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Arbeitsgemeinschaft: Ergodic Theory and Combinatorial Number Theory

Organised by
Vitaly Bergelson, Columbus
Nikos Frantzikinakis, Heraklion
Terence Tao, Los Angeles
Tamar Ziegler, Haifa

7th October – 13th October 2012

Abstract. The aim of this Arbeitsgemeinschaft was to introduce young researchers with various backgrounds to the multifaceted and mutually perpetuating connections between ergodic theory, topological dynamics, combinatorics, and number theory.

Mathematics Subject Classification (2000): 05D10, 11K06, 11B25, 11B30, 27A45, 28D05, 28D15, 37A30, 37A45, 37B05.

Introduction by the Organisers

In 1977 Furstenberg gave an ergodic proof of the celebrated theorem of Szemerédi on arithmetic progressions, hereby starting a new field, ergodic Ramsey theory. Over the years the methods of ergodic theory and topological dynamics have led to very impressive developments in the fields of arithmetic combinatorics and Ramsey theory. Furstenberg's original approach has been enhanced with several deep structural results of measure preserving systems, equidistribution results on nilmanifolds, the use of ultrafilters etc. Several novel techniques have been developed and opened new vistas that led to new deep results, including far reaching extensions of Szemerédi's theorem, results in Euclidean Ramsey theory, multiple recurrence results for non-abelian group actions, convergence results for multiple ergodic averages etc. These methods have also facilitated the recent spectacular progress on patterns in primes. The field of ergodic theory has tremendously benefited, since the problems of combinatorial and number-theoretic nature have given a boost to the in depth study of recurrence and convergence problems. The aim of this workshop was to expose wide circles of young mathematicians to the beautiful results and methods of this rapidly developing area of mathematics.

The Arbeitsgemeinschaft *Ergodic Theory and Combinatorial Number Theory*, organised by Vitaly Bergelson (Columbus), Nikos Frantzikinakis (Heraklion), Terence Tao (Los Angeles), Tamar Ziegler (Haifa), was held 7 October –13 October 2013. It was well attended with over 50 participants with broad geographic representation from all continents. The majority of the participants were graduate students and young postdocs with various mathematical backgrounds. We realize that the material we originally intended to cover in the one hour presentations was in most cases much too heavy. Nevertheless, all speakers made intelligent selections among the assigned material, and managed to present very good talks. The meeting also profited from the presence of some more senior researchers who interacted with the younger participants and had stimulating discussions. The traditional hike took place under foggy conditions and no one was lost.

Arbeitsgemeinschaft: Ergodic Theory and Combinatorial Number Theory

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Abstracts

Background in Ergodic Theory I

Dong Han Kim

Let $(X, \mathcal{B}, \mu)$ be a measure space. A transformation T is called measure preserving if $\mu(T^{-1}E) = \mu(E)$ for any measurable $E \in \mathcal{B}$. The following are examples of measure preserving systems:

- Rotations on the circle: $Tx = x + \theta \pmod 1$ on $X = [0, 1)$ preserves the Lebesgue measure.
- Homomorphism on the torus: $X = \mathbb{R}^n/\mathbb{Z}^n$ is the n-dimensional torus with Lebesgue measure and $Tx = Ax$ is an integral matrix A with nonzero determinant. The doubling map on the circle, $Tx = 2x \pmod 1$ on $[0, 1)$ with the Lebesgue measure is a special case.
- Shift space: $X = \prod_{n=1}^{\infty} \mathcal{A}$, $\mathcal{A}$ is a finite set called an alphabet, where $\sigma : X \to X$ is the left shift map $\sigma : (x_1x_2x_3 \dots) \mapsto (x_2x_3x_4 \dots)$ with a shift invariant (stationary) measure μ on X.

Given a measure preserving system (X, μ, T) we can choose a partition on X and thus associate a shift space. For example, the $(1/2, 1/2)$ Bernoulli system is isomorphic to the $2x$ map on the unit interval with the Lebesgue measure with the partition $\{[0, 1/2), [1/2, 1)\}$.

Let $T : (X, \mu) \to (X, \mu)$ be a probability measure preserving transformation and $E \subset X$ be a measurable set. The Poincaré Recurrence Theorem states that almost all point $x \in E$ returns infinitely often to E under the iteration by T.

A measure preserving transformation $T : (X, \mu) \to (X, \mu)$ is called ergodic if

$$T^{-1}E \circeq E \text{ implies } \mu(E) = 0 \text{ or } \mu(E) = 1.$$

Equivalently, there is no invariant function except for constant functions, or 1 is a simple eigenvalue of the operator $U_T : L^2(X, \mu) \to L^2(X, \mu)$, $U_T(f) = f \circ T$.

A rotation on the circle is ergodic if and only if the rotation angle θ is irrational. To see this let $f(x) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n x}$ be an invariant function. Then $f \circ T(x) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n (x+\theta)} = \sum_{-\infty}^{\infty} a_n e^{2\pi i n \theta} e^{2\pi i n x}$. Since $f \circ T(x) = f(x)$, we get $a_n e^{2\pi i n \theta} = a_n$ for all n. If θ is irrational, then $e^{2\pi i n \theta}$ cannot be 1 unless $n = 0$. Therefore, $a_n = 0$ for all $n \neq 0$, which implies that $f(x)$ is constant.

A homomorphism on the torus, $T(x) = Ax$ on $\mathbb{R}^n/\mathbb{Z}^n$ is ergodic if A has no root of unity as eigenvalues. In general, if G is a compact abelian group and $T : G \to G$ is a continuous surjective endomorphism, then T is ergodic with respect to the Haar measure if and only if the trivial character $\chi \equiv 1$ is the only $\chi \in \hat{G}$ that satisfies $\chi \circ T^n = \chi$ for some $n > 0$.

The Birkhoff ergodic theorem states that if $T : (X, \mu) \to (X, \mu)$ is ergodic and $f \in L^1(X, \mu)$, then for μ-almost every $x \in X$

$$\frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \to \int_X f \, d\mu.$$

If T is not ergodic, then the limit is an invariant function.

Let $\mathcal{M}(X)$ be the set of Borel probability measure on X. The map $\mu \mapsto \left(f \mapsto \int_X f\,d\mu\right)$ is a bijection between $\mathcal{M}(X)$ and $C(X)^*$. The weak*-topology on $\mathcal{M}(X)$ is the smallest topology such that $\mu \mapsto \int f\,d\mu$ is continuous for all $f \in C(X)$. If X is a compact metrisable space, then $\mathcal{M}(X)$ is compact in the weak*-topology.

Let X be a compact metric space and $T : X \to X$ be continuous and let $\mathcal{M}(X,T)$ be the set of T- invariant Borel probability measure. Then $\mathcal{M}(X,T)$ is non-empty, compact and convex. Moreover, μ is ergodic if and only if μ is an extreme point $\mathcal{M}(X,T)$ and two ergodic measures are mutually singular.

Let $M(X,T)$ be the set of T-invariant ergodic measures on X. The ergodic decomposition theorem says that

$$\mu = \int_{\mathcal{E}(X,T)} \nu d\lambda(\nu)$$

in the sense that for any $\mu \in \mathcal{M}(X,T)$, there exists a unique probability measure λ on the compact metric space $\mathcal{M}(X,T)$ such that $\lambda(\mathcal{E}(X,T)) = 1$ and

$$\int_X f\,d\mu(x) = \int_{\mathcal{E}(X,T)} \left(\int_X f\,d\nu(x)\right) d\lambda(\nu)$$

for any continuous map $f \in C(X)$.

If $M(X,T)$ has only one measure, T is said to be uniquely ergodic. For a uniquely ergodic T, the Birkhoff average converges to the same limits for every point. An irrational rotation is uniquely ergodic.

The upper Banach density of set S of integers is

$$d^*(S) := \limsup_{n_j - m_j \to \infty} \frac{|S \cap [m_j, n_j]|}{n_j - m_j}.$$

The theorem of Szemerédi states that any set of integers with positive upper Banach density contains arbitrarily long arithmetic progressions. Szemerédi's Theorem can be obtained by a multiple recurrence theorem in ergodic theory. This is a consequence of the following "correspondence principle" of Furstenberg: If S is a set of integers, then there exists an invertible probability preserving system $(X, \mathcal{X}, \mu, T)$ (in fact it is a shift on the space $X = \{0,1\}^{\mathbb{Z}}$ with an appropriate shift invariant measure), and a set $A \in \mathcal{X}$, with $\mu(A) = d^*(S)$, and such that

$$d^*(S \cap (S - n_1) \cap \ldots \cap (S - n_\ell)) \geq \mu(A \cap T^{-n_1}A \cap \cdots \cap T^{-n_\ell}A)$$

for every $n_1, \ldots, n_\ell \in \mathbb{Z}$ and $\ell \in \mathbb{N}$.

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Background in Ergodic Theory II

Markus Haase

The aim of this talk is to provide some more background material from ergodic theory. As usual, we let (X, T) be an invertible measure-preserving system on some probability space $X = (X, \Sigma_X, \mu_X)$. The corresponding Koopman operator on $\mathrm{L}^2(X)$ is — by abuse of notation — again denoted by T, i.e., $Tf = f \circ T$ for every function $f \in \mathrm{L}^2(X)$.

A. The spectral theorem. The Koopman operator T is a *unitary* operator on the Hilbert space $H = \mathrm{L}^2(X)$. To study such operators, the spectral theorem is a powerful tool, and in fact was employed by von Neumann [4] in his proof of the mean ergodic theorem. We sketch one possible approach.

Given a unitary operator T on some Hilbert space H one considers the commutative unital C^*-algebra

$$A := \{p(T, T^{-1}) \mid p \in \mathbb{C}[z, z^{-1}]\}$$

which, by the *Gelfand–Naimark theorem* is isomorphic (via Φ, say) to a C^*-algebra $\mathrm{C}(K)$. The compact space K is the *Gelfand space* of A; in our case it can be identified with $K = \sigma(T) \subseteq \mathbb{T}$, and under this identification $\Phi(T) = (z \mapsto z)$. By the *Riesz representation theorem* and since polynomials $\mathbb{C}[z, z^{-1}]$ are dense in $\mathrm{C}(K)$ (by the Stone–Weierstrass theorem), for each two vectors $x, y \in H$ there is a unique complex $\mu_{x,y} \in \mathrm{M}(K)$ such that

$$\langle T^n x, y\rangle = \int_K \Phi(T^n)\, \mathrm{d}\mu_{x,y} = \int_{\mathbb{T}} z^n\, \mathrm{d}\mu_{x,y} \qquad (n \in \mathbb{Z}).$$

A short computation yields $\|Sx\| = \|\Phi(S)\|_{\mathrm{L}^2(\mu_x)}$ for any $S \in A$ (we abbreviate $\mu_x = \mu_{x,x}$); hence the Hilbert space $\mathrm{L}^2(\mu_x)$ is isometrically isomorphic to the *cyclic subspace* $Z(x) := \mathrm{cl}_H Ax$ of H. (Under this isomorphy, $x \in Z(x)$ corresponds to the function $\mathbf{1}$, and T corresponds to multiplication by z.) As $Z(x)^\perp$ is invariant under T and T^{-1} one can use a Zorn argument to decompose H orthogonally as

$$H = \bigoplus\nolimits_\alpha Z(x_\alpha) \cong \bigoplus\nolimits_\alpha \mathrm{L}^2(K, \mu_{x_\alpha}) \cong \mathrm{L}^2\Big(\bigsqcup\nolimits_\alpha K_\alpha, \bigoplus\nolimits_\alpha \mu_{x_\alpha}\Big),$$

where each K_α is a disjoint copy of $K = \sigma(T)$. Hence (H, T) is unitarily equivalent to L^2 over a locally compact space together with a multiplication operator by a $\mathbb{T}$-valued continuous function.

B. Markov operators. Viewing a Koopman operator exclusively as a unitary operator disregards many structural features, in particular the *positivity.* Given probability spaces X, Y we call an operator $S : \mathrm{L}^2(Y) \to \mathrm{L}^2(X)$ a *Markov operator* if $S \geq 0$ (meaning that $Sf \geq 0$ whenever $f \geq 0$) and $S\mathbf{1} = \mathbf{1} = S'\mathbf{1}$. By a mild form of Riesz-Thorin interpolation, a Markov operator S is a contraction. Moreover, its adjoint S' is again a Markov operator, and the Markov operators form a compact convex set (for the weak operator topology). For the proof of the following theorem see [2, Section 13].

Theorem 1. *For a Markov operator $S : \mathrm{L}^2(Y) \to \mathrm{L}^2(X)$ the following assertions are equivalent:*

(i) $S(f \cdot g) = Sf \cdot Sg$ *for all* $f, g \in \mathrm{L}^\infty$*;*
(ii) $|Sf| = S|f|$ *for all* $f \in \mathrm{L}^2$*;*
(iii) $\|Sf\|_2 = \|f\|_2$ *for all* $f \in \mathrm{L}^2$*;*
(iv) *There is a Markov operator* $R : \mathrm{L}^2(X) \to \mathrm{L}^2(Y)$ *such that* $RS = I$*.*

Markov operators satisfying (i)-(iv) of Theorem A are called *Markov embeddings*.

C. Factors. Measure preserving systems (X, T) form the objects of a category with morphisms being the Markov operators S intertwining the Koopman operators (called *T-Markov operators*). Two T-Markov embeddings S_1, S_2 are *equivalent* if there is a T-Markov isomorphism Φ such that $S_1\Phi = S_2$.

Theorem 2. *For a dynamical system (X, T) the following objects canonically correspond to each other:*

1) *T-Markov embeddings $S : \mathrm{L}^2(Y) \to \mathrm{L}^2(X)$ modulo equivalence;*
2) *T-biinvariant Banach sublattices E of $\mathrm{L}^2(X)$ containing $\mathbf{1}$;*
3) *T-biinvariant unital C^*-subalgebras A of $\mathrm{L}^\infty(X)$ containing $\mathbf{1}$ modulo equivalence given by $A \sim B$ if $\mathrm{cl}_{\mathrm{L}^2} A = \mathrm{cl}_{\mathrm{L}^2} B$.*
4) *sub-σ-algebras Σ of Σ_X modulo equivalence given by equality modulo μ_X-null sets.*

[The correspondences are: $S \mapsto \mathrm{ran}(S)$ for 1)⇒2); $E \mapsto E \cap \mathrm{L}^\infty$ for 2)⇒3); $A \mapsto \{M \mid \mathbf{1}_M \in \mathrm{cl}_{\mathrm{L}^2} A\}$ for 3)⇒4); and $\Sigma \mapsto (X, \Sigma, \mu_X)$ for 4)⇒1).]

D. The Kronecker factor. Given a m.p.s (X, T), its *Kronecker factor* is

$$K(X) := \mathrm{cl}_{\mathrm{L}^2}\mathrm{span}\bigcup\nolimits_{\lambda\in\mathbb{T}} \ker(\lambda I - T).$$

It is easy to see that $\ker(\lambda - T) \cap \mathrm{L}^\infty$ is dense in $\ker(\lambda - T)$. Therefore, $K(X)$ is the L^2-closure of a T-biinvariant unital C^*-subalgebra of L^∞ and hence a factor. One says that (X, T) has *discrete spectrum* if $K(X) = \mathrm{L}^2(X)$.

Theorem 3 (Halmos–von Neumann [3]). *An ergodic m.p.s. (X, T) has discrete spectrum if and only if it is (Markov) isomorphic to a rotation on a compact monothetic group. This group is the dual of the group of eigenvalues of T, considered as a discrete subgroup of $\mathbb{T}$.*

[Sketch of proof: let $A = \mathrm{cl}_{\mathrm{L}^\infty}\mathrm{span}\bigcup_{\lambda\in\mathbb{T}} \ker(\lambda - T) \cap \mathrm{L}^\infty$; the Koopman operator T on A has relatively compact orbits hence the strong operator closure $G := \mathrm{cl}\{T^n \mid n \in \mathbb{Z}\}$ is a compact group of automorphisms of A; by Gelfand–Naimark $A \cong \mathrm{C}(K)$, and under this isomorphism each $S \in G$ is induced by a homeomorphism φ_S of K. The system $(K; \varphi_T)$ is minimal, and hence $S \mapsto \varphi_S(x_0)$ is a homeomorphism $G \to K$, where $x_0 \in K$ is any chosen point.]

E. Weakly mixing systems. We turn to a characterization of $K(X)^\perp$.

Lemma. *For $f \in \mathrm{L}^2(X)$, (X,T) an m.p.s., the following are equivalent:*

(i) $f \perp K(X)$;

(ii) *the spectral measure μ_f of f is continuous, i.e., has no atoms;*

(iii) $\frac{1}{n}\sum_{j=0}^{n-1} |\langle T^n f, f\rangle|^2 \to 0$;

(iv) $\mathrm{D}-\lim_n \langle T^n f, g\rangle = 0$ *for all $g \in \mathrm{L}^2$;*

(v) $\mathrm{D}-\lim_n T^n f = 0$ *in the weak topology;*

(vi) $0 \in \mathrm{cl}_w\{T^n f \mid n \geq 0\}$.

[For (i)⇒(ii) use the spectral theorem, for (ii)⇒(iii) see [1,p.59], implication (iii) ⇒(iv) is polarization and the Koopman-von Neumann lemma [1, Lemma 2.41], for (iv)⇒(v) reduce to the separable case and employ a diagonal argument, (v)⇒(vi) is trivial and (vi)⇒(i) straightforward.]

Theorem 4. *For a m.p.s. (X,T) the following assertions are equivalent.*

(i) (X,T) *is* weakly mixing, *i.e.,* $K(X) = \mathrm{span}\{\mathbf{1}\}$;

(ii) $\frac{1}{n}\sum_{j=0}^{n-1} |\langle T^j f, g\rangle - \int_X f \cdot \int_X g| \to 0$ *for all $f, g \in \mathrm{L}^2$;*

(iii) $(X \times X, T \times T)$ *is ergodic;*

(iv) $(X \times Y, T \times S)$ *is weakly mixing for each weakly mixing m.p.s.* (Y,S).

[(i)⇔(ii) follows from the lemma; for (ii)⇒(iii) use the mean ergodic theorem in its weak form; for (iii)⇒(ii) note that if $Tf = \lambda f$ and $|\lambda| = 1$ then $f \otimes \overline{f} \in \mathrm{fix}(T \otimes T)$; (iv)⇒(i) is clear and for (i)⇒(iv) one uses (ii) for the product system.]

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Ergodic Proof of the Polynomial Szemerédi Theorem I

Akos Magyar

In 1975, Szemerédi proved the following long standing conjecture of Erdös and Turàn:

Theorem. *Let Λ be a subset of the integers with positive upper density. Then Λ contains arbitrarily long arithmetic progressions.*

Szemerédi's proof was combinatorial in nature and intricate. In 1977 Furstenberg [3] gave an entirely different proof using ergodic theory. He showed that Szemerédi's theorem is equivalent to a statement about multiple recurrence of measure preserving systems and then proved this ergodic statement.

Theorem (Furstenberg [3]). *Let $(X, \mathcal{X}, \mu, T)$ be a finite measure preserving system and $A \in \mathcal{X}$ be a set with positive measure. Then for every $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that*

$$\mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0.$$

In order to establish such multiple recurrence property one usually analyzes the limiting behavior of some closely related ergodic averages. In the case of Szemerédi's one seeks to show that for every $f \in L^\infty(\mu)$ that is non-negative and not identically zero we have

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int f(x) \cdot f(T^n x) \cdots f(T^{kn}x)\ d\mu > 0. \tag{1}$$

Furstenberg proved this by first establishing a new structure theorem allowing one to decompose an arbitrary measure preserving system into component elements exhibiting one of two extreme types of behavior: *compactness*, characterized by regular, "almost periodic" trajectories, and *weak mixing*, characterized by irregular, "quasi-random" trajectories. On $\mathbb{T}$, these types of behavior are exemplified by rotations and by the doubling map, respectively.

We use the case $k = 2$ of (1) to illustrate the basic idea. Our goal is to show that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int f(x) \cdot f(T^n x) \cdot f(T^{2n}x)\ d\mu > 0. \tag{2}$$

An ergodic decomposition argument enables us to assume that our system is ergodic. We split f into "almost periodic" and "quasi-random" components. Let $\mathcal{K}$ be the closure in L^2 of the subspace spanned by the eigenfunctions of T, i.e. the functions $f \in L^2(\mu)$ that satisfy $f(Tx) = e^{2\pi i\alpha} f(x)$ for some $\alpha \in \mathbb{R}$. We write $f = g + h$, where $g \in \mathcal{K}$ and $h \perp \mathcal{K}$. It can be shown that $g, h \in L^\infty(\mu)$ and g is again nonnegative and g is not identically zero. We expand the average in (2) into a sum of eight averages involving the functions g and h. In order to show that the only non-zero contribution to the limit comes from the term involving g alone, it suffices to establish that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n g \cdot T^{2n} h = 0 \tag{3}$$

in $L^2(\mu)$, and similarly with h and g interchanged, and with $g = h$, which is similar. To establish (3), we use a Hilbert space van der Corput lemma on $x_n = T^n g \cdot T^{2n} h$. Some routine computations and a use of the ergodic theorem reduce the task to showing that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int h(x) \cdot h(T^{2n}x)\ d\mu \right| = 0.$$

But this is well known for $h \perp \mathcal{K}$ (in virtue of the fact that for $h \perp \mathcal{K}$ the spectral measure σ_h is continuous, for example).

We are left with the average (2) when $f = g \in \mathcal{K}$. In this case f can be approximated arbitrarily well by a linear combination of eigenfunctions, which easily implies that given $\varepsilon > 0$ one has $||T^n f - f||_{L^2(\mu)} \leq \varepsilon$ for a set of $n \in \mathbb{N}$ with bounded gaps. Using this fact and the triangle inequality, one finds that for a set of $n \in \mathbb{N}$ with bounded gaps,

$$\int f(x) \cdot f(T^n x) \cdot f(T^{2n} x)\, d\mu \geq \Big(\int f\, d\mu \Big)^3 - c \cdot \varepsilon$$

for a constant c that is independent of ε. Choosing ε small enough, we get (2).

For $k > 2$ the "quasi-random" component is not much harder to analyze. The structured component is the distal factor of the system, which can be built by successive isometric extensions. The proof that the multiple recurrence property is preserved by isometric extensions is much harder than the one described above. It ultimately depends on a clever use of van der Waerden's theorem on arithmetic progressions. Details, along this line of proof can be found in [2, 4, 6, 7].

