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C*-Algebras, Dynamics, and Classification

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Abstract. Classification is a central theme in mathematics, and a particularly rich one in the theory of operator algebras. Indeed, one of the first major results in the theory is Murray and von Neumann's type classification of factors (weakly closed self-adjoint algebras of operators on Hilbert space with trivial center), and one of its modern touchstones is the mid-1970s Connes-Haagerup classification of amenable factors with separable predual. Several significant themes in the classification theory of norm-separable C*-algebras have emerged since the work of Connes-Haagerup, and these were the focus of our workshop. They include Elliott's program to classify separable nuclear C*-algebras via K-theoretic invariants, the role of C*-algebras in the classification of orbit equivalence relations of discrete countable group actions, and the more recent contact between descriptive set theorists and operator algebraists which seeks to quantify the Borel complexity of the isomorphism relation for various natural classes of algebras.

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Introduction by the Organisers

The workshop ran in the traditional Oberwolfach style, with plenty of time reserved for interaction outside the regular program of lectures. We had 56 participants; there were 29 talks (45 or 30 minutes) and an informal additional lecture on Thursday night. We give an overview of the scientific background and the objectives of the workshop below.

A. K-theoretic rigidity and a conjecture for nuclear separable C*-algebras. The classification theory for norm-separable C*-algebras (norm-closed self-adjoint algebras of operators on Hilbert space) was begun by Glimm in 1960 when he classified uniformly hyperfinite (UHF) C*-algebras using what was later realized to be their K-theory. This was followed by the Bratteli-Elliott classification of approximately finite-dimensional (AF) C*-algebras by what would again, ultimately, turn out to be their K-theory. These results prompted Elliott to conjecture c. 1990 that separable nuclear C*-algebras are classifiable via invariants of a K-theoretic nature, a conjecture which has since been worked on extensively and with considerable success.

In the past five years the state of knowledge around Elliott's conjecture for simple C*-algebras has advanced rapidly, particularly in the case that the projections of the algebra separate its tracial functionals. For instance, we now know that the C*-algebras associated to minimal uniquely ergodic dynamics on finite-dimensional spaces are determined up to isomorphism by their graded ordered K-theory. At the center of these developments are the Jiang-Su algebra $\mathcal{Z}$ and the attendant property of $\mathcal{Z}$-stability (a C*-algebra A is $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$). This sort of tensorial absorption property is ubiquitous in operator algebra classification: Connes' proof that an amenable II$_1$ factor $\mathcal{M}$ with separable predual is the hyperfinite factor $\mathcal{R}$ proceeded by showing first that $\mathcal{M}\bar{\otimes}\mathcal{R} \cong \mathcal{M}$; the Kirchberg-Phillips classification of simple purely infinite C*-algebras relies heavily on the fact that any such algebra A satisfies $A \cong A \otimes \mathcal{O}_\infty$ for the Cuntz algebra $\mathcal{O}_\infty$. But not all simple separable nuclear C*-algebras are $\mathcal{Z}$-stable, in contrast with the tensorial absorption properties of factors and purely infinite algebras. Why so? Very roughly, the latter two classes of algebras are non-commutative generalizations of low-dimensional spaces, while general C*-algebras may exhibit characteristics of higher-, even infinite-dimensional topological spaces. Here as in the classical case, one expects many strong theorems to hold only for C*-algebras which are finite-dimensional in a suitable sense. This brings us to a conjecture that was a focus of the workshop, one that relates $\mathcal{Z}$-stability to topological and homological notions of finite-dimensionality for C*-algebras.

Conjecture 0.1. *Let A be a unital simple separable nuclear C*-algebra. The following are equivalent:*

(i) A has finite nuclear dimension;
(ii) $A \otimes \mathcal{Z} \cong A$;
(iii) A has strict comparison.

A detailed exposition of properties (i) and (iii) is beyond the scope of this introduction. Let us mention only that nuclear dimension generalizes the classical covering dimension of a space to the realm of C*-algebras, and that strict comparison means, roughly, that the pre-order on Hilbert modules over A given by inclusion up to isomorphism is determined by the rank of the modules as measured by traces. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are known, and (iii) $\Rightarrow$ (ii) holds under some additional conditions. The implication (ii) $\Rightarrow$ (i) so far is only known for classes where Elliott's classification conjecture holds, so that Elliott's program is a central facet of the conjecture. The equivalence of (ii) and (iii) would represent a broad generalization of Kirchberg's celebrated $\mathcal{O}_\infty$ stability theorem for nuclear simple separable purely infinite C*-algebras; in the case of finitely many extremal traces it has indeed recently been established by Matui and Sato.

One goal of the workshop was to make progress on this conjecture (and its implications for the structure theory of nuclear C*-algebras) both through Elliott's program and through work on the structure of Hilbert modules over nuclear C*-algebras. Promising results towards a generalization of the conjecture to the non-simple and non-unital case were also discussed. Another (closely related) theme was Kirchberg's program to classify nuclear purely infinite (not necessarily simple) C*-algebras and its connections with graph C*-algebras and with semigroup C*-algebras introduced recently by Cuntz and Li to study problems related to number theory.

B. Group actions and operator algebras. The study of orbit equivalence relations of group actions can be studied in the measurable, Borel, and topological settings, and all three areas have seen substantial progress in recent years. This workshop was concerned principally with the topological setting, and with the tight links to regularity properties as discussed above. The following questions formed perhaps a Leitmotiv.

Questions 0.2. *Let (X, G, α) be a dynamical system (with, say, X compact metrizable and G discrete, countable and amenable).*

- *(i) To what extent is the crossed product* C**-algebra $C(X) \rtimes_\alpha G$ determined by its Elliott invariant? In other words, when are transformation group* C**-algebras classifiable?*
- *(ii) To what extent does the crossed product* C**-algebra $C(X) \rtimes_\alpha G$ determine the underlying dynamical system?*

Progress on the first question is interesting since the classifying invariant is often computable. For the second question one would typically not expect the dynamical system to be determined up to isomorphism, but up to some suitable weaker notion (based on orbit equivalence), which in good cases will still be able to detect and isolate pertinent properties of the underlying dynamical system. Both questions are usually hard even when $G = \mathbb{Z}$.

When X is the Cantor set, $G = \mathbb{Z}$ and α is minimal, there are highly satisfactory answers to both questions: the transformation group C*-algebras are simple AT algebras, hence classified by their Elliott invariants (the latter essentially consisting of ordered K-groups, tracial state spaces and natural pairings between these), and they determine the underlying (Cantor minimal) system up to strong orbit equivalence; these results have only very recently been partially extended to the case of $\mathbb{Z}^d$-actions.

In the case that X is finite-dimensional, $G = \mathbb{Z}$, and α is minimal and uniquely ergodic, the first question has recently been settled; in this situation the transformation group C*-algebras are entirely determined by their ordered K-groups.

The focus here will be on ways to eliminate the trace space condition; we will also look at $\mathbb{Z}^d$-actions. As for the second question, strong orbit equivalence degenerates if the base space is connected, so one cannot expect the dynamical system to be determined up to strong orbit equivalence unless X is zero-dimensional. This prompts the question whether there is a higher rank version of the strong orbit equivalence relation; in a somewhat analogous context, a higher rank version of the Rokhlin property has been used successfully for the study of C*-dynamical systems.

C. Borel complexity and operator algebras. The classification of a category by invariants can only be reasonable if the invariants are somehow definable or calculable from the objects of the original category. For example, it is easily seen that there are at most continuum many non-isomorphic separable C*-algebras, and so it is possible, in principle, to assign to each isomorphism class of separable C*-algebras a unique real number, thereby classifying the separable C*-algebras completely up to isomorphism. Few mathematicians working in C*-algebras would find this a satisfactory solution to the classification problem for separable C*-algebras, let alone nuclear simple separable C*-algebras, since we do not obtain a way of computing the invariant, and therefore do not have a way of effectively distinguishing the isomorphism classes.

Since descriptive set theory is the theory of definable sets and functions in Polish spaces, it provides a natural framework for studying classification problems. In the past 30 years, a theory has been developed based on the observation that in many cases where the objects of the category are themselves either countable or separable, there is a natural standard Borel space which parametrizes (up to isomorphism) these objects. A classification problem is therefore a pair (X, E) consisting of a standard Borel space X, the (parameters for) objects to be classified, and an equivalence relation E, the relation of isomorphism among the objects in X. In most interesting cases, the equivalence relation E is easily definable from the elements of X, and is seen to be Borel or, at least, analytic.

Let (X, E) and (Y, F) be classification problems, in the above sense. A *Borel reduction* of E to F is a Borel function $f : X \to Y$ such that $xEy \iff f(x)Ff(y)$. If such a function f exists then we say that E is *Borel reducible* to F, and we write $E \leq_B F$. We think of E as being "at most as complicated" as F.

The application of this theory of Borel complexity to operator algebras has its roots in the work of Glimm and Effros from the 1960s, but has recently begun to take off. Sasyk and Törnquist have studied the Borel complexity of various classes of von Neumann factors, and proved, among other things, that even isomorphism of amenable type III factors is *turbulent*, a notion of very high Borel complexity introduced by Hjorth. Kerr, Li, and Pichot have obtained similar results for certain representation spaces and group actions on the hyperfinite II_1 factor, while Farah, Toms, and Törnquist have established turbulence for the isomorphism of nuclear simple separable C*-algebras (interestingly, the proof of this theorem depends on the validity of Elliott's conjecture for the so-called AI algebras).

Some questions in these directions addressed during the workshop include:

(1) Is the isomorphism relation for nuclear simple separable C*-algebras Borel reducible to the orbit equivalence relation of a Polish group action?
(2) Given a classification of a category of operator algebras by invariants, i.e., a theorem which guarantees the lifting of isomorphisms at the level of invariants to operator algebra isomorphisms, when is the lift Borel computable?
(3) What is the Borel complexity of the space of all separable C*-algebras? What about exact, or nuclear C*-algebras, or nuclear ones satisfying the Universal Coefficient Theorem?
(4) Can Borel complexity be employed to produce new examples of C*-algebras? (For example, nuclear C*-algebras which do not have locally finite nuclear dimension?)

Workshop: C*-Algebras, Dynamics, and Classification

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Abstracts

K-theory of crossed products by automorphic semigroup actions

Siegfried Echterhoff
(joint work with Joachim Cuntz and Xin Li)

Let $e \in P \subseteq G$ be a sub-semigroup of the countable group G. The left-reduced C*-semigroup algebra $C^*_\lambda(P)$ is defined as the sub-C*-algebra of the bounded operators $\mathcal{B}(\ell^2(P))$ generated by the set of isometries $\{V_p : p \in P\}$ with

$$V_p : \ell^2(P) \to \ell^2(P); V_p\delta_q = \delta_{pq},$$

where $\{\delta_q : q \in P\}$ denotes the standard orthonormal basis of $\ell^2(P)$.

An important example which motivated much of our work is given by the class of $ax+b$-semigroups $R \rtimes R^\times \subseteq K \rtimes K^\times$ in which R is the ring of integers in a number field K and $R^\times = R \setminus \{0\}$. In this case the semigroup C*-algebra has been studied extensively by Cuntz, Deninger, and Laca in [1], where the authors computed the KMS-states of the semigroup algebra $C^*_\lambda(R \rtimes R^\times)$ which turned out to have close connections to the Dedekind ζ-functions associated to the ideal classes for K. The results presented here have been motivated by the desire to understand the K-theory of these interesting algebras connected to number theory, but our results apply in many other interesting cases of sub-semigroups $P \subseteq G$.

Let $P \subseteq G$ be given. For $X \subseteq P$ let $E_X : \ell^2(P) \to \ell^2(X) \subseteq \ell^2(P)$ denote the orthogonal projection. Then

$$V_p E_X V_p^* = E_{pX} \quad \text{and} \quad V_p^* E_X V_p = E_{p^{-1}X}$$

where $p^{-1}X := \{q \in P : pq \in X\}$ denotes the inverse image of X under multiplication with p. Let $\mathcal{J}_P$ denote the smallest set of subsets $X \subseteq P$ which contains $\varnothing$ and P, and which is closed under finite intersections and the operations $X \mapsto pX, p^{-1}X$ as considered above. We call $\mathcal{J}_P$ the set of *constructible right ideals in* P. Since we assume that the unit e of G lies in P, we get $E_P = V_e \in C^*_\lambda(P)$, from which it then follows that $\{E_X : X \in \mathcal{J}_P\} \subseteq C^*_\lambda(P)$. In fact, $C^*_\lambda(P)$ equals the reduced semi-group crossed product $D_P \rtimes_\lambda P$ where $D_P \subseteq C^*_\lambda(P)$ denotes the C*-algebra generated by $\{E_X : X \in \mathcal{J}_P\}$.

Following ideas of Laca in the case of Ore semigroups (see [6]), which were extended to more general situations by Li ([8]), we want to dilate the semigroup action of P on D_P to an action of G on some algebra $D_{P\subseteq G}$. For this let $\mathcal{J}_{P\subseteq G}$ denote the G-saturation of $\mathcal{J}_P$ as subsets of G and let $D_{P\subseteq G}$ denote the commutative C*-subalgebra of $\mathcal{B}(\ell^\infty(G))$ generated by the orthogonal projections $\{E_Y : Y \in \mathcal{J}_{P\subseteq G}\}$. By an argument due to Fell the reduced crossed product $D_{P\subseteq G} \rtimes_\lambda G$ is faithfully represented in $\mathcal{B}(\ell^2(G))$ by the canonical representation in such a way that

$$C^*_\lambda(P) \subseteq E_P(D_{P\subseteq G} \rtimes_\lambda G)E_P, \tag{1}$$

where the orthogonal projection $E_P : \ell^2(G) \to \ell^2(P)$ is always a full projection in $D_{P\subseteq G} \rtimes_\lambda G$. We say that $P \subseteq G$ satisfies the *weak Toeplitz condition* if we have equality in (1) (this condition is implied by the stronger Toeplitz condition of [8, Definition 4.1]). It is shown in [8, Lemma 3.9] that the picture can be extended to crossed products by actions $\alpha : G \to \mathrm{Aut}(A)$ of G on a C*-algebra A. If $P \subseteq G$ satisfies the above (weak) Toeplitz condition we get

$$A \rtimes_\lambda P = (1_H \otimes E_P)((A \otimes D_{P\subseteq G}) \rtimes_\lambda G)(1_H \otimes E_P), \tag{2}$$

if we represent A faithfully and non-degenerately into $\mathcal{B}(H)$ and $(A \otimes D_{P\subseteq G}) \rtimes_\lambda G$ into $\mathcal{B}(H\hat{\otimes}\ell^2(G))$. Thus, the Toeplitz-condition implies that $K_*(A \rtimes_\lambda P) \cong K_*((A \otimes D_{P\subseteq G}) \rtimes_\lambda G)$ by Morita equivalence.

We want to use this picture for computing the K-theory of $C^*_\lambda(P)$ and, more generally, $A \rtimes_\lambda P$. For this we observe that $D_{P\subseteq G} \cong C_0(\Omega_{P\subseteq G})$ for some totally disconnected space $\Omega_{P\subseteq G}$. So we look to general actions of a countable group G on totally disconnected spaces Ω. In what follows we denote by $C_c^\infty(\Omega)$ the set of locally constant functions on Ω with compact supports. In order to explain our main result assume that there exists a linear basis $\mathcal{P} = \{p_i : i \in I\}$ of $C_c^\infty(\Omega)$ consisting of projections in $C_0(\Omega)$ such that $\mathcal{P}$ is G-invariant and closed under multiplication (up to 0). Then there is an action of G on the discrete space I such that $g \cdot p_i = p_{gi}$ for all $i \in I$. Moreover, there is a unique $*$-homomorphism

$$\mu : C_0(I) \to C_0(\Omega) \otimes \mathcal{K}(\ell^2(I))$$

which sends a Dirac-function $\delta_i \in C_0(I)$ to the projection $p_i \otimes d_i$ if $d_i : \ell^2(I) \to \mathbb{C}\delta_i \subseteq \ell^2(I)$ denotes the orthogonal projection to the subspace spanned by δ_i. The action of G on I induces a unitary representation $U : G \to \mathcal{U}(\ell^2(I))$ and the homomorphism μ becomes G-equivariant with respect to the given action on $C_0(I)$ and the action $\tau \otimes \mathrm{Ad}\, U$ on $C_0(\Omega) \otimes \mathcal{K}(\ell^2(I))$, where τ denotes the action on $C_0(\Omega)$. Thus μ determines a class $[\mu] \in KK^G\big(C_0(I), C_0(\Omega) \otimes \mathcal{K}(l^2(I))\big) \cong KK^G\big(C_0(I), C_0(\Omega)\big)$. Our central result is the following (see [4, §3]):

Theorem 1. *Let G, Ω, and $\{p_i : i \in I\}$ be as above and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of G on a C^*-algebra A. Assume in addition that G satisfies the Baum-Connes conjecture for $A \otimes C_0(I)$ and $A \otimes C_0(\Omega)$. Then the descent*

$$[\mathrm{id}_A \otimes \mu] \rtimes_\lambda G \in KK\big((A \otimes C_0(I)) \rtimes_\lambda G, (A \otimes C_0(\Omega)) \rtimes_\lambda G\big)$$

induces an isomorphism of K-theory groups

$$K_*\big((A \otimes C_0(I)) \rtimes_\lambda G\big) \cong K_*\big((A \otimes C_0(\Omega) \rtimes_\lambda G\big).$$

Since I is discrete, it follows from Green's imprimitivity theorem that there is a Morita equivalence

$$(A \otimes C_0(I)) \rtimes_\lambda G \sim_M \bigoplus_{[i]\in G\backslash I} A \rtimes_\lambda G_i,$$

where $G_i = \{g \in G : gi = i\}$ is the stabilizer of $i \in I$. Thus, if the conditions of the above theorem are satisfied, we obtain an isomorphism

$$K_*\big((A \otimes C_0(\Omega)) \rtimes_\lambda G\big) \cong \bigoplus_{[i]\in G\backslash I} K_*(A \rtimes_\lambda G_i).$$

The proof of the theorem uses a principle observed in [2] which, using the Baum-Connes assumption, allows to reduce the above theorem to the case of actions of finite groups and finite dimensional algebras, in which the result can be shown by some more or less elementary combinatorics. We refer to [4, §3] for more details. We should point out that by a seminal theorem of Higson and Kasparov ([5]) the Baum-Connes assumption is always satisfied if G is a-T-menable (or amenable). Moreover, under some extra condition which we don't explain here, we can even obtain KK-equivalence of the crossed products.

It turned out that the condition on the existence of a G-invariant and multiplicatively closed (up to 0) basis $\{p_i : i \in I\}$ of $C_c^\infty(\Omega)$ is quite restrictive for general actions on totally disconnected spaces. For example, it is never satisfied if an amenable group G acts minimally on the Cantor set Ω (see [4, Proposition 3.18]). But somehow surprisingly, the condition is very often satisfied for the dilated action of G on $\Omega_{P\subseteq G}$ if we start with a sub-semigroup $P \subseteq G$ as above. It is then implied by the following independence condition for $P \subseteq G$:

Definition. We say that $P \subseteq G$ satisfies the *independence condition* if the set $\mathcal{J}_{P\subseteq G}$ of constructible right P-ideals in G is independent in the following sense: If $X, X_1, \ldots, X_l \in \mathcal{J}_{P\subseteq G}$ such that $X = \cup_{i=1}^{l} X_i$, then $X = X_{i_0}$ for some $i_0 \in \{1, \ldots, l\}$.

Namely, if $P \subseteq G$ satisfies this independence condition, then $\{E_X : X \in I_{P\subseteq G}\}$ with $I_{P\subseteq G} := \mathcal{J}_{P\subseteq G} \smallsetminus \{\varnothing\}$ is a basis for $C_c^\infty(\Omega_{P\subseteq G})$ as desired. We then get

Theorem 2. Suppose that $P \subseteq G$ satisfies the (weak) Toeplitz condition and the independence condition. Suppose further that G acts on the C*-algebra A such that G satisfies the Baum-Connes conjecture for $A \otimes C_0(I_{P\subseteq G})$ and $A \otimes C_0(\Omega_{P\subseteq G})$. Then

$$K_*(A \rtimes_\lambda P) \cong \bigoplus_{[X] \in G \backslash I_{P\subseteq G}} K_*(A \rtimes_\lambda G_X)$$

where $G_X = \{g \in G : gX = X\}$. In particular, in case $A = \mathbb{C}$ we get

$$K_*(C_\lambda^*(P)) \cong \bigoplus_{[X] \in G \backslash I_{P\subseteq G}} K_*(C_\lambda^*(G_X)).$$

It is shown in [4] that many interesting classes of sub-semigroups $P \subseteq G$ satisfy these conditions. Among them are the quasi-lattice ordered semigroups $P \subseteq G$ which are characterized by the conditions $P \cap P^{-1} = \{e\}$ and for all $g \in G$ there exists a $p \in P$ with $P \cap gP = pP$. In this case G acts freely and transitively on $I_{P\subseteq G}$ and hence our results imply that $K_*(A \rtimes_\lambda P) \cong K_*(A)$ if G satisfies the Baum-Connes for $A \otimes C_0(\Omega_{P\subseteq G})$. For a number of other interesting applications we refer to [3, 4] and [9]. For our motivating example $R \rtimes R^\times \subseteq K \rtimes K^\times$ we get

Theorem 3. *Let R be a Dedekind domain and let $K = \mathcal{Q}(R)$ denote its quotient field. Let $R \rtimes R^\times \subseteq K \rtimes K^\times$ denote the corresponding $ax+b$-semigroup and let Cl_K denote the ideal class group of K. Then*

$$K_*(C_\lambda^*(R \rtimes R^\times)) \cong \bigoplus_{\gamma \in Cl_K} K_*(C_\lambda^*(I_\gamma \rtimes R^*)),$$

where $I_\gamma \subseteq R$ denotes a representative for γ and R^* denotes the group of units in R. In fact, the isomorphism is induced by a KK-equivalence between $C^*_\lambda(R \rtimes R^\times)$ and $\bigoplus_{\gamma \in Cl_K} C^*_\lambda(I_\gamma \rtimes R^*)$.

