also proven to be of important practical relevance for a variety of applications from formation control to distributed computation. From an engineering perspective, there remains a need to consider the *design* of consensus networks in conjunction with a notion of *performance*. This work contributes in that direction by first revealing how certain combinatorial properties of the underlying graph, namely the length of cycles, affect the $\mathcal{H}_2$ performance of the consensus network. This result is then used to motivate a synthesis problem for placing cycles in a graph using an ℓ_1 relaxation method.

1. Cycles, Performance, and the Edge Agreement Problem

We consider a general model of the consensus protocol that includes disturbances at both the process and measurement of the system. The closed-loop representation of this system takes the form

$$(1)\qquad \left\{ \begin{array}{rcl} \dot{x}(t) & = & -L(\mathcal{G})x(t) + \left[\begin{array}{cc} I & -E(\mathcal{G}) \end{array} \right] \left[\begin{array}{c} w(t) \\ v(t) \end{array} \right] \\ z(t) & = & E(\mathcal{G})^T x(t). \end{array} \right.$$

Here, $E(\mathcal{G})$ is the incidence matrix of a graph with arbitrary orientation, and $L(\mathcal{G}) = E(\mathcal{G})E(\mathcal{G})^T$ is the combinatorial graph Laplacian matrix [1]. The controlled variable $z(t)$ is taken to be the difference between neighboring states, and thus captures the notion of "agreement" when $z(t) = 0$.

A straightforward study of the $\mathcal{H}_2$ performance of the consensus system (1) is not possible due to the marginally stable eigenvalue at the origin. However, via a coordinate transformation, one can consider a minimal realization orthogonal to the agreement subspace and study the performance of the minimal system. An important result from [2] showed that there exists a transformation that preserves the combinatorial features encoded in the state matrix, and this leads to what is referred to as the *edge agreement problem*. This transformation is constructed using the *edge Laplacian* matrix, defined as [2]

$$(2)\qquad L_e(\mathcal{G}) \;=\; E(\mathcal{G})^T E(\mathcal{G}).$$

The minimal system can then be expressed as

$$(3)\quad \Sigma_e(\mathcal{G}) \left\{ \begin{array}{l} \dot{x}_{\mathcal{T}}(t) = -L_e^{\mathcal{T}} \mathcal{R}_{\mathcal{T}} \mathcal{R}_{\mathcal{T}}^T x_{\mathcal{T}}(t) + \left[\begin{array}{cc} E(\mathcal{T})^T & -L_e^{\mathcal{T}} \mathcal{R}_{\mathcal{T}} \end{array} \right] \left[\begin{array}{c} w(t) \\ v(t) \end{array} \right] \\ z(t) = x_{\mathcal{T}}(t). \end{array} \right.$$

Here, $L_e^{\mathcal{T}}$ is the edge Laplacian for some spanning graph $\mathcal{T} \subset \mathcal{G}$, and the rows of $\mathcal{R}_{\mathcal{T}} = [I\, T_{\mathcal{T}}]$ form a basis for the *cut-space* of the graph $\mathcal{G}$ [1].

The minimal system (3) is stable, and thus its $\mathcal{H}_2$ performance is well-defined [2]. It is straightforward to arrive at the following result,

Theorem 1 ([2])**.** *The $\mathcal{H}_2$ performance of the edge agreement problem (3) is given as*

$$\|\Sigma_e(\mathcal{G})\|_2^2 \quad = \quad \frac{1}{2}\mathbf{tr}[(\mathcal{R}_{\mathcal{T}}\mathcal{R}_{\mathcal{T}}^T)^{-1}] + n - 1. \tag{4}$$

An immediate consequence of this result is the following corollary, revealing how the addition of a single edge to a spanning tree affects the performance.

Corollary 2 ([3])**.** *The $\mathcal{H}_2$ performance of the edge agreement problem (3) for the graph $\mathcal{G} = \mathcal{T} \cup e$ is*

$$\|\Sigma_e(\mathcal{G})\|_2^2 = \|\Sigma_e(\mathcal{T})\|_2^2 - \frac{1}{2}(1 - l(c)^{-1}), \tag{5}$$

where $l(c)$ is the length of the cycle formed by adding the edge $e \notin \mathcal{T}$ to the spanning tree $\mathcal{T}$.

This result can be used to conclude that *long cycles* are preferable for improving the $\mathcal{H}_2$ performance. These results are extended to also show that *edge disjoint* cycles are also advantageous from an $\mathcal{H}_2$ perspective [3].

2. Design of Consensus Networks

The results of Theorem 1 and Corollary 2 can be used to consider synthesis problems for consensus networks. In particular, we can consider the problem of adding k edges to some skeletal tree structure with the objective of improving the $\mathcal{H}_2$ performance as much as possible. It is readily seen that such a problem is in fact combinatorial, and either specialized algorithms or certain convex relaxations are required to solve the problem. In this work, we consider an ℓ_1 relaxation of the problem.

The performance expressions can be converted to a convex problem by introducing weights on the edges. To promote sparsity of the relaxed solution, we also append a weighted ℓ_1-norm term in the objective function. The resulting minimization problem has the following form:

$$\begin{aligned} \min_{\gamma, w_i} \quad & -\alpha\gamma + (1-\alpha)\|m \circ w\|_1 \\ \text{s.t.} \quad & (\gamma - 1)I - T_{\mathcal{T}} W T_{\mathcal{T}}^T \leq 0 \\ & \textstyle\sum_i w_i = k \\ & 0 \leq w_i \leq 1. \end{aligned} \tag{6}$$

The matrix W contains the candidate edge weights on the diagonal, and the matrix $T_{\mathcal{T}}$ is taken from the cut-space of the graph $\mathcal{G}$.

The optimization problem given above is iterated using a re-weighted ℓ_1 algorithm, as proposed in [4]. While this scheme can not be expected to yield the optimal solution, in practice we find that it leads to sparse and integer solutions. An important feature of this problem, however, is the role that the weights m play in the optimization. Indeed, m can be chosen to promote, for example, long cycles or edge disjoint cycles. This parameter then falls under the realm of an engineering design choice and can be used to promote certain features of the solution of this algorithm.

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