In 1979, Furstenberg and Katznelson used similar techniques to prove a multidimensional extension of Szemerédi's theorem [5]. In 1996, Bergelson and Leibman [1] proved the following polynomial extension of Szemerédi's theorem.

Theorem. *Let Λ be a subset of the integers with positive upper density and $p_1, \ldots, p_k$ polynomials with integer coefficients and zero constant term. Then Λ contains patterns of the form $m, m + p_1(n), \ldots, m + p_k(n)$ for some $m, n \in \mathbb{N}$.*

In particular Λ contains patterns of the form $m, m + n^2, m + 2n^2$, and patterns of the form $m, m + n, m + n^2$. The proof follows the general strategy described above but there a few new obstacles to overcome. The more serious is that the coloristic result needed to prove polynomial multiple recurrence for distal systems was not available at that point. This is now known as the polynomial extension of van der Waerden's theorem and Bergelson and Leibman proved it by making clever use of topological dynamics tools coupled with an inductive scheme (PET induction) particularly designed for this polynomial setup.

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Uniformity Seminorms and Characteristic Factors

Ilya Khayutin

This lecture follows primarily the work of Host-Kra [3] and we present the structure theory of Host-Kra, and in a somewhat different framework of Ziegler, for probability measure preserving dynamical systems. This theory is used to prove the following theorem.

Theorem 1 ([3], [8]). *Let T be a measure preserving transformation of the probability measure space $(X, \mathcal{X}, \mu)$, $k \in \mathbb{N}$, and $f_1, f_2, \ldots, f_k \in L^\infty_\mu(X)$. Then the averages*

$$\frac{1}{N}\sum_{n=0}^{N-1} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{kn} f_k \tag{1}$$

converge in $L^2_\mu(X)$.

This approach goes back to Furstenberg's proof of Szemerédi's theorem, the following definition of characteristic factor has been formulated in [2].

A. Characteristic factors and structure theorem. The starting idea in studying the averages (1) goes back to Furstenberg's proof of Szemerédi's theorem. One looks for a factor of the system which is characteristic, i.e. convergence of the averages for the factor system implies the convergence of the averages for the original system, and which have some additional structure which allows us to prove convergence in the factor. To show that those characteristic factors have this algebraic or geometric structure is the deepest part of the structure theory. The following definition was first formulated in [2].

Definition (Characteristic Factor)**.** A factor $X \to Y$ is characteristic for the averages (1) for a given k if for $f_1, \ldots, f_k \in L^\infty_\mu(X)$ we have

$$\frac{1}{N}\sum_{n=0}^{N-1} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{kn} f_k -$$
$$\frac{1}{N}\sum_{n=0}^{N-1} T^n\,\mathbb{E}\big(f_1 \mid \mathcal{Y}\big) \cdot T^{2n}\,\mathbb{E}\big(f_2 \mid \mathcal{Y}\big) \cdot \ldots \cdot T^{kn}\,\mathbb{E}\big(f_k \mid \mathcal{Y}\big) \xrightarrow[N\to\infty]{L^2_\mu(X)} 0,$$

where $\mathcal{Y} \subseteq \mathcal{X}$ is the T-invariant σ-algebra which corresponds to the factor Y.

Equivalently, if for any $f_1, \ldots, f_k$ as above with $\mathbb{E}\big(f_i \mid \mathcal{Y}\big) = 0$ for some i, the multiple ergodic averages (1) converge to 0 in $L^2_\mu(X)$.

The key step in establishing convergence for the multiple ergodic averages (1) is following structure theorem which is the deepest and hardest result of this analysis.

Theorem 2 ([3], [8]). *Let $(X, \mathcal{X}, \mu, T)$ be an invertible ergodic measure preserving system. Then we have a chain of factors $X \to \ldots \to Z_k(X) \to Z_{k-1}(X) \ldots \to Z_1(X)$, s.t. for all $k \in \mathbb{N}$ the factor $Z_{k-1}(X)$ is isomorphic to an inverse limit of (k-1)-step nilsystems and is characteristic for the averages* (1).

B. Nilsystems. Nilsystems have some particularly nice rigidity properties which allow one to prove convergence of the averages (1).

Definition (Nilmanifold)**.** Let G be a k-step nilpotent Lie group, $\Gamma < G$ a uniform lattice, i.e. Γ is a discrete subgroup and G/Γ is compact, in particular there is a unique Haar left G-invariant probability measure on G/Γ, we call G/Γ a **k-step nilmanifold**.

Choosing $a \in G$, a **nilrotation** by a is the action $T\colon G/\Gamma \to G/\Gamma := x \mapsto ax$. We call the measure preserving system $(G, \mathcal{G}/\Gamma, m, T)$ a **nilsystem** where $\mathcal{G}/\Gamma$ is the Borel σ-algebra of G/Γ and m is the unique G-invariant measure.

Nilsystems exhibit some remarkable rigidity properties. In particular, each element of G acts Ad-unipotently.

Theorem 3 ([4], [5], [6], [7])**.** *Let $f \in \mathcal{C}(G/\Gamma)$, then the ergodic averages*

$$\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)$$

converge for all $x \in G/\Gamma$.

Note that if $X = G/\Gamma$ is a nilmanifold, so is $X^k = G^k/\Gamma^k$ for $k \in \mathbb{N}$. So if $a \in G$, look at the nilrotation on X^k given by $(a, a^2, \ldots, a^{k-1}) \in G^k$ and at the point $(x, x, \ldots, x) \in X^k$ for $x \in X$, we get by the previous theorem pointwise convergence of the averages (1) for continuous $f_1, \ldots, f_k \in \mathcal{C}(X)$. This implies L^2_μ convergence of the averages (1) for all $f_1, \ldots, f_k \in L^\infty_\mu(X)$.

The characteristic factors of the averages (1) are not nilsystems, but inverse limits of such. One gets L^2 convergence of the averages (1) for bounded measurable functions in an inverse limit of k-step nilsystems by approximation using functions measurable with respect to a k-step nilsystem factor.

C. Host-Kra cube spaces and measures. We define inductively the spaces $X^{[k]}$ which are instrumental in defining the Host-Kra characteristic factors. From now on we only consider invertible ergodic systems X. The reduction to this case from the general one is simple using the ergodic decomposition and the natural invertible extension.

The systems $X^{[k]}$ are constructed inductively from $(X, \mathcal{X}, \mu, T)$. Set $X^{[k]} = X^{2^k}$, $\mathcal{X}^{[k]}$ the corresponding product σ-algebra and $T^{[k]} = T^{2^k}$, i.e. $T^{[k]}$ acts as T in all coordinates.

Definition (Box Measures)**.** Define the measures $\mu^{[k]}$ inductively. Set $\mu^{[0]} = \mu$ and denote by $\mathcal{I}^{[k]}$ the σ-algebra of the $T^{[k]}$-invariant sets in $\mathcal{X}^{[k]}$. We define $\mu^{[k+1]}$ on $X^{[k+1]} := X^{[k]} \times X^{[k]}$ as the relatively independent joining of $(X^{[k]}, \mathcal{X}^{[k]}, \mu^{[k]}, T^{[k]})$ with itself over $\mathcal{I}^{[k]}$.

It is easy to see that $\mu^{[k]} = \mu^{2^k}$ for all k, i.e. $\mu^{[k]}$ is the product measure, if and only if X is weakly-mixing, equivalently the Kronecker factor is trivial. Otherwise, the systems $X^{[k]}$ are non-ergodic for $k \geq 1$.

D. Uniformity seminorms. Using the Host-Kra cube measure we define the following seminorms on $L^\infty_\mu(X)$.

Definition (Gowers-Host-Kra Seminorms)**.** For a real valued $f \in L^\infty_\mu(X)$ by

$$\|f\|_{U^k(X)}^{2^k} = \int_{X^{[k]}} \prod_{\epsilon\in\{0,1\}^k} f(x_\epsilon)d\mu^{[k]}(\mathbf{x}).$$

Proposition. *We have the following:*

(1) $\|\cdot\|_{U^k(X)}$ *is a seminorm on* $L^\infty_\mu(X)$ *for all* $k \geq 0$,
(2) *for all real valued* $f \in L^\infty_\mu(X)$: $\left|\int_X f d\mu\right| = \|f\|_{U^1(X)} \leq \|f\|_{U^2(X)} \leq \dots \leq \|f\|_{U^k(X)} \leq \dots \leq \|f\|_\infty$,
(3) X *is weak-mixing if and only if for all real valued* $f \in L^\infty_\mu(X)$: $\|f\|_{U^k(X)} = \left|\int_X f d\mu\right|$ *for all* $k \geq 1$.

Next we discuss the connection between the uniformity seminorms $\|\cdot\|_{U^k(X)}$ and the multiple ergodic averages (1).

Lemma. *For every* $k \in \mathbb{N}$ *and real valued* $f \in L^\infty_\mu(X)$ *we have*

$$\|f\|_{U^{k+1}(X)}^{2^{k+1}} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \|f\cdot T^n f\|_{U^k(X)}^{2^k}.$$

The proof follows by the mean ergodic theorem on the space $X^{[k]}$.

Lemma (Seminorms Majorize the Averages)**.** *Let* $(X, \mathcal{X}, \mu, T)$ *be ergodic and* $k \in \mathbb{N}$. *For any real valued* $f_1, \dots, f_k \in L^\infty_\mu(X)$ *with* $\|f_i\|_\infty \leq 1$ *for all* i, *the following holds*

$$\limsup_{N\to\infty} \left\| \frac{1}{N}\sum_{n=0}^{N-1} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{kn} f_k \right\|_{L^2} \leq \min_{1\leq l\leq k} l\cdot \|f_l\|_{U^k(X)}.$$

The proof follows inductively by the previous lemma and the Van der Corput lemma.

Corollary. *We can define factors* $Z_{k-1}(X)$ *of* X *such that*

$$\mathbb{E}(f \mid \mathcal{Z}_{k-1}) = 0 \Leftrightarrow \|f\|_{U^k(X)} = 0.$$

If $\mathcal{Z}_{k-1} \subseteq \mathcal{X}$ *is the* T*-invariant* σ*-algebra corresponding to the factor* $Z_{k-1}(X)$, *then those factors are characteristic for the averages* (1).

Corollary. *If* X *is weak-mixing then for all* $k \in \mathbb{N}$: $\|\cdot\|_{U^k(X)} = \left|\int_X \cdot d\mu\right|$, *hence we can take for all* k: $\mathcal{Z}_{k-1} = \{\emptyset, X\}$, *the trivial* σ*-algebra, which corresponds to the trivial one-point factor.*

This proves that for X weak-mixing, if $f_1, \dots, f_k \in L^\infty_\mu(X)$, then

$$\frac{1}{N}\sum_{n=0}^{N-1} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{kn} f_k \xrightarrow[N\to\infty]{L^2} \int_X f_1 d\mu \cdot \ldots \cdot \int_X f_k d\mu.$$

This theorem was first proved in [1]. Although this is not the shortest way to show this result, it demonstrates the relevance of the seminorms estimates to the matter at hand.

E. The Kronecker factor. Furstenberg showed in [1] the L^2 convergence of the averages of order $k = 2$ by proving that the Kronecker factor is characteristic for them. We have shown that the factor $Z_1(X)$ which we know to be characteristic for the averages of order 2 is actually the Kronecker factor. By this we have reproved Furstenberg's result by a somewhat lengthy argument. The proof uses the spectral theorem for compact operators to show that the invariant factor of $X^{[1]}$ is naturally isomorphic to the Kronecker of X, and then using the Fourier transform for compact abelian groups, $\mathcal{F}$, on the Kronecker we can show the that for every $f \in L^2_\mu(X)$: $\|f\|^2_{U^1(X)} = \|\mathcal{F}\,\mathbb{E}\,(f \mid \mathcal{K})\|^4_{L^4(K)}$ and hence $\mathbb{E}\,(f \mid \mathcal{Z}_1) = 0 \Leftrightarrow \mathbb{E}\,(f \mid \mathcal{K}) = 0$.

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Equidistribution of Polynomial Sequences on Nilmanifolds

Przemyslaw Mazur

Several recurrence and convergence results in ergodic theory reduce to recurrence and equidistribution properties of polynomial sequences on nilmanifolds. The aim of this talk is to give a brief overview of the basic qualitative equidistribution results of such sequences. Corresponding quantitative equidistribution results are also available and details can be found in [1].

A. Equidistribution in the Abelian case. Let $P\colon \mathbb{N} \to \mathbb{T}^d$ be a polynomial sequence on $\mathbb{T}^d$, i.e., a sequence of the form

$$P(n) = \alpha_0 + \alpha_1 n + \cdots + \alpha_k n^k$$

where $\alpha_0, \alpha_1, \ldots, \alpha_k \in \mathbb{T}^d$. A classical result of Weyl [5] describes completely the qualitative equidistribution properties of such sequences.

Theorem 1 (Abelian equidistribution criterion). *The sequence* $(P(n))$*, defined above, is equidistributed in* $\mathbb{T}^d$ *if and only if there exists* $i \in \{1, \dots, k\}$ *such that* $k \cdot \alpha_i \notin \mathbb{Z}$ *for every* $k \in \mathbb{Z}^d$ *with* $k \neq \mathbf{0}$.

The proof of Theorem 1 proceeds by first showing that the general sequence $(x(n))$ that takes values in $\mathbb{T}^d$ is equidistributed in $\mathbb{T}^d$ if and only if for every $k \in \mathbb{Z}^d$ with $k \neq \mathbf{0}$ one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \cdot x(n)} = 0. \tag{1}$$

This is known as Weyl's criteron and follows from the fact that trigonometric polynomials are dense in the space $C(\mathbb{T}^d)$. Combining this criterion with an elementary estimate of van der Corput one easily establishes the following useful result:

Theorem 2 (Abelian van der Corput criterion). *The sequence* $(x(n))$ *with values in* $\mathbb{T}^d$ *is equidistributed in* $\mathbb{T}^d$ *if for every* $h \in \mathbb{N}$ *and nonzero* $k \in \mathbb{Z}^d$ *the sequence* $(k \cdot (x(n+h) - x(n)))$ *is equidistributed in* $\mathbb{T}^d$.

The van der Corput criterion is particularly suitable for use for polynomial sequences. For instance, in order to show that the sequence $(n^2\alpha)$ is equidistributed in $\mathbb{T}$, it suffices to show that the sequence $(2nh\alpha)$ is equidistributed in $\mathbb{T}$ for every nonzero $h \in \mathbb{Z}$. Using Weyl's criterion one checks that these linear sequences are equidistributed if and only if α is irrational. The necessity of the conditions of Theorem 1 can be checked in a similar manner using the van der Corput criterion and induction on the degree of the polynomial P.

Using Theorem 1 it is not hard to get the following decomposition result (see [1, 4]):

Theorem 3 (Abelian factorization theorem). *Let* $(P(n)$ *be a polynomial sequence in* $\mathbb{Z}^d$*. There exists a decomposition* $P = P_r + P_e$*, such that all non-constant coefficients of the polynomial* P_r *have rational coordinates, and the sequence* $(P_e(n))$ *is equidistributed in a subtorus of* $\mathbb{T}^d$.

For example, if α is irrational and $P(n) = \left(\frac{n}{2} + n^2\alpha, n^2\left(\alpha + \frac{1}{3}\right)\right)$, then $P_r(n) = n(\frac{1}{2}, 0) + n^2(0, \frac{1}{3})$ and $P_e(n) = n^2(\alpha, \alpha)$. One can check that the sequence $(P_e(n))$ is equidistributed in the subtorus $\{(t,t), t \in \mathbb{T}\}$ of $\mathbb{T}^2$.

B. Equidistribution in the nilpotent case. The Abelian equidistribution results of the previous section admit extensions that cover polynomial sequences on nilmanifolds. A simple example of a non-Abelian nilmanifold to keep in mind is the following:

Example (Heisenberg nilmanifold)**.** Let G be the nilpotent group that consists of all upper triangular matrices of the form $\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right)$ with real entries. If we only allow integer entries we get a subgroup Γ of G that is discrete and cocompact. Then G/Γ is a nilmanifold.

We start with the simplest equidistribution result that covers linear sequences (see [2, 3]):

Theorem 4 (Nilpotent equidistribution criterion for linear sequences). *Let $X = G/\Gamma$ be a nilmanifold $a \in G$ and $x \in X$. The sequence $(a^n x)$ is equidistributed in X if and only if there exists no nontrivial character χ of X such that $\chi(a) = 1$.*

Note that a character χ of X factors through the Abelian nilmanifold $Z = X/[G,G]$. If $\pi\colon X \to Z$ is the projection map, then using the Abelian equidistribution result for linear sequences, we get that the sequence $(a^n x)$ is equidistributed in X if and only if the sequence $\big(\pi(a^n x)\big)$ is equidistributed in Z.

Applying the previous criterion to the Heisenberg nilmanifold we see that for $a = \left(\begin{smallmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{smallmatrix}\right)$ and $x \in X$, the sequence $(a^n x)$ is equidistributed in X if and only if the numbers 1, α, and β are rationally independent.

The proof of Theorem 4 is harder than the proof of related Abelian equidistribution results like the one given in Theorem 1. The reason is that linear combinations of characters in X are not dense in the space $C(X)$, and as a consequence Weyl's criterion is not applicable anymore. On the other hand, using a vertical character decomposition, and a non-Abelian counterpart of van der Corput's criterion, one can give a not very complicated proof of Theorem 4 (see [4]).

Next we give the equidistribution result that covers general polynomial sequences (see [2]).

Theorem 5 (Nilpotent equidistribution criterion for polynomial sequences). *Let $X = G/\Gamma$ be a nilmanifold with G connected, $a_1, \dots, a_l \in G$, $p_1, \dots, p_l \in \mathbb{Z}[t]$ and $g(n) = a_1^{p_1(n)} \cdots a_l^{p_l(n)}$. Then for every $x \in X$ the sequence $(g(n)x)$ is equidistributed in X if and only if there exists no nontrivial character χ of X such that $\chi \circ g$ is constant.*

Again one gets that the sequence $(g(n)x)$ is equidistributed in X if and only if the sequence $\big(\pi(g(n)x)\big)$ is equidistributed in $Z = X/[G,G]$. The proof of this result is carried out in [2] by cleverly reducing it to Theorem 4. Note that the case where the group G is not connected is more subtle, in this case the role of Z plays the nilmanifold $Z' = X/[G_0, G_0]$, where G_0 is the connected component of the identity element in G.

Using the previous equidistribution result one can get the following decomposition result (it is implicit in [2] and explicit in [1]):

Theorem 6 (Nilpotent factorization theorem). *Let $X = G/\Gamma$ be a nilmanifold, $(g(n))$ be as in Theorem 5, and $x \in X$. There exists a decomposition $g = c \cdot g_p \cdot g_e$, such that $c \in G$, the sequence $(g_r(n))$ is periodic, and the sequence $(g_e(n)x)$ is equidistributed in a subnilmanifold of X.*

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PET Induction and Seminorm Estimates

Konstantinos Tyros

In 1977, H. Furstenberg [2] gave a new proof of Szemerédi's Theorem [7] using ergodic theory. In particular, he proved that if $(X, \mathcal{X}, \mu, T)$ is an invertible measure preserving system and A is an element of Σ of positive measure, then

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(T^{-n}A \cap T^{-2n}A \cap ... \cap T^{-kn}A) > 0$$

every positive integer k and applying Furstenberg's Correspondence Principle he deduced Szemerédi's Theorem. This gives rise to the following natural set of questions. What can we say about the limit behavior of averages of the form

$$\frac{1}{N}\sum_{n=1}^{N}\int_X (f_1 \circ T^n) \cdot ... \cdot (f_k \circ T^{kn}) d\mu \tag{1}$$

with k a positive integer and $f_1, ..., f_k \in L^\infty(\mu)$, or even of averages of the form

$$\frac{1}{N}\sum_{n=1}^{N}\int_X (f_1 \circ T^{p_1(n)}) \cdot ... \cdot (f_k \circ T^{p_k(n)}) d\mu \tag{2}$$

where $p_1, ..., p_k$ are integer polynomial, i.e. polynomial taking integer values on the integers?

In 1987, V. Bergelson [1], assuming that the system is weakly mixing, treated the case where the averages are of the form (2), by developing the PET induction. In particular, if the polynomials appearing in (2) are essentially distinct, then the averages in (2) converge in $L^2(\mu)$ to the constant function $\prod_{i=1}^{k}\int_X f_i d\mu$. To each polynomial family we assign a type and we endow the set of all possible types by a well ordering. A use of the var der Corput inequality allows us to transform an average of the form (2) to a similar one involving a polynomial family of smaller type. The proof is completed by an induction (PET induction) on the type of the family of polynomials involved.

In 2005, B. Host and B. Kra [4] treated averages of the form (1) in an arbitrary invertible measure preserving system $(X, \mathcal{X}, \mu, T)$. The strategy of the proof is to construct a sequence of factors $(Z_l(X))_l$ of $(X, \mathcal{X}, \mu, T)$ that satisfyies the following properties:

(i) All averages of the form (1) involving functions defined on $Z_l(X)$ converge.

(ii) For every averages of the form (1) consisting of functions defined in $L^\infty(X)$, there exists some l such that the factor $Z_l(X)$ is characteristic for these averages.

Using a similar strategy, B. Host and B. Kra in [3] convergence of the averages (2). Moreover, under some minor additional assumptions they proved that the averages of the form (2) converge in $L^2(\mu)$. This additional assumption was later removed by A. Leibman in [5].

We give some additional details regarding the proof of convergence of the averages (2). By a result of A. Leibman [6] and the information provided by [4] concerning the structure of the factors $(Z_l(X))_l$ one can easily see that averages of the form (2) defined on one of these factors converge in L^2. Therefore, in order to complete the proof of the general case, B. Host and B. Kra proved that every average of the form (2) has a characteristic factor from $Z_l(X)$ for some $i \in \mathbb{N}$. This last result is obtained by a variation of the PET induction. Moreover, in the case that the involved polynomials are linear, upper bounds for the limit of the norm of the corresponding averages are provided, in terms of some seminorms defined in [4].

In this talk we gave a detailed presentation of the PET induction as designed by V. Bergelson in [1], as well as a description of the needed modifications of this method as made by B. Host and B. Kra in [3]. Moreover, we presented the aforementioned seminorm estimates.

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Ergodic Proof of the Polynomial Szemerédi Theorem II

Tanja Eisner

The goal of this talk is to prove the following characterization of polynomials which are "good" for the polynomial Szemerédi theorem.