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The Cuntz semigroup of close C^*-algebras

Stuart White

(joint work with Francesc Perara, Andrew Toms and Wilhelm Winter)

The uniform distance between two operator algebras A and B concretely represented on the same Hilbert space $\mathcal{H}$ was introduced by Kadison and Kastler using the Hausdorff metric: $d(A,B)$ is defined to be the infimum of those $\gamma > 0$ with the property that every operator x in the unit ball of one algebra can be approximated by an operator y in the unit ball of the other algebra such that $\|x - y\| < \gamma$. Close operator algebras are naturally produced by small unitary perturbations uAu^* of a fixed algebra A (for a unitary u close to the identity operator or at least close to the commutant of A) and Kadison and Kastler conjectured that all sufficiently close pairs of von Neumann algebras arise in this fashion.[1]

Conjecture (Kadison-Kastler)**.** Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $M, N \subset \mathcal{B}(\mathcal{H})$ are von Neumann algebras with $d(M,N) < \delta$, then there exists a unitary u on $\mathcal{H}$ with $uMu^* = N$ and $\|u - 1_{\mathcal{H}}\| < \delta$.

The conjecture was established in the late 1970's by Christensen, Johnson and Raeburn-Taylor when one of M or N is injective. However, for C^*-algebras, there are examples due to Johnson and Choi-Christensen which show the need for further refinement of the conjecture. This leads to the following question.[2]

[1] In fact the strong form of the conjecture stated above implies a positive answer to Kadison's similarity problem: see [1].

[2] which has been resolved positively when one algebra is nuclear.

Question. Let $\mathcal{H}$ be a separable Hilbert space. Does there exist $\delta > 0$ such that every pair $A, B \subset \mathcal{B}(\mathcal{H})$ of separable C^*-algebras with $d(A,B) < \delta$ are (spatially) isomorphic?

With the aim of establishing isomorphisms between close algebras, Kadison-Kastler's original work [2] examined properties and invariants of close operator algebras, showing that close von Neumann algebras have the same type decomposition structure. This theme has been continued subsequently: in particular, Khoshkam has shown that close nuclear C^*-algebras have isomorphic K-theory ([4]). His argument also works outside the nuclear setting provided that the two algebras are *stably* or *completely* close: i.e. $A\otimes\mathcal{K}$ is close to $B\otimes\mathcal{K}$ as a subalgebra of $\mathcal{B}(\mathcal{H})\otimes\mathcal{K}$, or equivalently $d_{cb}(A,B) := \sup_n d(A\otimes M_n, B\otimes M_n)$ is small.[3]

To see this, suppose that A and B are unital C^*-algebras with $d_{cb}(A,B) < \gamma$. Then given $n \in \mathbb{N}$ and a projection $p \in M_n(A)$, we can find a projection $q \in M_n(B)$ with $\|p - q\| < \frac{\gamma}{\sqrt{2}}$.[4] The map $[p]_0 \mapsto [q]_0$ provides an isomorphism between the Murray-von Neumann semigroups of A and B which induces an isomorphism $K_0(A) \cong K_0(B)$. This map is well defined is as the Murray-von Neumann equivalence class of of a projection is stable under a small perturbation: if $\|p_1 - p_2\| < 1$ in a unital C^*-algebra A, then $p_1 \sim_u p_2$ in A.

The main result of this report is that under the same hypotheses, the Cuntz semigroup of A, $\mathrm{Cu}(A)$, is isomorphic to that of B.

Theorem 1 ([5])**.** Suppose that A and B are C^*-algebras acting non-degenerately on the same Hilbert space with $d_{cb}(A,B) < 1/42$. Then there exists a scale preserving isomorphism $\mathrm{Cu}(A) \to \mathrm{Cu}(B)$.

This is somewhat surprising: in contrast to projections the Cuntz-class of a positive element is sensitive to small perturbations, so an approach similar to that used in [4] can not work. Further, the Cuntz semigroup is a very refined invariant which captures a wealth of information about a C^*-algebra: all of this transfers to completely close algebras.

Very rapidly increasing sequences

The key tool used to prove Theorem 1 is the notion of *very rapidly increasing sequences* in the Cuntz semigroup.

On $(A\otimes\mathcal{K})_+$, write $a \precsim b$ if $v_n b v_n^* \to a$ for some sequence (v_n) in $A\otimes\mathcal{K}$. Write $a \sim b$ if $a \precsim b$ and $b \precsim a$. The Cuntz semigroup of A, $\mathrm{Cu}(A)$, is $(A\otimes\mathcal{K})_+/\sim$ equipped with the operation $\langle a\rangle + \langle b\rangle := \langle a\oplus b\rangle$ and the order $\langle a\rangle \le \langle b\rangle$ iff $a \precsim b$. A key result of Coward, Elliott and Ivanescu is the existence of suprema of countable upward directed sets in $\mathrm{Cu}(A)$. Now define $x \ll y$ in $\mathrm{Cu}(A)$ if whenever $(y_n)_n$ is a sequence in $\mathrm{Cu}(A)$ with $y \le \sup_n y_n$, there exists some n with $x \le y_n$. A sequence (y_n) in $\mathrm{Cu}(A)$ with $y_n \ll y_{n+1}$ for all n is called *rapidly increasing.*

[3]As nuclear C^*-algebras have the similarity property, there is an estimate $d_{cb}(A,B) < Kd(A,B)$ whenever A (or B) is nuclear for some universal $K > 0$.

[4]This estimate is obtained by defining a self-adjoint unitary $u = 2p - 1 \in M_n(A)$ which can be approximated by a self-adjoint unitary $v \in M_n(B)$ with $\|u - v\| < \sqrt{2}\gamma$. Now take $q = (v+1)/2$.

Definition. Say that a rapidly increasing sequence $(a_n)_{n=1}^\infty$ of positive contractions in $A\otimes\mathcal{K}$ is *very rapidly increasing* if given $\varepsilon > 0$ and $n \in \mathbb{N}$, there exists $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, there exists a contraction $v \in A\otimes\mathcal{K}$ with $\|(va_mv^*)a_n - a_n\| < \varepsilon$.

Given a positive contraction a, there is a canonical very rapidly increasing sequence $(g_{2^{-(n+1)},2^{-n}}(a))_{n=1}^\infty$ with supremum a: thus we can view the Cuntz semigroup as the suprema of very rapidly increasing sequences.[5] In a tracial sense, very rapidly increasing sequences behave increasingly like projections, giving the following stability facts.[6]

(i) Let $(a_n)_{n=1}^\infty$ be a very rapidly increasing sequence of contractions. Then for every $0 < \lambda < 1$, the sequence $(\langle(a_n - \lambda)_+\rangle)_{n=1}^\infty$ is upward directed and $\sup_n\langle(a_n - \lambda)_+\rangle = \sup_n\langle a_n\rangle$.

(ii) Any two very rapidly increasing sequences with the same suprema in $\mathrm{Cu}(A)$ can be intertwined (after telescoping) to a single very rapidly increasing sequence.

(iii) Suppose A, B are C^*-algebras with $d(A,B) < \alpha$ for $\alpha < 1/27$. Consider a very rapidly increasing sequence $(a_n)_{n=1}^\infty$ of positive contractions in A and another positive contraction $a \in A$ with $\langle a\rangle \leq \sup_n\langle a_n\rangle$. Then, given any positive contractions $b_n, b \in B$ with $\|a_n - b_n\|, \|a - b\| < 2\alpha$ for all n, there exists n_0 such that

$$\langle(b - 18\alpha)_+\rangle \ll \langle(b_n - \gamma)_+\rangle \ll \langle(b_n - 18\alpha)_+\rangle \tag{1}$$

for $n \geq n_0$ and $18\alpha < \gamma < 2/3$.

When $d_{cb}(A,B) < \alpha < 1/27$, applying the last fact repeatedly to $A\otimes\mathcal{K}$ and $B\otimes\mathcal{K}$, we obtain a well defined map $\mathrm{Cu}(A) \to \mathrm{Cu}(B)$ given by

$$\sup_n\langle a_n\rangle \mapsto \sup_n\langle(b_n - 18\alpha)_+\rangle$$

whenever (a_n) is a very rapidly increasing sequence of positive contractions in $A\otimes\mathcal{K}$ and b_n are positive contractions in $B\otimes\mathcal{K}$ with $\|b_n - a_n\| < 2\alpha$.[7] There is more work to be done to show that this map is surjective, though when $d_{cb}(A,B) < 1/42$ we do obtain an inverse to (1) by reversing the roles of A and B in the construction.

As $\mathcal{Z}$-stable C^*-algebras[8] have the similarity property [3], there exists a constant $K > 0$ such that $d_{cb}(A,B) \leq K\cdot d(A,B)$ whenever A is $\mathcal{Z}$-stable. Thus, whenever A and B are close algebras with $A \cong A\otimes\mathcal{Z}$, we have an isomorphism $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$, so B has the Cuntz semigroup of a $\mathcal{Z}$-stable C^*-algebra. A similar result holds when A is stable; in this case (provided A has stable rank one), we can use the resulting isomorphism between $\mathrm{Cu}(A)$ and $\mathrm{Cu}(B)$ to show that any sufficiently close algebra B is also stable. The analogous statement for $\mathcal{Z}$-stable algebras remains open.

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[5] Here $g_{2^{-(n+1)},2^{-n}}$ is the piecewise linear function on $\mathbb{R}$ with $g_{2^{-(n+1)},2^{-n}}(t) = 0$ for $t \leq 2^{-(n+1)}$ and $g_{2^{-(n+1)},2^{-n}}(t) = 1$ for $t \geq 2^{-n}$ and linear in between.

[6] Here, $(x-\varepsilon)_+$ denotes the function $h_\varepsilon(x)$ where $h_\varepsilon(t) = \max(t-\varepsilon, 0)$.

[7] One use of fact (iii) shows that $(\langle b_n - 18\alpha)_+\rangle)_n$ is upward directed — it is critical here that we remove the same amount 18α from both sides in (1) — so this supremum exists.

[8] those A which absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially, i.e. $A \cong A\otimes\mathcal{Z}$.

Question. Does there exist $\delta > 0$, such that whenever A is a separable C^*-algebra which absorbs $\mathcal{Z}$ tensorially and B is another C^*-algebra with $d(A,B) < \delta$, then $B \cong B \otimes \mathcal{Z}$?

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The generator rank for C^*-algebras

Hannes Thiel

The generator problem asks which C^*-algebras are singly generated, i.e., generated as a C^*-algebra by one of its elements. More generally, for a given C^*-algebra A one wants to determine the minimal number of generators, i.e., the minimal k such that A contains k elements that are not contained in any proper sub-C^*-algebra.

It is often convenient to consider self-adjoint generators, which only leads to a minor variation of the original generator problem, since two self-adjoint elements a, b generate the same sub-C^*-algebra as the element $a + ib$. Given a C^*-algebra A, let us denote by $\mathrm{gen}(A)$ the minimal number of *self-adjoint* generators for A, and set $\mathrm{gen}(A) = \infty$ if A is not finitely generated, see [2]. By definition, A is singly generated if and only if it is generated by two self-adjoint elements, that is, if and only if $\mathrm{gen}(A) \leq 2$.

For more details on the minimal number of self-adjoint generators we refer the reader to [2] and [5]. We just note that for a compact, metric space X, it is easy to see that $\mathrm{gen}(C(X)) \leq k$ if and only if X can be embedded into $\mathbb{R}^k$.

The problem with computing the invariant $\mathrm{gen}(\_)$ is that it does not behave well with respect to inductive limits, i.e., in general we do not have $\mathrm{gen}(A) \leq \liminf_n \mathrm{gen}(A_n)$ if $A = \varinjlim A_n$ is an inductive limit. This is unfortunate since many C^*-algebras are given as inductive limits, e.g., AF-algebras or approximately homogeneous algebras (AH-algebras).

To see an example where the minimal number of generators increases when passing to an inductive limit, let $X \subset \mathbb{R}^2$ be the topologists sine-curve. Then X can be embedded into $\mathbb{R}^2$ but not into $\mathbb{R}^1$, and therefore $\mathrm{gen}(C(X)) = 2$. However, X is an inverse limit of spaces X_n that are each homeomorphic to the interval, i.e., $X_n \cong [0,1]$. Therefore $C(X) \cong \varinjlim_n C(X_n)$, with $\mathrm{gen}(C(X)) = 2$, while $\mathrm{gen}(C(X_n)) = 1$ for all n.

To get a better behaved theory, instead of counting the minimal number of self-adjoint generators, we will count the minimal number of "stable" self-adjoint generators. This is the underlying idea of our definition of the generator rank of a C^*-algebra. Given a C^*-algebra A, and $k \geq 1$, we let A^k_{sa} denote the space of self-adjoint k-tuples in A, and we let $\mathrm{Gen}_k(A)_{\mathrm{sa}} \subset A^k_{\mathrm{sa}}$ be the subset of tuples that generate A.

Definition 1 ([3, Definition 2.2])**.** Let A be a unital C^*-algebra. The *generator rank* of A, denoted by $\mathrm{gr}(A)$, is the smallest integer $k \geq 0$ such that $\mathrm{Gen}_{k+1}(A)_{\mathrm{sa}}$ is dense in A^{k+1}_{sa}. If no such k exists, we set $\mathrm{gr}(A) = \infty$.

Given a non-unital C^*-algebra A, let $\widetilde{A}$ denote its minimal unitization, and set $\mathrm{gr}(A) := \mathrm{gr}(\widetilde{A})$.

Thus, while "$\mathrm{gen}(A) \leq k$" records that $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is not empty, "$\mathrm{gr}(A) \leq k - 1$" records that $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is dense. This indicates why the generator rank is usually much larger than the minimal number of self-adjoint generators. The payoff, however, is that the generator rank is much easier to compute.

The definition of the generator rank is analogous to that of the real rank as introduced by Brown and Pedersen, [1]. This explains the index shift of the definition, and with this index shift one obtains the general estimate $\mathrm{rr}(A) \leq \mathrm{gr}(A)$, see [3, Proposition 2.5].

The most interesting value of the generator rank for A is one, which means exactly that the (single) generators are dense in A. One can show that $\mathrm{Gen}_k(A)_{\mathrm{sa}}$ is a G_δ-subset of A^k_{sa} for each k (although not necessarily dense). It follows that $\mathrm{gr}(A) \leq 1$ if and only if the generators form a dense G_δ-subset of A, which means that the generic element of A is a generator.

The generator rank has many of the permanence properties that are also satisfied by other noncommutative dimension theories, see [4]. In particular, it does not increase when passing to ideals, quotients or inductive limits. Thus, the generator rank is indeed better behaved than the theory of counting the minimal number of self-adjoint generators. However, while it easy to see that for unital C^*-algebras A, B we have $\mathrm{gen}(A \oplus B) = \max\{\mathrm{gen}(A), \mathrm{gen}(B)\}$, the analog question for the generator rank seems surprisingly hard:

Question 2. Given two separable C^*-algebras A and B, do we have $\mathrm{gr}(A \oplus B) = \max\{\mathrm{gr}(A), \mathrm{gr}(B)\}$?

It is easy to see that every finite-dimensional C^*-algebra has generator rank one. We get the following consequence:

Corollary 3 ([3, Corollary 3.3])**.** Let A be a separable AF-algebra. Then A has generator rank at most one. In particular, A is singly generated.

We also compute the generator rank of commutative and homogeneous C^*-algebras. If X is a compact, metric space, then:

$$\begin{aligned} \mathrm{gr}(C(X)) &= \dim(X \times X), \\ \mathrm{gr}(C(X, M_n)) &= \left\lceil \frac{\dim(X) + 1}{2n - 2} \right\rceil, \qquad \text{for } n \geq 2. \end{aligned}$$

This allows us to show that a unital, separable AH-algebra has generator rank one if it is either simple with slow dimension growth, or when it tensorially absorbs a UHF-algebra, see [3, Corollary 4.30]. The following natural questions remain open:

Question 4. Let A be unital, separable C^*-algebra that tensorially absorbs the Jiang-Su algebra. Does it follow that A has generator rank at most one?

Question 5. Let A be unital, separable, real rank zero, stable rank one, nuclear C^*-algebra. Does it follow that A has generator rank at most one?

Note that every II_1-factor M acting on a separable Hilbert space contains a weakly dense sub-C^*-algebra A that is unital, separable and has real rank zero and stable rank one. Thus, a positive answer to Question 5 without the assumption of nuclearity would imply that every II_1-factor M is singly generated (as a von Neumann algebra). It is known that this would imply that every separably acting von Neumann algebra M is singly generated, which is a long-standing open question first asked by Kadison in 1967.

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The complexity of the relation of unitary equivalence of automorphisms of separable C^*-algebras

Martino Lupini

If A is a separable C*-algebra, denote by $\mathrm{Aut}\,(A)$ the Polish group of automorphisms of A endowed with the topology of pointwise convergence, and $\mathrm{Inn}\,(A)$ the Borel subgroup of inner automorphisms. Two automorphisms α, β of A are said to be unitarily equivalent if $\alpha \circ \beta^{-1} \in \mathrm{Inn}\,(A)$. This defines a Borel equivalence relation $E_A^{u.e.}$ on $\mathrm{Aut}(A)$. The main result presented here concerns the Borel complexity of the equivalence relation $E_A^{u.e.}$.

The study of Borel complexity of Borel (or analytic) equivalence relations on standard Borel spaces is one of the main applications of descriptive set theory (see [2] for an introduction to this subject). If E and E' are two analytic equivalence relations on standard Borel spaces X and X' respectively, E is said to be Borel reducible to E' if there is a Borel map $f : X \to X'$ such that, for every $x, y \in X$, xEy if and only if $f(x)\, E' f(y)$. This offers a notion of comparison that allows one to confront the complexity of different equivalence relations. Some distinguished equivalence relations can be used as benchmark of complexity. Among these are the relation $=_{\mathbb{R}}$ of equality of real numbers and the relation $\simeq_{\mathcal{C}}$ of isomorphism within some class of countable structures $\mathcal{C}$. An analytic equivalence relation E is called smooth if it is Borel reducible to $=_{\mathbb{R}}$, and classifiable by countable structures if it is Borel reducible to $\simeq_{\mathcal{C}}$ for some class $\mathcal{C}$ of countable structures. Since $=_{\mathbb{R}}$ is Borel reducible to $\simeq_{\mathcal{C}}$ for any class $\mathcal{C}$ of countable structures with uncountably many isomorphism classes, a smooth equivalence relations is, in particular, classifiable by countable structrues. Smooth equivalence relations are rare and have the lowest Borel complexity. An example of smooth equivalence relation is the relation of unitary equivalence of irreducible representation of a given separable type I C*-algebra. Much wider is the class of equivalence relations that are classifiable by countable structures. This can be regarded as the class of equivalence relations for which one can hope to find "easy" complete invariants, such as (ordered) groups, rings, modules, etc. An example of such relation is, for example, the relation of isomorphism of AF algebras or Kirchberg algebras.

In [3], John Phillips proved that the relation $E_A^{u.e.}$ of unitary equivalence of automorphisms of a separable non-continuous trace C*-algebra is not smooth. The main result presented here is the following one, which is a strengthening of Phillips' result.

Theorem. *If A is a non-continuous trace separable C*-algebra, then the relation $E_A^{u.e.}$ of unitary equivalence of automorphisms of A is not classifiable by countable structures.*

The main tool in the proof of the theorem is the following non-claissifiability criterion.

Criterion. Suppose that E is an analytic equivalence relation on the standard Borel space X. Assume moreover that there is a Borel function $f : (0,1)^{\mathbb{N}} \to X$ such that, for any $x, y \in (0,1)^{\mathbb{N}}$, if $x - y \in \ell_1$, then $f(x)\, E f(y)$, and for any comeager subset C of $(0,1)^{\mathbb{N}}$ there are $x, y \in C$ such that $f(x) \not{E}\, f(y)$. Then, E is not classifiable by countable structures.

The proof of this criterion can be deduced from Hjorth's theory of turbulence and, in particular, from the fact that the action of ℓ_1 on $\mathbb{R}^{\mathbb{N}}$ by translation is turbulent (cf. Proposition 3.25 of [1]). An introduction to the theory of turbulence can by found in [1] (Chapter 3).

In [3], Phillips also shows that, if A is a unital C*-algebra with continuous trace, then the relation $E_A^{u.e.}$ is smooth. Together with the main result here, this implies that there is a dichotomy in the Borel complexity of the relation of unitary equivalence of automorphisms of a separable unital C*-algebra A: Either such a relation is smooth, or it is non classifiable by countable structures. It would be interesting to know if the same dichotomy holds for non-unital C*-algebras.