Theorem. (Bergelson, Leibman, Lesigne [2]) *Let $p_1, \ldots, p_r : \mathbb{Z} \to \mathbb{Z}$, $r \in \mathbb{N}$, be polynomials. Then the following assertions are equivalent.*

(i) *For every dense set* $E \subset \mathbb{Z}$ *there exist* $a, n \in \mathbb{Z}$ *such that*

$$a, a + p_1(n), \dots, a + p_r(n) \in E.$$

(ii) *The polynomials* $p_1, \dots, p_r$ *are* jointly intersective, *i.e.,*

$$\forall k \in \mathbb{Z} \setminus \{0\} \ \exists n \in \mathbb{Z} : p_1(n), \dots, p_r(n) \in k\mathbb{Z}.$$

Here, by a dense set we mean a set with positive upper density. Note that (ii) is a special case of (i) for the dense sets $k\mathbb{Z}$, $k \neq 0$.

The following conditions on polynomials $p_1, \dots, p_r$ ensure that they are jointly intersective.

- $p_1, \dots, p_r$ have a joint integer root. The case that every p_j vanishes at zero corresponds to the classical polynomial Szemerédi theorem of Bergelson and Leibman [1]. However, there are jointly intersective polynomials without integer and even rational roots.
- $p_1, \dots, p_r$ are multiples of the same intersective polynomial q. In fact, this condition is necessary and sufficient for a family of polynomials of one variable to be jointly intersective.

The method of the proof of the non-trivial implication (ii)⇒(i) consists of several steps which we sketch here. First, the Furstenberg correspondence principle leads to the following reformulation of (i):

(i') For every measure preserving system (X, μ, T) and every measurable set $A \subset X$ with $\mu(A) > 0$,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-p_1(n)}A \cdots \cap T^{-p_r(n)}A) > 0.$$

The second step is to use the modern theory of characteristic factors [3] and a limiting argument to reduce the problem to nilsystems, i.e., systems of the form $(G/\Gamma, \mu, T)$, where G is a nilpotent Lie group, Γ is a cocompact discrete subgroup of G, μ is the Haar measure and T is a left rotation on G/Γ by an element of G.

In the next step one considers a larger nilmanifold and uses a differential geometry argument to translate (i') to the following recurrence property of polynomial sequences: For a nilmanifold $X = G/\Gamma$, a polynomial sequence in G of the form $g(n) = a_1^{p_1(n)} \cdots a_r^{p_r(n)}$ and $x \in X$,

$$x \in \overline{\mathrm{Orb}_g(x)},$$

where $\mathrm{Orb}_g(x) = \{g(n)x,\ n \in \mathbb{Z}\}$.

Note that at this point, the classical polynomial Szemerédi theorem is already proved since $p_j(0) = 0$ for every j implies $x = g(0)x$. To show the assertion for general jointly intersective polynomials, one first needs to study certain algebraic properties of such polynomials for orbits on tori. Here, the idea is to pass to a sublattice of $\mathbb{Z}$ on which intersective polynomials behave nicely so that the closure of the orbit is a subtorus. When this is done, one uses the following result of Leibman [4]: An orbit under a polynomial sequence is dense in X if and only if its projection is dense in the so-called horizontal torus $G/[G,G]\Gamma$. The induction on the dimension of the nilmanifold finishes the proof.

Remarks. 1) The above approach gives an alternative proof of Szemerédi's theorem corresponding to the case $p_1(n) = n, \ldots, p_r(n) = rn$. It seems to be the only known proof of Szemerédi's theorem which can be generalized to jointly intersective polynomials. In particular, it is not clear how to prove the van der Waerden type result of the above theorem using other methods.

2) By the same methods one can show that (ii) implies that the set

$$E \cap (E - p_1(n)) \cdots \cap (E - p_r(n))$$

is syndetic for every $E \subset \mathbb{Z}$ with positive upper Banach density.

The following analogue of the multidimensional polynomial Szemerédi theorem due to Bergelson and Leibman [1] is open.

Conjecture. *Let $p_1, \ldots, p_r : \mathbb{Z}^m \to \mathbb{Z}^k$ be polynomials. Then the following assertions are equivalent.*

(i) *For every dense set $E \subset \mathbb{Z}^k$ there exist $a \in \mathbb{Z}^k$ and $n \in \mathbb{Z}^m$ so that $a, a + p_1(n), \ldots, a + p_r(n) \in E$.*

(ii) *The polynomials $p_1, \ldots, p_r$ are jointly intersective, i.e., for every finite index subgroup $\Lambda \subset \mathbb{Z}^k$ there exists $n \in \mathbb{Z}^m$ so that $p_1(n), \ldots, p_r(n) \in \Lambda$.*

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The Structure of Multiple Correlation Sequences and Applications

Michael Bateman

A. The results. We discuss here the paper [2] of Bergelson, Host, and Kra, which proves a strong generalization of special cases of the Furstenberg multiple recurrence theorem and simultaneously generalizes a theorem of Khintchine. Precisely:

Theorem 1. *Suppose $(X, \mathcal{X}, \mu, T)$ is an invertible ergodic measure preserving system. Then for every $\epsilon > 0$, the set of n such that*

$$\mu(A \cap T^n A \cap T^{2n} A \cap T^{3n} A) > \mu(A)^4 - \epsilon$$

is syndetic. Similarly, for every $\epsilon > 0$, the set of n such that

$$\mu(A \cap T^n A \cap T^{2n} A) > \mu(A)^3 - \epsilon$$

is syndetic.

In light of this theorem, the following fact is perhaps rather surprising:

Theorem 2. *For every $l \in \mathbb{N}$, there exists an ergodic invertible measure preserving system $(X, \mathcal{X}, \mu, T)$ and set A, such that*

$$\mu(A \cap T^n A \cap T^{2n} A \cap T^{3n} A \cap T^{4n} A) < \mu(A)^l$$

for every n.

The point is that there is a significant difference between the 5-term recurrence theorems and the 2,3, and 4 -term recurrence theorems. We note that the 2-term theorem is due to Khintchine, and strengthens the Poincaré recurrence theorem. Also, we note that Theorem 1 generalizes the cases $k = 3, 4$ of the Furstenberg recurrence theorem:

Theorem 3 (Furstenberg [3]). *Suppose $(X, \mathcal{X}, \mu, T)$ is an invertible measure preserving system. Then for any k, and any A with $\mu(A) > 0$,*

$$\liminf_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} \mu(A \cap T^n A \cap \cdots \cap T^{(k-1)n} A) > 0.$$

We note that Furstenberg's theorem already tells us that

$$\{n \colon \mu(A \cap T^n A \cap T^{2n} \cap T^{3n} A) > 0\}$$

is syndetic, so the major gain in the theorem of [2] is providing the lower bound of $\mu(A)^4 - \epsilon$ (in the case of 4-term recurrence) rather than merely a lower bound of 0.

B. Proof of positive results. The proof of Theorem 1 requires studying the following multiple correlation sequences:

$$I(f, k, n) = \int f(x)\, f(T^n x) \cdots f(T^{(k-1)n} x)\; d\mu.$$

We imagine that k and f are fixed, so that the numbers $I(f, k, n)$ form a sequence in the parameter n. Another main theorem of [2] that ultimately relies on the structure theorem from [4] is:

Theorem 4. *The sequence $I(f, k, n)$ is equal to a $(k-1)$-step nilsequence plus a sequence that converges to zero in uniform density.*

We say that a sequence $\{a_n\}_n$ converges to zero in uniform density if

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N} |a_n| = 0.$$

The point is that the behavior of the numbers $I(f, k, n)$ is captured by the behavior of a nilsequence up to a negligible error. Theorem 1 then follows from Theorem 4 together with rather explicit knowledge of the nilsequence guaranteed by the theorem. The analogous calculations do not work in the length 5 and higher cases.

C. Proof of negative results. Theorem 1 does not hold without the assumption of ergodicity. Consider the system on the 2-torus given by

$$T(x,y) = (x, y+x),$$

and consider a set $A = \mathbf{T} \times B$, where the set B will have properties specified momentarily. We then have

$$\begin{aligned} \mu(A \cap T^n A \cap T^{2n} A) &= \int\int \mathbf{1}_B(y)\mathbf{1}_B(y+nx)\mathbf{1}_B(y+2nx)dxdy \\ &= \int\int \mathbf{1}_B(y)\mathbf{1}_B(y+x)\mathbf{1}_B(y+2x)dxdy \end{aligned}$$

where the second equality follows from a simple change of variables and the first equality follows from the definition of T. The point is that this expression quantifies the presence of three term arithmetic progressions in B. If we define B to be a union of small intervals that is a scaled down version of a set of integers without three-term arithmetic progressions (such integer sets are given in [1]), then the above quantities will be small. Wrestling with the numerology a bit proves that ergodicity is required in Theorem 3, since the above system is not ergodic.

Importantly, the same method is used to provide a counterexample for the case of 5-term progressions. Instead of creating a system that allows us to capture facts about 3APs, we create an ergodic system so that the number

$$\mu(A \cap T^n A \cap T^{2n} A \cap T^{3n} A \cap T^{4n} A)$$

captures the number of 5-tuples $a_1, \dots, a_5$ that satisfy $f(j) = a_j$ for some integer quadratic polynomial f. An example of Ruzsa shows that there are large sets of integers without any such 5-tuples, and we shrink this example to live inside the torus as before. Also, as before, some wrestling with the numerology proves Theorem 2.

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Euclidean Ramsey Theory

Brian Cook

The basic question in Euclidean Ramsey Theory can be described as follows: Given a k-point configuration in $\mathbb{R}^n$, say $K = \{x_1, \dots, x_k\}$, and a natural number r, does every r-coloring of $\mathbb{R}^n$ contain a monochromatic K' which is congruent to K. A fixed configuration is said to be Ramsey if there exists an n_0 such that this question has an affirmative answer for all values of r provided that n sufficiently large.

The complete classification of configurations which are Ramsey is unknown, however there are two basic results in this direction. Define a brick to be the vertices of a rectangular parallelepiped, and a spherical set to be one whose elements lie on a common sphere. It is known that any subset of a brick is Ramsey and any non-spherical set is not Ramsey, see [4]. As all bricks are in fact spherical, it is natural to conjecture that Ramsey sets are those that are spherical. This classification is still an open problem.

Katznelson and Weiss provide the first instance of a density type result in Euclidean Ramsey Theory, and do so via ergodic methods. Let $K = \{0, 1\}$. Given a set $E \subset \mathbb{R}^2$ of positive upper Banach density, i.e., there exists a sequence of squares in the plane $\{S_i\}$ such that $m(S_i) \to \infty$ with

$$\limsup \frac{m(E \cap S_i)}{m(S_i)} > 0,$$

we have a congruent copy of λK provided that λ is sufficiently large. In other words, every sufficiently large distance appears in A. This result first appears in print in [1], where Bourgain shows this and that a similar result holds for a n-point configuration $K = \{x_1, \dots, x_n\} \in \mathbb{R}^n$ which spans a $(n-1)$-dimensional subspace. The methods used in the proof are those from harmonic analysis.

Also in [1], Bourgain provides a counterexample showing that no density type result can hold for the configuration $K = \{0, 1, 2\} \in \mathbb{R}^n$. This example is extended to all non-spherical configurations by Graham [3]. Turning back to ergodic methods, Furstenberg, Katznelson, and Weiss prove that the result for three term progressions fails by a δ. Define E_δ to be the collection of points within distance δ of the set E. With $K = \{0, u, v\} \subset \mathbb{R}^2$, and $E \subset \mathbb{R}^2$ of positive upper Banach density, there exists λ_0 such that given any $\lambda > \lambda_0$ and any $\delta > 0$ there exists a configuration K' in E_δ such that K' is congruent to λK.

The proof given in [2] translates the problem to an associated problem in a dynamical system via a correspondence principle. They attach to the a given set E in the plane a $\mathbb{R}^2$ measure preserving system $(X, \mathcal{B}, \mu, T)$ and a subset $\tilde{E} \subset X$ such that the recurrence property

$$\mu(\tilde{E} \cap T_u^{-1}\tilde{E} \cap T_v^{-1}\tilde{E}) > 0 \qquad (1)$$

implies the recurrence property of the thickened set E_δ

$$E_\delta \cap (E_\delta - u) \cap (E_\delta - v) \neq \emptyset.$$

The problem of showing that (1) holds is carried out by reducing to the case when the system $(X, \mathcal{B}, \mu, T)$ is a Kronecker action.

Ziegler [6] extends this method to show that a similar result holds for $(k+1)$-point configurations in $\mathbb{R}^k$. This is again a reduction to the case of Kronecker actions, meaning that the Kronecker factor is characteristic for the associated averages. Following the advent of the Host-Kra structure Theorem [5], Ziegler in [7] obtains the full result for arbitrary configurations in any dimension.

For nontrivial triangular configurations in the plane, it is still unknown whether the thickening of the set E is necessary.

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Recurrence for Random Sequences

Benjamin Krause

A subset of the positive integers, $S \subset \mathbb{Z}^+$ is said to be *ergodic* if for any ergodic measure preserving system, $(X, \mathcal{X}, \mu, T)$, and any $f \in L^2(X, \mathcal{X}, \mu, T)$, the averages $\frac{1}{|S \cap [1,N]|} \sum_{n \in S, n \leq N} T^n f$ converge in the L^2 sense to $\int f$ (i.e. the orthogonal projection of f onto set of functions fixed by T). In particular, Von Neumann's familiar mean ergodic theorem states that $\mathbb{Z}^+$ is an ergodic sequence.

The spectral theorem provides a concrete framework for studying norm convergence of exotic ergodic averages: $S = \{s_1, s_2, \dots\} \subset \mathbb{Z}^+$ is *ergodic* precisely when the exponential sums

$$\frac{1}{|S \cap [1,N]|} \sum_{n \in S, n \leq N} e^{2\pi i n \beta}$$

converge pointwise as $N \to \infty$ for every $\beta \in \mathbb{T} \sim [0,1)$ to the function $1_{\{0\}}(\beta)$.

A serious obstruction to pointwise convergence of the above exponential sums is the sparseness of the set S: under the lacunary assumption $\frac{s_{j+1}}{s_j} \geq 5$, for instance, one can find $\alpha = \alpha(S) \in [0,1)$ so that $\{\frac{1}{t} \sum_{j \leq t} e^{2\pi i s_j \alpha}\}$ fails to converge (cf. e.g. [5], §2.3).

In §8 of his celebrated paper, [2], Bourgain uses a probabilistic argument to show, however, the existence of very sparse ergodic sets S.

Proposition ([2], Proposition 8.2 (i))**.** *Suppose that* $\{Y_n\} : \Omega \to \{0,1\}$ *are independent random variables with expectations* $\{\sigma_n\}$*, and let* $S^\omega = \{n : Y_n(\omega) = 1\}$*. If* $\lim_n n\sigma_n = \infty$*, then* S^ω *is almost surely ergodic.*

For instance if $\sigma_n = \frac{\log n}{n}$, then almost surely we have $\frac{|S^\omega \cap [1,N]|}{(\log N)^2} \to \frac{1}{2}$.

To prove this result, Bourgain uses concentration of measure technique to compare the randomly generated sums

$$\frac{1}{|S^\omega \cap [1,N]|} \sum_{n \in S^\omega, n \leq N} e^{2\pi i n \beta} = \frac{1}{|S^\omega \cap [1,N]|} \sum_{n \leq N} Y_n(\omega) e^{2\pi i n \beta}$$

with the associated deterministic sums, $\frac{1}{\sum_{n \leq N} \sigma_n} \sum_{n \leq N} \sigma_n e^{2\pi i n \beta}$, whose convergence follows by summation by parts.

We sketch the proof of this comparison step. Set $W_N := \sum_{n \leq N} \sigma_n$, and express $W_N = C(N)^2 \log N$, where $C(N) \to \infty$ as $N \to \infty$.

By the law of large numbers, $\frac{|S^\omega \cap [1,N]|}{W_N} = \frac{\sum_{n \leq N} Y_n(\omega)}{W_N} \to 1$ almost surely, so it suffices to show that

$$\lim_{N \to \infty} \sup_{\beta \in \mathbb{T}} \left| \frac{1}{W_N} \sum_{n \leq N} (Y_n(\omega) - \sigma_n) e^{2\pi i n \beta} \right| =: \lim_{N \to \infty} \sup_{\beta \in \mathbb{T}} |P_N(\omega, \beta)| = 0$$

almost surely. By Borel-Cantelli, this in turn may be accomplished by exhibiting an increasing sequence $c(N) \to \infty$ so that

$$\mathbb{P}\left(\sup_{\beta \in \mathbb{T}} |P_N(\omega, \beta)| \geq \frac{1}{c(N)} \right)$$

is summable.

Temporarily ignoring the supremum, one uses Chernoff's inequality ([6], Theorem 1.8) to estimate, for fixed $\beta \in \mathbb{T}$,

$$\mathbb{P}\left(|P_N(\omega, \beta)| \geq \frac{1}{c(N)} \right) \lesssim \max\{ e^{-\frac{W_N^2}{4c(N)^2 V_N}}, e^{-\frac{W_N}{2c(N)}} \} \leq e^{-\frac{W_N}{4c(N)^2}},$$

where $V_N := \mathrm{Var}\left(\sum_{n \leq N} (Y_n(\omega) - \sigma_n) e^{2\pi i n \beta} \right)$ is bounded by W_N.

This estimate says that for large N, for any given $\beta \in \mathbb{T}$, $P_N(\omega, \beta)$ is overwhelmingly likely to be $o(c(N)^{-1})$. This alone is not enough to conclude the proof, since one first must take an uncountable supremum over $\beta \in \mathbb{T}$ before applying any concentration estimate; in particular, a crude union bound is ineffective. But each $P_N(\omega, \beta)$ is a (smooth) trigonometric polynomial of degree at most N. In particular, by Bernstein's polynomial inequality [1], one may estimate

$$\sup_{\beta \in \mathbb{T}} |P_N(\beta)| \leq 10 \max_{\beta \in \Delta_N} |P_N(\beta)|,$$

where $\Delta_N \subset \mathbb{T}$ is a $10N$-element net.

Consequently, for all sufficiently large N,

$$\mathbb{P}\left(\sup_{\beta\in\mathbb{T}}|P_N(\beta)|\geq\frac{1}{c(N)}\right)\leq\mathbb{P}\left(\max_{\beta\in\Delta_N}|P_N(\beta)|\geq\frac{10}{c(N)}\right)=$$

$$\mathbb{P}\left(\bigcup_{\beta\in\Delta_N}|P_N(\beta)|\geq\frac{10}{c(N)}\right)\leq 10N\sup_{\beta\in\mathbb{T}}\mathbb{P}\left(|P_N(\beta)|\geq\frac{10}{c(N)}\right)\leq$$

$$Ne^{-\frac{W_N}{Kc(N)^2}}=e^{\log N(1-\frac{C(N)^2}{Kc(N)^2})},$$

where $K>0$ is a large fixed number chosen to absorb various constants. Finally, choosing $c(N)^2=\frac{C(N)}{K}$, so that $(1-\frac{C(N)^2}{Kc(N)^2})=1-C(N)\to-\infty$ completes the proof.

Bourgain's method of "comparison and concentration" is actually robust enough to prove fruitful in the study of questions concerning *pointwise* convergence along randomly generated subsets of the integers:

Proposition ([2], Proposition 8.2 (ii))**.** *For any ergodic measure-preserving system, (X,μ,T), and any $f\in L^2(X,\mu,T)$, if, in the above notation, $\sigma_n>\frac{\log\log n}{n}$, then*

$$\frac{1}{|S^\omega\cap[1,N]|}\sum_{n\leq N}Y_n(\omega)f(T^nx)\to\int_X f\ d\mu$$

μ-almost everywhere, with probability 1.

Two particularly exciting modern directions in the study of convergence along (sparse) randomly generated subsets are

- Pointwise convergence of ergodic averages for $f\in L^1(X,\mu,T)$;
- Convergence of *multiple* ergodic averages of the form

$$\frac{1}{N}\sum_{n\leq N}f(T^{a_n(\omega)}x)g(S^{a_n(\omega)}x),$$

  where T,S are commuting shifts, and $a_n(\omega)$ are randomized versions of fractional powers n^α, with $\alpha>0$.

See [4] and [3], respectively, for further discussion.

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Recurrence for Hardy Sequences

Joanna Kułaga-Przymus

A Hardy field is a field consisting of germs of real-valued functions at infinity that is closed under differentiation. A union of all Hardy fields $\mathcal{H}$ is a large class of functions which includes e.g. the so-called logarithmico-exponential functions $\mathcal{LE}$ (i.e. functions which can be "built" using real constants, the exponential function, the logarithmic function, addition, multiplication, division and composition of functions - as long as the function is defined for large arguments). A list of key properties satisfied by functions from $\mathcal{H}$ includes the following:

- every $a \in \mathcal{H}$ eventually has a constant sign,
- every $a \in \mathcal{H}$ is eventually monotone,
- for $a \in \mathcal{H}, b \in \mathcal{LE}$ there exists a Hardy field containing both a and b,
- for $a \in \mathcal{H}$, $b \in \mathcal{LE}$ the following limit always exists: $\lim_{x\to\infty} \frac{a(x)}{b(x)} \in \mathbb{R} \cup \{\pm\infty\}$ (hence one can use l'Hospital's rule for calculating limits of quotients).

Recurrence (as well as convergence) problems which classically involved arithmetic progressions [12], and later also polynomial sequences [2, 3], can be generalized to the case of sequences obtained by evaluating functions from Hardy fields at integers. Known results on (single and multiple) recurrence involve such sequences with an additional growth condition (the growth is at most polynomial) which stay away from constant multiples of polynomials with integer coefficients:

Theorem (Special Case [9], General Case [7])**.** *Let $T\colon (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ be an invertible measure-preserving system. Fix $k \geq 1$, $A \in \mathcal{B}$ with $\mu(A) > 0$ and let $a \in \mathcal{H}$ have at most polynomial growth. Then*

$$\mu(A \cap T^{[a(n)]}A \cap T^{2[a(n)]}A \cap \cdots \cap T^{k[a(n)]}A) > 0$$

whenever $|a(t) - cp(t)| \to \infty$ for any $c \in \mathbb{R}$ and any $p \in \mathbb{Z}[t]$.

Examples of such sequences are $[n^{\sqrt{2}}]$, $[n \log n]$, $[n^3\sqrt{3} + n + 1]$, $[n^2 + \log\log n]$.

One can also prove an L^2-convergence result concerning averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T^{[a(n)]}f_1 \cdot T^{2[a(n)]}f_2 \cdot \ldots \cdot T^{k[a(n)]}f_k, \tag{1}$$

where $f_1, \ldots, f_k \in L^\infty(X,\mathcal{B},\mu)$:

Theorem (Case $k = 1$ [5], General Case [7])**.** *Fix $k \geq 1$. Let $a \in \mathcal{H}$ have at most polynomial growth. The averages* (1) *converge if and only if one of the following holds:*

- $|a(t) - cp(t)| \succ \log t$ *for all* $c \in \mathbb{R}$ *and* $p \in \mathbb{Z}[t]$,
- $a(t) - cp(t) \to d$ *for some* $c, d \in \mathbb{R}$ *and* $p \in \mathbb{Z}[t]$,
- $|a(t) - t/m| \ll \log t$ *for some* $m \in \mathbb{Z}$.

The proof of the above theorem for $k = 1$ follows from an equidistribution result for Hardy sequences [4] and the spectral theorem. The main tools used in the proof of the general case include the structure theorem [11], equidistribution results on nilmanifolds [6] obtained using quantitative equidistribution [10], and the so-called PET induction [1].

There are many open problems in this area, see [7, 8] for a detailed list. They involve e.g. averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T^{[a_1(n)]}f_1 \cdot T^{[a_2(n)]}f_2 \cdot \ldots \cdot T^{[a_k(n)]}f_k, \tag{2}$$

where $a_i \in \mathcal{H}$ are from the same Hardy field and grow at most polynomially fast or averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{[a_1(n)]}f_1 \cdot T_2^{[a_2(n)]}f_2 \cdot \ldots \cdot T_k^{[a_k(n)]}f_k,$$

where $T_1, \ldots, T_k \colon (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ are commuting measure-preserving transformations. Sufficient (and necessary) conditions for recurrence and for convergence are the most "obvious" types of results one could ask for. It is not unreasonable to expect to obtain conditions similar to the ones for the more "standard" averages (1). Here is a sample:

Question ([7])**.** *Fix* $k \geq 1$. *Let* $a_i \in \mathcal{H}$ *for* $1 \leq i \leq k$ *have at most polynomial growth. Is it true that the averages* (2) *converge if and only if for every* $a \in$ *span*$(a_1, \ldots, a_k)$*, other than the trivial combination, one of the following holds:*

- $|a(t) - cp(t)| \succ \log t$ *for all* $c \in \mathbb{R}$ *and* $p \in \mathbb{Z}[t]$,
- $a(t) - cp(t) \to d$ *for some* $c, d \in \mathbb{R}$ *and* $p \in \mathbb{Z}[t]$,
- $|a(t) - t/m| \ll \log t$ *for some* $m \in \mathbb{Z}$?

The convergence of averages (2) was proven to hold under some stronger assumptions in [7].

A yet further possible direction is to try to use the "Szemerédi type" results and look for the corresponding Hardy-field patterns in the primes. One can also ask questions about recurrence and convergence along sequences of the form $[p_n^c]$, where $c > 0$ is not an integer and p_n is the n-th prime number.

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Van der Corput Sets

Manfred G. Madritsch

A. Historical background. Starting point of the theory of van der Corput sets was the famous paper by Weyl [10], who considered uniformly distributed sequences. In particular, in this article he investigated polynomial sequences and proved that a sequence of the form $(p(n))_{n\geq 1}$ with p a polynomial is uniformly distributed if and only if $p-p(0)$ has at least one irrational coefficient. The proof of this theorem is based on applications of the Cauchy-Schwarz inequality. Together with Weyl's criterion this leads to the investigation of differences in the arguments of exponential sums. These differences motivated van der Corput [9] to consider them as such and to prove the following statement.

Theorem. *Let $(u_n)_{n\geq 1}$ be a sequence in $\mathbb{T}$. If for all $h\geq 1$ the sequence $(u_{n+h}-u_n)_{n\geq 1}$ is uniformly distributed, then the sequence $(u_n)_{n\geq 1}$ is uniformly distributed.*

This statement simplifies and shortens Weyl's proof by using his idea to a greater extend. The central tool of Van der Corput's proof is the following inequality, which has many applications apart from uniform distribution.

Theorem. *Let $(u_n)_{n\geq 1}$ be a complex valued sequence such that $u_n=0$ for $n<1$ and $n>N$. If H is a positive integer then*

$$\Big|\sum_{n\leq N} u_n\Big|^2 \leq \frac{N+H-1}{H}\sum_{|h|\leq H}\left(1-\frac{|h|}{H}\right)\sum_{n\leq N} u_{n+h}\overline{u_n}.$$

B. Van der Corput sets. Van der Corput's theorem states that it suffices to consider the sequences of differences $(u_{n+h}-u_n)_{n\geq 1}$ for $h\in\mathbb{N}$. However, it can be shown that it suffices to consider only a subset of the natural numbers. This motivates the following definition.

Definition. Let $H \subset \mathbb{N}$ be a set. Then we call H a van der Corput set (or vdC set) if it has the following property: given a sequence $(u_n)_{n\geq 1}$ in $\mathbb{T}$, if for all $h \in H$ the sequences $(u_{n+h} - u_n)_{n\geq 1}$ are uniformly distributed, then the sequence $(u_n)_{n\geq 1}$ is uniformly distributed.

Let us call (vdC) the class of all van der Corput sets. These sets were considered from the aspects of pseudo-polynomials and uniform distribution in [1, 4, 6]. The main interest for the field of ergodic theory originates from the following result of Kamae and Mendès-France [7].

Theorem. *Let H be a van der Corput set. Then we have the following property: given a positive measure σ on $\mathbb{T}$ such that for all $h \in H$, $\widehat{\sigma}(h) = 0$, then $\sigma(\{0\}) = 0$.*

Some years later Ruzsa [8] proved further equivalent definitions of van der Corput sets.

Theorem. *Let H be a van der Corput set. Then the following properties hold.*

(1) *H is correlative, i.e. meaning that whenever $(\alpha_n)_{n\geq 1}$ is a sequence of complex numbers satisfying*

$$\sum_{n\leq N} |\alpha_n|^2 = \mathcal{O}(N) \quad \textit{and}$$

$$\sum_{n\leq N} \alpha_{n+h}\overline{\alpha_n} = o(N) \quad (h \in H),$$

then

$$\sum_{n\leq N} \alpha_n = o(N).$$

(2) *Let $\varepsilon > 0$, then there is a polynomial $P(x) = \sum_{h\in H} a_h \cos hx$ with $a_h \in \mathbb{R}$ satisfying*

$$P(x) \geq 0, \quad P(0) = 1, \quad a_0 \leq \varepsilon.$$

(3) *H is forcing continuity for positive measures (FC^+), i.e. if $\sigma \in M_+(\mathbb{T})$ is a positive measure on the circle and $\widehat{\sigma}(h) = 0$ for $h \in H$, then σ is continuous.*

As a corollary he obtained the following result.

Corollary. *If I is an infinite subset of $\mathbb{Z}$, then the set of differences $H := \{n-m : n, m \in I \text{ and } n \neq m\}$ is a vdC set.*

C. Van der Corput sets and recurrence. The main goal of this talk is to draw the relations of van der Corput sets with recurrence sets and sets forcing continuity.

Definition. We call a set $H \subset \mathbb{Z}$ Poincaré or recurrent if whenever $(X, \mathfrak{B}, \mu, T)$ is a dynamical system and A is a measurable set of positive measure, then

$$\mu(T^{-h}A \cap A) > 0$$

for some $h \in H$.

Let us call (P) the class of all recurrent sets. Since it is not so easy to relate van der Corput sets and recurrent sets directly, we will make a detour to intersective sets.

Definition. We call a set $H \subset \mathbb{Z}$ intersective if for all subsets $S \subset \mathbb{Z}$ with positive upper density we have

$$H \cap (S - S) \neq \emptyset.$$

Luckily for us Bertrand-Mathis [3] was able to show that the notions of intersectivity and recurrence coincide. Now the following result by Kamae and Mendes-France [7] connects the notion of van der Corput set and intersective set.

Theorem. *If there exists a set S with nonzero upper density such that $H \cap (S - S) = \emptyset$, then H cannot be a van der Corput set.*

Now together with the different equivalent definitions of van der Corput sets of Ruzsa we obtain the following sequence of implications:

$$(FC+) \Longrightarrow (vdC) \Longrightarrow (P).$$

Ruzsa [8] conjectured that also the inverse implications are true, however this was disproved by Bourgain [5].

D. Enhanced vdC. In this final section of the talk we want to repair this chain of implications. To this end Bergelson and Lesigne [2] introduced the notion of enhanced vdC sets. In the same paper they showed that enhanced vdC sets share many properties of vdC sets like:

(1) *Ramsey property.* If $H = H_1 \cup H_2$ is an enhanced vdC set, then at least one of the sets H_1 and H_2 is enhanced vdC.
(2) *Sets of differences.* Let $H \in \mathbb{N}$. Suppose that, for all $n > 0$ there exists $a_1 < a_2 < \cdots < a_n$ such that $\{a_j - a_i : 1 \leq i < j \leq n\} \subset H$. Then H is an enhanced vdC set.

Furthermore they were able to show that the notions of enhanced vdC set and FC^+ set coincide. However for the case of enhanced vdC sets many questions are open and the interested reader may find some of them in the paper of Bergelson and Lesigne [2].

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Markov Processes and Ramsey Theory for Trees

Jakub Konieczny

In 2003, H. Furstenberg and B. Weiss published the paper *"Markov Processes and Ramsey Theory for Trees"*, where they established new results in Ramsey theory through application of novel recurrence properties for Markov processes [1]. The aim of the talk is to summarize their work.

A fundamental question in Ramsey theory is to ascertain how large (in terms of cardinality) a subset of a structured set must be, in order to ensure that some structure is preserved in the subset. One of the most fundamental results in Ramsey theory is van der Waerden's theorem:

Theorem (van der Waerden). *For any fixed length h and number of colors r, there exists a lower bound on the length H such that if we color an arithmetic progression of length at least H with r colors, then there exists a monochromatic arithmetic sub-progression of length h.*

We consider a generalization of this theorem to the context of trees. For this purpose, we need to lay down some definitions. To begin with, a *tree* is defined to be an acyclic graph with a distinguished vertex called *root*, so that every vertex is connected with the root by exactly one simple path. When two vertices are connected by an edge, the one that lies further from the root is said to be a *descendant* of the closer one, and vertices that have no descendants are referred to as *leaves*. All considered trees will have *uniform height*, meaning that all leaves lie at the same distance, called the *height* of the tree, from the root.

If edges of a tree are all assigned letters from a fixed finite alphabet A, there is a natural way of associating words over A with vertices of the tree. In fact, it is convenient to define trees ab initio as subsets of words satisfying certain natural conditions. A *binary* tree is a full tree over two letter alphabet, so in particular all vertices except for leaves have exactly two descendants.

In order to consider Ramsey theory for trees, one also needs to introduce a notion of substructure. The appropriate definition turns out to be quite subtle. We define an arithmetic binary subtree to be a subset consisting of evenly spaced vertices, structured in the shape of a binary tree, in a way so that any vertex is connected by a path of fixed length to its two descendants. Additionally, we require a property referred to as *immediate branching*, which says that the initial edges of the two paths connecting a given vertex with its two descendants are always assigned some two fixed, distinct letters of the alphabet. For a binary tree, this simply means that the two paths leading to the descendants are disjoint.

The first of the discussed theorems is an analogue of van der Waerden's theorem in the realm of trees.

Theorem A. *For any fixed height h and number of colors r, there exists some lower bound on the height H such that if we color a binary tree of height at least H with r colors, then there exists a monochromatic arithmetic binary subtree of height h.*

The next result generalizes Theorem A in a way that is similar to the way that Szemerédi's theorem generalizes van der Waerden's theorem. To state it, we need to introduce the notion of the *density* of a subset of a tree. If the tree has height L, then any maximal path p connecting the root and a leaf consists of $L+1$ vertices, so for a subset S of the tree we can define the density of its restriction to p to be $D_p(S) = \frac{|S \cap p|}{L+1}$. The density of S is then defined to be the average of $D_p(S)$ over all maximal paths p. With this definition, the higher a vertex is placed in a tree, the larger weight is attached to it.

Theorem B. *For any fixed height h and lower bound on density $\delta > 0$, there exists some lower bound on the height H such that if we take any subset of a binary tree of height at least H with density at least δ, then the subset contains an arithmetic binary subtree of height h.*

The last of the discussed theorems concerns trees that are not necessarily binary. To control the behavior of considered trees, an additional parameter called *branching* is introduced. For a tree with uniform height L, branching is the unique

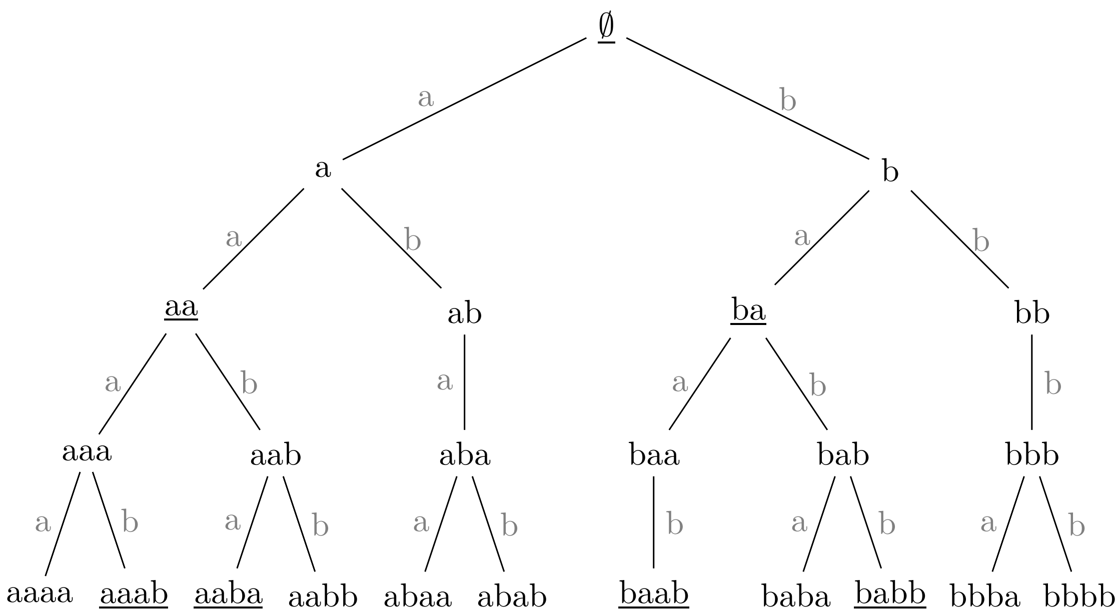

Figure 1. An example of a tree over a two letter alphabet $A = \{a, b\}$. The underlined vertices form an arithmetic binary subtree. The subtree vertex *baab* cannot be replaced by *baba* (immediate branching), nor by *baa* (equal spacing).

number $\alpha > 0$ such that the tree has $2^{\alpha L}$ leaves. A motivating example is a binary tree, whose branching is equal to 1.

Theorem C. *For any fixed height h, lower bound on the branching $\beta > 0$, and upper bound on the degree s, there exists a lower bound on the height H such that any tree of height at least H, branching at least β and vertex degree at most $s+1$ contains an arithmetic binary subtree of height h.*

The proofs of these theorems are very similar to analogous proofs of multiple recurrence results in classical ergodic theory. Theorem A is an immediate consequence of Theorem B, since for any coloring with r colors, one can find a monochromatic set with density at least $\frac{1}{r}$.

To prove Theorem B, we reduce it to a version of the Multiple Recurrence Theorem for Markov processes by means of an explicit construction. The space for the Markov system consists of infinite labeled Markov trees, which can be thought of as trees with transition probabilities attached to edges and additional labels attached to vertices. This is a far reaching generalization of the space of $\{0,1\}$-valued sequences. Transition probabilities for the Markov system are chosen in a way that reflects random walks on the trees. These transitions roughly correspond to the shift map on $\{0,1\}^{\mathbb{N}}$. We also introduce an invariant measure, acquired through a limit transition, so as to encode existence of certain positive density subtrees. Application of the Multiple Recurrence Theorem to thus constructed system directly proves Theorem B.

The proof of Theorem C follows along similar lines. One encodes the behavior of positive density trees in a Markov system on the space of labeled Markov trees. A crucial difference is that one needs to introduce the notion of entropy, together with some continuity properties, in order to ensure that the constructed Markov system is sufficiently non-degenerate.

The difficult step is to prove the version of the Multiple Recurrence for Markov systems. One first reduces the general problem to a special class of *endomorphic* Markov systems, which are analogous to invertible dynamical systems. The reduction follows from the facts that any Markov system is a *factor* of an endomorphic Markov system, and that the claim for a factor follows from the claim for the larger system. The proof of the Multiple Recurrence in the special case is based on a construction of a dynamical system that can be used to approximate the Markov system, thus reducing the problem to the grounds of the classical ergodic theory.

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Convergence Results Involving Multiple Ergodic Averages

Sun Wenbo

A. History. In [13], the following theorem was proved:

Theorem 1. *Let G be a nilpotent group of measure preserving transformations of a probability space $(X, \mathcal{X}, \mu)$. Let $f_1, \ldots, f_l \in L^\infty(\mu)$ and $p_{i,j}$ be integer valued polynomials. Then for every $T_1, \ldots, T_l \in G$, the average*

$$\frac{1}{N}\sum_{n=1}^{N}\prod_{j=1}^{d}(T_1^{p_{1,j}(n)} \ldots T_l^{p_{l,j}(n)})f_j$$

converges in $L^2(\mu)$.

We give a historical account of this problem. For $l = 1, p_{1,i}(n) = jn$, for $j = 1, \ldots, d$, Theorem 1 was first proved in the paper [8] by Host and Kra and subsequently Ziegler gave a different proof in [15]. In particular they proved the following:

Theorem 2. *Let $(X, \mathcal{X}, \mu, T)$ be an invertible measure preserving space, and $f_1, \ldots, f_d \in L^\infty(\mu)$. Then the limit*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_1(T^n x) \ldots f_d(T^{dn}x)$$

exists in $L^2(\mu)$.

The next natural step was to extend Theorem 2 from the linear case to the polynomial case, i.e. prove mean convergence for the averages

$$\frac{1}{N}\sum_{n=1}^{N} f_1(T^{p_1(n)}x) \ldots f_d(T^{p_d(n)}x)$$

which is the special case of Theorem 1 for $l = 1$. Bergelson proved it for weakly mixing systems in [2]. In [6], Furstenberg and Weiss proved a special case for $d = 2, p_1(n) = n, p_2(n) = n^2$. The general case was proved by Host, Kra [9] and Leibman [10]. A further natural step was to extend the previous result to the case of commutative transformations, i.e. prove mean convergence for the averages

$$\frac{1}{N}\sum_{n=1}^{N} f_1(T_1^{p_1(n)}x) \ldots f_d(T_d^{p_d(n)}x)$$

when $T_iT_j = T_jT_i$, which is another special case of Theorem 1. When all the polynomials are equal to n, Conze and Lesigne [4] proved this for the case $d = 2$. In [14], Zhang proved extended this result for $d = 3$ under the assumption $T_iT_j^{-1}(i \neq j)$ is totally ergodic, and Frantizikinakis and Kra [5] extended this result for general d under the same assumption. Later, the ergodicity assumptions were dropped by Tao [11]. After that, alternative proofs were given by Austin [1], Host [7] and Towsner [12]. For nonlinear polynomials, Chu, Frantzikinakis and

Host [3] proved mean convergence under the assumption that the polynomials p_i have distinct degrees. Finally, Walsh proved Theorem 1 in [13].

We should also remark that Walsh's proof uses little ergodic theory and is similar in spirit to the proof of convergence for commuting transformations with linear iterates given by Tao in [11]. Unfortunately, although such arguments are particularly suitable for proving convergence, they give little information for the limiting function, so in particular it is not clear if such arguments can be used to prove multiple recurrence results. On the other hand the more complicated ergodic arguments provide information for the limiting function (and in some cases give an explicit description) that can be used to exhibit multiple recurrence.

Below we explain some ideas used in the Proof of Theorem 1.

B. L-Reducible functions. We let$\overline{g} = (g_1, \dots, g_d), g_i(n) = T_1^{p_{1,i}(n)} \dots T_l^{p_{l,i}(n)}$. To prove Theorem 1, we define the notion of an L-reducible function:

Definition. Given$L \in \mathbb{N}$ and $\epsilon > 0$, we say that $\sigma \in L^\infty(\mu)$ with $\|\sigma\|_{L^\infty(\mu)} \leq 1$ is an (L,ϵ)-*reducible function* (or L-reducible function) with respect to $\overline{g}$, if there exist $M > 0, b_0, \dots, b_{d-1} \in L^\infty(\mu), \|b_i\|_{L^\infty(\mu)} \leq 1$, such that for all $0 \leq l \leq L$, we have

$$\|g_d(l)\sigma - \frac{1}{M}\sum_{m=1}^{M} < g_d|1_G >_m (l) b_0 \prod_{i=1}^{d-1} < g_d|g_i >_m (l) b_i\|_{L^\infty(\mu)} \leq \epsilon,$$

where $< g|h >_m (n) = g(n)g^{-1}(n+m)h(n+m)$.

By direct computation, we get the following two facts:

(i) If f_d is orthogonal to any L-reducible function ($L \ll N$), then the previous averages converge to 0.

(ii) If f_d is the linear combination of some L-reducible functions ($L \geq N$), then we can replace the family of functions $\overline{g} = (g_1, \dots, g_d)$ in the average (1) with a new family $\overline{g'} = (g_1, \dots, g_{d-1}, < g_d|1_G >_m, < g_d|g_1 >_m, \dots, < g_d|g_{d-1} >_m)$. We call the procedure of transforming from $\overline{g}$ to $\overline{g'}$ a *step*.

C. Structure theorem. Motivated by a traditional method used to prove convergence for multiple ergodic averages, we want to decompose f_d into two parts: one part is orthogonal to any L-reducible function and another is the linear combination of some L-reducible functions. Unfortunately, we cannot do so because the L in the previous observations cannot be the same one (one is larger than n and one is smaller than n). Nevertheless, we can decompose f_d into three parts instead of two. We can write f_d as $f_d = f_{d,1} + f_{d,2} + f_{d,3}$, where $f_{d,1}$ can be viewed as "orthogonal to any L_1-reducible function", $f_{d,2}$ can be viewed as "the linear combination of some L_2-reducible functions", and $f_{d,3}$ is an error term ($L_1 \ll N \leq L_2$).