Problem 1. Suppose that A is a separable C*-algebra such that $E_A^{u.e.}$ is non-smooth. Is it true that $E_A^{u.e.}$ is non-classifiable by countable structures?

It should be observed that, in the non-unital setting, continuous trace does not imply smoothness of the relation of unitary equivalence of automorphisms. There is in fact an example of a separable C*-algebra with continuous trace A such that $E_A^{u.e.}$ is even not classifiable by countable structures. It would be interesting to know exactly for which C*-algebras A the relation $E_A^{u.e.}$ is smooth or, respectively, classifiable by countable structure.

Problem 2. Characterize the C*-algebras A for which $E_A^{u.e.}$ is smooth or, respectively, classifiable by countable structures.

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Tracial approximation and classification of C^*-algebras of generalized tracial rank 1

Guihua Gong

(joint work with Huaxin Lin and Zhuang Niu)

For a simple C*-algebras A, if any given finite set $F \subset A$ can be approximated arbitrarily well by a subalgebra $B \subset A$ of the form $B = \bigoplus_{i=1}^n M_{k_i}(\mathbb{C})$ (such A is called an AF algebra) or of the form $B = \{\bigoplus_{i=1}^n M_{k_i}(\mathbb{C})\} \bigoplus \{\bigoplus_{j=1}^m M_{l_j}(C[0,1])\}$ (such A is called an AI algebra), then A is classified by Elliott in the early stage of the classification theory. But for most simple C*-algebras A, such approximation can not be done for arbitrarily given finite subset $F \subset A$. On the other hand, it is much easier to approximate a "large portion" of any given finite subset $F \subset A$ by a class of good C*-algebras such as finite dimensional algebras, or interval algebras. The following definition was given by Lin:

Definition 1. Let $\mathscr{S}$ be a class of "good" unital C*-algebra. A is called TA$\mathscr{S}$, if for any finite set $F \subset A$, $0 \neq a \in A_+$, and $\epsilon > 0$, $\exists B \in \mathscr{S}$, $B \subset A$ with $\mathbf{1}_B = p$ such that

(1) $\|px - xp\| < \epsilon \quad \forall x \in F$;

(2) $\operatorname{dist}(pxp, B) < \epsilon \quad \forall x \in F$;

(3) $\mathbf{1}_a - p$ is unitarily equivalent to a projection in the hereditary subalgebra of A generated by a.

(This property is called Decomposition Theorem by Elliott-Gong, which predates Lin's definition.)

If we choose $\mathscr{S}$ to be a Elliott-Thomsen building block with trivial K_1-group defined as follows, then we refer TA$\mathscr{S}$ algebras as algebras with generalized tracial rank 1.

Definition 2. Let F_1, F_2 be two finite dimensional C*-algebras, and $\alpha_0, \alpha_1 \colon F_1 \to F_2$ two unital homomorphisms,

$$A(F_1, F_2, \alpha_0, \alpha_1) = \{(f, a) \in C([0,1], F_2) \oplus F_1;\ \alpha_0(a) = f(0), \alpha_1(a) = f(1)\}$$

is called Elliott-Thomsen building block.

In this talk, we described the complete classification of $\mathcal{Z}$-stable (here $\mathcal{Z}$ is the Jiang-Su algebra) simple nuclear separable C*-algebra A with UCT, under the condition that $A \otimes \mathrm{UHF}$ is of generalized tracial rank 1. This class of C*-algebras may include all $\mathcal{Z}$-stable simple unital ASH algebras – at least it is proved that it covers Elliott invariants of all $\mathcal{Z}$-stable unital ASH algebras.

Semigroup C^*-algebras, K-theory, and classification

Xin Li

1. The construction, conditions, and examples

Let P be a subsemigroup of a group G. For every $p \in P$, $V_p : \ell^2(P) \to \ell^2(P)$, $\varepsilon_x \mapsto \varepsilon_{px}$ defines an isometry. The reduced semigroup C*-algebra of P is given as follows:

Definition 1. $C_r^*(P) := C^*(\{V_p \colon p \in P\}) \subseteq \mathcal{L}(\ell^2(P))$.

In the study of semigroup C*-algebras (see [6], [7]), the following two conditions turn out to be very important:

Let $\mathcal{J} := \left\{p_1^{-1} q_1 \cdots p_n^{-1} q_n P \colon p_i, q_i \in P\right\} \cup \{\varnothing\}$.

Definition 2. $\mathcal{J}$ is independent if for all X, X_1, ..., X_n in $\mathcal{J}$,

$$X = \bigcup_{i=1}^{n} X_i \Rightarrow X = X_i \text{ for some } 1 \leq i \leq n.$$

Definition 3. $P \subseteq G$ is Toeplitz if for every $g \in G$ with $E_P \lambda_g E_P \neq 0$, there exist $p_1, q_1, \ldots, p_n, q_n$ in P such that $E_P \lambda_g E_P = V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n}$. Here λ is the left regular representation of G, and E_P is the orthogonal projection in $\mathcal{L}(\ell^2(G))$ onto the subspace $\ell^2(P)$ of $\ell^2(G)$.

The following lemma explains why the Toeplitz condition is useful:

Lemma 4. If $P \subseteq G$ is Toeplitz, then $C_r^*(P) \sim_M C_0(\Omega_{P\subseteq G}) \rtimes_r G$, where $\Omega_{P\subseteq G}$ is a totally disconnected locally compact space on which G acts.

Here is a list of examples of subsemigroups of groups for which our two conditions (independence and Toeplitz) hold:

- $\mathbb{N}_0 \subseteq \mathbb{Z}$, $\mathbb{N}_0^n \subseteq \mathbb{Z}^n$, $\mathbb{N}_0^{*n} \subseteq \mathbb{Z}^{*n}$;
- Artin-Tits monoids in right-angled Artin-Tits groups or Artin-Tits groups of finite type, e.g. Braid monoids in Braid groups;
- Examples from number theory: Let K be a number field, let R be the ring of integers in K, and consider the $ax+b$-semigroup $R \rtimes R^\times$ as a subsemigroup of the $ax+b$-group $K \rtimes K^\times$ over K.

2. K-theory

To compute K-theory for semigroup C*-algebras, the strategy is as follows: The Toeplitz condition allows us to pass over to reduced group crossed products attached to dynamical systems $G \curvearrowright \Omega$ with Ω totally disconnected (see Lemma 4). The independence condition gives further restrictions on $G \curvearrowright \Omega$. Let us make precise what sort of dynamical systems we want to study:

Let $G \curvearrowright \Omega$ be a dynamical system with G countable and Ω locally compact, totally disconnected and second countable. $\mathcal{CO}(\Omega)$ denotes the collection of compact open subsets of Ω.

Definition 5. $\mathcal{V} \subseteq \mathcal{CO}(\Omega) \setminus \{\emptyset\}$ is called a G-invariant basis if

(1) $\mathcal{V}$ is closed under finite intersections $\neq \emptyset$,
(2) $G \cdot \mathcal{V} = \mathcal{V}$,
(3) $R(\mathcal{V}) = \mathcal{CO}(\Omega)$, where $R(\mathcal{V})$ is the set-theoretical ring generated by $\mathcal{V}$, i.e. the smallest family of subsets of Ω containing $\mathcal{V}$ and closed under finite unions and complements,
(4) $\mathcal{V}$ is independent in the same sense as in Definition 2, i.e., a compact open set in $\mathcal{V}$ cannot be decomposed into finitely many strictly smaller compact open sets in $\mathcal{V}$.

Here is our general K-theoretic result for crossed products attached to dynamical systems with such an invariant basis:

Theorem 6 (j. w. with Cuntz and Echterhoff)**.** Let $G \curvearrowright \Omega$ be a dynamical system as above and assume that $\mathcal{V} \subseteq \mathcal{CO}(\Omega) \setminus \{\emptyset\}$ is a G-invariant basis. If G satisfies the Baum-Connes conjecture with coefficients, then $K_*(C_0(\Omega) \rtimes_r G) \cong \bigoplus_{[V]\in G\backslash \mathcal{V}} K_*(C_r^*(G_V))$. Here $G_V = \{g \in G\colon\ g \cdot V = V\}$.

This gives a formula for $K_*(C_r^*(P))$ if we assume the independence and the Toeplitz condition. We refer the reader to [1] and [2] for more details. Moreover, this K-theoretic formula is not only interesting in the context of semigroup C*-algebras, but it applies in other situations as well. For example, M. Norling observed that the formula can be used to compute K-theory for inverse semigroup C*-algebras [8].

3. Classification

Using our general K-theoretic results, we obtain the following classification results for semigroup C*-algebras:

The first result is about semigroup C*-algebras for Artin monoids in right-angled Artin groups. These C*-algebras have been studied in [3], [4] and [5].

Theorem 7. Let (A_Γ, A_Γ^+) and $(A_{\Gamma'}, A_{\Gamma'}^+)$ be graph-irreducible right-angled Artin groups in the sense of [4] (we assume that Γ and Γ' are graphs with countably many vertices). The semigroup C*-algebras $C_r^*(A_\Gamma^+)$ and $C_r^*(A_{\Gamma'}^+)$ are isomorphic if and only if (exactly) one of the following conditions hold:

- Both Γ and Γ' consist of only one vertex; in that case, $C_r^*(A_\Gamma^+)$ and $C_r^*(A_{\Gamma'}^+)$ are isomorphic to the Toeplitz algebra;
- Both Γ and Γ' have countably infinitely many vertices; in that case, $C_r^*(A_\Gamma^+)$ and $C_r^*(A_{\Gamma'}^+)$ are isomorphic to the Cuntz algebra $\mathcal{O}_\infty$;
- Both Γ and Γ' have at least two but finitely many vertices, and the Euler characteristics in the sense of [4], [5] coincide: $\chi(\Gamma) = \chi(\Gamma')$.

The proof of this result uses a recent result of S. Eilers, G. Restorff and E. Ruiz concerning the classification of graph C*-algebras.

The second classification result is concerned with semigroup C*-algebras of $ax + b$-semigroups over rings of integers in number fields.

Theorem 8. Let K and L be finite Galois extensions of $\mathbb{Q}$, and let R and S be the rings of integers in K and L. Assume that K and L have the same number of roots of unity. Then $C_r^*(R \rtimes R^\times) \cong C_r^*(S \rtimes S^\times)$ if and only if $K \cong L$.

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Untitled

Nicola Watson

Nicola Watson presented the recent results of her work on the classification of nuclear, simple C^*-algebras, relating the properties of having finite nuclear dimension, (rationally) real rank zero and being TAF.

On groups with quasidiagonal C^*-algebras

José R. Carrión
(joint work with Marius Dadarlat and Caleb Eckhardt)

In [7] Lance provided a C^*-algebraic characterization of amenability for discrete groups by proving that a discrete group Γ is amenable if and only if its reduced C^*-algebra, $C_r^*(\Gamma)$ is nuclear. Later Rosenberg showed that if $C_r^*(\Gamma)$ is quasidiagonal, then Γ is amenable [5]. The converse to Rosenberg's theorem remains open, namely: if Γ is a discrete, amenable group, is $C_r^*(\Gamma)$ quasidiagonal [11]?

Our main results are the following.

First, if Γ is not amenable, then the modulus of quasidiagonality of $C_r^*(\Gamma)$ is controlled by the number of pieces in a paradoxical decomposition of Γ. The modulus of quasidiagonality measures how badly a C*-algebra violates quasidiagonality [9]. This may be regarded as a quantitative version of Rosenberg's previously mentioned result, which may be rephrased to say that the modulus of quasidiagonality does not vanish for some subset of a non-amenable group.

We call a group that embeds in $\prod_{k=1}^{\infty} M_{n(k)}(\mathbb{C}) \big/ \sum_{k=1}^{\infty} M_{n(k)}(\mathbb{C})$ for some increasing sequence of positive integers $(n(k))$ an *MF group*, in analogy with the MF algebras of Blackadar and Kirchberg [2]. Our second result states that $C_r^*(\Gamma)$ is quasidiagonal if and only if Γ is amenable and MF. We expand the class of amenable groups with quasidiagonal C^*-algebras beyond the class of groups that are locally embeddable into finite groups in the sense of Vershik and Gordon [10] (so-called LEF groups). It applies, for example, to the topological full groups associated to Cantor minimal systems [3, 8, 6, 4] and gives us examples of simple amenable groups with quasidiagonal C^*-algebras. It also applies to an example of Abels that provides an amenable group that is not LEF [1]. We observe that if a group is not LEF, then it cannot be a union of residually finite groups and one cannot obtain quasidiagonality using previously known techniques based on finite approximation properties of the group. However, we have that Γ/Δ is MF if Γ is residually finite and Δ is a central subgroup of Γ.

Our third result concerns strong quasidiagonality. If Γ and Λ are amenable groups such that Γ is non-torsion and Λ has a finite dimensional representation other than the trivial one, then $C^*(\Lambda \wr \Gamma)$ has a non-finite quotient and therefore cannot be strongly quasidiagonal. We have, therefore, that the C^*-algebra of the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is quasidiagonal (even residually finite dimensional) but has a non-finite quotient.

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A Dixmier-Douady theory for strongly self absorbing C^*-algebras

Marius Dadarlat

(joint work with Ulrich Pennig)

Let X be a compact metrizable space. For a separable C*-algebra D we denote by $\mathscr{C}_D(X)$ the set of all separable continuous field C*-algebras over X with fibers abstractly isomorphic to D. In particular, if A is a separable continuous field C*-algebra over X with all fibers isomorphic to the compact operators $\mathbb{K}$ on an infinite dimensional separable Hilbert space, we write $A \in \mathscr{C}_{\mathbb{K}}(X)$.

Dixmier and *Douady* [1] proved the following theorems:

Theorem 1. Suppose that X has finite covering dimension. Then $A \in \mathscr{C}_{\mathbb{K}}(X)$ is locally trivial if and only if for each point $x \in X$, there is closed neighborhood V of x and a projection $p \in A(V)$ such that $\mathrm{rank}(p(v)) = 1$ for all $v \in V$.

Theorem 2. The isomorphism classes of locally trivial fields in $\mathscr{C}_{\mathbb{K}}(X)$ form a group under the operation of tensor product. Moreover this group is isomorphic to $\check{H}^3(X,\mathbb{Z})$.

We give generalizations of these results to continuous field C*-algebras with fibers $D \otimes \mathbb{K}$, where D is a fixed strongly self-absorbing C*-algebra. The class of strongly self-absorbing C*-algebras was introduced by Toms and Winter [3]. A separable unital C*-algebra D is strongly self-absorbing if the the $*$-homomorphism $\ell : D \to D \otimes D$, $\ell(d) = d \otimes 1_D$ is approximately unitarily equivalent to some $*$-isomorphism $D \to D \otimes D$. The only known strongly self-absorbing C*-algebras are the Jiang-Su algebra $\mathscr{Z}$, the UHF-algebras of infinite type, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ and any tensor product of those algebras.

Let D be a separable strongly self-absorbing C*-algebra (the case $D = \mathbb{C}$ is allowed). $K_0(D)$ has a natural ring structure. We denote by $K_0(D)^\times$ the invertible elements in this ring and by $K_0(D)^\times_+$ the group of positive invertibles.

Theorem A. Suppose that X has finite covering dimension and let $A \in \mathscr{C}_{D\otimes\mathbb{K}}(X)$. Then A is locally trivial if and only if for each point $x \in X$, there is closed neighborhood V of x and a projection $p \in A(V)$ such that $[p(v)] = [1_D]$ in $K_0(D)$, for all $v \in V$.

The latter condition means that for each $v \in V$, there is an isomorphism $\phi : D \otimes \mathbb{K} \to A(V)$ such that $\phi_*[1_D \otimes e] = [p(v)]$ where $e \in \mathbb{K}$ is a rank-one projection.

Using results from [2] we prove that the group $\mathrm{Aut}(D)$ is contractible with respect to the point-norm topology. Moreover, we employ some classical delooping results from algebraic topology due to G. Segal and P. May to show that $\mathrm{Aut}(D \otimes \mathbb{K})$ is an infinite loop space and hence $\mathrm{Aut}(D \otimes \mathbb{K})$ is the 0-space in the spectrum of a generalized cohomology theory $E^*_D(X)$. Suppose that X is connected with base point x_0. If $D \neq \mathbb{C}$, then there isomorphisms of multiplicative (abelian) groups

$$[X, \mathrm{Aut}(D \otimes \mathbb{K})] \cong E^0_D(X) \cong K_0(D)^\times_+ \oplus K_0\left(C_0(X \setminus x_0) \otimes D\right).$$

The group operation on the latter group is induced by the multiplication in the ring $K_0(C(X) \otimes D)$.

Theorem B. Let X be a finite connected CW complex. The isomorphism classes of locally trivial fields in $\mathscr{C}_{D\otimes\mathbb{K}}(X)$ form an abelian group under the operation of tensor product. Moreover this group is isomorphic to $E^1_D(X)$.

If $D = \mathbb{C}$, then of course $E^1_{\mathbb{C}}(X) \cong H^3(X, \mathbb{Z})$, this is the Dixmier-Douady case.

In general one may use the Atiyah-Hirzebruch spectral sequence for specific computations. For illustration, we have the following.

If $D = UHF$ is the universal UHF-algebra then

$$E^1_{UHF}(X) \cong H^1(X, \mathbb{Q}^\times_+) \times H^3(X, \mathbb{Q}) \times H^5(X, \mathbb{Q}) \times \cdots$$

If $D = \mathscr{Z}$ is the Jiang-Su algebra and if assume that $H^*(X, \mathbb{Z})$ is torsion free, then

$$E^1_{\mathscr{Z}}(X) \cong H^3(X, \mathbb{Z}) \times H^5(X, \mathbb{Z}) \times \cdots$$

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Uniqueness and Existence Theorems

Huaxin Lin

One of the most important tools used in the Elliott program is the Elliott intertwining argument. It has been employed for classification of AF-algebras, simple A$\mathbb{T}$-algebras of real rank zero, for example. It was used by almost all authors who provided a classification theorem. The Elliott intertwining argument includes two types of subarguments, the so-called uniqueness and existence theorems. These are the integral part of the Elliott intertwining argument. A Uniqueness Theorem has the following form: Let A and B be two unital C^*-algebras. Suppose that $\phi_1, \phi_2 : A \to B$ are two unital monomorphisms. Suppose also that

$$Inv(\phi_1) = Inv(\phi_2).$$

Then there exists a sequence of unitaries $u_n \in B$ such that

$$\lim_{n\to\infty} u_n^*\phi_2(a)u_n = \phi_1(a) \text{ for all } a \in A.$$

An Existence Theorem has the following form: Let $\Phi : Inv(A) \to Inv(B)$ be a map. Then there exists a unital homomorphism $\phi : A \to B$ such that $Inv(\phi) = \Phi$.

W. Winter ([7]) provided a remarkable method which provides a new approach to the Elliott program. In fact, Winter's method makes it possible to classify C^*-algebras which may not have finite tracial rank. As in the Elliott intertwining argument, Winter's method also requires a uniqueness and existence theorem. But this time, it requires a much finer uniqueness and existence theorem. Given two unital homomorphisms $\phi : A \otimes M_{\mathbf{p}} \to B \otimes M_{\mathbf{p}}$ and $\psi : A \otimes M_{\mathbf{q}} \to A \otimes M_{\mathbf{q}}$, consider

$$\begin{aligned} \phi_0 &= \phi \otimes \mathrm{id}_{M_{\mathbf{q}}} : A \otimes M_{\mathbf{p}} \otimes M_{\mathbf{q}} \to B \otimes M_{\mathbf{p}} \otimes M_{\mathbf{q}} \text{ and} \\ \phi_1 &= \psi \otimes \mathrm{id}_{M_{\mathbf{p}}} : A \otimes M_{\mathbf{p}} \otimes M_{\mathbf{q}} \to B \otimes M_{\mathbf{p}} \otimes M_{\mathbf{q}}, \end{aligned}$$

where $M_{\mathbf{p}}$ and $M_{\mathbf{q}}$ are UHF-algebras with supernatural numbers $\mathbf{p}$ and $\mathbf{q}$ of infinite type which are relatively prime. When are they asymptotically unitarily equivalent, i.e., when is there a continuous path of unitaries $\{u(t) : t \in [0,1)\} \subset B \otimes M_{\mathbf{p}} \otimes M_{\mathbf{q}}$ such that

$$\lim_{t\to 1} u(t)^*\phi_0(a)u(t) = \phi_1(a) \text{ for all } a \in A \otimes M_{\mathbf{p}} \otimes M_{\mathbf{q}}?$$

A uniqueness theorem is needed here. If ϕ_1 and ϕ_2 are asymptotically unitarily equivalent, then $[\phi_1] = [\phi_2]$ in $KK(A,B)$, $(\phi_1)_T = (\phi_2)_T$ and $\phi_1^\ddagger = \phi_2^\ddagger$ are the same, where $\phi_T, \psi_T : T(A) \to T_f(C(X))$, where $T_f(C(X))$ is the faithful tracial state space of $C(X)$, and
$\phi^\ddagger$, $\psi^\ddagger : U(M_n(C(X)))/CU(M_n(C(X))) \to U(M_n(A))/CU(M_n(A))$ are the continuous homomorphisms induced by ϕ and ψ. But there are more. Namely, the rotation related maps $\mathcal{R}_{\phi,\psi}$. Using the Basic Homotopy Lemma among many other related results, one has the following:

Theorem 1. ([1] and [4]) Let A be a unital AH-algebra and let B be a unital separable simple amenable C^*-algebra with $TR(B) \le 1$. Suppose that $\phi, \psi : A \to B$

are two unital monomorphisms such that

$$[\phi] = [\psi] \text{ in } KK(A,B),\ \phi_T = \psi_T,\ \phi^\ddagger = \psi^\ddagger \text{ and}$$
$$\overline{R_{\phi,\psi}} = 0.$$

Then there exists a continuous path of unitaries $\{u(t) : t \in [0,1)\} \subset B$ such that

$$\lim_{t\to 1} u(t)^*\phi(a)u(t) = \psi(a) \text{ for all } a \in A.$$

We also need an existence theorem. Realizing an element in $KK(A,B)$ by a homomorphism in the case that $K_i(A)$ is not finitely generated and has torsion is quite different from the case that $K_i(A)$ is torsion free or finitely generated. The issue is that functor KK does not to preserve inductive limits. As it turns out, it requires also a version of uniqueness theorem as well as a version of the Basic Homotopy Lemma ([2]).

Theorem 2. [4] (and [6] in the case that $TR(A) = 0$) Let C be a unital AH-algebra and let A be a unital separable simple C^*-algebra with $TR(A) \le 1$. Suppose that $\kappa \in KK(C,A)$ with $\kappa(K_0(C)_+ \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$ and $\lambda : T(A) \to T_f(C)$ is a continuous affine map which is compatible with κ. Then there exists a monomorphism $\phi : C \to A$ such that $([\phi], \phi_T) = (\kappa, \lambda)$.

Moreover, for any $\eta \in Hom(K_1(C), \mathrm{Aff}(T(A))/\mathcal{R}_0$, there is another monomorphism $\psi : C \to A$ with $([\psi], \psi_T) = (\kappa, \lambda)$ such that

$$\overline{R_{\phi,\psi}} = \eta.$$

There are examples ([3]) that X is a one-dimensional compact metric space, A is a UHF-algebra and $\kappa \in KL(C(X), A)$ with $\kappa([1_{C(X)}]) = [1_A]$, $\kappa(K_0(C(X)) \setminus \{0\}) \subset K_0(A)_+ \setminus \{0\}$, but no unital homomorphism $\phi : C(X) \to A$ such that $[\phi] = [\kappa]$! Therefore, the word "compatible" is important and necessary.

This results in the following classification theorem:

Theorem 3. ([4]) Let A and B be two unital separable simple amenable C^*-algebras which are of rational tracial rank at most one which satisfy the UCT. Then $A \cong B$ if and only if

$$(K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A) \cong (K_0(B), K_0(B)_+, [1_B], K_1(B), T(B), r_B).$$

The rest of the talk is part of a joint work with Guihua Gong and Zhuang Niu. With the building blocks of Elliott and Thomsen, a more general class of amenable simple C^*-algebras can be introduced. A C^*-algebra A in this class will be written as $GTR(A) \le 1$ (generalized tracial rank at most one). We have the following uniqueness theorem.

Theorem 4. (Gong–L–Niu—2012) Let A_1 and B be two unital separable simple amenable C^*-algebras which satisfy the UCT. Let $A = A_1 \otimes U$ for some infinite dimensional UHF-algebra U such that $GTR(A) \le 1$. Suppose that $\phi, \psi : A \to B$ are two unital monomorphisms. Then ϕ and ψ are asymptotically unitarily equivalent if and only if

$$[\phi] = [\psi] \text{ in } KK(A,B), \phi_T = \psi_T,$$
$$\phi^\ddagger = \psi^\ddagger \text{ and } \overline{R_{\phi,\psi}} = 0.$$

The following two statements serve as existence theorems.

Theorem 5. (Gong–Lin –Niu–2012) Let A_1 and B_1 be unital separable amenable simple C^*-algebras which satisfy the UCT, let $A = A_1 \otimes U_1$ and $B = B_1 \otimes U_2$, where U_1 and U_2 are two infinite dimensional UHF-algebras. Suppose that $GTR(A) \le 1$ and $GTR(B) \le 1$. Suppose also that $(\kappa, \lambda, \gamma)$ is a compatible triple as above. Then there exists a unital monomorphism $\phi : A \to B$ such that $([\phi], \phi_T, \phi^\ddagger) = (\kappa, \lambda, \gamma)$.

Theorem 6. (Gong–Lin –Niu–2012) Given a unital monomorphism $\phi : A \to B$ and given an element $R \in \mathrm{Hom}(K_1(A), \overline{\rho_B(K_0(B))})/\mathcal{R}_0$. There exists a unital monomorphism $\psi : A \to B$ such that

$$[\psi] = [\phi], \;\; \psi_T = \phi_T, \;\; \psi^\ddagger = \phi^\ddagger$$

$$\overline{R_{\phi,\psi}} = R.$$

Combining the existence theorems as above, using Winter's method, we have the following:

Theorem 7. (Gong–Lin—Niu) Let A and B be two unital separable simple amenable $\mathcal{Z}$-stable C^*-algebras which satisfy the UCT. Suppose that $GTR(A \otimes M_p) \le 1$ and $GTR(B \otimes M_p) \le 1$ for any UHF-algebra M_p of infinite type. Then $A \cong B$ are isomorphic if and only if

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

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Topological full groups of étale groupoids

Hiroki Matui

In this talk I described some recent results about topological full groups of étale groupoids on Cantor sets. Let G be an essentially principal étale groupoid whose unit space $G^{(0)}$ is a Cantor set. The topological full group $[[G]]$ of G is a subgroup of $\mathrm{Homeo}(G^{(0)})$ consisting of all homeomorphisms of $G^{(0)}$ whose graph is 'contained' in the groupoid G as a compact open subset. Clearly $[[G]]$ is a countable group. There exists a natural short exact sequence:

$$1 \longrightarrow U(C(G^{(0)})) \longrightarrow N(C(G^{(0)}), C_r^*(G)) \longrightarrow [[G]] \longrightarrow 1,$$

where $N(C(G^{(0)}), C_r^*(G))$ denotes the group of unitaries in $C_r^*(G)$ which normalize $C(G^{(0)})$.

The following theorem says that when G is minimal, $[[G]]$ (and its certain normal subgroups) 'remembers' the groupoid G. We let $[[G]]_0$ denote the kernel of the index map $I : [[G]] \to H_1(G)$ and let $D([[G]])$ denote the commutator subgroup of $[[G]]$.

Theorem 1. For $i = 1, 2$, let G_i be a minimal and essentially principal étale groupoid whose unit space is a Cantor set. The following are equivalent.

(1) G_1 is isomorphic to G_2 as an étale groupoid.
(2) $[[G_1]]$ is isomorphic to $[[G_2]]$ as a group.
(3) $[[G_1]]_0$ is isomorphic to $[[G_2]]_0$ as a group.
(4) $D([[G_1]])$ is isomorphic to $D([[G_2]])$ as a group.

Furthermore, under some additional assumptions, we can prove that $D([[G]])$ is simple. One may think of $D([[G]])$ as an analogue of alternating groups.

Theorem 2. Let G be a minimal groupoid as above. Suppose that G is either almost finite or purely infinite. Then $D([[G]])$ is simple.

Let $\varphi : \mathbb{Z} \to \mathrm{Homeo}(X)$ be a minimal action of $\mathbb{Z}$ on a Cantor set X. For φ one can associate the transformation groupoid G_φ. The groupoid G_φ is known to be almost finite.

Theorem 3. For G_φ as above, the following hold.

(1) The abelianization $[[G_\varphi]]_{\mathrm{ab}}$ is isomorphic to $(H_0(G_\varphi) \otimes \mathbb{Z}_2) \oplus \mathbb{Z}$.
(2) $D([[G_\varphi]])$ is finitely generated if and only if φ is expansive.
(3) $D([[G_\varphi]])$ is never finitely presented.

Moreover, K. Juschenko and N. Monod recently proved the following theorem.

Theorem 4. For G_φ as above, $[[G_\varphi]]$ is amenable.

This provides the first examples of finitely generated simple amenable infinite groups.

Next, we consider étale groupoids arising from one-sided shifts of finite type. Let (X, σ) be a one-sided irreducible shift of finite type. Then one can define the étale groupoid G by

$$G = \left\{(x,n,y) \in X \times \mathbb{Z} \times X \mid \exists k,l \in \mathbb{N},\ n = k-l,\ \sigma^k(x)=\sigma^l(y)\right\}.$$

It is easy to see that G is purely infinite and minimal. V. V. Nekrashevych observed that when (X,σ) is the full shift over n symbols, the topological full group $[[G]]$ is canonically isomorphic to the Higman-Thompson group $V_{n,1}$. Thus, $[[G]]$ is regarded as a generalization of the Higman-Thompson group.

In the following theorem, for a clopen subset $Y \subset X$, we let $G|Y = \{g \in G \mid r(g), s(g) \in Y\}$ denote the reduction of G to Y. The groupoid $G|Y$ is also purely infinite and minimal.

Theorem 5. Let G be as above and let $Y \subset X$ be a clopen set.

(1) $[[G|Y]]$ (and $[[G|Y]]_0$ and $D([[G|Y]])$) 'remembers' $G|Y$.
(2) $D([[G|Y]])$ is simple.
(3) $[[G|Y]]$ has the Haagerup property.
(4) The abelianization $[[G|Y]]_{\mathrm{ab}}$ is isomorphic to $(H_0(G) \otimes \mathbb{Z}_2) \oplus H_1(G)$.
(5) $[[G|Y]]$ is of type F_∞, and hence is finitely presented.
(6) $[[G|Y]]_0$ and $D([[G|Y]])$ are finitely generated.

We remark that, in this setting, $H_0(G)$ is a finitely generated abelian group and $H_1(G)$ is isomorphic to the torsion-free part of $H_0(G)$.

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Large subalgebras, crossed products, and the Cuntz semigroup

N. Christopher Phillips

Large subalgebras and centrally large subalgebras are useful for obtaining information about the structure of transformation group C*-algebras. They are large enough to give information about the algebra they are contained in, but can often be chosen to be small enough that their structure is much more easily accessible.

In the following, we use $a \precsim_A b$ to mean that a is Cuntz subequivalent to b with respect to the algebra A. (Cuntz subequivalence with respect to a subalgebra $B \subset A$ might be different from Cuntz subequivalence with respect to A.)

Definition 1. Let A be an infinite dimensional simple unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.

(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1-g)c_j,\, c_j(1-g) \in B$.
(4) $g \precsim_B y$.
(5) $\|(1-g)x(1-g)\| > 1 - \varepsilon$.

We emphasize that the Cuntz subequivalence in (4) is relative to B, not A.

Definition 2. Let A be an infinite dimensional simple separable unital C*-algebra. A unital subalgebra $B \subset A$ is said to be *centrally large* in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, $x \in A_+$ with $\|x\| = 1$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

(1) $0 \leq g \leq 1$.
(2) For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
(3) For $j = 1, 2, \ldots, m$ we have $(1-g)c_j,\, c_j(1-g) \in B$.
(4) $g \precsim_B y$.
(5) $\|(1-g)x(1-g)\| > 1 - \varepsilon$.
(6) For $j = 1, 2, \ldots, m$ we have $\|ga_j - a_jg\| < \varepsilon$.

The difference from the definition of a large subalgebra (Definition 1) is the approximate commutation condition in (6). In particular, a centrally large subalgebra is large.

The main easily described example is as follows; it is originally due to Putnam. We have used a different convention from that used elsewhere, where one usually takes

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, uC_0(X \setminus Y)\big).$$

Definition 3. Let X be a locally compact Hausdorff space and let $h\colon X \to X$ be a homeomorphism. Let $u \in C^*(\mathbb{Z}, X, h)$ be the canonical unitary which implements the action. Let $Y \subset X$ be a nonempty closed subset. Define

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, C_0(X \setminus Y)u\big) \subset C^*(\mathbb{Z}, X, h).$$

We call this subalgebra the *Y-orbit breaking subalgebra* of $C^*(\mathbb{Z}, X, h)$.

Theorem 4. Let X be a compact Hausdorff space and let $h\colon X \to X$ be a minimal homeomorphism. Let $Y \subset X$ be a compact set such that $h^n(Y) \cap Y = \varnothing$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then $C^*(\mathbb{Z}, X, h)_Y$ is a centrally large subalgebra of $C^*(\mathbb{Z}, X, h)$.

We give some of the relations between a simple unital C*-algebra A and a large subalgebra B.

Theorem 5. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a large subalgebra. Then:

(1) B is simple.
(2) A is purely infinite if and only if B is purely infinite.
(3) The restriction map $T(A) \to T(B)$ on the tracial state spaces is bijective.
(4) The restriction map $QT(A) \to QT(B)$ on the spaces of normalized 2-quasitraces is bijective.

(5) If one deletes from the Cuntz semigroups $\mathrm{Cu}(A)$ and $\mathrm{Cu}(B)$ the classes of the projections, the inclusion map $B \to A$ induces a bijection on what remains (the classes of the purely positive elements).
(6) A and B have the same radius of comparison.
(7) A has strict comparison of positive elements if and only if B has strict comparison of positive elements.

There is a kind of converse to Theorem 5(1): if one drops simplicity of A from the definition, but B is simple, then it follows that A is simple. However, Definitions 1 and 2 are clearly unsuitable when A is not simple. For example, if A is simple and $B \subset A$ is a proper subalgebra which is large in A, then $B \oplus B$ should be large in $A \oplus A$. But Definition 1 is not satisfied.

Theorem 5(5) is false without the exclusion of the classes of the projections. In particular, $K_0(A)$ and $K_0(B)$ can be quite different.

We get more if we assume that B is centrally large in A. The following theorem is joint work with Dawn Archey.

Theorem 6. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. If B has stable rank one, then so does A.

Conjecture 7. Let A be an infinite dimensional simple separable unital C*-algebra, and let $B \subset A$ be a centrally large subalgebra. If B is tracially Z-stable in the sense of Hirshberg and Orovitz, then so is A.

Tracial Z-stability is a kind of weakening of Z-stability.

The proof of the conjecture is nearly done. Hirshberg and Orovitz, using work of Matui and Sato, show that a simple separable unital nuclear tracially Z-stable algebra is in fact Z-stable. In particular, if A is nuclear and B is Z-stable, then so is A.

Theorem 4 is what is needed for applications to crossed products by minimal homeomorphisms. As long as $Y \neq \varnothing$, the Y-orbit breaking subalgebra of $C^*(\mathbb{Z}, X, h)$ will be a direct limit of recursive subhomogeneous C*-algebras with topological dimension equal to the dimension of X. In particular, if X is finite dimensional, then the Y-orbit breaking subalgebra is a direct limit of recursive subhomogeneous C*-algebras with no dimension growth. If Y meets each orbit at most once, so that Theorem 4 applies, then $C^*(\mathbb{Z}, X, h)_Y$ is simple (Theorem 5(1)), and therefore Z-stable. (The consequences gotten this way are not new.) If X is not finite dimensional, then at least the computation of the radius of comparison of $C^*(\mathbb{Z}, X, h)$ is reduced to the computation of the radius of comparison of a simple direct limit of recursive subhomogeneous C*-algebras.

The original motivation for large subalgebras is the following result.

Theorem 8. Let $d \in \mathbb{Z}_{>0}$. Let X be a compact metric space, and let h be a free minimal action of $\mathbb{Z}^d$ on X. Suppose that one of the following holds:

(1) X is the Cantor set.
(2) X is a compact smooth manifold and the action is via diffeomorphisms.

Then there is a centrally large subalgebra $B \subset C^*(\mathbb{Z}^d, X, h)$ such that B is a direct limit of recursive subhomogeneous C*-algebras with no dimension growth.

It is likely that case (2) can be generalized to continuous actions on finite dimensional compact metric spaces, using methods of Kulesza.

Since B is Z-stable, one concludes that $C^*(\mathbb{Z}^d, X, h)$ has strict comparison of positive elements and, assuming Conjecture 7, that $C^*(\mathbb{Z}^d, X, h)$ is in fact Z-stable.

The subalgebra B is an analog of the orbit breaking subalgebra of $C^*(\mathbb{Z}, X, h)$. Unlike for the orbit breaking subalgebra, no easy known formula for B is known. Even the construction appears to depend on finite dimensionality.

A significant amount (but not all of the conclusion here) was already known when X is the Cantor set. Although not stated in these terms, the proofs depended on a suitable centrally large subalgebra and results related to those above. The large subalgebra is AF, making it possible to prove that the order on projections over $C^*(\mathbb{Z}^d, X, h)$ is determined by traces directly from the corresponding fact about the subalgebra.

In case (2), nothing was previously known. Even when $C^*(\mathbb{Z}, X, h)$ has many projections, there seems to be no reason to expect B to have many projections. Since $C^*(\mathbb{Z}, X, h)$ has strict comparison of positive elements, it follows that the order on projections over $C^*(\mathbb{Z}^d, X, h)$ is determined by traces. Strangely, however, the proof ultimately depends on Theorem 5(5), which is about the part of the Cuntz semigroup which remains after the classes of the projections are deleted.

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Classification of crossed product C^*-algebras and mean dimension for topological dynamical systems

Taylor Hines

(joint work with Andrew Toms and N. Christopher Phillips)

The mean topological dimension was introduced by Gromov [2] and later studied by Lindenstrauss and Weiss [4] as a way of measuring both the underlying size as well as the 'chaotic-ness' of a topological dynamical system. In this talk, we show how the mean dimension relates to the classification program for C^*-algebras by giving evidence for why the mean dimension of a topological dynamical system should roughly equal twice the radius of comparison of the corresponding crossed product C^*-algebra. This evidence is based on an example constructed by Giol and Kerr [1], who show that positive mean dimension is related to positive strict comparison, as well as the result of Lin and Phillips [3] that such a crossed product C^*-algebra can be approximated by an inverse limit of recursive subhomogeneous algebras. The evidence not only suggests a method for computing the radius of comparison (a purely algebraic invariant) in terms of the mean dimension (a purely dynamic invariant) but also that the mean dimension zero systems are the ones that have classifiable crossed products.

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Towards a model for free Cantor minimal $\mathbb{Z}^2$-systems

Thierry Giordano
(joint work with Ian Putnam and Christian Skau)

1. Introduction

The work of R. Hermann, I. Putnam and C. Skau [HPS] used ideas from operator algebras to construct a complete model for minimal actions of the group $\mathbb{Z}$ on a compact, totally disconnect metrizable space having no isolated points, i.e. a Cantor set. The data (a Bratteli diagram, with an order structure on the edge set of the diagram) is basically combinatorial and the two great features of the model are that it contains, in a reasonably accessible form, the orbit structure of the resulting Cantor dynamical system and also cohomological data proveded either from the K-Theory of the associated C*-algebras or more directly from the dynamics via group cohomology. This led to a complete classification of such systems up to orbit equivalence [GPS1]. This was the first extension of a famous program initiated by Henry Dye [D] in the study of orbit equivalence in ergodic theory to the topological situation (See also [CFW], [OW]).

The classification in [GPS1] was extended to include minimal actions of $\mathbb{Z}^2$ in [GMPS1] and minimal actions of finitely generated abelian groups in [GMPS2]. However the original model of [HPS] had not been extended, which has handicapped the general understanding of these actions. Notice also that the higher dimensional case has applications to the study of quasicrystals.

2. The cohomology of free minimal actions of $\mathbb{Z}^2$ on the Cantor set

Let us review some properties of the cohomology of a free minimal action (X,φ) of $\mathbb{Z}^2$ on the Cantor set. Recall (see for example [HF]) that we consider this cohomology as the group cohomology of $\mathbb{Z}^2$ with coefficient module $C(X,\mathbb{Z})$ but with no preferred choice of projective resolution.

We then have:

1) $H^0(X,\varphi) = \{f \in C(X,\mathbb{Z})\,; f = f \circ \varphi\} = \mathbb{Z}$, as the system is minimal.

2) $H^2(X,\varphi) = C(X,\mathbb{Z})/\{f - f\circ\varphi\,;\ f \in C(X,\mathbb{Z})\}$ is the group of co-invariants of (X,φ). Notice that $H^2(X,\varphi)$ is not necessarily torsion free.

To describe the 1-cocycles and the 1-coboundaries, let us introduce the following notation:

(i) If $\psi \in \mathrm{Homeo}(X)$ and $f \in C(X,\mathbb{Z})$, we define $\partial_\psi f$ by

$$\partial_\psi f(x) = f \circ \psi(x) - f(x), \quad x \in X\,.$$

(ii) To a 1-cocycle θ, we then assign the two functions f and g in $C(X,\mathbb{Z})$ defined by $f(x) = \theta(x,(1,0))$ and $g(x) = \theta(x,(0,1))$, $x \in X$.

Using the cocycle relation and denoting by $\alpha = \varphi^{(1,0)}$ and $\beta = \varphi^{(0,1)}$ the two canonical generators of the $\mathbb{Z}^2$-action φ, it is easy to show that

$Z^1(X,\varphi) = \{(f,g)\,;\, f,g \in C(X,\mathbb{Z}) \text{ and} \partial_\beta f = \partial_\alpha g\}$ and
$B^1(X,\varphi) = \{(\partial_\alpha h, \partial_\beta h)\,;\, h \in C(X,\mathbb{Z})\}$.