D. Induction. The sketch of the proof of Theorem 1 is as follow: we first decompose $f_d = f_{d,1} + f_{d,2} + f_{d,3}$ as mentioned before. Then the $f_{d,1}$ part will be small by observation (i) and so is the $f_{d,3}$ part as it is the error term. By observation (ii), for the $f_{d,2}$ part, we can reduce the original family $\overline{g}$ to a new family $\overline{g'}$. If we continue this procedure, we can reduce the family $\overline{g'}$ to another family $\overline{g''}$. It can be proved that this reduction will stop after finitely many steps when the family becomes trivial. This finishes the proof as the convergence result is obvious for a trivial family.

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Probabilistic Properties of Multiple Ergodic Averages

Evgeny Verbitskiy

Nonconventional ergodic theorems (also known as "multiple ergodic theorems") establish convergence of expressions like

$$S_N^{(k)} = \frac{1}{N}\sum_{n=1}^{N} f_1(T^n x)\dots f_k(T^{nk}x),$$

where $T : X \to X$ is a measure-preserving transformation.

Recently, there has been a surge of new results establishing finer probabilistic properties of non-conventional averages: namely, Central Limit theorems, Large

Deviations results, and results on the validity of multifractal formalism for non-conventional averages.

A. Central limit theorems. Principle results are due to Kifer and Varadhan [8, 10, 12, 11], who obtained central limit theorems for both discrete time expressions of the form

$$\frac{1}{\sqrt{N}}\sum_{n=1}^{[Nt]}\big(F\big(X(q_1(n)),\ldots,X(q_\ell(n))\big)-\bar{F}\big)$$

and similar expressions in the continuous time where the sum is replaced by an integral. Here $X(n), n\geq 0$ is a sufficiently mixing stationary vector process with some moment conditions, F is a continuous function with polynomial growth and Lipschitz-regularity properties, $\bar{F}=\int Fd(\mu\times\cdots\times\mu)$, μ is the distribution of $X(0)$. The functions $q_i(n)$, $i=1,\ldots,l$, are positive functions assuming integer values on integers with some growth conditions which are satisfied, for instance, when q_i's are polynomials of increasing degrees. It is interesting to mention the following phenomenon: if $q_i(n)=in$ for $i\leq k\leq \ell$ while for $i>k$, the polynomials $q_i(n)$ grow faster than linear, then $\{q_1,\ldots,q_k\}$ and $\{q_{k+1},\ldots,q_l\}$ have different effect on the limiting process.

B. Large deviations results. Principle results are due to Kifer & Varadhan [12] and Carinci, Chazotte, Giardina, and Redig [1]. Suppose μ is a translation invariant measure on $\Sigma=\{-1,1\}^{\mathbb{N}}=\{(x_n)_{n\in\mathbb{N}}:\ x_n\in\{-1,1\}\}$. Does the rate function

$$I^{(k)}(a)=\lim_{\epsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\log\mu\Big(S_N^{(k)}\in[a-\epsilon,a+\epsilon]\Big)$$

exist and have nice properties? The natural candidate for I is the Legendre transform of the "free-energy"

$$F(\lambda)=\lim_{n\to\infty}\frac{1}{n}\mathbb{E}_\mu\exp\Big(\lambda n S_n^{(k)}\Big).$$

provided this limit exists and is differentiable. For **Bernoulli measures** μ (i.e., $\{x_n\}$ form an iid sequence of random variables), existence of rate functions has been established by Kifer & Varadhan [12]. For $\Sigma=\{-1,1\}^{\mathbb{N}}$ and $f(x)=x_1$ for $x=(x_1,x_2,\ldots)$, the rate function has been identified explicitly in [1] by making link to Ising models of statistical mechanics. Kifer and Varadhan announced that in the forthcoming paper existence of large deviations rate functions will be established for **Markov** μ's.

C. Multifractal analysis of nonconventional averages. Initiated by the paper of Fan, Liao, and Ma [2], but also Kifer [9] , the study of the multiple ergodic average from a point view of multifractal analysis have attracted much attention. The major achievements have been made by Fan, Kenyon, Peres, Schmeling, Seuret, Solomyak, Wu and et al. ([3, 4, 5, 6, 7, 14, 13, 15]).

Consider symbolic space $\Sigma = \{-1,1\}^{\mathbb{N}}$, and for $\theta \in [-1,1]$, consider also the level sets

$$B_\theta := \Big\{x \in X : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k x_{2k} \cdots x_{\ell k} = \theta\Big\}.$$

Then Fan-Liao-Ma [2] showed that

$$\dim_H(B_\theta) = 1 - \frac{1}{\ell} + \frac{1}{\ell} H\Big(\frac{1+\theta}{2}\Big),$$

where $H(t) = -t\log_2 t - (1-t)\log_2(1-t)$ is the entropy function.

For the symbolic space $\Sigma = \{0,1\}^{\mathbb{N}}$, one can similarly consider level sets

$$A_\alpha := \Big\{(\omega_k)_1^\infty \in \Sigma : \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \omega_k\omega_{2k} = \alpha\Big\} \qquad (\alpha \in [0,1]).$$

In [2], the authors asked to compute the Hausdorff dimension of the sets A_α's. It turned out to be a substantially more difficult question. As a first step, they also suggested to study a subset of A_0:

$$A_0' := \Big\{(\omega_k)_1^\infty \in \Sigma : \omega_k\omega_{2k} = 0 \quad \text{for all } k \geq 1\Big\}.$$

One can show that A_0 and A_0' have the same Hausdorff dimension.

The Hausdorff dimension of A_0' was later given by Kenyon, Peres and Solomyak [7]:

$$\dim_H A = -\log(1-p),$$

where $p \in [0,1]$ is the unique solution of the equation

$$p^2 = (1-p)^3.$$

Enlightened by the idea of [7], the question about A_α was finally answered by Peres and Solomyak [13], and independently by Fan, Schmeling and Wu [3, 4]: for any $\alpha \in [0,1]$,

$$\dim_H A_\alpha = -\log_2(1-p) - \frac{\alpha}{2}\log_2 \frac{q(1-p)}{p(1-q)},$$

where $(p,q) \in [0,1]^2$ is the unique solution of the system

$$\begin{cases} p^2(1-q) = (1-p)^3, \\ 2pq = \alpha(2+p-q). \end{cases}$$

Multifractal problems lead to interesting combinatorial and probabilistic questions. At the present moment, only a very limited class of functions f can be treated: for example, one can evaluate multifractal spectra corresponding to averages

$$\frac{1}{n}\sum_{i=1}^{n} x_i x_{2i} \ldots x_{ki}, \quad x \in \{-1,1\}^{\mathbb{N}},$$

but not the spectra of

$$\frac{1}{n}\sum_{i=1}^{n} (x_i x_{2i} + x_i x_{2i} x_{3i}).$$

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Szemerédi and van der Waerden Theorems for Commuting Actions of Non-Commutative Groups

Phu Chung

In 1977, Furstenberg established an ergodic theorem for $\mathbb{Z}$-actions which implies the theorem of Roth on arithmetic progressions, the first nontrivial case of Szemerédi's theorem [3, 4].

Let G be a countable group. Let $\{T_g\}_{g\in G}$ and $\{S_g\}_{g\in G}$ be commuting measure preserving actions of G on a probability space $(X, \mathcal{B}, \mu)$. In this talk, I will present the proof of Bergelson, McCutcheon and Zhang [2], concerning an extension of the ergodic Roth theorem for amenable groups and two of its applications: a multiple recurrence theorem and a van der Waerden type theorem in this amenable setting.

A group G is called amenable if there exists a sequence $\{F_n\}_{n\in\mathbb{N}}$ of nonempty finite subsets of G such that $\forall g \in G$, $\frac{|gF_n \cap F_n|}{|F_n|} \to 1$ as $n \to \infty$. Amenability for discrete groups is preserved by the following processes: taking subgroups, forming quotient groups, forming group extensions, forming upward directed unions of amenable groups.

The class of amenable groups contains finite groups, abelian groups, nilpotent groups, solvable groups, groups with polynomial growth and more general, groups with sub-exponential growth. A free group with two generators F_2 is a non-amenable group.

Theorem. *Let G be a countable amenable group. Let $\{F_n\}_{n\in\mathbb{N}}$ be a left Folner sequence of G. Then for any $u, v \in L^2(X, \mathcal{B}, \mu)$, the following limit*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{g\in F_n} u(T_g x)v(S_g T_g x)$$

exists in the L^1 norm.

A set $A \subset G$ is right (left) syndetic if there is some finite set $F \subset G$ such that $\bigcup_{f\in F} Af = G$ $(\bigcup_{f\in F} fA = G)$.

Corollary. *With the above assumptions, for any $A \in \mathcal{B}$ with $\mu(A) > 0$, there exists $\lambda > 0$ such that the set*

$$E_\lambda = \{g \in G : \mu(A \cap T_g^{-1} A \cap (S_g T_g)^{-1}(A)) > \lambda\}$$

is both left and right syndetic.

Corollary (Three dimensional van der Waerden-type theorem)**.** *Let G be a countable amenable group and $r \in \mathbb{N}$. For any finite partition $G \times G \times G = \bigcup_{i=1}^r C_i$, the set*

$$\{g \in G : \text{there exist } i, 1 \le i \le r, \text{ and } (a,b,c) \in G\times G\times G \text{ such that} \\ \{(a,b,c), (ag,b,c), (ag,bg,c), (ag,bg,cg)\} \subset C_i\}$$

is both left and right syndetic.

Minimal idempotent ultrafilters were used successfully by Bergelson and McCutcheon [1] to get a stronger version of the first corollary for general countable groups (not necessary amenable).

Let $(G, \cdot)$ be a countable group. We denote by βG the space of ultrafilters on G. The group operation $\cdot$ on G extends naturally to βG by the rule $A \in p \cdot q \Leftrightarrow \{x \in G : Ax^{-1} \in p\} \in q$, where $Ax^{-1} = \{y \in G : yx \in A\}$ for any $p, q \in \beta G$, and $A \subset G$. Then $(\beta G, \cdot)$ becomes a semigroup.

Let $\mathcal{K}$ be the union of the minimal right ideals of βG. Then $\mathcal{K}$ is a two-sided ideal and indeed the smallest two-sided ideal. A minimal idempotent is an idempotent ultrafilter p belonging to the minimal ideal $\mathcal{K}$. A set $A \subset G$ is a right central* set if $A \in p$ for any minimal idempotent p. We say that $A \subset G$ is an inverse right central* set if $A^{-1} = \{g^{-1} : g \in A\}$ is a right central* set. Note that if $A \subset G$ is right central*, then A is right syndetic and if A is inverse right central* then A is left syndetic.

Theorem. *Let G be a countable group. Let $\{T_g\}_{g\in G}$ and $\{S_g\}_{g\in G}$ be commuting measure preserving actions of G on a probability space $(X,\mathcal{B},\mu)$. Then for any $A\in\mathcal{B}$ with $\mu(A)>0$, there exists $\lambda>0$ such that the set*

$$E_\lambda=\{g\in G:\mu(A\cap T_g^{-1}A\cap(S_gT_g)^{-1}(A))>\lambda\}$$

is both (right) central and inverse (right) central*.*

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Recurrence Beyond Nilpotent Groups

Michael Björklund

A. General comments. Let G be a group equipped with a finite and symmetric generating set S. For simplicity we can assume that S is of the form $\{g_1^{\pm},g_2^{\pm}\}$, where g_1 and g_2 are elements of infinite order. If G is abelian, then the ergodic Roth's theorem asserts that for every ergodic probability measure preserving action G on a standard Borel probability space (X,μ) and for every measurable subset $B\subset X$ of positive measure, there exists $n\neq 0$ such that

$$\mu(B\cap g_1^nB\cap g_2^nB)>0.$$

In particular, there exists $n\neq 0$ such that $\mu(g_1^nB\cap g_2^nB)>0$. We stress that this weaker version is an easy consequence of Poincaré Recurrence Theorem since we can rewrite the last expression as $\mu(B\cap(g^{-1}g_2)^nB)$, and thus apply the latter theorem for a single transformation.

This strategy fails if G is not abelian, and it is the aim of this talk to investigate to which extent the conclusion fails for different classes for groups. More specifically, we will consider probability measure preserving group actions of non-abelian groups G on standard Borel probability measure spaces (X,μ) and ask about the existence of a positive measure Borel set $B\subset X$ such that

$$\mu(g_1^nB\cap g_2^nB)=0,\quad\forall\, n\neq 0.$$

It is not too hard to build such examples for very non-abelian groups such as free groups on at least two generators; however, as we will see in a very special case and discuss in greater generality, this situation cannot occur for nilpotent groups. However, it does occur already for non-nilpotent solvable groups. It is an open problem to determine whether this can occur for groups of intermediate growth.

B. Special cases. We are only going to discuss this topic for the following two groups. Let $A \in SL_2(\mathbb{Z})$ and define

$$G_A = \mathbb{Z} \ltimes_A \mathbb{Z}^2,$$

which as a set can be identified with $\mathbb{Z}^3$, and the group multiplication is defined by

$$(m, x)(n, y) = (m + n, x + A^n y), \quad m, n \in \mathbb{Z} \ x, y \in \mathbb{Z}^2.$$

If A is the identity matrix we recover $\mathbb{Z}^3$, if A is unipotent, e.g.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then G_A is nilpotent, and if A is hyperbolic, e.g.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

then G_A is solvable, but not nilpotent.

We can distinguish between three special elements in G_A, namely

$$a = (1, 0) \quad b = (0, e_2) \quad c = (0, e_1),$$

where e_i, $i = 1, 2$, denote the standard basis in $\mathbb{Z}^2$. In the second case above, these elements satisfy the relations

$$ab = cba \quad ac = ca \quad \text{and} \quad bc = cb,$$

and in the third case, we have

$$ab = cba \quad ac = c^2ab \quad \text{and} \quad bc = cb.$$

Note that if G_A acts by measure-preserving actions on a standard Borel probability measure space (X, μ) and if $B \subset X$ is a positive measure Borel set, then

$$\mu(a^n B \cap b^n B) = \mu(B \cap c^{\gamma_n}(a^{-1}b)^n B)$$

for all n, where γ_n is an integer sequence. If A is unipotent, then γ_n is a quadratic sequence, and thus van der Corput methods (as in the ergodic proof of Sarközy's Theorem) can be utilized. This approach can be further developed for general nilpotent groups, and it can be proved that the analogue of Roth's theorem holds here too ([1]).

If A is hyperbolic, then γ_n grows exponentially and we don't expect any good recurrence properties (for a general action). To construct an explicit example we will use induction of sub-actions. The idea is to consider an isometric action of $\mathbb{Z}^2$ on $\mathbb{T}^1$ (and so given by an element $\tau \in HOM(\mathbb{Z}^2, \mathbb{T})$ with special properties and then induce to a full G_A-action. The special properties can be briefly encoded as the non-recurrence of the element τ under the action of the transpose A^* on $\mathbb{T}^2$; this non-recurrence holds due to the hyperbolicity of A (and thus A^*).

The general construction of such counter-examples is due to Bergelson and Leibman and can be found in [2].

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Sumset Phenomenon and Ergodic Theory

Dominik Kwietniak

This note surveys some recent results on properties of sumsets of infinite sets.

A. Introduction. In 1920, H. Steinhaus [9] proved that the set of distances between the points of a Lebesgue measurable set $A \subset \mathbb{R}$ of positive measure fills up an interval $[0,\varepsilon)$ for some $\varepsilon > 0$. Steinhaus' result has the following refinement, known as Steinhaus Lemma:

Theorem. *Let G be a locally compact group with identity e and let λ denote a left Haar measure on G. If A is a λ-measurable subset of G such that $0 < \lambda(A) < \infty$, then the set AA^{-1} contains an open neighborhood of e.*

A generic "sumset phenomenon" result says that *the sumset of two **large** sets has nontrivial **structure***, where notions of "largeness" and "structure" depend again on the context. The first result of this type for the integers is the one of Jin [8], who showed by methods of non-standard analysis that whenever A and B are sets of integers having positive upper Banach density, the sumset $A+B$ is piecewise syndetic.

B. Notation. Let G be a group. If A and B are subsets of G and $c \in G$, then we write $Ac = \{ac : a \in A\}$, $AB = \{ab : a \in A, b \in B\}$, $A^{-1} = \{a^{-1} : a \in A\}$. If $+$ is used to denote the group operation in G, as we always do if G is abelian, then we denote these sets by $A + c$, $A + B$, and $-A$, respectively. An *interval* of length k is any set I of k consecutive integers. A set $A \subset \mathbb{Z}$ is *syndetic* if for some positive integer k any interval of length k contains an element of A. A set $A \subset \mathbb{Z}$ is *thick* if A contains an interval of length k for any positive integer k. We say that A is *piecewise syndetic* if $A + [0,k]$ is thick for some $k \in \mathbb{N}$. By $|X|$ ($X \triangle Y$, respectively) we denote the cardinality of a set X (the symmetric difference of X and Y, respectively). The relative density of a set A with respect to a finite set B is defined as $D(A|B) = |A \cap B|/|B|$. The *upper Banach density* of a set $A \subset \mathbb{Z}$ is the number

$$d^*(A) = \limsup_{k\to\infty} \max_{n\in\mathbb{Z}} D(A|\{n, n+1, \dots, n+k-1\}).$$

C. Sumset phenomenon for $\mathbb{Z}$. We say that a set $A \subset \mathbb{Z}$ is *large* if $d^*(A) > 0$.

Theorem (Jin 2002 [8]). *If $A, B \subset \mathbb{Z}$ are both large, then $A + B$ is piecewise syndetic.*

This result was strengthened by Bergelson, Furstenberg and Weiss [3]. They replaced the conclusion that $A + B$ is piecewise syndetic by the conclusion that $A + B$ is *piecewise Bohr*. Recall that the *Bohr compactification* of a discrete countable group G is a compact Hausdorff topological group $\mathrm{b}G$ that may be canonically associated to G. It is the largest compact Hausdorff group $\mathrm{b}G$ such that there exists a non-necessarily one-to-one homomorphism $\iota\colon G \to \mathrm{b}G$ with a dense image. It can be proved that every discrete countable group has a unique (up to natural isomorphism) Bohr compactification. A set $B \subset G$ is a *Bohr set* if B contains $\iota^{-1}(U)$ for some nonempty open set $U \subset \mathrm{b}G$. If G is abelian and $\widehat{G}$ denotes the group of characters of G, then one can prove that $\mathrm{b}G = \overline{\iota(G)}$, where $\iota\colon G \to \mathbb{T}^{\widehat{G}}$ by $g \mapsto \{\gamma(g)\}_{\gamma\in\widehat{G}}$. Hence, for an abelian G a set $B \subset G$ is Bohr if there exists an integer $n > 0$, characters $\gamma_1, \ldots, \gamma_n \in \widehat{G}$, and a nonempty open set $U \subset \mathbb{T}^n$ such that $\{g \in G : (\gamma_1(g), \ldots, \gamma_n(g)) \in U\}$ is a nonempty subset of B. For $G = \mathbb{Z}$ a set B is a Bohr set if and only if there exists a *Kronecker* system $(Z, \mathcal{Z}, m_Z, R_\alpha)$ and a nonempty open set $U \subset Z$ such that $\{n \in \mathbb{Z} : R_\alpha^n(e) \in U\}$. A set $B \subset \mathbb{Z}$ is *piecewise Bohr* if it is an intersection of a Bohr set and a thick set. Since every Bohr set is syndetic, piecewise Bohr sets are piecewise syndetic.

Theorem (Bergelson, Furstenberg and Weiss 2006 [3]). *If $A, B \subset \mathbb{Z}$ are both large, then $A + B$ is piecewise Bohr.*

Further refinement comes from Griesmer [6, 7].

Definition. Let $\nu = \{\nu_j\}$ be a sequence of probability measures on $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$. Define an *upper density* of a set $A \subset \mathbb{Z}$ with respect to ν as

$$d_\nu(A) = \limsup_{j\to\infty} \nu_j(A).$$

Definition. A sequence $\nu = \{\nu_j\}$ of probability measures on $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$ is an *equidistributed averaging sequence* if for every $\theta \in (0, 2\pi)$ we have

$$\lim_{j\to\infty} \int \exp(in\theta)\mathrm{d}\nu_j(n) = 0.$$

Theorem (Griesmer 2012 [7]). *Let $\nu = \{\nu_j\}$ be an equidistributed averaging sequence and let $A, B \subset \mathbb{Z}$.*

(1) *If A is large and $d_\nu(B) > 0$, then $A + B$ is piecewise Bohr.*
(2) *If A is large and $d_\nu(B) = 1$, then $A + B$ is thick.*

D. Sumset phenomenon for countable amenable groups. It is natural to ask whether the above theorems are valid in a more general setting. It turns that the last three theorems hold in all countable amenable groups [2, 7]. Those are the groups in which it is possible to define an appropriate notion of density generalizing upper Banach density. A countable discrete group G is *amenable*, if there exists a left invariant, finitely additive, probability measure μ defined for all subsets of G. Equivalently, a countable group G is amenable if there is a Følner sequence in G. A *Følner sequence* in a group G is a sequence $F_1, F_2, \ldots$ of finite subsets of G such that for any $g \in G$ we have

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0.$$

Given a Følner sequence $F = \{F_n\}$ and $A \subset G$ we define an *upper density of* A with respect to F by setting $d_F(A) := \limsup D(A|F_n)$ $(n \to \infty)$. The *upper density* $d^*(A)$ of a set $A \subset G$ is defined as the supremum of $d_F(A)$ over all Følner sequences F in G.

We say that a set $S \subset G$ is *syndetic* if there is a finite set $F \subset G$ such that $F \cdot S = G$. A set $T \subset G$ is *thick* if for each finite set $F \subset G$ there exits a $g \in G$ such that $Fg \subset T$. A set $B \subset G$ is *piecewise syndetic* (*piecewise Bohr*, respectively) if it is an intersection of a syndetic set (Bohr set, respectively) and a thick set. Every piecewise Bohr set is piecewise syndetic, but not conversely.

With all necessary changes the last three theorems hold in the setting of countable amenable groups (see [6, p. 41] for the definition of an *ergodic averaging scheme* generalizing Definition).

E. Concluding remarks and open problems. The original proof of Jin's theorem [8] used nonstandard analysis. A nice and elementary proof of Jin's result may be found in [2]. The proofs in [2, 3, 7] rely on ergodic theory. Recently, Beiglböck [1] provided a beautiful short proof of Jin's theorem and its generalization. The proof is valid for an arbitrary countable amenable semigroup.