Let us state some properties of the first group of cohomology:

3) $H^1(X,\varphi)$ is a torsion free group.

4) The group $\mathbb{Z}^2$, realized as the subgroup generated by $[(\mathbf{1}_X,0)]$ and $[0,(\mathbf{1}_X)]$ is canonically imbedded in $H^1(X,\varphi)$. We do not know if this inclusion is always strict, for a free, minimal $\mathbb{Z}^2$-action on the Cantor set.

Definition. For an invariant probability measure μ of a free, minimal $\mathbb{Z}^2$-action (X,φ) on the Cantor set, let $\tau^1_\mu : H^1(X,\varphi) \to \mathbb{R}^2$ and $\tau^2_\mu : H^2(X,\varphi) \to \mathbb{R}$ denote the two group homomorphisms given by:

$$\tau^1_\mu([(f,g)]) = (\int_X f d\mu, \int_X g d\mu) \quad \text{and} \quad \tau^2_\mu([f]) = \int_X f d\mu\,.$$

Definition. Let (X,φ) be a free, minimal action of $\mathbb{Z}^2$ on the Cantor set. For any pair of cocycles $(f_i,g_i) \in Z^1(X,\varphi)$, let us define the continuous function

$$(f_1,g_1) \wedge (f_2,g_2) = f_1 \cdot g_2 \circ \beta - g_1 \cdot f_2 \circ \alpha$$

We then have:

Proposition. Keeping the above notation, then the map defined above is an associative, anti-commutative $\wedge$-product from $H^1(X,\varphi)$ to $H^2(X,\varphi)$.

If for any two vectors $(a_i,b_i) \in \mathbb{R}^2$, we consider the standard wedge product $(a_1,b_1) \wedge (a_2,b_2)$ given by $a_1 b_2 - b_1 a_2 \in \mathbb{R}$, we then have:

Proposition. Let (X,φ) be a free, minimal action of $\mathbb{Z}^2$ on the Cantor set with a unique, ergodic, invariant probability measure μ. For any pair $[(f_1,g_1)],[(f_2,g_2)] \in H^1(X,\varphi)$, we have:

$$\tau^2_\mu([(f_1,g_1)] \wedge [(f_2,g_2)]) = \tau^1_\mu([(f_1,g_1)]) \wedge \tau^1_\mu([(f_2,g_2)]).$$

Example. Let $0 < \alpha < \beta < 1$ be two rationally independent irrational numbers. We consider the natural action of $\mathbb{Z}^2$ on the circle $\mathbb{R}/\mathbb{Z}$ by rotation by α and by β. Disconnecting the circle along the $\mathbb{Z}^2$-orbit of 0, we get the Cantor set X and the two rotations R_α and R_β extends as homeomorphisms of X. In [HF], Hunton and

Forrest show that the three 1-cocycles $(\mathbf{1}_X, 0)$, $(0, \mathbf{1}_X)$ and $(\chi_{[0,\alpha]}, \chi_{[0,\beta]})$ generate H^1.

For the unique ergodic invariant measure μ on X, we have that $\tau^1_\mu(H^1)$ is the dense subgroup of $\mathbb{R}^2$ generated by $(1,0), (0,1)$ and (α, β).

We do not know if for any ergodic invariant probability measure μ of a free, minimal $\mathbb{Z}^2$-action (X, φ) on the Cantor set, $\tau^1_\mu(H^1)$ is a dense subgroup of $\mathbb{R}^2$. However it will be one of the assumption we will need for the construction of our model below.

Before showing an interesting consequence of this assumption, let us recall that a *Denjoy homeomorphism* is an aperiodic homeomorphism of the circle which is not conjugate to a pure rotation. By a *Denjoy system* we mean a Denjoy homeomorphism restricted to its unique invariant Cantor set (See [GPS1] and [PSS]).

Theorem. Let α be an irrational number satisfying no integral equations and assume that for any ergodic invariant probability measure μ of a free, minimal $\mathbb{Z}^2$-action (X, φ) on the Cantor set, $\tau^1_\mu(H^1)$ is a dense subgroup of $\mathbb{R}^2$.

Then there is no free, minimal $\mathbb{Z}^2$-action (X, φ) on the Cantor set, with a unique, invariant probability measure μ such that $\mu(C(X, \mathbb{Z})) = \mathbb{Z} + \alpha\mathbb{Z}$. In other words, there is no free, minimal $\mathbb{Z}^2$-action on the Cantor set orbit equivalent to a Denjoy system.

3. A model of free minimal actions of $\mathbb{Z}^2$ on the Cantor set

If (X, φ) is a Cantor minimal $\mathbb{Z}$-system, then its first cohomology group is equal to the quotient of $C(X, \mathbb{Z})$ by the subgroup $\{ f - f \circ \varphi^{-1} \ ; \ f \in C(X, \mathbb{Z}) \}$.

Hence, Hermann, Putnam and Skau's result [HPS] and Effros, Handelman and Shen's characterization of dimension groups [EHS] show that any simple dimension group is the first cohomology group of a Cantor minimal system.

Recall that for any two vectors $(a_i, b_i) \in \mathbb{R}^2$, the standard wedge product $(a_1, b_1) \wedge (a_2, b_2)$ is given by $a_1 b_2 - b_1 a_2$. Then we extend the dimension one above result to free, minimal, uniquely ergodic $\mathbb{Z}^2$-action (X, φ) on the Cantor set as follows:

Theorem. Let G^1 and G^2 be two torsion free, countable abelian groups, endowed with an associative, anti-commutative $\wedge$-product from G^1 into G^2 and let $\tau^1 : G^1 \to \mathbb{R}^2$ and $\tau^2 : G^2 \to \mathbb{R}$ be two group homomorphisms.

If $\tau^1(G^1)$ contains $\mathbb{Z}^2$ as the subgroup generated by $\{(1,0), (0,1)\}$ and is a dense subgroup of $\mathbb{R}^2$and if for any $a, b \in G^1$, $\tau^2(a \wedge b) = \tau^1(a) \wedge \tau^1(b)$.

Then there exists a uniquely ergodic, free, minimal $\mathbb{Z}^2$-action (X, φ) on the Cantor set such that $H^i(X, \varphi) \cong G^i$ for $i = 1, 2$.

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Mean dimension and von Neumann-Lück dimension

Hanfeng Li

(joint work with Bingbing Liang)

Gromov introduced mean dimension for continuous actions of countable amenable groups on compact metrizable spaces [1], as a dynamical analogue of the covering dimension of compact metrizable spaces. It was studied in detail by Lindenstrauss and Weiss [3]. Let a countable amenable group Γ act continuously on a compact metrizable space X. For each finite open cover $\mathcal{U}$ of X, set

$$\mathrm{ord}(\mathcal{U}) = \max_{s\in X} \sum_{U\in\mathcal{U}} 1_U(x) - 1,$$

and set

$$\mathcal{D}(\mathcal{U}) = \min_{\mathcal{V}\succ\mathcal{U}} \mathrm{ord}(\mathcal{V}),$$

where $\mathcal{V} \succ \mathcal{U}$ means that a finite open cover $\mathcal{V}$ of X is finer than $\mathcal{U}$ in the sense that every element of $\mathcal{V}$ is contained in some element of $\mathcal{U}$. For any finite open covers $\mathcal{U}$ and $\mathcal{V}$ of X, denote by $\mathcal{U} \vee \mathcal{V}$ the open cover of X consisting of $U \cap V$ for all $U \in \mathcal{U}$ and $V \in \mathcal{V}$. The function defined on the set of all nonempty finite subsets of Γ sending F to $\mathcal{D}(\bigvee_{s\in F} s^{-1}\mathcal{U})$ satisfies the conditions of the Ornstein-Weiss lemma [7] [3, Theorem 6.1], and hence $\frac{\mathcal{D}(\bigvee_{s\in F} s^{-1}\mathcal{U})}{|F|}$ converges to some real number t when F becomes more and more left invariant in the sense that for any $\delta > 0$, there exist some $\varepsilon > 0$ and nonempty finite subset K of Γ such that $|\frac{\mathcal{D}(\vee_{s\in F} s^{-1}\mathcal{U})}{|F|} - t| < \delta$ for every nonempty finite subset F of Γ satisfying $|KF \setminus F| < \varepsilon|F|$. Denote this limit t by $\mathrm{mdim}(\mathcal{U})$. The *mean topological dimension* of the action of Γ on X, denoted by $\mathrm{mdim}(X)$, is defined as $\sup_{\mathcal{U}} \mathrm{mdim}(\mathcal{U})$ for $\mathcal{U}$ ranging over all finite open covers of X.

For any countable group Γ, Lück defined a dimension for every (left) module $\mathcal{M}$ of the integral group ring $\mathbb{Z}\Gamma$ [4, 5]. Here $\mathbb{Z}\Gamma$ consists of finitely supported $\mathbb{Z}$-valued functions on Γ. We write the elements of $\mathbb{Z}\Gamma$ as $\sum_{s\in\Gamma} f_s s$, where $f_s \in \mathbb{Z}$ is zero for all but finitely many $s \in \Gamma$. Then the algebraic structure of $\mathbb{Z}\Gamma$ is defined by

$$\sum_{s\in\Gamma} f_s s + \sum_{s\in\Gamma} g_s s = \sum_{s\in\Gamma} (f_s + g_s)s, \quad (\sum_{s\in\Gamma} f_s s)(\sum_{t\in\Gamma} g_t t) = \sum_{s\in\Gamma} \Big(\sum_{t\in\Gamma} f_{st^{-1}} g_t \Big) s.$$

Consider the left regular representation of Γ on $\ell^2(\Gamma)$ given by $\lambda_(s \sum_{t\in\Gamma} g_t t) = \sum_{t\in\Gamma} g_{s^{-1}t} t$. The left group von Neumann algebra $\mathcal{L}\Gamma$ is the strong operator closure of all complex-coefficient linear combinations of λ_s for all $s \in \Gamma$. And $\mathbb{Z}\Gamma$ embeds into $\mathcal{L}\Gamma$ naturally by $\sum_{s\in\Gamma} f_s s \mapsto \sum_{s\in\Gamma} f_s \lambda_s$. The von Neumann trace tr of $\mathcal{L}\Gamma$ is defined by $\mathrm{tr}(a) = \langle a\delta_e, \delta_e\rangle$, where δ_e denotes the element of $\ell^2(\Gamma)$ taking value 1 at the identity element of Γ and value 0 everywhere else. One can also extend the trace tr to $M_n(\mathcal{L}\Gamma)$ for any $n \in \mathbb{N}$ by $\mathrm{tr}((a_{jk})_{1\le j,k\le n}) = \sum_{j=1}^n \mathrm{tr}(a_{jj})$. For each finitely generated projective (left) $\mathcal{L}\Gamma$-module $\tilde{\mathcal{M}}$, its dimension $\dim(\tilde{\mathcal{M}})$ is defined as $\mathrm{tr}(P)$ for any $P \in M_n(\mathcal{L}\Gamma)$ for some $n \in \mathbb{N}$ with $P^2 = P$ and $\tilde{\mathcal{M}} \cong (\mathcal{L}\Gamma)^n P$. For any (left) $\mathcal{L}\Gamma$-module $\tilde{\mathcal{M}}$, its von Neumann-Lück dimension, denoted by $\dim(\tilde{\mathcal{M}})$, is defined as $\sup_{\tilde{\mathcal{N}}} \dim(\tilde{\mathcal{N}})$ for $\tilde{\mathcal{N}}$ ranging over all finitely generated projective submodules of $\tilde{\mathcal{M}}$. For any (left) $\mathbb{Z}\Gamma$-module $\mathcal{M}$, its *von Neumann-Lück dimension*, denoted by $\dim(\mathcal{M})$, is defined as $\dim(\mathcal{L}\Gamma \otimes_{\mathbb{Z}\Gamma} \mathcal{M})$.

For any discrete abelian group G, denote by $\hat{G}$ the Pontrjagin dual of G, which is a compact abelian group. For for any countable group Γ and any countable $\mathbb{Z}\Gamma$-module $\mathcal{M}$, the module structure of $\mathcal{M}$ corresponds to an action of Γ on the countable discrete abelian group $\mathcal{M}$ by automorphisms, which in turn corresponds to an action of Γ on the compact metrizable abelian group $\hat{\mathcal{M}}$ by continuous automorphisms, a so-called *algebraic action*. One natural question is the relation between various L^2-invariants of the module $\mathcal{M}$ and dynamical invariants of the action $\Gamma \curvearrowright \hat{\mathcal{M}}$. Our main result is:

Theorem 1. Let Γ be a countable amenable group and $\mathcal{M}$ be a countable left $\mathbb{Z}\Gamma$-module. Then the mean topological dimension of the action $\Gamma \curvearrowright \hat{\mathcal{M}}$ is equal to the von Neumann-Lück dimension of $\mathcal{M}$.

We give two applications to dynamics.

Corollary 2. Let Γ be a countable amenable group. Consider an equivariant short exact sequence of compact metrizable abelian groups

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

with Γ-actions by continuous automorphisms. Then

$$\mathrm{mdim}(X_2) = \mathrm{mdim}(X_1) + \mathrm{mdim}(X_3).$$

For a continuous action of a countable amenable group Γ on a compact metrizable space X, given any compatible metric ρ on X, Lindenstrauss and Weiss also introduced a metric mean dimension for the action [3], as a dynamical analogue of the lower box dimension of a compact metric space. For a finite open cover $\mathcal{U}$ of X, set $\mathrm{mesh}(\mathcal{U},\rho) = \max_{U\in\mathcal{U}} \mathrm{diam}(U,\rho)$. For any nonempty finite subset F of Γ, define a new metric ρ_F on X by $\rho_F(x,y) = \max_{s\in F}\rho(sx,sy)$. For any $\varepsilon > 0$, the function defined on the set of all nonempty finite subsets of Γ sending F to $\log\min_{\mathrm{mesh}(\mathcal{U},\rho_F)<\varepsilon}|\mathcal{U}|$ also satisfies the conditions of the Ornstein-Weiss lemma, and hence $\frac{\log\min_{\mathrm{mesh}(\mathcal{U},\rho_F)<\varepsilon}|\mathcal{U}|}{|F|}$ converges to some limit, denoted by $S(X,\varepsilon,\rho)$, when F becomes more and more left invariant. The *metric mean dimension* of the action $\Gamma \curvearrowright X$ with respect to ρ, denoted by $\mathrm{mdim}_\rho(X)$, is defined as $\varliminf_{\varepsilon\to 0}\frac{S(X,\varepsilon,\rho)}{|\log\varepsilon|}$.

The Pontrjagin-Schnirelmann theorem [8, 6] says that for any compact metrizable space, its covering dimension is equal to the minimal value of the lower box dimensions over all compatible metrics. A natural question is whether the dynamical analogue holds. Lindenstrauss and Weiss showed that for any continuous action of a countable amenable group on a compact metrizable space, its mean topological dimension is no bigger than the metric mean dimension for any compatible metric [3]. Lindestruass showed that if a continuous action of $\mathbb{Z}$ on a compact metrizable space has a nontrivial minimal factor, then its mean topological dimension is equal to the minimal value of the metric mean dimensions over all compatible metrics [2].

Corollary 3. For any action of a countable amenable group Γ on a compact metrizable abelian group X by continuous automorphisms, one has $\mathrm{mdim}(X) = \mathrm{mdim}_\rho(X)$ for some compatible metric ρ on X being translation invariant in the sense that $\rho(x+y,x+z) = \rho(y,z)$ for all $x,y,z\in X$.

We also give an application to L^2-invariants. Recall that the *rank* of a discrete abelian group G is defined as the dimension of real vector space $\mathbb{Q}\otimes_{\mathbb{Z}} G$.

Corollary 4. Let Γ be a countably infinite amenable group, and $\mathcal{M}$ be a countable left $\mathbb{Z}\Gamma$-module. If $\mathcal{M}$ has finite rank as a discrete abelian group, then $\dim(\mathcal{M}) = 0$.

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Combinatorial independence, amenability, and sofic entropy

David Kerr
(joint work with Hanfeng Li)

Inspired by the Elton-Pajor theorem in Banach spaces and by work in the local theory of dynamical entropy as initiated by Blanchard, we developed in [5] a systematic approach to the study of combinatorial independence in dynamics that permits one to treat in a unified way various phenomena associated with randomness like weak mixing and entropy. We showed for example that a continuous action $G \curvearrowright X$ of a countable amenable group on a compact metric space has positive entropy if and only if it has a nondiagonal IE-pair, with the notion of IE-pair being defined as follows. Given a pair (A_0, A_1) of subsets of X, a set $J \subseteq G$ is an *independence set* for (A_0, A_1) if the collection $\{(s^{-1}A_0, s^{-1}A_1)\}_{s\in J}$ is independent in the sense that for every finite set $F \subseteq J$ and function $\omega : F \to \{0,1\}$ the intersection $\bigcap_{s\in F} s^{-1}A_{\omega(s)}$ is nonempty. The *independence density* of (A_0, A_1) is the largest $q \geq 0$ such that every finite set $F \subseteq G$ has a subset of cardinality at least $q|F|$ which is an independence set for (A_0, A_1) (in [5] we formulated independence density using Følner sets, but it turns out to give the same quantity [8]). A pair $(x_0, x_1) \in X \times X$ is an *IE-pair* if for all neighbourhoods U_0 and U_1 of x_0 and x_1, respectively, the pair (U_0, U_1) has positive independence density. More generally for every $k \geq 1$ one defines IE-k-tuples, the set of which is written $\mathrm{IE}_k(X, G)$. Note that these definitions do not require G to be amenable.

Recent seminal work by Bowen [1] initiated the development of an entropy theory for actions of groups which satisfy the very weak finite approximation property of soficity [6, 7, 4]. In [6], for example, we established a variational principle and used it to compute the sofic topological entropy of certain principle algebraic actions of residually finite groups in terms of the Fuglede-Kadison determinant in the group von Neumann algebra, yielding a formula which is consistent with previous work on algebraic actions. The goal of the present project has been to undertake an analysis of combinatorial independence as it relates to topological entropy in this broadened sofic framework [8].

Let $G \curvearrowright X$ be a continuous action of a countable sofic group on a compact metric space and $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}$ a sofic approximation sequence for G, meaning that

(i) $\displaystyle\lim_{i\to\infty} \frac{1}{d_i}\big|\{v \in \{1,\dots,d_i\} : \sigma_{i,st}(v) = \sigma_{i,s}\sigma_{i,t}(v)\}\big| = 1$ for all $s, t \in G$,

(ii) $\lim_{i\to\infty} \frac{1}{d_i}\left|\{v \in \{1,\dots,d_i\} : \sigma_{i,s}(v) \neq \sigma_{i,t}(v)\}\right| = 1$ for all distinct $s,t \in G$,

and $d_i \to \infty$ as $i \to \infty$. Given a dynamically generating continuous pseudometric ρ on X, we define on the set of all maps from a finite set $\{1,\dots,d\}$ to X the pseudometric

$$\rho_2(\varphi,\psi) = \left(\frac{1}{d}\sum_{v=1}^{d}(\rho(\varphi(v),\psi(v)))^2\right)^{1/2}.$$

For a nonempty finite set $F \subseteq G$, a $\delta > 0$, and a map σ from G to $\mathrm{Sym}(d)$ for some $d \in \mathbb{N}$, we write $\mathrm{Map}(\rho,F,\delta,\sigma)$ for the set of all maps $\varphi : \{1,\dots,d\} \to X$ such that $\rho_2(\varphi\sigma_s, \alpha_s\varphi) < \delta$ for all $s \in F$, where α_s is the transformation $x \mapsto sx$ of X. Writing $N_\varepsilon(\cdot)$ to mean the maximum cardinality of an ε-separated set, we then define the *topological entropy* of the action to be

$$h_\Sigma(X,G) = \sup_{\varepsilon>0}\inf_F\inf_{\delta>0}\limsup_{i\to\infty}\frac{1}{d_i}\log N_\varepsilon(\mathrm{Map}(\rho,F,\delta,\sigma_i)),$$

a quantity that does not depend on ρ.

In this framework we define a Σ-IE-pair as an externalization to Σ of the internal concept of IE-pair from the first paragraph, with the points in the sofic approximation space playing the role that group elements did before. We say that a set $J \subseteq \{1,\dots,d\}$ is a *(ρ,F,δ,σ)-independence set* for a pair (A_0,A_1) of subsets of X if for every function $\omega : J \to \{0,1\}$ there exists a $\varphi \in \mathrm{Map}(\rho,F,\delta,\sigma)$ such that $\varphi(v) \in A_{\omega(v)}$ for every $v \in J$. Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. We say that (A_0,A_1) has *positive upper independence density* over Σ if there exists a $q > 0$ such that for every nonempty finite set $F \subseteq G$ and $\delta > 0$ the set of all i for which (A_0,A_1) has a (ρ,F,δ,σ_i)-independence set of cardinality at least qd_i is a member of $\mathcal{U}$. As before we now define Σ-IE-pairs and more generally Σ-IE-k-tuples, the collection of which is denoted by $\mathrm{IE}_k^\Sigma(X,G)$. As in the amenable case, it turns out that $h_\Sigma(X,G) > 0$ if and only if there is a nondiagonal Σ-IE-pair in $X \times X$.