This short survey does not exhaust the subject. There is a preprint of di Nasso [5] applying nonstandard analysis tools, and a forthcoming paper of Björklund and Fish [4] contains a wealth of new results.

Let $m_{b\mathbb{Z}}$ be the Haar measure on $b\mathbb{Z}$.

Question (Question 5.1 of [7]). *Let $A \subset \mathbb{Z}$, and let $\tilde{A}$ denote the closure of A in $b\mathbb{Z}$. Which, if any, of the following implications hold?*

(1) *If $m_{b\mathbb{Z}}(\tilde{A}) > 0$ and $d^*(B) > 0$,then $A + B$ is piecewise syndetic.*
(2) *If $m_{b\mathbb{Z}}(\tilde{A}) > 0$ and $d^*(B) > 0$, then $A + B$ is piecewise Bohr.*
(3) *If $\tilde{A} = b\mathbb{Z}$ and $d^*(B) > 0$, then $A + B$ is thick.*

Question (Question 5.2 of [7]). *If $A \subset \mathbb{Z}$ is dense in $b\mathbb{Z}$, is A a set of recurrence, that is, for every measure preserving system $(X, \mathcal{X}, \mu, T)$ and every set $D \in \mathcal{X}$ with $\mu(D) > 0$, there exists $n \in A$ such that $\mu(A \cap T^{-n}A) > 0$?*

Question (Question 5.3 of [7]). *Suppose $A \subset \mathbb{Z}$ has the property that $A + B$ is thick whenever $d^*(B) > 0$. Must the following be true? "For all ergodic measure preserving system $(X, \mathcal{X}, \mu, T)$ and every set $D \in \mathcal{X}$ with $\mu(D) > 0$, we have $\mu\big(\bigcup_{a\in A} T^a D\big) = 1$."*

Question (Question 5.4 of [7]). *Suppose $A \subset \mathbb{Z}$ has the property that $A + B$ is piecewise syndetic (alternatively, piecewise Bohr) whenever $d^*(B) > 0$. What can be said about A?*

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Topological Dynamics and Ramsey Theory

Bálint Farkas

The following classical result of B. L. van der Waerden confirms a conjecture of Baudet [6].

Theorem 1 (van der Waerden). *Let $r \in \mathbb{N}$, and let*

$$\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r$$

be a partition (an r-coloring). Then for every $k \in \mathbb{N}$ there is a monochromatic arithmetic progression of length k, i.e., there is $j_0 \in \{1, \ldots, r\}$ and there are $a, n \in \mathbb{N}$, $n \neq 0$ such that

$$a, a+n, a+2n, \ldots, a+(k-1)n \in C_{j_0}.$$

In [3] H. Furstenberg and B. Weiss realized how effectively methods from topological dynamics can be used to prove this and similar statements in Ramsey theory. That very paper opened a whole new area of research, whose very essentials this talk is concerned with.

The topological dynamical formulation of van der Waerden's theorem reads as follows:

Theorem 2 (van der Waerden, topological version). *Let X be compact space, and let $T : X \to X$ be a homeomorphism of X. Then for every open cover*

$$X = U_1 \cup U_2 \cup \cdots \cup U_r$$

and for every $k \in \mathbb{N}$, there is $j_0 \in \{1, 2, \ldots, r\}$, and there is $n \in \mathbb{N}$, $n \neq 0$, such that

$$U_{j_0} \cap \bigcap_{i=1}^{k-1} T^{-ni} U_{j_0} \neq \emptyset. \tag{1}$$

To see how this result follows from van der Waerden's theorem, we induce a coloring of $\mathbb{N}$ by means of the given open cover and an element $x \in X$: Define

$$\chi(n) := \min\{j \in \{1, \ldots, r\} : T^n x \in U_j\}.$$

Now, an arithmetic progression $a, a+n, \ldots, a+(k-1)n$ of length k in color j_0 yields that $T^a x$ belongs to the intersection given in (1). In turn, van der Waerden's theorem can be deduced from the topological version above by studying the recurrence properties of the shift transformation on a suitable subsystem of the space $\{1, \ldots, r\}^{\mathbb{N}}$ of all r-colorings.

In [1] V. Bergelson and A. Leibman generalized Theorem 2 for polynomial expressions of commuting homeomorphisms. An *integral polynomial* is a polynomial $p \in \mathbb{Z}[x]$ with $p(0) = 0$. Given $T_1, T_2, \ldots, T_t : X \to X$ commuting homeomorphisms and integral polynomials $p_1, p_2, \ldots, p_t$ we call

$$g(n) := T_1^{p_1(n)} T_2^{p_2(n)} \ldots T_t^{p_t(n)}$$

a *polynomial expression.*

Theorem 3 (Bergelson, Leibman). *Let X be a compact space, let $T_1, \ldots, T_t$ be commuting homeomorphisms of X, and let A be a finite set of non-trivial polynomial expressions of these homeomorphisms. Then for every open cover*

$$X = U_1 \cup U_2 \cup \cdots \cup U_r$$

there is $j_0 \in \{1, 2, \ldots, r\}$ and there is $n \in \mathbb{N}$, $n \neq 0$, such that

$$U_{j_0} \cap \bigcap_{g \in A} g^{-1}(n) U_{j_0} \neq \emptyset.$$

The proof of this result uses the Polynomial Exhaustion Technique due to Bergelson, now classically termed as *PET-induction*, which is a powerful tool for proving statements about finite subsets of integral polynomials by induction along a suitable well-ordered set (in this case, the set of weight functions). In the talk I showed some details of the proof.

Remark.

(1) The merit of the topological dynamical approach becomes apparent, when we pass to minimal subsystems. In fact, it is enough to prove Theorem 3 for minimal systems, i.e., when there are no nontrivial closed subsets of X invariant under the action of the transformations. In this case, the following is true: For every nonempty open $V \subseteq X$ there is an $n \in \mathbb{N}$, $n \neq 0$, with

$$V \cap \bigcap_{g \in A} g^{-1}(n) V \neq \emptyset.$$

(2) If $T_1, \ldots, T_t$ act minimally on X and X is metric, then the set of $x \in X$ for which there exists $(n_k) \subseteq \mathbb{N}$ with $g(n_k)x \to x$ for all $g \in A$ as $k \to \infty$ is a dense G_δ set, in particular, it is residual. This explains why results as Theorem 2 may be called *multiple recurrence theorems.*

(3) If we set $A := \{g_j(n) = T^{jn} : j = 1, \dots, t\}$ for some given homeomorphism $T : X \to X$, we obtain the first straightforward generalization of Birkhoff's recurrence theorem (corresponding to the case $t = 1$), i.e., the topological van der Waerden theorem. The case when $A = \{g_j(n) := T_j^n : j = 1, \dots, t\}$ for some commuting homeomorphisms $T_1, T_2, \dots, T_t : X \to X$ is also due to Furstenberg and Weiss [3].

(4) Leibman [5] proved that Theorem 3 remains valid even if $T_1, T_2, \dots, T_t$ generate a nilpotent group. The proof is verbatim the same as the one for Theorem 3 as soon as one defines suitable weights of systems of polynomial expressions (by using bases in nilpotent groups).

(5) The proof of Theorem 3 yields that the set of good $n \in \mathbb{N}$ has a special structure. A subset S of $\mathbb{N}$ is called an *IP set* if there is a sequence $(n_k) \subseteq \mathbb{N}$ such that all nonrepeating, nonempty finite sums formed from the elements of the sequence belong to S. Now, if $T_1, \dots, T_t$ act minimally on X, then there is a $j_0 \in \{1, \dots, r\}$ such that set

$$\Big\{n : U_{j_0} \cap \bigcap_{g \in A} g^{-1}(n) U_{j_0} \neq \emptyset\Big\}$$

intersects every IP set nontrivially, i.e., it is a so-called IP* set.

That IP* sets are in a certain sense large, is the consequence of Hindman's theorem [4].

Theorem 4 (Hindman, additive version). *If*

$$\mathbb{N} = C_1 \cup C_2 \cup \ldots \cup C_r,$$

then there is $j_0 \in \{1, \dots, r\}$ such that C_{j_0} is an IP set.

Theorem 5 (Hindman, set theoretic version). *Let $\mathcal{F}$ be the set of nonempty, finite subsets of $\mathbb{N}$. If*

$$\mathcal{F} = C_1 \cup C_2 \cup \ldots \cup C_r,$$

then there is $j_0 \in \{1, \dots, r\}$ and a sequence (α_i) of disjoint finite nonempty subsets of $\mathbb{N}$ such that α_i together with any finite union of them belongs to C_{j_0}.

These two formulations can easily be seen to be equivalent, by considering the binary expansion of natural numbers and the binary sequence then as a characteristic sequence of a subset of $\mathbb{N}$. The additive version of Hindman's theorem can be conveniently proved by using enveloping semigroups of topological dynamical systems, idempotents therein and the notion of *proximality*, see [3].

Remark. (1) Hindman's theorem yields that IP sets are partition regular. This in turn gives that the dual family of IP*-sets is closed under taking finite intersections, and in fact it is a filter, see [2, Ch. 8]. Note that a subset $S \subseteq \mathbb{N}$ is an IP* set if and only if it is contained in every idempotent ultrafilter over $\mathbb{N}$.

(2) The set $\mathcal{F}$ is directed under the relation: $\alpha < \beta$ iff $\max\alpha < \min\beta$. Given X a compact metric space, a net $(x_\alpha)_{\alpha\in\mathcal{F}} \subseteq X$ and a sequence $(\alpha_j)_{j\in\mathbb{N}} \subseteq \mathcal{F}$ of pairwise disjoint sets, we set $\widehat{x}_\alpha := x_{\bigcup_{j\in\alpha}\alpha_j}$. Then $(\widehat{x}_\alpha)_{\alpha\in\mathcal{F}}$ is called a *sub-IP-net* of $(x_\alpha)_{\alpha\in\mathcal{F}}$. Hindman's theorem is equivalent to the following: Every net $(x_\alpha)_{\alpha\in\mathcal{F}}$ in a compact metric space has a convergent sub-IP-net. The deduction of this statement from the set theoretic version of Hindman's theorem can be carried out along the same lines as the proof of the Bolzano–Weierstraß theorem, but now by using Theorem 5 as the structured infinite pigeon-hole principle.

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Ultrafilters and Coloring Problems

Andreas Koutsogiannis

The goal of this talk is to give applications of ultrafilters to partition Ramsey theory.

The first principle of Ramsey theory is that when we have an infinite highly organized structure (for example a semigroup), then, for any finite coloring of this structure we can find arbitrarily large (sometimes even infinite) monochromatic highly organized substructures. The second principle is that there is a notion of largeness such that any large set contains these highly organized substructures.

The first of the classical results of Ramsey theory are due to Hilbert, Schur and van der Waerden (see [4]).

Hilbert's Lemma states that we can find infinite monochromatic translations of a set of finite sums.

Theorem 1 (Hilbert, 1892). *For any finite coloring $\mathbb{N} = \bigcup_{i=1}^r C_i$, $r \in \mathbb{N}$, and for any $n \in \mathbb{N}$, there exists $1 \le i_0 \le r$ such that for some x_j, $j = 1, \ldots, n$ and for infinitely many t we have*

$$t + FS((x_j)_{j=1}^n) = t + \left\{ \textstyle\sum_{j\in\alpha} x_j : \emptyset \neq \alpha \subseteq \{1, \ldots, n\} \right\} \subseteq C_{i_0}.$$

Schur allows one to omit the translates on the finite sums when $n = 2$.

Theorem 2 (Schur, 1916). *If $\mathbb{N} = \bigcup_{i=1}^r C_i$, $r \in \mathbb{N}$, there exist $1 \le i_0 \le r$ and $x, y, z \in C_{i_0}$ such that $x + y = z$.*

The following result is due to van der Waerden and it is one of the most fundamental results of Ramsey theory.

Theorem 3 (van der Waerden, 1927). *Whenever the natural numbers are finitely partitioned, one of the cells of the partition contains arbitrary long arithmetic progressions.*

One may wonder why it is called Ramsey theory. The reason is that Ramsey's theorem is a more general structural result, not depending on the arithmetical structure of $\mathbb{N}$.

Theorem 4 (Ramsey, 1930). *Let M be an infinite subset of $\mathbb{N}$ and $k,\ r \in \mathbb{N}$. If $[M]^k = \{m_1 < \ldots < m_k :\ m_i \in M,\ 1 \leq i \leq k\}$ and we have $[M]^k = \bigcup_{i=1}^r C_i$, then there exist $1 \leq i_0 \leq r$ and an infinite subset L of M such that $[L]^k \subseteq C_{i_0}$.*

We define the *filters* and *ultrafilters* of the set of the natural numbers and we extend the addition (resp. the multiplication) of $\mathbb{N}$ to the set of ultrafilters of $\mathbb{N}$, $\beta\mathbb{N}$, which is the Stone-Čech Compactification of $\mathbb{N}$. For $X \neq \emptyset$ the *ultrafilters* of X are the finite additive $\{0,1\}$-valued measures on the subsets of X (see [1], [5]).

Via a fundamental result of Ellis, we prove the existence of idempotent ultrafilters and we use this fact in order to give a proof of Hindman's theorem a la Poincaré recurrence (see [1], [2], [5]).

Theorem 5 (Hindman, 1974) *Let $\mathbb{N} = \bigcup_{i=1}^r C_i$, $r \in \mathbb{N}$, be a finite coloring of the set of the natural numbers. Then, there exist an infinite sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ and $1 \leq i_0 \leq r$ such that*

$$FS((x_n)_{n\in\mathbb{N}}) = \{\textstyle\sum_{i\in\alpha} x_i :\ \emptyset \neq \alpha \subseteq \mathbb{N},\ |\alpha| < \infty\} \subseteq C_{i_0}.$$

At this point, note that Hindman's result is a generalization of the results of Hilbert and Schur. In addition, we prove that a subset of the natural numbers is a member of an idempotent ultrafilter if and only if it contains an *IP-set* (i.e. the finite sums of an infinite sequence). Hence, we can say that Hindman's theorem is the density version of itself.

Since there exist IP-sets that do not even contain arithmetic progressions of length 3 (take for example $FS((10^n)_{n\in\mathbb{N}}))$), not every idempotent may reveal something about van der Waerden's theorem. In order to prove van der Waerden's theorem we first prove the existence of a *minimal idempotent* (see [2]) according to the partial order between idempotents

$$\mu \leq \nu \iff \mu * \nu = \nu * \mu = \mu, \text{ where}$$
$$\mu * \nu(A) = \mu(\{n \in \mathbb{N} :\ \nu(\{m \in \mathbb{N} :\ n + m \in A\}) = 1\}) \text{ for every } \mu, \nu \in \beta\mathbb{N},$$
$$A \subseteq \mathbb{N},$$

showing that a subset of the natural numbers which is a member of a minimal idempotent contains arbitrary long arithmetic progressions (these sets are called *central*).

In order to mix the two structures, that of the addition and the multiplication, of $\mathbb{N}$, we prove (see [3]) the existence of an additively and multiplicatively central set in $\mathbb{N}$.

Theorem 6 (Bergelson, Hindman, 1990). *For any finite partition* $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, *there exists* $1 \leq i_0 \leq r$ *such that* C_{i_0} *is both additively and multiplicatively central.*

So, for every finite partition of the set of natural numbers, there exists a combinatorial rich cell of the partition that contains arbitrary large arithmetic progressions, arbitrary large geometric progressions, an additively IP-set, as well as a multiplicatively IP-set.

Finally, we stated some more coloring results concerning words, the proof of which uses the minimal idempotents in an essential way.

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Ultrafilters and Ergodic Theory

Jason Rute

This talk is a survey on how ultralimits (or p-limits) can be used to prove recurrence results in ergodic theory, which in turn have combinatorial consequences such as the following.

Theorem 1. *Partition* $\mathbb{N}$ *into* $A_1 \cup \ldots \cup A_k$. *Then there is some* i $(1 \leq i \leq k)$ *and some* $x, y, z \in A_i$ *such that* $x - y = z^2$.

The main idea is that recurrence results can be strengthened by replacing averaging limits with ultralimits, while the proofs remain similar.

A. Background on ultrafilters. Recall the following definitions and facts.

Definition. An *ultrafilter* p on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ such that the following hold: (1) For all $A, B \subseteq \mathbb{N}$, if $A \in p$ and $A \subseteq B$, then $B \in p$. (2) For all $A, B \subseteq \mathbb{N}$, if $A \in p$ and $B \in p$, then $A \cap B \in p$. (3) $\varnothing \notin p$. (4) If $\mathbb{N} = A_1 \cup \ldots \cup A_k$, then $A_i \in p$ for some i $(1 \leq i \leq k)$.

The collection of ultrafilters on $\mathbb{N}$ is denoted $\beta\mathbb{N}$, and is homeomorphic (under an appropriate topology) to the Stone-Čech compactification of $\mathbb{N}$. Informally, an ultrafilter is a measure of largeness. Some set A is p-large, or has p-measure one if $A \in p$.

Definition. Addition on $\mathbb{N}$ is extended to $\beta\mathbb{N}$ via

$$\forall A \subseteq \mathbb{N} \quad A \in (p+q) \Leftrightarrow \{n \in \mathbb{N} \mid (A-n) \in p\} \in q$$

where $A - n = \{m \in \mathbb{N} \mid m + n \in A\}$.

Definition ([4])**.** Given $p \in \beta\mathbb{N}$, say that

(1) p is *idempotent* if $p + p = p$.
(2) p is *essential idempotent* if p is idempotent and for all $A \in p$, A has positive upper Banach density, i.e.

$$\limsup_{|N-M|\to\infty} \frac{1}{N-M} |A \cap [M, \dots, N-1]| > 0.$$

(3) p is *minimal idempotent* if p is idempotent and p belongs to a minimal right ideal of $(\beta\mathbb{N}, +)$.

Let $A \subseteq \mathbb{N}$. Say that A is an "IP", D, C set if, respectively, $A \in p$ for *some* idempotent, essential idempotent, minimal idempotent $p \in \mathbb{N}$. Say that A is a IP^*, D^*, C^* set if, respectively, $A \in p$ for *all* idempotent, essential idempotent, minimal idempotent $p \in \mathbb{N}$. (C sets are also called *central sets.* What I refer to as "IP" sets can also be characterized combinatorially via Hindman's theorem.)

The previous definitions are measures of "largeness" and they satisfy the following implications

$$IP^* \Rightarrow D^* \Rightarrow C^* \Rightarrow C \Rightarrow D \Rightarrow \text{“}IP\text{”}.$$

B. Ultralimits. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence from a compact Hausdorff space X. Let $p \in \beta\mathbb{N}$.

Definition. Say that $p\text{-}\lim_{n\in\mathbb{N}} x_n = a$ if and only if for all neighborhoods U of a, we have $\{n \in \mathbb{N} \mid x_n \in U\} \in p$. This is the *ultralimit* of (x_n) under p. (An alternate notation is $\lim_{n\to p} x_n$; here $n \to p$ in the topology of $\beta\mathbb{N}$.)

Proposition 1. *The ultralimit $p\text{-}\lim_{n\in\mathbb{N}} x_n$ exists (because X is compact) and is unique (because X is Hausdorff).*

This next proposition is a consequence of ultrafilter addition.

Proposition 2. *Given $p, q \in \beta\mathbb{N}$,*

$$p\text{-}\lim_{n\in\mathbb{N}} q\text{-}\lim_{m\in\mathbb{N}} x_{n+m} = q{+}p\text{-}\lim_{k\in\mathbb{N}} x_k.$$

Hence if $p \in \beta\mathbb{N}$ is idempotent, then

$$p\text{-}\lim_{n\in\mathbb{N}} p\text{-}\lim_{m\in\mathbb{N}} x_{n+m} = p\text{-}\lim_{k\in\mathbb{N}} x_k.$$

C. A refinement of a theorem of Sárközy and Furstenberg. Using ultralimits, one can state and prove the following refinement of a theorem of Sárközy [8] and Furstenberg [7].

Theorem 2 ([1]). *Let $q \in \mathbb{Q}[n]$ such that $q(\mathbb{Z}) \subseteq \mathbb{Z}$, and $q(0) = 0$. Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system and $A \in \mathcal{B}$. For any idempotent $p \in \beta\mathbb{N}$,*

$$p\text{-}\lim_{n\in\mathbb{N}} \mu(A \cap T^{q(n)}A) \geq \mu(A)^2.$$

Also, since p is an arbitrary idempotent, for all $\varepsilon > 0$,

$$\{n \in \mathbb{N} \mid \mu(A \cap T^{q(n)}A) > \mu(A)^2 - \varepsilon\} \text{ is } IP^*. \tag{1}$$

Theorem 1 follows from Theorem 2 by the Furstenberg correspondence principle and the fact that one of the A_i in the partition is both "IP" and has positive upper Banach density (see [2]). This is stronger than the Sárközy and Furstenberg result which only shows that there is a monochromatic x, y such that $x - y$ is a perfect square. The added strength comes from (1) which lets us choose some z in the same part as x and y.

The proof of Theorem 2, however, is similar to more classical versions using averaging limits: The space of L^2 functions is decomposed into two orthogonal subspaces. One decomposition is handled using an idempotent ultralimit version of the van der Corput trick, while the other makes heavy use of Proposition 2.

D. Ultralimits and factors. Using stronger ultrafilters, we can characterize the Kronecker factor and weak mixing.

Proposition 3 ([2, 4]). *Let $(X, \mathcal{B}, \mu, T)$ be an invertible measure preserving system. Let $p \in \beta\mathbb{N}$ be an essential (or minimal) idempotent. Then the Kronecker factor is*

$$\left\{ f \in L^2 \;\middle|\; p\text{-}\lim_{n\in\mathbb{N}} T^n f = f \right\}.$$

(Here the ultralimit is in the weak-topology. Hence the unit ball is compact and the ultralimit exists.)

Idempotent ultrafilters can be used to characterize mild mixing as well (see [2]).

Using essential idempotents and Proposition 3, one can prove a version of Szemerédi's theorem for generalized polynomials [6].

All the ultralimit definitions and facts on $\mathbb{N}$ also extend to any countable group G. Hence, one can use minimal idempotents in βG with a version of Proposition 3 to prove the following nonamenable, noncommutative version of Roth's theorem for groups.

Theorem 3 ([5]). *Let G be a countable group. Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $(T_g)_{g\in G}$ and $(S_g)_{g\in G}$ be measure preserving actions of G such that $T_g S_h = S_h T_g$ for all $g, h \in G$. Then all $A \in \mathcal{B}$ and for all $\varepsilon > 0$,*

$$\{g \in \mathbb{N} \mid \mu(A \cap T_g^{-1}A \cap S_g^{-1}T_g^{-1}A) > \varepsilon\} \text{ is } C^*.$$

For further information on ultralimits in ergodic theory and additive combinatorics, see the surveys [1, 2, 3].

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Multiple Recurrence for Generalized Polynomials

Pavel Zorin-Kranich

The extension by Furstenberg and Katznelson [7] of Furstenberg's multiple recurrence theorem [6] can be formulated as follows. For any commuting measure-preserving transformations $T_1, \dots, T_k$ on a probability space (X, μ) and every $A \subset X$ with $\mu(A) > 0$, for any linear functions $g_i : \mathbb{Z} \to \mathbb{Z}$, $i = 1, \dots, k$, the multiple recurrence set

$$R := \{n \in \mathbb{Z} : \mu(T_1^{-g_1(n)} A \cap \dots \cap T_k^{-g_k(n)} A) > 0\}$$

is syndetic. Furstenberg and Katznelson later showed that R is in fact IP* [8] using methods that can be called "ergodic theory without averaging", in particular IP-limits instead of Cesàro limits. This turned out to be a powerful tool, used for instance in their proof of the density Hales-Jewett theorem [9].

Another generalization has been found by Bergelson and Leibman who showed that R remains syndetic under the assumption that $g_i : \mathbb{Z} \to \mathbb{Z}$ are polynomials vanishing at zero [3]. The first step to combining this result with the above has been made by Bergelson, Furstenberg and McCutcheon in the case $k = 2$ [1]. The main result of their paper states that, for an arbitrary unitary operator U, polynomial $g : \mathbb{Z} \to \mathbb{Z}$ vanishing at zero and IP system (n_α) the weak limit

$$\text{w-IP}_\alpha\text{-lim} U^{g(n_\alpha)}$$

is an orthogonal projection provided that it exists. Since by Hindman's theorem [10] the existence can be assumed without loss of generality (i.e. upon passing to a sub-IP-system), a simple Hilbert space argument shows that R is IP*. This result has been extended to higher values of k by Bergelson and McCutcheon [4] who developed an appropriate version of the Furstenberg-Katznelson structure theory for measure preserving systems. Roughly speaking, in this theory the above Hilbert space projection result was utilized to construct compact sub-extensions of extensions that are not weakly mixing.

The proof of the above Hilbert space projection result relies on a certain algebraic property of the map $\alpha \mapsto g(n_\alpha)$, namely that is an *FVIP system*, i.e. it belongs to a finitely generated group of VIP systems (IP analogues of polynomials) closed under discrete derivatives. The notion of an FVIP system has gained importance after Bergelson, Håland and McCutcheon [2] found new examples that come from *admissible generalized polynomials*. These are, roughly speaking, functions $\mathbb{Z} \to \mathbb{Z}$ obtained from the identity function $n \mapsto n$ using standard algebraic operations and the composition of an $\mathbb{R}$-linear combination with the nearest integer function. Thus for instance $\lfloor an^2 + \lfloor bn + 1/2 \rfloor + 1/3 \rfloor$ is an admissible generalized polynomial while $\lfloor \pi n \rfloor$ is not since $\lfloor \cdot + 1/3 \rfloor$ "looks like" the nearest integer function $\lfloor \cdot + 1/2 \rfloor$ while $\lfloor \cdot \rfloor$ does not. It is arguably the main result of the aforementioned article [2] that for every IP system (n_α) and every admissible generalized polynomial g there exists a sub-IP-ring on which $g(n_\alpha)$ is an FVIP system.

For an FVIP system (g_α) and an arbitrary unitary operator U the weak limit

$$P = \text{w-IP}_\alpha\text{-lim} U^{g_\alpha}$$

is an orthogonal projection provided that it exists. This allowed McCutcheon to extend the multiple recurrence theorem to admissible generalized polynomials [12] along the lines of his earlier joint work with Bergelson [4], obtaining that R is IP* under the assumptions that the functions g_i are admissible generalized polynomials.

All above results generalize to the case when $T_1, \dots, T_k$ generate a nilpotent group as shown by Leibman [11] and the speaker [13]. It should be noted that P fails to be a projection for a general VIP system (see [1] for a counterexample), but it is conjectured that the limit is always a positive operator. If true, such a result could be the first step on the way to a joint extension of the density Hales-Jewett theorem and the polynomial multiple recurrence results.

Another approach to multiple recurrence for generalized polynomials is due to Bergelson and McCutcheon [5] and uses ultralimits instead of IP-limits. It is not yet fully explored, in particular only the case $T_1 = \dots = T_k$ has been treated in a satisfactory generality. The first step in this argument is again a Hilbert space projection theorem, this time stating not only that $P = p\text{-}\lim U^{g(n)}$ is a projection but also that its range is the Kronecker factor. It is also applicable to a class of functions g larger than the admissible generalized polynomials. The fact that the range of P is the Kronecker factor greatly simplifies some of the further steps, for instance one can reuse the classical Furstenberg structure theory. These improvements come at a price: one has to work with essential idempotent ultrafilters p, leading for instance to the conclusion that R is D* (a property weaker than IP* but still stronger than syndeticity).

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Ultrafilters, Nonstandard Analysis and Characteristic Factors

Heinrich-Gregor Zirnstein

A. Introduction. The aim of this talk was to give a short introduction to nonstandard analysis [1, 6] and Loeb measures [3] motivated by the recent application of these techniques to ergodic theory. For instance, Towsner [5] gives a proof of the convergence of multiple ergodic averages by using nonstandard intervals to construct measure preserving systems. Likewise, Szegedy [2] proves structure theorems in additive number theory by working with ultrapowers of the finite groups $\mathbb{Z}/N\mathbb{Z}$.

B. Ultrapowers. The original motivation for nonstandard mathematics was to make rigorous the notions of *infinitely large* and *infinitesimally small numbers*. This can be done by enlarging the set of natural numbers $\mathbb{N}$ to the set of *nonstandard natural numbers* $^*\mathbb{N}$ as follows:

Definition (Ultrapower)**.** Pick a non-principal ultrafilter p on $\mathbb{N}$. The *ultrapower* ${}^*V$ of a set V is defined as the set of equivalence classes of sequences

$$ {}^*V := \prod_{n\to p} V := (\prod_{n\in\mathbb{N}} V)/\sim $$

where two sequences are considered equal if they are equal for "most" indices

$$ (a_n) \sim (b_n) \iff \{n : a_n = b_n\} \in p. $$

The constant sequences embed $V \subseteq {}^*V$. Furthermore, the ultrapower shares many properties with the original set, *all* operations and relations on V can be extended to ${}^*V$ so that any expression that makes sense over V, like $a+b>c$, also makes sense over ${}^*V$. In fact, the ultrapower has essentially the same properties as the original set:

Theorem (Transfer principle)**.** *Let φ be a logical formula with quantifiers that range of the sets $V, \mathcal{P}(V), \ldots$. Let ${}^*\varphi$ denote the formula obtained when replacing the ranges of quantifiers with the ultrapowers ${}^*V, {}^*\mathcal{P}(V), \ldots$. Then, φ holds true if and only if ${}^*\varphi$ holds true.*

For example, the induction principle for standard numbers

$$ \varphi = \forall.S \in \mathcal{P}(\mathbb{N}).\exists s \in \mathbb{N}.\forall x \in \mathbb{N}. s \in S \wedge (x \in S \implies s \leq x) $$

is equivalent to the induction principle for the nonstandard numbers

$$ {}^*\varphi = \forall.S \in {}^*\mathcal{P}(\mathbb{N}).\exists s \in {}^*\mathbb{N}.\forall x \in {}^*\mathbb{N}. s \in S \wedge (x \in S \implies s \leq x). $$

Note that ${}^*\mathcal{P}(V)$ is the ultrapower of the powerset, which is much smaller than the collection $\mathcal{P}({}^*V)$ of all subsets, ${}^*\mathcal{P}(V) \subsetneq \mathcal{P}({}^*V)$. Sets of the former form are called *internal sets.* Thus, the nonstandard induction principle does not apply to all subsets of the nonstandard numbers, only to the internal subsets. Exercise: the set $\mathbb{N}$ is not internal. Likewise, the *internal functions* are defined as sequences of functions.

The first example of an element of ${}^*\mathbb{N}$ that is not in $\mathbb{N}$ is given by the (equivalence class of the) sequence $c \in {}^*\mathbb{N}$ with $c_n = n$. As this sequences eventually grows larger than any constant sequence, it is larger than any standard number, i.e. we have

$$ \forall m \in \mathbb{N}.\ c > m $$

Numbers with this property are called *unbounded*, they are "infinitely large".

We can also introduce the notion of infinitesimal numbers.

Definition. Let $r \in {}^*\mathbb{R}$ be a hyperreal number.

- The hyperreal number r is called *bounded* if it can be bounded by a standard number
$$ \exists L \in \mathbb{R}. |r| < L. $$
- The hyperreal number r is called *infinitesimal* if it is smaller than every standard number
$$ \forall \varepsilon \in \mathbb{R}.\ |r| < \varepsilon. $$

- Every bounded number has a *standard part*

$$\operatorname{st}(r) := \text{ unique } x \in \mathbb{R} \text{ such that } x - r \text{ is infinitesimal}$$

C. The Loeb measure. Having infinite numbers at our disposal, we can also make sense of integrals as infinite sums. The connection to classical measure theory is given by the construction of the Loeb measure.

Theorem. *(Loeb measure) Let $c \in {}^*\mathbb{N}$ be an unbounded nonstandard number and consider the interval of natural numbers $[1, c] \subset {}^*\mathbb{N}$. There exists a σ-algebra $\mathcal{L}$ and a unique measure ν on $\mathcal{L}$, the* Loeb measure*, with the following properties*

- *The σ-algebra $\mathcal{L}$ contains all internal subsets of this unbounded interval.*
- *The Loeb measure counts elements*

$$\nu(A) = \operatorname{st}\left(\frac{|A|}{c}\right) \qquad \text{for } A \text{ internal.}$$

- *The σ-algebra $\mathcal{L}$ is complete in the measure-theoretic sense*

$$A \subseteq B, B \in \mathcal{L}, \nu(B) = 0 \implies A \in \mathcal{L}.$$

Given a measure, we can define measurable and integrable functions in the usual way. For internal functions, the integral is just an unbounded sum.

Lemma. *Let $f : [1, c] \to {}^*\mathbb{R}$ be a finitely bounded internal function. Then, the function* $\operatorname{st} \circ f$ *is integrable and we have*

$$\int \operatorname{st}(f(x))d(x) = \operatorname{st}\left(\frac{1}{c}\sum_{k=1}^{c} f(k)\right).$$

D. A universal measure preserving system. In a sense, the shift on the nonstandard interval $[1, c]$ is the prototypical measure preserving system.

Definition. Let $c \in {}^*\mathbb{N}$ be unbounded. Consider the interval $[1, c]$ and the measure preserving system $([1, c], \mathcal{L}, \nu, T)$ with the transformation

$$T(n) = (n + 1) \mod c.$$

This system is called a *universal system.*

Proposition. *Every standard Borel space $([0, 1], \mathcal{B}, \mu)$ with a measure preserving transformation T is a factor of the universal system.*

Essentially, this just means that the integral in any measure preserving system can be represented by an infinite sum, very much in the spirit of the pointwise ergodic theorem, which tells us that

$$\frac{1}{c}\sum_{n=0}^{c} f(T^n x) \to \int f d\mu,$$

at least in the case where the transformation is ergodic. Thus, the universality can be deduced from the pointwise ergodic theorem, but this is not required, see [4] for a proof from first principles.

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Nilspace factors of ultra product groups and Gowers norms

Balázs Szegedy

Furstenberg's correspondence principle provides a classical example for a limit theory. Assume that for every $i \in \mathbb{N}$ we have a $0-1$ sequence S_i of length i. We say that $\{S_i\}_{i=1}^{\infty}$ is **convergent** if the frequency of every fixed pattern (for example 0110) converges as i goes to infinity. Furstenberg constructed a natural limit object for a convergent sequence $\{S_i\}_{i=1}^{\infty}$ in the form of a measure preserving system $(\{0,1\}^{\mathbb{Z}}, \mathcal{B}, \mu, T)$ where μ is a probability measure on the Borel σ-algebra $\mathcal{B}$ of the compact space $\{0,1\}^{\mathbb{Z}}$ that is invariant under the coordinate-wise shift T. This correspondence creates a very fruitful link between additive combinatorics and dynamical systems.

Dynamical systems in general can be very chaotic. A major tool to isolate manageable structures in them is the concept of factor. Let $(X, \mathcal{A}, \mu, T)$ be a measure preserving system. A **factor** of this system is given by a sub-σ-algebra $\mathcal{B} \subset \mathcal{A}$ with $T^{-1}(\mathcal{B}) \subseteq \mathcal{B}$. A factor is particularly useful if the system $(X, \mathcal{B}, \mu, T)$ is simple enough to analyze but rich enough to carry information about a given question. Let $f \in L^{\infty}(X, \mathcal{A}, \mu)$ be a function. The goal is to make sure that the projection $\mathbb{E}(f|\mathcal{B})$ is similar to f regarding a given property. Philosophically, we can say that $f_s := \mathbb{E}(f|\mathcal{B})$ is the **structured part** and $f_r := f - f_s$ is the noise (or quasi-random part) of f. One of the oldest examples is the **Kronecker factor** $\mathcal{K}$. It is the unique biggest factor which is isomorphic to a compact abelian group with a shift. Furstenbergs's ergodic theoretic proof of Roth's theorem on 3-term arithmetic progressions relies on the fact that projections to this factor preserve averages related to 3-term arithmetic progressions. An important point of view in the subject was brought in by Host and Kra [4] who developed a way of characterizing factors using norms. The norms they introduced are called Host-Kra semi norms and are close relatives of the so-called Gowers norms. Let f be a function on a finite abelian group A. The k-th Gowers norm [3] of f is defined by

$$\|f\|_{U_k}^{2^k} = \mathbb{E}\Big(\Delta_{t_1}(\Delta_{t_2}(\dots \Delta_{t_k} f(x) \dots))\Big)$$

where $\Delta_t f(x) = f(x)\overline{f(x+t)}$ and the expected value is taken over independent random choices of $t_1, t_2, \dots, t_k$ and x.

Our goal is to point out that many useful ideas in ergodic theory can be transferred to a more general picture in the frame of which dynamical systems are replaced by various limit constructions. We continue by describing a convenient framework for limit structures using ultra products.

Let ω be a non-principal ultra filter on $\mathbb{N}$. If $\{X_i\}_{i=1}^{\infty}$ is growing sequence of finite sets then the ultra product space $X = \prod_\omega X_i$ is the set of equivalence classes of $\prod_i X_i$ such that $(x_1, x_2, \dots) \sim (x'_1, x'_2, \dots)$ if and only if $\{i : x_i = x'_i\}$ is a set in ω. Let $\mathcal{Q}$ denote the family of subsets $Y \subseteq X$ which have the form $Y = \prod_\omega Y_i$ where $Y_i \subset X_i$ for every $i \in \mathbb{N}$. Sets in $\mathcal{Q}$ generate a **σ-topology** $\mathcal{T}$ (see [8]) on X that is similar to an ordinary topology but only countable unions of open sets are guaranteed to be open. With this topology X is compact in the sense that if it is covered by a countable union of open sets then there is a finite sub-system which already covers X. We will also consider the σ-algebra $\mathcal{A}$ generated by $\mathcal{T}$. There is a unique **probability measure** μ on $(X, \mathcal{A})$ which has the property that $\mu(Y) = \lim_\omega \mu_i(Y_i)$ for every $Y \in \mathcal{Q}$ where μ_i is the uniform measure on X_i.

The space X equipped with the topology $\mathcal{T}$ and probability space structure $(X, \mathcal{A}, \mu)$ serves as a pre-**limit object** of the spaces X_i. Typically it is assumed that the sets X_i are models of a fixed axiom system (graphs, hypergraphs, groups, fields, vectorspaces, etc...). In this case X is a measurable and topological structure satisfying the same axioms. A **factor of** X is a sub-σ-algebra of $\mathcal{A}$ which preserves certain algebraic structures. The precise definition depends always on the specific context. Limit theories arise when we study projections to these factors. It was demonstrated in [2] that both Szemerédi's regularity lemma [10] and the Hypergraph regularity theory [6] can be obtained by projections of ultra product graphs (resp. hypergraphs) to certain factors. In particular it produces a new proof of Szemerédi's famous theorem [9] on arithmetic progressions. In the rest of this paper we outline an application of the limit framework to groups.

Assume that each space X_i is a finite group G_i and G is the ultra product group. The group G is similar to a compact topological group in the sense that the maps $(x, y) \to xy$ and $x \to x^{-1}$ are continuous in the corresponding σ-topologies.

If each G_i (an so G) is abelian then one can define the Gowers norms U_k on $L^\infty(G)$ in a similar way as for finite groups however they are only semi-norms. We say that a function $f \in L^\infty(G)$ is **k-degree noise** if $\|f\|_{U_{k+1}} = 0$. A function $f \in L^\infty(G)$ is **k-degree structured** if it is orthogonal to every k-degree noise. It is proved in [7] (see also [8]) that for each k there is a σ-algebra $\mathcal{F}_k$ such that $L^\infty(\mathcal{F}_k)$ is the set of k-degree structured functions. It is easy to see that $\{\mathcal{F}_i\}_{i=1}^{\infty}$ is a growing sequence of σ-algebras.

The Hilbert-space $L^2(\mathcal{F}_k)$ is a module over the function algebra $L^\infty(\mathcal{F}_{k-1})$. We denote by $\hat{G}_k$ the set of shift invariant rank one modules over $L^\infty(\mathcal{F}_{k-1})$. It is proved in [7] (see also [8]) that

$$L^2(\mathcal{F}_k) = \bigoplus_{W \in \hat{A}_k} W$$

where the components are orthogonal. It follows that every function $f \in L^2(\mathcal{F}_k)$ has a unique decomposition $f = \sum_{W \in \hat{A}_k} P_W(f)$ where $P_W(f)$ is the projection of f to W. We interpret this as the **k-th order Fourier decomposition** of f. The set $\hat{G}_k$ is an abelian group with respect to point-wise multiplication. We say that $\hat{G}_k$ is the **k-th order dual group** of G.

A further, deeper analysis of k-degree structured functions can be given by using a family of algebraic structures called parallelepiped structures or nil-spaces [5],[1]. It turns out that $\mathcal{F}_k$ is composed of nilspace factors [8] but we omit the details here. This inifinite theory can be used to give inverse theorems for the Gowers norms as described in [8].

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Möbius and Dynamics

Seonhee Lim

The Möbius function $\mu : \mathbb{N} \to \mathbb{R}$ is a function defined by $\mu(n) = (-1)^q$ if $n = p_1 \cdots p_q$ is a product of q distinct primes and $\mu(n) = 0$ otherwise.

The Möbius function is believed to behave randomly. For instance, the Prime Number Theorem is equivalent to

$$\sum_{n \leq N} \mu(n) = o(n),$$

and the Riemann hypothesis is equivalent to $\sum_{n \leq N} \mu(n) = O(n^{1/2+\epsilon})$, for any $\epsilon > 0$.

Another property of the Möbius function is that it has high complexity in the sense that it is realized by values of a function on a system of positive entropy.

P. Sarnak conjectured that the Möbius function is asymptotically orthogonal to any deterministic sequence [6].

Definition. A sequence a_n is deterministic if there exist a compact space X and a continuous map $T : X \to X$ whose topological entropy $h_{top}(T)$ is zero, such that

$$a_n = f(T^n x),$$

for some continuous function f on X and a point $x \in X$.

Conjecture (Sarnak Conjecture [6]). *For any deterministic sequence a_n,*

$$\sum_{n \le N} \mu(n) a_n = o(N).$$

Below, we give some examples of deterministic sequences for which this conjecture has been verified.

A. Background. Let $\mathbb{H}^2$ be the hyperbolic plane. The group $PSL_2(\mathbb{R})$ acts simply transitively on the unit tangent bundle $T^1(\mathbb{H}^2)$, thus we can identify $PSL_2(\mathbb{R})$ with $T^1(\mathbb{H}^2)$. Under this identification, the geodesic flow $\{g^t\}$ on the unit tangent bundle $T^1(\mathbb{H}^2)$ corresponds to multiplying the matrix $\left(\begin{smallmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{smallmatrix}\right)$ on the right, where as a horocycle flow $\{u^s\}$ corresponds to multiplying the matrix $\left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right)$ on the right. One can easily see that nearby vectors diverge exponentially under the geodesic flow, which is a property of a hyperbolic flow. In contrast, nearby vectors diverge polynomially under the horocycle flow, a property of a parabolic flow. These properties are reflected on the fact that the geodesic flow has positive entropy whereas the horocycle flow has zero entropy. Here, by entropy we mean topological entropy. It is the supremum of measure-theoretic entropies over all flow-invariant probability measures (variational principle). It is defined as follows:

Definition. Let us define a family of new metrics:

$$d_N(x, y) = \max_{0 \le t \le N} d(T^t(x), T^t(y)).$$

These metrics can be considered as metrics on the "spaces of orbit segments of length N". A subset $S \subset X$ is (N, ϵ)-separated if $d_N(x, y) > \epsilon$, for any $x, y \in S$. Let $S_{N,\epsilon}$ be a maximal (N, ϵ)-separated set. The volume entropy is the exponential growth rate of the number of "distinguishable orbit segments":

$$h_{top}(T) = \lim_{\epsilon \to 0} \limsup_{N \to \infty} \frac{|S_{N,\epsilon}|}{N}.$$

The topological entropy of the geodesic flow is equal to the exponential volume growth of balls (volume entropy), which is $n - 1$ for $\mathbb{H}^n$, thus 1 for hyperbolic surfaces. The topological entropy of the horocycle flow is zero. Let us use the fact that for $s > 0$,

$$\begin{pmatrix} s^{1/2} & 0 \\ 0 & s^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s^{-1/2} & 0 \\ 0 & s^{1/2} \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^s.$$

Since the topological entropy is invariant under conjugacy, we have

$$h_{top}(u) = h_{top}(u^s) = sh_{top}(u).$$

The last equality follows by the definition of topological entropy. Assuming that the topological entropy of the horocycle flow is finite, we conclude that it must be zero.