Given actions $G \curvearrowright X$ and $G \curvearrowright Y$ we have the product formula

$$\mathrm{IE}_k(X \times Y, G) = \mathrm{IE}_k(X,G) \times \mathrm{IE}_k(Y,G).$$

for IE-tuples. We do not know whether the same formula holds for Σ-IE-tuples. However, if we consider the measure-preserving action of G on the commutant of the ultraproduct Loeb space $\prod_{\mathcal{U}}\{1,\dots,d_i\}$ associated to the sofic approximation sequence $\Sigma = \{\sigma_i : G \to \mathrm{Sym}(d_i)\}$, then the ergodicity of the commutant of this action in the space of all automorphisms arising from elements of $\mathrm{Sym}(d_i)$ implies the product formula. This ergodicity can fail for sofic approximation sequences of nonamenable groups due to observations of Elek and Paunescu, but we show that it holds for residually finite groups when Σ is built from finite quotients, and also for amenable groups with Σ arbitrary. The latter was achieved by establishing a refinement of the Rokhlin lemma for sofic approximations.

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Poly-$\mathbb{Z}$ group actions on Kirchberg algebras

Masaki Izumi

(joint work with Hiroki Matui)

Our goal is to classify outer actions of a poly-$\mathbb{Z}$ group on a Kirchberg algebra. So far we have obtained a complete classification result for Hirsch length less than or equal to 3, in the sense that we have enough obstructions to distinguish any two outer actions belonging to different KK-trivial cocycle conjugacy classes.

A discrete group G is said to be *poly-$\mathbb{Z}$* if there exists a normal series $\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_N = G$, with $G_{i+1}/G_i \cong \mathbb{Z}$ for all $i = 0, 1, \cdots, N-1$. The number $h(G) = N$ is called the *Hirsch length* of G, which is independent of the choice of the normal series as above, and coincides with the cohomological dimension of G.

A typical example of a poly-$\mathbb{Z}$ group is $\mathbb{Z}^N$. More generally, the class of poly-$\mathbb{Z}$ groups includes every finitely generated torsion free nilpotent group and every cocompact lattice of a simply connected solvable Lie group. For $h(G) = 1$, there exists only one poly-$\mathbb{Z}$ group $\mathbb{Z}$. For $h(G) = 2$, there exist exactly two poly-$\mathbb{Z}$ groups $\mathbb{Z}^2$ and $\mathbb{Z} \rtimes_{-1} \mathbb{Z}$, the fundamental group of the Klein bottle. For $h(G) = 3$, there exist infinitely many poly-$\mathbb{Z}$ groups, e.g. $\mathbb{Z}^2 \rtimes_\Gamma \mathbb{Z}$ with $\Gamma \in GL(2, \mathbb{Z})$.

Throughout this note, we use the following notation: α and β are outer actions of a poly-$\mathbb{Z}$ group G on a unital Kirchberg algebra A. We denote by $A^\flat$ the quotient C^*-algebra $C^b([0,\infty), A)/C_0([0,\infty), A)$. We often identify an element in $A^\flat$ with one of its representatives in $C^b([0,\infty), A)$. We denote $A_\flat = A^\flat \cap A'$, where A is identified with the set of constant functions. The action α induces G-actions on $A^\flat$ and $A_\flat$, which will be denoted by the same symbol α.

A family of unitaries $\{a_g\}_{g\in G}$ in A is said to be an α-cocycle if they satisfy the cocycle relation $a_g \alpha_g(a_h) = a_{gh}$. When $\{a_g\}_{g\in G}$ is an α-cocycle, then α^a defined by $\alpha^a_g = \operatorname{Ad} a_g \circ \alpha_g$ is a G-action too, which is called a cocycle perturbation of α. We say that α and β are *KK-trivially cocycle conjugate* if there exist $\gamma \in \operatorname{Aut}(A)$ with $KK(\gamma) = KK(\mathrm{id})$ and an α-cocycle $\{a_g\}_{g\in G}$ satisfying $\beta_g = \gamma \circ \operatorname{Ad} a_g \circ \alpha_g \circ \gamma^{-1}$.

The first classification result for poly-$\mathbb{Z}$ group actions on Kirchberg algebras was obtained by H. Nakamura [3] for $G = \mathbb{Z}$ in 2000. He showed that α and β are KK-trivially cocycle conjugate if and only if $KK(\alpha_1) = KK(\beta_1)$.

A partial classification result for $G = \mathbb{Z}^2$ was obtained by Izumi-Matui [2] in 2010. We introduced an invariant $\Phi(\alpha) \in KK^1(A,A)$ for $\mathbb{Z}^2$-actions α satisfying $KK(\alpha_g) = KK(\mathrm{id})$, and showed that Φ gives rise to a one-to-one correspondence between the set of KK-trivial cocycle conjugacy classes of α with $KK(\alpha_g) = KK(\mathrm{id})$ and $\{x \in KK^1(A,A);\ K_0(1)\hat{\otimes}_A x = 0 \in K_1(A)\}$. Since $KK^1(A,A)$ as a $\mathbb{Z}^2$-module induced by α in this case is trivial, we can identify $KK^1(A,A)$ with the cohomology group $H^2(\mathbb{Z}^2, KK^1(A,A))$, and the invariant $\Phi(\alpha)$ is identified with the primary obstruction $\mathfrak{o}^2(\alpha,\mathrm{id})$ discussed below.

We go back to the general case, and try to seek necessary conditions for α and β to be KK-trivially cocycle conjugate, that is $\beta_g = \gamma\circ \mathrm{Ad}\, a_g \circ \alpha_g \circ \gamma^{-1}$. An obvious necessary condition is $KK(\alpha_g) = KK(\beta_g)$. A little less obvious necessary condition is that β should be continuously approximated by cocycle perturbations of α. Namely since $KK(\gamma) = KK(\mathrm{id})$, there exists a continuous family of unitaries $\{w(t)\}_{t\geq 0}$ in A satisfying $\gamma(x) = \lim_{t\to\infty} \mathrm{Ad}\, w(t)(x)$ for any $x \in A$. Thus setting $u_g(t) = w(t)a_g\alpha_g(w(t)^*)$, which is an α-cocycle for each t, we obtain.

$$\beta_g(x) = \lim_{t\to\infty} \mathrm{Ad}\, w(t)\circ \mathrm{Ad}\, a_g \circ \alpha_g \circ \mathrm{Ad}\, w(t)^*(x) = \lim_{t\to\infty} \mathrm{Ad}\, u_g(t)\circ \alpha_g(x).$$

It turns out that these two conditions are sufficient. In fact, the second condition can be relaxed a little bit.

Theorem 1. If there exists an α-cocycle $\{u_g\}_{g\in G}$ in $U(A^\flat)$ satisfying the condition $\beta_g = \lim_{t\to\infty} \mathrm{Ad}\, u_g(t)\circ\alpha_g$, then α and β are KK-trivially cocycle conjugate.

The proof of Theorem 1 is an induction argument by the Hirsch length, and it is technically very complicated. It requires the following facts as basic ingredients.

Theorem 2. Let the notation be as above.

(1) There exists an outer asymptotically representable action of G on the Cuntz algebra $\mathcal{O}_\infty$. (See [2] for the definition of asymptotic representability).
(2) If A is strongly self-absorbing, there exists a unique outer action of G on A up to KK-trivial cocycle conjugacy. In particular, such an action is asymptotically representable.
(3) Let μ be an outer action of G on $\mathcal{O}_\infty$. Then α on A and $\alpha\otimes\mu$ on $A\otimes\mathcal{O}_\infty$ are cocycle conjugate.

Theorem 1 alone does not classify actions except for $G = \mathbb{Z}$, and we need a criterion for the existence of an α-cocycle $\{u_g\}_{g\in G}$ in $U(A^\flat)$ satisfying the condition in Theorem 1. Assume that $KK(\alpha_g) = KK(\beta_g)$ holds for every $g \in G$. Then there exists $u_g \in U(A^\flat)$ satisfying $\beta_g = \lim_{t\to\infty} \mathrm{Ad}\, u_g(t)\circ\alpha_g$. Let $w_{g,h} = u_g\alpha_g(u_h)u_{gh}^*$, which is a unitary in $U(A_\flat)$, and let $\sigma_g = \mathrm{Ad}\, u_g \circ \alpha_g|_{A_\flat}$. Then the pair (σ, w) is a cocycle action of G on $A_\flat$, and the following two conditions are equivalent: (i) $\{u_g\}_{g\in G}$ can be chosen to form an α-cocycle, (ii) (σ, w) is equivalent to a genuine action. For (ii), one can easily find an obstruction. We denote by $\mathfrak{o}^2(\alpha,\beta)$ the cohomology class of $[K_1(w_{g,h})]$ in $H^2(G,K_1(A_\flat))$, which does not depend on the choice of $\{u_g\}_{g\in G}$. Unless $\mathfrak{o}^2(\alpha,\beta)$ vanishes, (ii) never holds. We call $\mathfrak{o}^2(\alpha,\beta)$ the *primary obstruction* for α and β to be KK-trivially cocycle conjugate. For the coefficient module $K_1(A_\flat)$, we have the following description. Note that Dadarlat [1] showed $\pi_n(\mathrm{Aut}(A\otimes\mathbb{K})_0)\cong KK^n(A,A)$ for $n\geq 1$.

Theorem 3. For each finite CW-complex X, there exists an isomorphism from $[X,U(A_\flat)]_0$ onto $[X,\mathrm{Map}(S^1,\mathrm{Aut}(A\otimes\mathbb{K}))]_0$, which is natural in X. In particular, the isomorphism for $X=S^{n-1}$ yields $K_n(A_\flat)\cong\pi_n(\mathrm{Aut}(A\otimes\mathbb{K})_0)$ for $n\geq 1$.

When $\mathfrak{o}^2(\alpha,\beta)=0$, we may and do choose $\{u_g\}_{g\in G}$ so that $w_{g,h}\in U(A_\flat)_0$, and choose a continuous path $\{\tilde{w}_{g,h}(s)\}_{s\in[0,1]}$ in $U(A_\flat)_0$ from 1 to $w_{g,h}$. Then $K_1(\sigma_g(\tilde{w}_{h,k})\tilde{w}_{g,hk}\tilde{w}^*_{gh,k}\tilde{w}^*_{g,h})\in K_1(SA_\flat)=\pi_1(U(A_\flat)_0)\cong\pi_2(\mathrm{Aut}(A\otimes\mathbb{K})_0)$, and they give rise to an element $\mathfrak{o}^3(\alpha,\beta,u)\in H^3(G,KK(A,A))$. We call $\mathfrak{o}^3(\alpha,\beta,u)$ the *secondary obstruction*, which does not depend on the choice of $\{\tilde{w}_{g,h}\}_{g,h\in G}$ while it may depend on the choice of $\{u_g\}_{g\in G}$.

Theorem 4. Assume $KK(\alpha_g)=KK(\beta_g)$.

(1) Assume $h(G)=2$. Then α and β are KK-trivially cocycle conjugate if and only if $\mathfrak{o}^2(\alpha,\beta)=0$.

(2) Assume $h(G)=3$. Then α and β are KK-trivially cocycle conjugate if and only if $\mathfrak{o}^2(\alpha,\beta)=0$ and $\mathfrak{o}^3(\alpha,\beta,u)=0$ for some choice of $\{u_g\}_{g\in G}$.

Theorem 4 shows that when $h(G)\leq 3$, the number of KK-trivial cocycle conjugacy classes of outer G-actions α on A with given $KK(\alpha_g)$ is bounded by $\#H^2(G,KK^1(A,A))\times\#H^3(G,KK(A,A))$. While it is a subtle problem to decide the exact ranges of $\mathfrak{o}^2$ and $\mathfrak{o}^3$ in general, we can decide them in the case of the Cuntz algebra $A=\mathcal{O}_n$. Note that $KK(\gamma)=KK(\mathrm{id})$ for any $\gamma\in\mathrm{Aut}(\mathcal{O}_n)$.

Theorem 5. Assume that $h(G)\leq 3$ and n is finite.

(1) There exist exactly $\#H^2(G,\mathbb{Z}/(n-1))$ cocycle conjugacy classes of outer G-actions on $\mathcal{O}_n$.

(2) There exist exactly $\#H^2(G,\mathbb{Z}/(n-1))\times\#H^3(G,\mathbb{Z}/(n-1))$ cocycle conjugacy classes of outer cocycle actions of G on $\mathcal{O}_n$.

Let BG be the classifying space, and let EG be its universal cover. Then we have $H^{n+1}(G,K_n(A_\flat))\cong H^{n+1}(BG,\pi_n(\mathrm{Aut}(A\otimes\mathbb{K})_0))$, which suggests that our obstructions might come from those for the existence of a section of a fiber bundle over BG with a fiber $\mathrm{Aut}(A\otimes\mathbb{K})_0$. Indeed it is the case. We denote by $\mathcal{P}_\alpha$ the principal $\mathrm{Aut}(A)$-bundle over BG defined by $(EG\times\mathrm{Aut}(A))/G$ with a G-action $g(x,\gamma)=(gx,\alpha_g\circ\gamma)$. Replacing A with $A\otimes\mathbb{K}$ and α_g with $\alpha^s_g=\alpha_g\otimes\mathrm{Ad}\,\rho_g$, where ρ is the right regular representation of G, we obtain a principal $\mathrm{Aut}(A\otimes\mathbb{K})$-bundle over BG, denoted by $\mathcal{P}^s_\alpha$.

Conjecture 6. The following two conditions are equivalent:

(1) The two actions α and β are KK-trivially cocycle conjugate.

(2) $\mathcal{P}^s_\alpha$ and $\mathcal{P}^s_\beta$ are isomorphic by a base point preserving map.

The implication from (1) to (2) is always true. Note that (2) never holds unless $KK(\alpha_g) = KK(\beta_g)$. Assuming this condition, we can construct a fiber bundle $\mathcal{I}^s_{\alpha,\beta} = (EG \times \mathrm{Aut}(A \otimes \mathbb{K})_0)/G$ with a G-action $g(x,\gamma) = (gx, \beta^s_g \circ \gamma \circ \alpha^s_{g^{-1}})$. Then (2) holds if and only if $\mathcal{I}^s_{\alpha,\beta} \to BG$ has a section, and $\mathfrak{o}^2$ (resp. $\mathfrak{o}^3$) is identified with the primary (resp. secondary) obstruction for the existence of the section. Thus Theorem 4 shows that the implication from (2) to (1) is true for $h(G) \leq 3$.

When A is strongly self-absorbing, the homotopy groups of $\mathrm{Aut}(A)$ are trivial (see [1]). This implies that $\mathcal{P}_\alpha$ and $\mathcal{P}_\beta$ are trivial bundles, and (2) holds. Therefore Theorem 2,(2) shows that the conjecture is true for such A.

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Isomorphism of separable C*-algebras is below a group action

Asger Törnquist

In this talk, I reported in recent joint work with George Elliott, Ilijas Farah, Vern Paulsen, Christian Rosendal and Andrew Toms. The question was raised in previous work by Farah-Toms-Tornquist if the isomorphism relation of general separable C*-algebras can be distinguished from the isomorphism relation for nuclear simple separable C*-algebras using Borel reducibility as a complexity measure. It was further considered that a distinguishing feature could be that isomorphism of nuclear simple separable C*-algebras is "below a group action", whereas, potentially, the general isomorphism problem might not be. Here, being "below a group action" means that the isomorphism relation is Borel reducible to an orbit equivalence relation induced by a Borel action of a Polish group on a standard Borel space. The new result is that isomorphism of separable C*-algebras is below a group action, thus ruling out this as a way of distinguishing the general isomorphism problem from its restriction to various natural subclasses. Additionally, similar techniques also allow us to prove that complete isomettry and n-isometry for (separable) operator spaces and operator systems also are below a group action.

Classification of graph C^*-algebras with one non-trivial ideal

Efren Ruiz

(joint work with Søren Eilers, Takeshi Katsura, Gunnar Restorff and Mark Tomforde)

The magnificent recent progress of the classification theory for simple C^*-algebras has few direct consequences for general C^*-algebras, even for those with finite ideal lattices. Furthermore, it is not even clear what kind of K-theoretical invariant to use in such a context.

When there is just one non-trivial ideal, however, there is a canonical choice of invariant. Associated to every extension $0 \to B \to E \to A \to 0$ of nonzero C^*-algebras is the standard six term exact sequence of K-groups, $K_{\mathrm{six}}(E)$,

$$\begin{tikzcd} K_0(B) \arrow[r] & K_0(E) \arrow[r] & K_0(A) \arrow[d] \\ K_1(A) \arrow[u] & K_1(E) \arrow[l] & K_1(B) \arrow[l] \end{tikzcd}$$

providing a necessary condition for two extensions to be isomorphic.

Many of the classification results using the six term exact sequence of K-groups involve C^*-algebras whose simple sub-quotients are of the same type. Moreover, most results were achieved by using the standard Elliott intertwining argument. In [7], Rørdam used a completely different technique to classify a certain class C^*-algebras with one non-trivial ideal. He considered C^*-algebras A with one non-trivial ideal I such that I and A/I are separable nuclear purely infinite simple C^*-algebras in the bootstrap category of Rosenberg and Schochet. Employing the fact that every separable nuclear purely infinite simple C^*-algebra in the bootstrap category is strongly classified by KK-theory, in the sense that every invertible element of $\mathrm{KK}(A, B)$ lifts to a $*$-isomorphism from A to B, and the fact that every essential extension of A by B is absorbing, Rørdam showed that the six term sequence is, indeed, a complete stable isomorphism invariant in this case.

The author with Eilers and Restorff generalized Rørdam's results in [4] and [3]. We have provided a framework for classifying nonsimple C^*-algebras whose simple sub-quotients are not necessarily of the same type. In particular, in [4] we showed that the six term exact sequence in K-theory is a complete stable isomorphism invariant for C^*-algebras A with one non-trivial ideal I satisfying the following:

(1) I and A/I are strongly classified by KK-theory and
(2) the extension $0 \to I \to A \to A/I \to 0$ is full.

This has allowed for the classification of certain nonsimple C^*-algebras in which there are ideals and quotients of mixed type (some finite and some infinite).

In [5], Eilers and Tomforde, used the techniques from the theory of graph C^*-algebras, to show that the machinery of [4] can be used to classify graph C^*-algebras with one non-trivial ideal up to stable isomorphism. Building on their work, the author with Eilers and Restorff in [2] showed that the six term exact sequence in K-theory together with the scale of the quotient is a complete isomorphism invariant for the class of non-unital graph C^*-algebras with one non-trivial ideal. Thus, nearly completing the classification of graph C^*-algebras with one non-trivial ideal. The only remaining case is the case where the C^*-algebra is unital. Unfortunately, the techniques in [5] and [2] can not be used in the unital case. The results in these papers relied on the fact that every non-unital full extension absorbs any extension which is not the case for unital extensions. Recently, the author with Eilers and Restorff obtained existence and uniqueness theorems for unital graph C^*-algebras with one non-trivial ideal. One can then use the standard Elliott intertwining argument to classify unital graph C^*-algebras with one non-trivial ideal. As a consequence, isomorphisms of the invariant can be lifted to an isomorphism on the associated graph C^*-algebras.

To complete the classification of graph C^*-algebras with one non-trivial ideal, one needs to determine the range of the invariant. Recent results of Eilers, Katsura, Tomforde, and West [1] and the author with Eilers, Katsura, and Tomforde, we now can determine the range of the invariant. Hence, completing the classification of graph C^*-algebras with one non-trivial ideal.

We can use the above results to determine when a unital C^*-algebra with one non-trivial ideal is a graph C^*-algebra. Our results can also be used to determine when an extension of two simple graph C^*-algebras is again a graph C^*-algebra.

Theorem 1. If A is a unital C^*-algebra with one non-trivial ideal I such that A has real rank zero,

(1) if A/I is AF, then $A/I \cong \mathsf{M}_n$;
(2) if I is AF, then $I \cong \mathbb{K}$;
(3) $K_*(I)$ and $K_*(A/I)$ are finitely generated;
(4) $K_1(I)$ and $K_1(A/I)$ are free groups;
(5) $\operatorname{rank}(K_1(I)) \leq \operatorname{rank}(K_0(I))$, and
(6) $\operatorname{rank}(K_1(A/I)) \leq \operatorname{rank}(K_0(A/I))$,

then there exists a graph E such that $A \cong C^*(E)$.

Theorem 2. Let A be a C^*-algebra with one non-trivial ideal I such that I and A/I are graph C^*-algebras. Suppose the following holds:

(1) A has real rank zero.
(2) If $K_0(A/I)_+ = K_0(A/I)$ and $K_0(I)_+ \neq K_0(I)$, then I is stable and $K_0(A)_+ = K_0(A)$.
(3) If A is a unital C^*-algebra, then
  (a) $K_0(A)$ is finitely generated
  (b) $\operatorname{rank}(K_1(A)) \leq \operatorname{rank}(K_0(A))$
  (c) $K_0(I)_+ \neq K_0(I)$ implies that $K_0(I) \cong \mathbb{Z}$.

Then there exists a graph E such that $A \cong C^*(E)$.

We end with some open problems.

Problem 3. Find an algebraic invariant that will be a complete stable isomorphism invariant for the class of graph C^*-algebras with finitely many ideals.

It has been conjectured by the author with Eilers and Restorff that ideal related ordered K-theory is the right invariant to consider. Another invariant that one could consider is the reduced filtered K-theory introduced by Restorff in [6].

Problem 4. Find an algebraic invariant that will be a complete isomorphism invariant for the class of graph C^*-algebras with finitely many ideals.

One can also consider permanence properties of graph C^*-algebras like that of Theorem 2. When is an extension A of graph C^*-algebras again a graph C^*-algebra? It will not be the case that A is always a graph C^*-algebra. As Theorem 2 indicates, there are K-theoretical obstructions.

Problem 5. Let A be a C^*-algebra and let I be an ideal of A. Suppose I and A/I are graph C^*-algebras. Determine when A is a graph C^*-algebra? Can the obstructions be describe using K-theoretical data?

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The Kadison-Singer problem

Charles Akemann

Charles Akemann talked about the development of the Kadison-Singer problem. He presented an approach to the problem using the notion of a paving by projections.

One-parameter continuous fields of Kirchberg algebras with rational K-theory

Rasmus Bentmann
(joint work with Marius Dadarlat)

This work is concerned with the classification of continuous fields of Kirchberg algebras over the unit interval by K-theoretic invariants. To give some background, we first recall related work by a number of different authors. We then state our main result and close with some remarks concerning filtrated K-theory with (generalized) coefficients.

In [6], M. Dadarlat and R. Meyer proved a universal *multi*-coefficient theorem (UMCT) for separable $\mathrm{C}(X)$-algebras over a totally disconnected compact metrizable space X. As a consequence, by Kirchberg's classification theorem [9], separable continuous fields over such spaces whose fibres are stable Kirchberg algebras satisfying the universal coefficient theorem (UCT) are classified by an invariant the authors call "*filtrated* K-*theory with coefficients*." This result is also implicitly contained in [7].

The invariant filtrated K-theory with coefficients comprises the K-theory with coefficients (the Λ-modules defined in [5], also called *total* K-*theory*) of all distinguished subquotients of the given field, along with the action of all natural maps between these groups. It is demonstrated in [6], generalising an observation from [3], that coefficients are necessary for such a classification result over any infinite metrizable compact space.

First classification results for continuous fields of Kirchberg algebras over the interval where proven by Dadarlat and Elliott in [4]. It is shown that, for $d \in \{0,1\}$, separable stable continuous fields whose fibers are Kirchberg algebras satisfying the UCT and having torsion-free K_d-group and trivial K_{d+1}-group are classified by the so-called K_d-*sheaf*.

The purpose of our work is to relax the K-theoretic assumptions made in [4]. We allow both K_0 and K_1 of the fibers to be non-zero. In order to avoid coefficients, we do however assume that the K-groups of all subquotients are divisible (or free). We call this property K-*divisibility* (K-*freeness*). For instance, all fields which are stable under tensoring with the universal UHF-algebra are K-divisible.

Our invariant is the following: the *filtrated* K-*theory* of a $\mathrm{C}([0,1])$-algebra A consists of the abelian groups $\mathrm{K}_*\big(A(I)\big)$ for all (open, half-open or closed) subintervals $I \subseteq [0,1]$ together with the six-term sequence maps

$$\mathrm{K}_*\big(A(J)\big) \to \mathrm{K}_*\big(A(I)\big) \to \mathrm{K}_*\big(A(I \setminus J)\big) \to \mathrm{K}_{*+1}\big(A(J)\big)$$

for every subinterval $I \subseteq [0,1]$ and every relatively open subinterval $J \subseteq I$.

Our main result then reads as follows:

Theorem. *Filtrated* K-*theory is a complete invariant for separable,* K-*divisible continuous* $\mathrm{C}([0,1])$-*algebras whose fibers are stable Kirchberg algebras satisfying the* UCT.

An analogous result holds for K-free algebras. As Takeshi Katsura pointed out during the workshop, our approach also works under the alternative assumption that, for $d = 0$ or $d = 1$, we have $\mathrm{K}_d\big(A(I)\big) = 0$ for all subintervals $I \subseteq [0,1]$. It remains open whether the same is true for the K-theoretic assumptions in Dadarlat-Elliott's theorem.

Besides Kirchberg's classification result for non-simple nuclear purely infinite C^*-algebras [9] (and the work of several people showing in combination that Kirchberg's theorem is applicable), our approach is based on two crucial ingredients:

- the work in [6], which relates E-theory over a second countable space X with the corresponding version of KK-theory and with E-theory groups over finite approximating spaces of X;
- the universal coefficient theorem for C^*-algebras over so-called *accordion spaces* from [1] (generalizing results from [13, 2, 11, 10]) including a description of projective and injective objects in the target category of filtrated K-theory.

The relevance of accordion spaces in this framework is due to the fact that sufficiently many finite approximating spaces of the interval are accordion spaces.

In order to remove the K-divisibility/K-freeness condition from the previous classification result one expects, as indicated earlier, to need some version of filtrated K-theory with coefficients for C^*-algebras over the interval. This requires, to begin with, the correct definition of filtrated K-theory with coefficients for C^*-algebras over accordion spaces. It was observed in [8] that, already over the two-point Sierpiński space S, the naïve candidate for such a definition—using the corresponding six-term sequence of Λ-modules—produces an invariant which lacks desired properties such as a UMCT.

We argue that, in order to give a fully satisfactory definition of filtrated K-theory with coefficients for C^*-algebras over S, one has to allow all finitely generated indecomposable exact six-term sequences of abelian groups as coefficients—just like Dadarlat and Loring choose all finitely generated indecomposable abelian groups as coefficients to make their UMCT work [5].

It is easy to see that there is a countable number of isomorphism classes of such six-term sequences. However, unlike in the case of abelian groups, it is not possible to list them in a nice way; more precisely, it follows from the main result in [12] that their classification is at least as complicated as the classification of modules over the free associative $\mathbb{Z}/p$-algebra with two non-commuting generators for every prime p.

This wildness phenomenon seems to make filtrated K-theory with (generalized) coefficients as sketched above virtually impossible to compute explicitly, limiting its rôle in the theory to a rather theoretical one.

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Do phantom Cuntz-Krieger algebras exist?

Sara Arklint

(joint work with Rasmus Bentmann, Takeshi Katsura, Gunnar Restorff and Efren Ruiz)

The Cuntz and Cuntz-Krieger algebras are historically and in general of great importance for our understanding of simple and nonsimple purely infinite C^*-algebras as they were not only the first constructed examples of such but are also very tangible due to the combinatorial nature of their construction, [6].

The Cuntz algebras and the simple Cuntz-Krieger algebras can be identified as the UCT Kirchberg algebras with a specific type of K-theory. A similar characterization for Cuntz-Krieger algebras with finitely many ideals is desirable. We conjecture such a characterization and report on partial confirmations of the conjecture. The results rely heavily on the deep results by Kirchberg on ideal-related KK-theory, [9].

Definition 1. We say that a C^*-algebra A *looks like a Cuntz-Krieger algebra* if

(1) A is unital, separable, nuclear, purely infinite,
(2) A has real rank zero,
(3) $X = \mathrm{Prim}(A)$ is a finite space,
(4) for all $x \in X$, $K_*(A(x))$ is finitely generated, $K_1(A(x))$ is a free group, $\mathrm{rank}(K_0(A(x))) = \mathrm{rank}(K_1(A(x)))$,
(5) and $A(x)$ is UCT for all $x \in X$.

If a C^*-algebra A looks like a Cuntz-Krieger algebra without being isomorphic to a Cuntz-Krieger algebra, we call it a *phantom Cuntz-Krieger algebra.*

Of course, all Cuntz-Krieger algebras of real rank zero look like Cuntz-Krieger algebras. Note that up to stable isomorphism, a real rank zero extension of C^*-algebras that look like Cuntz-Krieger algebras will look like a Cuntz-Krieger algebra. Hence if phantom Cuntz-Krieger algebras do not exist, the definition provides a characterization of the Cuntz-Krieger algebras of real rank zero and of extensions of such.

The Cuntz-Krieger algebras can be viewed as graph algebras, and within the realm of graph algebras, phantom Cuntz-Krieger algebras do not exist.

Theorem 2 ([3])**.** Let E be a countable directed graph. If $C^*(E)$ is unital and $\operatorname{rank}(K_0(C^*(E))) = \operatorname{rank}(K_1(C^*(E)))$, then $C^*(E)$ is a Cuntz-Krieger algebra.

Using this, one can show the following which gives a positive answer to a question raised by Elliott.

Theorem 3 ([3])**.** Let A be a unital C^*-algebra. If A is stably isomorphic to a Cuntz-Krieger algebra, then A is a Cuntz-Krieger algebra.

As a corollary, corners of Cuntz-Krieger algebras are Cuntz-Krieger algebras, [3]. Since Cuntz-Krieger algebras are semiprojective, this is a special case of a conjecture posed by and confirmed in the graph algebra case by Eilers-Katsura: that unital corners of semiprojective C^*-algebras are semiprojective, [7].

The obvious approach for establishing nonexistence of phantom Cuntz-Krieger algebras is through K-theoretical classification. For instance, note that if a C^*-algebra looks like a Cuntz-Krieger algebra and has vanishing K-groups, then it is $\mathcal{O}_2$-absorbing and by the work of Kirchberg, [9], has to be a Cuntz-Krieger algebra. Furthermore, a simple C^*-algebra A that looks like a Cuntz-Krieger algebra will by Szymański's theorem, [14], have the same K-groups as a Cuntz-Krieger algebra B of real rank zero. So since such C^*-algebras are UCT Kirchberg algebras, it follows from the Kirchberg-Phillips classification theorem, [10], that A and B are isomorphic, and simple phantom Cuntz-Krieger algebras do not exist.

The case with one nontrivial ideal was dealt with similarly by Eilers-Katsura-Tomforde-West, [8]. The invariant used in this case consists of the six-term exact sequence in K-theory related to the ideal and was originally introduced by Rørdam, [13]. Completeness of this invariant up to unital isomorphism follows from the ideal-related UCT by Bonkat, [5], and Eilers-Katsura-Tomforde-West establish its range for graph algebras and Cuntz-Krieger algebras. For general primitive ideal spaces, the generalization of Rørdam's invariant is needed.

Definition 4. For a C^*-algebra A, its (concrete) *filtered K-theory* $\mathrm{FK}(A)$ consists of the groups and maps

$$\begin{tikzcd}
K_*(J/I) \arrow[rr] & & K_*(K/I) \arrow[dl] \\
& K_*(K/J) \arrow[ul, "\circ" marking] &
\end{tikzcd}$$

occuring in all six-term exact sequences for all extensions of subquotients $0 \to J/I \to K/J \to K/J \to 0$ in A.

The *reduced filtered K-theory* $\mathrm{FK}_{\mathcal{R}}(A)$ consists of only some of these groups and maps and was introduced by Restorff to classify real rank zero Cuntz-Krieger algebras up to stably isomorphism, [12]. This invariant seems most appropriate for working with graph algebras, and by applying the range result by Eilers-Katsura-Tomforde-West one can establish its range.

Theorem 5 ([1])**.** Let A be a C^*-algebra that looks like a Cuntz-Krieger algebra. Then there exists a Cuntz-Krieger algebra B of real rank zero and with $\mathrm{Prim}(A) \cong \mathrm{Prim}(B)$, together with an isomorphism $\mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ that sends $[1_A]$ in $K_0(A)$ to $[1_B]$ in $K_0(B)$.

The completeness of (reduced) filtered K-theory is a far deeper result and more difficult to achieve. Generalizing the results by Bonkat and Restorff, Meyer-Nest established a ideal-related UCT for filtered K-theory in the case with linear ideal lattice, [11]. Unfortunately, Meyer-Nest also provided a counterexample to classification with filtered K-theory, and Bentmann-Köhler, [4], used their methods to show that filtered K-theory is a complete invariant for so-called Kirchberg X-algebras exactly when the primitive ideal space X is a so-called accordion space. However, none of the constructed counterexamples have the K-theory of a C^*-algebra that looks like a Cuntz-Krieger algebra. For combinatorical reasons, five of the six connected four-point spaces that are not accordion spaces have a more manageable filtered K-theory, and using this we were able to establish completeness for (reduced) filtered K-theory in these cases under extra assumptions on the K-theory.

Theorem 6 ([9, 11, 4, 2, 1])**.** Let A and B be C^*-algebras that look like Cuntz-Krieger algebras. Assume that $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ are homeomorphic and of the type described above. Then any isomorphism $\mathrm{FK}_{\mathcal{R}}(A) \to \mathrm{FK}_{\mathcal{R}}(B)$ that sends $[1_A]$ in $K_0(A)$ to $[1_B]$ in $K_0(B)$, can be lifted to a $*$-isomorphism $A \to B$.

As a corollary, phantom Cuntz-Krieger algebras do not exist under these assumptions on the primitive ideal space. It seems possible that the reduced filtered K-theory is complete for all C^*-algebras that look like Cuntz-Krieger algebras independent of primitive ideal space but a strategy of proof different than the one in [2, 1] will be needed.

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A Connection between Easy Quantum Groups, Varieties of Groups, and Reflection Groups

Moritz Weber

(joint work with Sven Raum)

We present a link between easy quantum groups, discrete groups, and combinatorics. By this, we infer new connections between quantum isometry groups, tensor categories, C^*-algebras, reflection groups, varieties of groups, and the combinatorics of partitions. More precisely, we consider easy quantum groups [2] and find a relation to subgroups of the infinite free product $\mathbb{Z}_2^{*\infty}$ of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. By this, we obtain a link to varieties of groups, which yields a "quantum invariant" for varieties of groups on the one hand, and a statement on the complexity of easy quantum groups on the other. Moreover, we obtain a triangular relationship between easy quantum groups, categories of partitions, and discrete groups. Also, we obtain a large number of new quantum isometry groups.

The talk refers to an article, which will appear soon [5].

Easy quantum groups. Let $G \subseteq O_n$ be an orthogonal Lie group and consider the C^*-algebra $C(G)$ of continuous functions on G. It is generated by the coordinate functions and may be seen as the following universal C^*-algebra:

$$C(G) = C^*\left(u_{ij}, 1 \le i,j \le n \mid u_{ij} = u_{ij}^*, (u_{ij}) \text{ is orthogonal}, u_{ij}u_{kl} = u_{kl}u_{ij}, (R_G)\right)$$

Here, (R_G) are certain relations on the generators u_{ij}, and (u_{ij}) is the matrix formed by the generators u_{ij}. The *liberation* G^+ of G is a compact quantum group given by the universal C^*-algebra

$$C(G^+) = C^*\left(u_{ij}, 1 \le i,j \le n \mid u_{ij} = u_{ij}^*, (u_{ij}) \text{ is orthogonal}, (R_G)\right),$$

where we omit the commutativity of the generators u_{ij}. It is equiped with a comultiplication which turns it into a Hopf algebra.

In this sense, Wang [7, 8] constructed the *free orthogonal* and the *free symmetric quantum group*, liberating the groups O_n and S_n. The intertwiner spaces of S_n, S_n^+, O_n and O_n^+ admit a combinatorial description by means of partitions. In their 2009 article [2], Banica and Speicher initiated a systematic study of *easy quantum groups*, i.e. of compact quantum groups whose intertwiner spaces have "a nice combinatorics" – they are given by categories of partitions [2, 9]. A partition is given by k upper and l lower points which may be connected by lines. Denote by $P(k,l)$ the set of all such partitions. A category of partitions $\mathcal{C}$ consists of a collection of subsets $D(k,l) \subset P(k,l)$, for all $k,l \in \mathbb{N}$, that is closed under operations reflecting the properties of a tensor category (i.e. of an intertwiner space). By [2], a compact quantum subgroup G of O_n^+ is called *easy*, if its intertwiner space is spanned by linear maps indexed by a category of partitions $\mathcal{C}$, i.e.:

$$\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}\{T_p \mid p \in D(k,l)\} \quad \text{for all } k,l \in \mathbb{N}$$

It is a consequence of the seminal work by Woronowicz [10] that the correspondence between easy quantum groups and their categories of partitions is one-to-one, thus easy quantum groups are completely determined by their categories of partitions.

The approach of Banica and Speicher constitutes a constructive view on the liberation of groups but it goes far beyond it. It is a source of a large number of new examples of intermediate quantum subgroups of S_n^+ and O_n^+, or more general of S_n and O_n^+. Furthermore, it has become a useful link between quantum groups, combinatorics and free probability theory (Köstler, Speicher, Curran, ...). At the same time, easy quantum groups give rise to interesting operator algebras (Vaes, Vergnioux, Brannan, Freslon, ...). Parts of the easy quantum groups were classified in [2, 1, 9], but the full classification remained an open problem. Roughly speaking, we show that it is not feasible.

A map between easy quantum groups and subgroups of $\mathbb{Z}_2^{*\infty}$. Let $S_n \subseteq G \subseteq O_n^+$ be an easy quantum group. By definition, its intertwiner space is given by a category of partitions $\mathcal{C}$. We restrict to those categories, which contain the partition $(1,4)(2,3,5,6) \in P(0,6)$ on six points (i.e. the first and the fourth point are connected by a line, and the remaining four by another), but not the partition $(1)(2) \in P(0,2)$ on two points. Those categories are called *simplifiable hyperoctahedral*. Note that non-hyperoctahedral categories of partitions are completely classified [1, 9] – our class of categories is a subclass of the hyperoctahedral categories, the non-classified case.

We label the partitions in $\mathcal{C}$ by letters $a_1, a_2, \ldots$ in order to obtain words. Mapping these words to the infinite free product $\mathbb{Z}_2^{*\infty}$ (where now $a_i^2 = e$) yields the following main result.

Theorem 1 ([5])**.** There is a lattice isomorphism F between simplifiable hyperoctahedral categories of partitions and S_0-invariant subgroups of E, where E is the subgroup of $\mathbb{Z}_2^{*\infty}$ consisting of all words of even length.

Here, S_0 is the subsemigroup of $\mathrm{End}(\mathbb{Z}_2^{*\infty})$ generated by finite identifications of letters and conjugation by any letter. Based on this map F, we deduce several consequences.

Consequences for easy quantum groups and varieties of groups. The elements $x_k := a_1 a_{k+1}$ in $\mathbb{Z}_2^{*\infty}$ give rise to a free basis. Hence, the map F also yields a correspondence to proper S-invariant subgroups of $\mathbb{F}_\infty$, where S is analogous to S_0. This class of subgroups of $\mathbb{F}_\infty$ contains the lattice of fully characteristic subgroups of $\mathbb{F}_\infty$, which in turn is anti-isomorphic to the lattice of *varieties of groups* [3]. A variety of groups is the class of all groups that satisfy a given set of identical relations (choose a set R of words in $\mathbb{F}_\infty$; the variety given R is the class of all groups, in which any choice of elements fulfills all the relations from R). By Ol'shanskii [4], there are uncountably many varieties of groups. Thus:

Theorem 2 ([5])**.** There is an injection of lattices of varieties of groups into the lattice of easy quantum groups, hence there are uncountably many pairwise non-isomorphic easy quantum groups.

Our theorem also gives a way of dealing with varieties of groups by means of compact quantum groups resp. by the combinatorics of partitions.

Consequences for quantum isometry groups. Quantum isometry groups were first studied by Bichon, Bhowmick, and Goswami, see also the work of Banica, Skalski, Soltan, and others. The idea is to consider quantum group (co-)actions on non-commutative spaces. We reveal the following triangular correspondence:

$$\begin{array}{ccccc} \text{categories of partitions} & & \overset{\alpha}{\longleftrightarrow} & & \text{easy quantum groups} \\ & \beta \nwarrow\!\!\searrow & \text{discrete groups} & \swarrow\!\!\nearrow \gamma & \end{array}$$

The map α is given by Woronowicz resp. by Banica and Speicher's approach to easy quantum groups, whereas β is given by our map F. More precisely, for any $n \in \mathbb{N}$ we map a simplifiable hyperoctahedral category of partitions $\mathcal{C}$ to the quotient of the n-fold free product $\mathbb{Z}_2^{*n}$ by $F(\mathcal{C})_n$, where $F(\mathcal{C})_n$ consists of words only involving the letters $a_1, \ldots, a_n$. The map γ is basically given by the quantum isometry group G of the discrete group, and in the converse direction by the diagonal group inside of $C(G)$.

Consequences for C^*-algebras. The link between varieties of groups and C^*-algebras can also be investigated from a purely C^*-algebraic point of view, since to each variety of groups we assign a C^*-algebra.

The computation of the K-theory of these C^*-algebras – or in general of the easy quantum groups resp. of their C^*-algebras – is an open problem. Voigt [6] computed the K-theory for O_n^+, but the other cases are unknown.

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Trace spaces of simple nuclear C^*-algebras with finite-dimensional extreme boundary

Yasuhiko Sato

We mainly consider unital separable simple nuclear C*-algebra A with many extremal traces. Recently, we prove that if the trace space of A has compact finite-dimensional extreme boundary then there exist unital embeddings of matrix algebras into a certain central sequence algebra of A which is determined by the uniform topology on the trace space. As an application, it is shown that if furthermore A has strict comparison then A absorbs the Jiang-Su algebra tensorially.

In [1], B. Blackadar introdueced the notion of strict comparison by using the uniform topology on the trace space of C*-algebras. M. Rørdam adapted that strict comparison in order to apply Goodearl-Handelman's Hahn-Bnach type theorem [7], and he proved that $\mathcal{Z}$-absorption implies strict comparison in [15], [16]. Therefore, in the study of the Jiang-Su algebra it becomes necessary to obtain the uniform structure on the trace space of C*-algebras. The following is the main result of this note.