Let us consider the orbit closure of the horocycle map. Let us restrict to $\Gamma = SL_2(\mathbb{Z})$ for simplicity. There are obvious orbit closures, namely finite orbits and closed orbits (corresponding to finite points on a horizontal line, and a horizontal line). There are also dense orbits since the horocycle map is ergodic. Ratner's theorem with a simple calculation says that these are the only possibilities. More generally, for a Lie group G, a lattice subgroup Γ and a group U generated by unipotent elements, the orbit closure of U is the orbit of a closed subgroup $H \leq G$.

Next let us consider the following question. Suppose the orbit of x under the horocycle map u is dense in $X = \Gamma\backslash G$. Is the orbit of (x, x) under (u^p, u^q) dense in $X \times X$? For simplicity, let us assume $p = q = 1$. Then we are asking what are the self-joinings of μ. There is one that always exists, namely the trivial joining $\mu \times \mu$. There is also the diagonal joining, which is the pushforward of μ by $x \mapsto (x, x)$. If $g \in G$ is a commensurator element ($g \in Comm(\Gamma)$), i.e. $\Gamma' := \Gamma \cap g\Gamma g^{-1} \subset \Gamma$ is of finite index, then a finite cover joining exists, which is the pushforward of μ' by $\Gamma' x \mapsto (\Gamma x, \Gamma gx)$. Ratner's joining classification theorem says that for the unipotent flow u, the trivial joining and the finite cover joinings are the only possibilities.

B. Sarnak's conjecture and the Bourgain-Sarnak-Ziegler theorem. One reason to believe the conjecture of Sarnak is that it is implied by the Chowla conjecture.

There are cases for which the Sarnak conjecture is known to hold. Some of the cases are:

1. Kronecker flow: G is a compact Abelian group, $g \in G$, and $T(x) = xg$. [3]
2. Homogeneous flow on a compact nilmanifold: $X = \Gamma\backslash N$ where N is a nilpotent Lie group and Γ is a cocompact discrete subgroup of N. [4]
3. Unipotent flow on $X = \Gamma\backslash SL(2, \mathbb{R})$, where Γ is a lattice subgroup. [2]
4. Weak-mixing systems with minimal self-joining property [2]

We remark that Green and Tao were motivated in [4] to verify the Sarnak conjecture for nilflows because of its relation to the Hardy-Littlewood conjecture. They established a stronger form of Sarnak's conjecture for nilmanifolds (with rate $O(\frac{1}{\log^A N})$ for any $A > 0$) and used it in conjunction with the inverse Gowers norm theorem of Green-Tao-Ziegler [5] to verify a special case of the Hardy-Littlewood conjecture, namely that the number of arithmetic progressions of primes of length k is asymptotic to a specific multiple of $\frac{N}{\log^k N}$.

Background explained in the previous section is related to the Sarnak conjecture by the following theorem.

Theorem (Bourgain-Sarnak-Ziegler)**.** *If* $\sum_{n\leq N} f(T^{pn}x)f(T^{qn}x) = o(N)$, *then* $\sum_{n\leq N}\mu(n)a_n = o(N)$.

Now let us sketch the proof of the Sarnak conjecture for the horocycle flow (following Bourgain-Sarnak-Ziegler). By Ratner's joining classification theorem, we know that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n\leq N} f(T^{pn}x)f(T^{qn}x) = \int_{X\times X}(f,f)(T,T)(x,x)d\nu,$$

for some ν which is either the trivial joining μ or a finite cover joining. If ν is the trivial joining, then the above expression equals $\int_X f d\mu \int_X f d\mu$.

(1) If $\int f d\mu = 0$, then it is zero.

(2) If not, then $f = f_1 + c$ where $\int f d\mu = 0$ and $c = \int f d\mu$. Then $\sum \mu(n)f(n) = \sum \mu(n)f_1(n) + \sum \mu(n)C \to 0$, since the first part goes to zero by part (1) and the second part by the Prime Number Theorem.

The remaining case is when ν is a finite cover joining. This is the technical part of the paper of Bourgain-Sarnak-Ziegler. They showed that for most (all but finite) (p, q), this case does not happen, using the fact that $g \in Comm(\Gamma)$.

We remark that the assumption of the Bourgain-Sarnak-Ziegler theorem is very strong, for example, it implies the asymptotic orthogonality of $f(T^n x)$ not only with the Möbius function but also with any bounded multiplicative function.

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Green-Tao and Tao-Ziegler Theorems I

Benny Löffel

This talk is the first of two talks concerning the Green-Tao and Tao-Ziegler Theorems. Green and Tao proved the following result.

Theorem (Green-Tao)**.** *The primes contain infinitely many arithmetic progressions of length k for any k.*

The aim of this first talk is to give an overview of the proof of this theorem as presented in [1], whereas the second talk will focus on some specific aspects of this proof and also discuss the generalization to polynomial progressions, which is a result of Tao and Ziegler [2].

A. A quantitative Szemerédi theorem. One might want to prove this theorem by applying Szemerédi's theorem. Of course, this is not possible directly, since the primes do not have positive upper density in the natural numbers. However, Green and Tao used a "transference principle" which allows them to modify the problem in such a way, that one actually can apply a quantitative version of Szemerédi's theorem to conclude the proof.

We use the following notation. We consider a fixed k and try to find arithmetic progressions of length k in the primes $\leq N$, where we let N go to infinity. For technical reasons we will not work in the interval $[1, N] = \{1, \dots, N\}$, but in the ring $\mathbb{Z}/N\mathbb{Z}$. We say that an element $p \in \mathbb{Z}/N\mathbb{Z}$ is prime if the corresponding element in $[1, N]$ is a prime number. All quantities which appear below depend on N, except mentioned otherwise.

The goal is to apply the following (quantitative) form of Szemerédi's theorem.

Theorem (Szemerédi)**.** *Let $f\colon \mathbb{Z}_N \longrightarrow \mathbb{R}$ be a function such that $0 \leq f \leq 1$ and*

$$\mathbb{E}\big(f(x)|x \in \mathbb{Z}_N\big) \geq \delta \tag{1}$$

for some $\delta > 0$. Then

$$\mathbb{E}\big(f(x)f(x+r)\dots f(x+(k-1)r)|x, r \in \mathbb{Z}_N\big) \geq c(k, \delta) - o(1) \tag{2}$$

for some constant $c(k, \delta)$ which does not depend on f or N.

To see why this theorem implies Szemerédi's theorem, consider f to be the characteristic function of some dense set. In our case, we would like to choose f to be the characteristic function of the primes in the interval $[1, N]$, written as a function $f\colon \mathbb{Z}_N \longrightarrow \mathbb{R}$. Then (2) tells us that there are arithmetic progressions of length k in the primes $\leq N$ (for N large enough). However, this f does not satisfy (1) and so it would be desirable to have a function $g\colon \mathbb{Z}_N \longrightarrow \mathbb{R}$ which satisfies the assumptions of Szemerédi's theorem and such that $E_f \approx E_g$, where

$$E_f = \mathbb{E}\big(f(x)f(x+r)\dots f(x+(k-1)r)|x, r \in \mathbb{Z}_N\big),$$
$$E_g = \mathbb{E}\big(g(x)g(x+r)\dots g(x+(k-1)r)|x, r \in \mathbb{Z}_N\big).$$

To achieve this, Green and Tao use a transference principle.

B. The transference principle. The transference principle gives us such a function g whenever f can be bounded above by some "pseudorandom measure" $\nu\colon \mathbb{Z}_N \longrightarrow \mathbb{R}_+$. A function $\nu\colon \mathbb{Z}_N \longrightarrow \mathbb{R}_+$ is called a measure if $\mathbb{E}(\nu) = 1 + o(1)$. A measure ν is called pseudorandom if it satisfies a "linear forms condition" and a "correlation condition", which we do not discuss further here.

First of all, we have the problem that the primes are not random enough, so we cannot bound their characteristic function by a pseudorandom measure. This regularity arises from the fact that the primes are not uniformly distributed among the equivalence classes modulo p, for any (small) prime number p. To remove this regularity, we use the so called W-trick. We define

$$W = \prod_{p \leq w(N)} p$$

where $w(N)$ goes slowly to infinity as $N \to \infty$. The trick is now, that we do not try to find arithmetic progressions in the set

$$\{n \mid 1 \leq n \leq N,\ n \text{ is prime}\},$$

but in the set

$$\{n \mid 1 \leq n \leq N,\ Wn+1 \text{ is prime}\} \tag{3}$$

which is much more random. Clearly, any arithmetic progression in the set (3) leads to an arithmetic progression in the primes, so it suffices to find arithmetic progressions in the set (3).

For this, we define the function $f\colon \mathbb{Z}_N \longrightarrow \mathbb{R}$ by

$$f(n) = \begin{cases} c_{k,W} \log(Wn+1) & \text{if } Wn+1 \text{ is prime and } \varepsilon_k N \leq n \leq 2\varepsilon_k N, \\ 0 & \text{otherwise.} \end{cases}$$

where $c_{k,W}$ is some constant depending only on k and W and ε_k is some quantity which is chosen such that we have no problems with wrap-arounds.[1]

Note that the support of f is not dense, but f is unbounded as $N \to \infty$. A simple calculation using the prime number theorem in arithmetic progressions shows, that $E_f > \delta$ for some $\delta > 0$ and for all N large enough. Based on work of Goldston and Yıldırım, Green and Tao proved that there exists a pseudorandom measure $\nu\colon \mathbb{Z}_N \longrightarrow \mathbb{R}$ such that $f(x) \leq \nu(x)$ for all $x \in \mathbb{Z}_N$.

The rough idea of the transference principle is now as follows: Whenever f can be bounded by some pseudorandom measure ν, there exists a bounded function g (say $0 \leq g \leq 1$), which behaves almost as f in the sense that $f-g$ is small in some appropriate norm $\|\cdot\|$. Green and Tao have chosen this norm such that whenever $\|f-g\|$ is small so is $\mathbb{E}(f) - \mathbb{E}(g)$ and $E_f - E_g$. To make all this precise, one has to do quite a bit of work. This is discussed in more detail in the second talk.

Actually, in their paper [1], Green and Tao used this transference principle implicitly. In [2], Tao and Ziegler stated the transference principle the first time explicitly, and subsequently Gowers [3] and Reingold, Trevisan, Tulsiani and Vadhan [4] gave (independently) simpler proofs of this principle.

To give a lower bound for E_f, it suffices now to find a lower bound for E_g. But this is easy: Since we know that $\mathbb{E}(f) > \delta$ and $\mathbb{E}(f) \approx \mathbb{E}(g)$ by the remarks above, we get that $\mathbb{E}(g) > \delta$. Then the assumptions of Szemerédi's theorem are satisfied and hence $E_f \approx E_g > c(k,\delta) - o(1)$. This concludes the proof of the Green-Tao theorem.

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Green-Tao and Tao-Ziegler Theorems II

Rene Rühr

This is a summary of the second talk on the Green-Tao theorem ([2]) which states that there are infinitely many progressions of arbitrary length inside the primes. It is proven by a generalization of a quantitative Szemerédi theorem for $\mathbb{Z}_N$ mimicking ergodic theoretic ideas on which we focus:

Theorem (Szemerédi's theorem relative to a pseudorandom measure)**.** *Let f be a non-negative function on $\mathbb{Z}$ that is pointwise bounded by a pseudorandom measure ν. Assume that $\mathbb{E}[f(x)|x \in \mathbb{Z}] \geq \delta$. Then*

$$\mathbb{E}\left[f(x)f(x+m)\dots f(x+(k-1)m)|x,m \in \mathbb{Z}\right] > c(\delta,k) - o_{k,\delta}(1)$$

where $c(\delta,k)$ does not depend on N and $o_{k,\delta}(1)$ is a sequence that goes to zero as $N \to \infty$.

The pseudorandom measure ν is a function[1] that satisfies the normalization

$$\mathbb{E}[\nu(x)|x \in \mathbb{Z}] = 1 + o(1)$$

and two properties, the linear form condition and the correlation condition that are defined (or motivated) during the talk. Suppose that f satisfies the assumption of the theorem. Then the "generalized Koopman-von Neumann structure theorem" allows us to decompose the function f into a uniform part f_U and an anti-uniform part $f_{U^\perp}$ with respect to the Gowers uniformity norm $\|\cdot\|_{U^{k-1}}$:

$$0 \leq f_U + f_{U^\perp} \leq f$$

so that $f_{U^\perp}$ is essentially bounded by 1 and $\|f_U\|_{U^{k-1}}$ is small. As $f_{U^\perp}$ has asymptotically the same expectation as f, we can apply a quantitative version of Szemerédi's theorem [3] to it. On the other hand the "generalized von Neumann theorem" tells us that the average along arithmetic progressions is controlled by the Gowers uniformity norm and so the average

$$\mathbb{E}[f_{U^\perp}(x)f_{U^\perp}(x+m)\dots f_{U^\perp}(x+(k-1)m)|x,m \in \mathbb{Z}]$$

approximates the desired term

$$\mathbb{E}[f(x)f(x+m)\dots f(x+(k-1)m)|x,m \in \mathbb{Z}]$$

asymptotically. We prove the special case for $k=3$, which already includes the main ideas and motivates the linear form condition which states that the measure ν satisfies

$$\mathbb{E}[\nu \circ \phi_1(y) \dots \nu \circ \phi_L(y)|y \in \mathbb{Z}^K] = 1 + o(1)$$

[1] The functions f and ν are actually sequences of functions defined on $\mathbb{Z}_N$ for any N.

where the ϕ_l are arbitrary linear forms $\phi_l : \mathbb{Z}_N^K \to \mathbb{Z}_N$ whose coefficients satisfy some minor properties only depending on k (in particular $K = O(k)$).

Theorem (Generalized von Neumann Theorem)**.** *Let f_0, f_1, f_2 be functions on $\mathbb{Z}_N$ bounded pointwise in modulo by a pseudorandom measure ν, then*

$$\mathbb{E}[f_0(x)f_1(x+m)f_2(x+2m)|x, m \in \mathbb{Z}_N] = \|f_0\|_{U^2} + o(1).$$

We finish by sketching the proof of the following transference principle

Theorem (Generalized Koopman-von Neumann structure theorem)**.** *Let f be a non-negative function bounded by a pseudorandom measure ν and let $\varepsilon > 0$ be sufficiently small. Then there exists a σ-algebra $\mathcal{B}$ and a measurable set $\Omega \in \mathcal{B}$ such that the following three conditions hold.*

$$\begin{aligned} \textit{Smallness Condition:} \quad & \mathbb{E}[\mathbf{1}_\Omega(x)\nu(x)|x \in \mathbb{Z}_N] = o_\varepsilon(1) \\ \nu \textit{ is uniformly distributed:} \quad & \|\mathbf{1}_{\Omega^c}\mathbb{E}[\nu - 1|\mathcal{B}]\|_{L^\infty} = o_\varepsilon(1) \\ \textit{Gowers uniformity:} \quad & \|\mathbf{1}_{\Omega^c}(f - \mathbb{E}[f|\mathcal{B}])\|_{U^{k-1}} \leq \varepsilon^{\frac{1}{2^k}} \end{aligned}$$

We set $f_U = \mathbf{1}_{\Omega^c}(f - \mathbb{E}[f|\mathcal{B}])$ which corresponds to the uniform part of f and define $f_{U^\perp} = \mathbf{1}_{\Omega^c}\mathbb{E}[f|\mathcal{B}]$ so that the second property indeed implies that $f_{U^\perp} = 1 + o(1)$.

The Green-Tao theorem has been generalized in [4] to show that there are infinitely many *polynomial progressions* inside the primes. This paper is also the first to give explicitly an abstract structure theorem whose proof has been simplified in [1] using a finite-dimensional Hahn-Banach theorem.

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Reporter: Nikos Frantzikinakis

Participants

Andrey Alpeev
St. Petersburg State University
Chebyshev Laboratory
14 Liniya V.O. D 29
199178 St. Petersburg
RUSSIAN FEDERATION

Dr. Michael Bateman
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Vitaly Bergelson
Department of Mathematics
Ohio State University
Columbus, OH 43210
UNITED STATES

Dr. Michael Björklund
Departement Mathematik
ETH-Zentrum
Rämistr. 101
8092 Zürich
SWITZERLAND

Rex Cheung
Department of Mathematics
Yale University
10 Hillhouse Avenue
New Haven CT 06511
UNITED STATES

Dr. Nhan-Phu Chung
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig
GERMANY

Dr. Brian Cook
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Dr. Tanja Eisner
Korteweg-de Vries Institute for
Mathematics
Science Park 904
1098 XH Amsterdam
NETHERLANDS

Dr. Daniel El-Baz
21 Sunningdale Clifton
Bristol BS8 2NF
UNITED KINGDOM

Prof. Dr. Gerd Faltings
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Balint Farkas
Fachbereich Mathematik
Bergische Universität Wuppertal
Gaußstraße 20
41119 Wuppertal
GERMANY

Dr. Mikolaj Fraczyk
ul. Zadole 30
40-719 Katowice
POLAND

Prof. Dr. Nikos Frantzikinakis
Department of Mathematics
University of Crete
Knossou Ave., POBox 2208
71409 Heraklion, Crete
GREECE

Prof. Dr. Alexander Gamburd
Department of Mathematics
The Graduate Center (CUNY)
365 Fifth Avenue
New York NY 10016-4309
UNITED STATES

Prof. Dr. Roland Gunesch
Fachbereich Mathematik
TU Darmstadt
Schloßgartenstr. 7
64289 Darmstadt
GERMANY

Dr. Markus Haase
Delft Institute of Applied Mathematics
Delft University of Technology
Mekelweg 4
2628 CD Delft
NETHERLANDS

Dr. Hiep Han
Instituto de Matematica e
Estatistica
Universidade de Sao Paulo (IME-USP)
Rua do Matao 1010
Sao Paulo 05508-090 - SP
BRAZIL

Dr. Elad Horev
Department Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Adam Kanigowski
Wydzial Matematyki UMK
Nicholas Copernicus University
ul. Chopina 12/18
87 100 Torun
POLAND

Dimitris Karageorgos
Leof. Stratarchou
Alexandrou Papagou 115
Zografou
15773 Athens
GREECE

Dr. Tom M. W. Kempton
Mathematisch Instituut
Universiteit Utrecht
Budapestlaan 6
P. O. Box 80.010
3508 TA Utrecht
NETHERLANDS

Ilya Khayutin
Institute of Mathematics
The Hebrew University
Givat-Ram
91904 Jerusalem
ISRAEL

Dr. Dong Han Kim
Department of Mathematics Education
Dongguk University
26 Pil-dong 3-ga, Jung-gu
Seoul 100 715
KOREA, REPUBLIC OF

Jakub Konieczny
Jagiellonian University
Institute of Mathematics
Lojasiewicza 6
30-348 Krakow
POLAND

Dr. Andreas Koutsogiannis
Department of Mathematics
University of Athens
Panepistemiopolis
157 84 Athens
GREECE

Benjamin Krause
535 Kelton Avenue, Apartment #9
Los Angeles, CA 90024
UNITED STATES

Dr. Alexander P. Kreuzer
Fachbereich Mathematik
Arbeitsgruppe Logik
TU Darmstadt
Schlossgartenstr. 7
64289 Darmstadt
GERMANY

Dr. Joanna Kulaga-Przymus
Wydzial Matematyki UMK
Nicholas Copernicus University
ul. Chopina 12/18
87 100 Torun
POLAND

Dr. Akshat Kumar
4639 Gerrilyn Way, Apt. 228
Livermore, CA 94550
UNITED STATES

Dr. Dominik Kwietniak
Jagiellonian University
Institute of Mathematics
Lojasiewicza 6
30-348 Krakow
POLAND

Prof. Dr. Seonhee Lim
Department of Mathematical Sciences
Seoul National University
Seoul 151-747
KOREA, REPUBLIC OF

Benny Löffel
Departement Mathematik
ETH-Zentrum
Rämistr. 101
8092 Zürich
SWITZERLAND

Dr. Neil Lyall
Department of Mathematics
University of Georgia
Athens, GA 30602
UNITED STATES

Dr. Manfred Madritsch
Equipe Theorie de Nombres / IECN
Université de Lorraine
B.P. 70239
54506 Vandoeuvre-les-Nancy Cedex
FRANCE

Prof. Dr. Akos Magyar
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Dr. Romanos Malikiosis
Division of Mathematical Sciences
Nanyang Technological University
21 Nanyang Link
Singapore 637371
SINGAPORE

Dr. Freddie Manners
Montford House
Montford, Shrewsbury
Shropshire SY4 1AA
UNITED KINGDOM

Dr. Benjamin Matschke
I.H.E.S.
Le Bois Marie
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Przemyslaw Mazur
ul. Krolewska 47/39
30-040 Krakow
POLAND

Dr. Rudi Mrazovic
Milana Resetara 15
10090 Zagreb
CROATIA

Prof. Dr. Rainer Nagel
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Rene Rühr
Departement Mathematik
ETH-Zentrum
Rämistr. 101
8092 Zürich
SWITZERLAND

Jason Rute
Department of Mathematical Sciences
Carnegie Mellon University
6113 Wean Hall
Pittsburgh PA 15213
UNITED STATES

Dr. Oliver Sargent
Department of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
UNITED KINGDOM

Dr. Wenbo Sun
Department of Mathematics
Northwestern University
2033 Sheridan Road
Evanston, IL 60208-2730
UNITED STATES

Prof. Dr. Balazs Szegedy
Department of Mathematics
University of Toronto
40 St George Street
Toronto, Ont. M5S 2E4
CANADA

Prof. Dr. Terence Tao
UCLA
Department of Mathematics
Los Angeles, CA 90095-1555
UNITED STATES

Tatchai Titichetrakun
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Ioannis Tsekouras
Department of Mathematics
University of Crete
Knossou Ave., POBox 2208
71409 Heraklion, Crete
GREECE

Dr. Konstantinos Tyros
Department of Mathematics
University of Toronto
40 St George Street
Toronto, Ont. M5S 2E4
CANADA

Prof. Dr. Evgeny Verbitskiy
Mathematisch Instituut
Universiteit Leiden
Postbus 9512
2300 RA Leiden
NETHERLANDS

Prof. Dr. Tamar Ziegler
Department of Mathematics
Technion - Israel Institute of
Technology
Haifa 32000
ISRAEL

Pavel Zorin-Kranich
Korteweg-de Vries Instituut
Universiteit van Amsterdam
Postbus 94248
1090 GE Amsterdam
NETHERLANDS

Heinrich-Gregor Zirnstein
Lutherstr. 18
04315 Leipzig
GERMANY