Theorem 1. Let A be a unital separable simple infinite-dimensional nuclear C*-algebra with at least one tracial state. Suppose that the extreme boundary of $T(A)$ is a compact finite-dimensional space. Then for any $k \in \mathbb{N}$ there exists a unital embedding of the k by k matrix algebra into a variant of the central sequence algebra of A defined by

$$A' \cap \left(l^\infty(\mathbb{N}, A)/\{(a_n)_n \in l^\infty(\mathbb{N}, A) : \lim_{n\to\infty} \max_{\tau\in T(A)} \tau(a_n^* a_n) = 0\} \right).$$

Here, we denote by $T(A)$ the set of tracial states of A which is called *trace space* in [6]. And, in what follows we denote by $\partial_{\mathrm{e}}(T(A))$ the extreme boundary of $T(A)$. As a main application of this theorem, we present the following result. Once we know the above theorem, the proof of this corollary can be obtained in the same way as the proof of [14, Theorem 1.1].

Corollary 2. If A and $T(A)$ satisfy the same conditions in the above theorem, then the following are equivalent:

(i) $A \otimes \mathcal{Z} \cong A$.
(ii) A has strict comparison.
(iii) Any completely positive map from A to A can be excised in small central sequences.
(iv) A has property (SI).

By using the next Proposition, M. Dadarlat and A. Toms investigated the dimension functions on the compact finite-dimensional extreme boundary of trace spaces, (in the proof of [3, Lemma 4.4]). This result was essentially based on the works by D. A. Edward [5], J. Cuntz, G. and K. Pedersen [2], and H. Lin [12]. The starting point of our proof is this proposition.

Proposition 3. Let A be a unital separable simple infinite-dimensional C*-algebra with at least one tracial state. Suppose that $\partial_{\mathrm{e}}(T(A))$ is compact. Then for any positive function $f \in C(\partial_{\mathrm{e}}(T(A))$ there exists a sequence a_n, $n \in \mathbb{N}$ of positive elements in A such that

$$\lim_{n\to\infty} \max_{\tau\in\partial_{\mathrm{e}}(T(A))} |\tau(a_n) - f(\tau)| = 0 \quad \text{and} \quad \|a_n\| \leq \|f\| \text{ for } n \in \mathbb{N}.$$

Our first aim is to study central sequences with a kind of uniform structure on trace spaces. For this purpose, the following lemma plays a central role. In the lemma, let us remark that the required finite unitaries $\{u_i\}$ are heavily depend on double-dealing of a finite subset F of A. So this lemma is much weaker than the strong amenability which was defined by B. Johnson. [11].

Lemma 4. Let A be a unital nuclear C*-algebra. Then for any finite subset F of A and $\varepsilon > 0$ there exist unitaries $u_1, u_2, ..., u_N$ of A such that

$$\|[\frac{1}{N}\sum_{i=1}^{N} \mathrm{Ad}\, u_i(a), b]\| < \varepsilon, \quad \text{for all } a, b \in F.$$

Corollary 5. Let A be a unital separable nuclear C*-algebra with at least one tracial state. Then for any $a \in A$ there exists a central sequence $a_n \in A$, $n \in \mathbb{N}$ such that $\|a_n\| \leq \|a\|$ and

$$\tau(a) = \tau(a_n) \quad \text{for any } \tau \in T(A) \text{ and } n \in \mathbb{N}.$$

Combining this result with Proposition 3, we could obtain a simple main technical tool Lemma 6 concerning multiplicativity and orthogonality on the compact extreme boundary. In the following, (i) is a variant of [13, Lemma 4.6] and (ii) is a version of [14, Lemma 3.3] for the uniform topology on $\partial_{\mathrm{e}}(T(A))$.

Lemma 6. Let A be a unital separable simple infinite-dimensional C*-algebra. Suppose that $\partial_{\mathrm{e}}(T(A))$ is compact. Then the following hold:

(i) For any central sequence $(f_n)_n \in A_\infty$ and $a \in A$, it follows that

$$\lim_{n\to\infty} \max_{\tau\in\partial_{\mathrm{e}}(T(A))} |\tau(f_n a) - \tau(f_n)\tau(a)| = 0.$$

(ii) Moreover, if A is nuclear, for mutually orthogonal positive functions $f_i \in C(\partial_{\mathrm{e}}(T(A)))$, $i = 1, 2, ..., N$ there exist central sequences $(a_{i,n})_n$, $i = 1, 2, ..., N$ of positive maps in A such that

$$\lim_{n\to\infty} \max_{\tau\in\partial_{\mathrm{e}}(T(A))} |\tau(a_{i,n}) - f_i(\tau)| = 0 \quad \text{for } i = 1, 2, ..., N,$$

$$\text{and} \lim_{n\to\infty} \|a_{i,n}a_{j,n}\| = 0 \quad \text{for } i \neq j.$$

Due to the above technical tools we can obtain the following proposition. The proof of the main theorem is straightforward from this proposition, and the proof of this proposition was inspired by techniques for $C(X)$-algebras from [8, Theorem 4.6] and [4, Theorem 0.1]. Recently, these techniques were developed by A. Toms and W. Winter to show $\mathcal{Z}$-absorption of the crossed product C*-algebras by minimal homeomorphisms on a compact finite dimensional space [18]. Our proof as well as theirs heavily relies on the condition of finite covering dimenion. It might be interesting that the number of completely positive maps corresponds to the covering dimension of $\partial_{\mathrm{e}}(T(A))$ in this proposition.

Proposition 7. Let A be a unital separable simple nuclear C*-algebra. Suppose that $\partial_{\mathrm{e}}(T(A))$ is compact and $d = \dim(\partial_{\mathrm{e}}(T(A))) < \infty$. Then for any $k \in \mathbb{N}$ there exist order zero completely positive maps $\varphi_l : M_k \to A_{\mathrm{t}\infty}$, $l = 0, 1, ..., d$ such that

$$\sum_{l=0}^{d} \varphi_l(1_k) = 1 \quad \text{and} \quad [\varphi_l(a), \varphi_m(b)] = 0 \quad \text{for } l \neq m, \quad a, b \in M_k.$$

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The Homotopy Lifting Theorem for Semiprojective C^*-Algebras

Bruce Blackadar

We prove a complete analog of the Borsuk Homotopy Extension Theorem for arbitrary semiprojective C*-algebras:

Theorem. [Borsuk Homotopy Extension Theorem] Let X be an ANR, Y a compact metrizable space, Z a closed subspace of Y, (φ_t) $(0 \le t \le 1)$ a uniformly continuous path of continuous functions from Z to X (i.e. $h(t,z) = \varphi_t(z)$ is a homotopy from φ_0 to φ_1). Suppose φ_0 extends to a continuous function $\bar{\varphi}_0$ from Y to X. Then there is a uniformly continuous path $\bar{\varphi}_t$ of extensions of the φ_t to functions from Y to X (i.e. $\bar{h}(t,y) = \bar{\varphi}_t(y)$ is a homotopy from $\bar{\varphi}_0$ to $\bar{\varphi}_1$).

In particular, any function from Z to X homotopic to an extendible function is extendible. The theorem also works for metrizable spaces which are not necessarily compact when phrased in the homotopy language; we have stated it in the version which can potentially be extended to noncommutative C*-algebras. The theorem can be regarded as giving a "universal cofibration property" for maps into ANR's.

There is a direct analog of (compact) ANR's in the category of (separable) noncommutative C*-algebras: the semiprojective C*-algebras. Many of the results about ANR's carry through to semiprojective C*-algebras with essentially identical proofs (just "turning arrows around"). However, Borsuk's proof of the Homotopy Extension Theorem is not one of these: the proof simply does not work in the noncommutative case. The underlying reason is that in a metrizable space, every closed set is a G_δ, but this is false in the primitive ideal space of a separable noncommutative C*-algebra in general.

We can, however, by a different argument obtain a complete analog of the Borsuk Homotopy Extension Theorem for arbitrary semiprojective C*-algebras:

THEOREM. [HOMOTOPY LIFTING THEOREM] Let A be a semiprojective C*-algebra, B a C*-algebra, I a closed ideal of B, (φ_t) $(0 \le t \le 1)$ a point-norm continuous path of *-homomorphisms from A to B/I. Suppose φ_0 lifts to a *-homomorphism $\bar{\varphi}_0 : A \to B$, i.e. $\pi_I \circ \bar{\varphi}_0 = \varphi_0$. Then there is a point-norm continuous path $(\bar{\varphi}_t)$ $(0 \le t \le 1)$ of *-homomorphisms from A to B beginning at $\bar{\varphi}_0$ such that $\bar{\varphi}_t$ is a lifting of φ_t for each t, i.e. the entire homotopy lifts. In particular, φ_1 lifts to a *-homomorphism from A to B.

COROLLARY. Let A be a semiprojective C*-algebra, B a C*-algebra, I a closed ideal of B, φ a *-homomorphism from A to B/I. If φ is homotopic to a *-homomorphism from A to B/I which lifts to B, then φ lifts to B.

In the course of the proof we obtain some other results about semiprojective C*-algebras which are of interest: a partial lifting theorem with specified quotient, a lifting result for homomorphisms close to a liftable homomorphism, and that sufficiently close homomorphisms from a semiprojective C*-algebra are homotopic.

THEOREM. [SPECIFIED QUOTIENT PARTIAL LIFTING THEOREM] Let A be a semiprojective C*-algebra, B a C*-algebra, (J_n) an increasing sequence of closed ideals of B with $J = [\cup J_n]^-$, I another closed ideal of B, and $\varphi : A \to B/J$ and $\tilde{\varphi} : A \to B/I$ *-homomorphisms with $\pi_{I+J} \circ \varphi = \pi_{I+J} \circ \tilde{\varphi}$. Then for some sufficiently large n there is a *-homomorphism $\psi : A \to B/J_n$ such that $\pi_J \circ \psi = \varphi$ and $\pi_{I+J_n} \circ \psi = \pi_{I+J_n} \circ \tilde{\varphi}$.

THEOREM. [CLOSE LIFTING THEOREM] Let A be a semiprojective C*-algebra generated by a finite or countable set $\mathcal{G} = \{x_1, x_2, \dots\}$ with $\lim_{j\to\infty} \|x_j\| = 0$ if $\mathcal{G}$ is infinite. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that, whenever B is a C*-algebra, I a closed ideal of B, φ and ψ *-homomorphisms from A to B/I with $\|\varphi(x_j) - \psi(x_j)\| < \delta$ for all j and such that φ lifts to a *-homomorphism $\bar{\varphi} : A \to B$ (i.e. $\pi_I \circ \bar{\varphi} = \varphi$), then ψ also lifts to a *-homomorphism $\bar{\psi} : A \to B$ with $\|\bar{\psi}(x_j) - \bar{\varphi}(x_j)\| < \epsilon$ for all j. (The δ depends on ϵ, A, and the set $\mathcal{G}$ of generators, but not on the B, I, φ, ψ.)

COROLLARY. [CLOSE HOMOTOPY THEOREM] Let A be a semiprojective C*-algebra generated by a finite or countable set $\mathcal{G} = \{x_1, x_2, \dots\}$ with $\lim_{j\to\infty} \|x_j\| = 0$ if $\mathcal{G}$ is infinite. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that, whenever B is a C*-algebra, φ_0 and φ_1 *-homomorphisms from A to B with $\|\varphi_0(x_j) - \varphi_1(x_j)\| < \delta$ for all j, then there is a point-norm continuous path (φ_t) $(0 \le t \le 1)$ of *-homomorphisms from A to B connecting φ_0 and φ_1 with $\|\varphi_t(x_j) - \varphi_0(x_j)\| < \epsilon$ for all j for any

$t \in [0,1]$. (The δ depends on ϵ, A, and the set $\mathcal{G}$ of generators, but not on the B, φ_0, φ_1.)

In fact, for any $\epsilon > 0$, a δ that works for the Close Lifting Theorem also works for the Close Homotopy Theorem.

Finally, we discuss ℓ-open and ℓ-closed C*-algebras.

If A and B are C*-algebras, denote by $\mathrm{Hom}(A,B)$ the set of *-homomorphisms from A to B, endowed with the point-norm topology. $\mathrm{Hom}(A,B)$ is separable and metrizable (if A and B are separable). If A and B are unital, let $\mathrm{Hom}_1(A,B)$ be the set of unital *-homomorphisms from A to B. $\mathrm{Hom}_1(A,B)$ is a clopen subset of $\mathrm{Hom}(A,B)$ (since a projection close to the identity in a C*-algebra is equal to the identity).

If $A = C(X)$ and $B = C(Y)$, then $\mathrm{Hom}_1(A,B)$ is naturally homeomorphic to X^Y, the set of continuous functions from Y to X, endowed with the topology of uniform convergence (with respect to any fixed metric on X, or with respect to the unique uniform structure on X compatible with its topology).

Examples show that $\mathrm{Hom}(A,B,I)$ is neither open nor closed in $\mathrm{Hom}(A,B/I)$ in general. We seek conditions on A insuring that $\mathrm{Hom}(A,B,I)$ is always open or closed in $\mathrm{Hom}(A,B/I)$ for any B and I.

Definition. Let A be a separable C*-algebra.

(i) A is called *ℓ-open* if, for every pair (B,I), the set $\mathrm{Hom}(A,B,I)$ is open in $\mathrm{Hom}(A,B/I)$.
(ii) A is *ℓ-closed* if, for every pair (B,I), the set $\mathrm{Hom}(A,B,I)$ is closed in $\mathrm{Hom}(A,B/I)$.

The next result is an immediate corollary of the Close Lifting Theorem:

Corollary. Every semiprojective C*-algebra is both ℓ-open and ℓ-closed.

The converse is at least very nearly true in the commutative category, and I conjecture it holds in general.

Although there is no obvious direct proof that an ℓ-open C*-algebra is ℓ-closed, I do not know an example of a C*-algebra which is ℓ-open but not ℓ-closed, and I conjecture that none exist. There are ℓ-closed C*-algebras which are not ℓ-open. I do not have a good idea how to characterize ℓ-closed C*-algebras.

We conclude with some examples.

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Reduction of the dimension of nuclear C^*-algebras

Luis Santiago

The notion of covering dimension of a topological space has led to different dimension theories for C*-algebras; for instance, the stable rank, real rank, decomposition rank, and nuclear dimension. Each of these dimension theories have had important applications to the theory of C*-algebras. It was shown in [5] and [6] that simple separable nonelementary unital C*-algebras with finite decomposition rank or more generally with finite nuclear dimension absorb the Jiang-Su algebra tensorially. As a consequence, new classification results have been obtained for simple C*-algebras. For C*-algebras that can be written as direct limits of subhomogeneous algebras one can also associate a dimension, namely the infimum over all such direct limit decompositions of the supremum of the covering dimension of the spectrum of all the C*-algebras appearing on the given direct limit. For instance, it was shown in [3] that for simple separable unital AH-algebras with very slow dimension growth this dimension is at most three. This result was used in [1] to classify this class of C*-algebras.

In this work we study different notions of dimension for certain C*-algebras of the form $A \otimes B$, where B is simple, nuclear, either projectionless or unital with no nonzero projections but its unit, and with a specified direct limit decomposition. We are particularly interested in two cases: the first case is when B is the C*-algebra $\mathcal{W}$ constructed in [4] and the second is when B is the Jiang-Su algebra $\mathcal{Z}$. The C*-algebra $\mathcal{W}$ is a simple separable nuclear C*-algebra that is stably finite, stably projectionless, has a unique tracial state, and has trivial K-groups. This algebra should be considered as a stably finite analog of the Cuntz algebra $\mathcal{O}_2$. It should play central role in the classification of projectionless C*-algebras.

Let A be a C*-algebra and let $\mathrm{T}(A)$ denote the cone of lower semicontinuous traces on A_+ with values in $[0, \infty]$ (note that the traces are not required to be densely finite). It has been shown in [2] that $\mathrm{T}(A)$ belongs to the category of compact Hausdorff non-cancellative cones with jointly continuous addition and jointly continuous scalar multiplication. Our main motivation for studying C*-algebras of the form $A \otimes \mathcal{W}$ is the following conjecture of Leonel Robert:

Conjecture 1. If A and B are separable nuclear C*-algebras then

$$\mathrm{T}(A) \cong \mathrm{T}(B) \quad \Longleftrightarrow \quad A \otimes \mathcal{W} \otimes \mathcal{K} \cong B \otimes \mathcal{W} \otimes \mathcal{K},$$

where the isomorphism between $\mathrm{T}(A)$ and $\mathrm{T}(B)$ is assumed to be a linear homeomorphism.

This conjecture has been shown to be true for AF-algebras and for $\mathcal{O}_2$-absorbing algebras. In fact, in the $\mathcal{O}_2$-absorbing case this conjecture is nothing more than Kirchberg Classification Theorem of $\mathcal{O}_2$-absorbing algebras. As a consequence of Theorem 4 below we obtained the following result: if A is a separable direct limit of homogeneous C*-algebras, then the tensor product $A \otimes \mathcal{W}$ is a direct limit of a sequence of 1-dimensional noncommutative CW-complexes (these are subhomogeneous algebras of 1-dimensional spectrum). This result reduces the proof of

Robert's Conjecture for direct limits of homogeneous algebras to prove a classification result for direct limits of 1-dimensional noncommutative CW-complexes. This result also implies that the decomposition rank and the nuclear dimension of $A \otimes \mathcal{W}$ is one. Another consequence of Theorem 4 is that if A is a C*-algebra in the class $\mathcal{A}$ defined below then the stable rank of $A \otimes \mathcal{W}$ is one. In particular, the stable rank of the tensor product of $\mathcal{W}$ with a direct limit of separable type I C*-algebras is one.

Definition 2. Let $\mathcal{A}$ be a class of C*-algebras. We say that a C*-algebra B is locally contained in $\mathcal{A}$ if for every $\epsilon > 0$ and every finite subset F of B there exists a C*-algebra $A \in \mathcal{A}$ and a *-homomorphism $\varphi\colon A \to B$ such that the distance from x to $\varphi(A)$ is less than ϵ for every $x \in F$.

Definition 3. Let us denote by $\mathcal{A}$ the smallest class of C*-algebras that satisfies the following properties:

(i) $\mathrm{C}_0(X) \in \mathcal{A}$ for every locally compact space X.
(ii) If $A \in \mathcal{A}$ then $A \otimes \mathrm{M}_n(\mathbb{C}) \in \mathcal{A}$ for every $n \in \mathbb{N}$.
(iii) If $A \in \mathcal{A}$ then every hereditary sub-C*-algebra of A belongs to $\mathcal{A}$. In particular, every closed two-sided ideal of A belongs to $\mathcal{A}$.
(iv) If $A \in \mathcal{A}$ then every quotient of A belongs to $\mathcal{A}$.
(v) If $A, C \in \mathcal{A}$ and if
$$0 \to A \to B \to C \to 0$$
is an exact sequence of C*-algebras then $B \in \mathcal{A}$.
(vi) If A is locally contained in $\mathcal{A}$ then $A \in \mathcal{A}$.

The following theorem is our main result:

Theorem 4. Let B be a direct limit of a system of (nonunital) recursive subhomogeneous algebras with no dimension growth. The following statements hold:

(i) If B has a finite number of ideals then $\mathrm{sr}(B) = 1$, if and only if, for every ideal I of B the index map $\delta\colon \mathrm{K}_1(A/I) \to \mathrm{K}_0(I)$ is trivial.
(ii) If B is simple, projectionless, and $\mathrm{K}_0(B) = \mathrm{K}_1(B) = 0$ then $\mathrm{sr}(A \otimes B) = 1$ for every C*-algebra $A \in \mathcal{A}$.
(iii) If B is simple, $\mathrm{K}_1(B) = 0$, and B is either projectionless or it is unital and its only non-zero projection is its unit then $\mathrm{sr}(A \otimes B) = 1$ for every C*-algebra A that is approximately contained in the class of (nonunital) recursive subhomogeneous algebras with 1-dimensional spectrum.

Moreover, if B is a simple direct limit of a sequence of 1-dimensional noncommutative CW-complexes with $\mathrm{K}_0(B) = \mathrm{K}_1(B) = 0$, and A is approximately contained in the class of C*-algebras that are stably isomorphic to a commutative C*-algebra, then $A \otimes B$ is approximately contained in the class of C*-algebras that are stably isomorphic to 1-dimensional noncommutative CW-complexes. In particular the decomposition rank and the nuclear dimension of $A \otimes B$ is one. If in addition A is separable then $A \otimes B$ can be written as an inductive limit of C*-algebras that are stably isomorphic to 1-dimensional noncommutative CW-complexes.

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Central sequences of C^*-algebras and tensorial absorption of $\mathcal{Z}$

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(joint work with Mikael Rørdam)

A year ago, the following *remarkable result* was proved:

Theorem 1 (Matui–Sato)**.** Let A be a unital, separable, simple, non-elementary, stably finite, nuclear C*-algebra, and suppose that $\partial_e T(A)$ is finite. Then the following are equivalent:

(i) $A \cong A \otimes \mathcal{Z}$,
(ii) A has strict comparison (i.e., $\mathrm{Cu}(A)$ is almost unperforated),
(iii) Every cp map $A \to A$ can be excised in small central sequences,
(iv) A has property (SI).

The equivalence of (1) and (2) for every non-elementary separable simple nuclear C*-algebra A has been conjectured by A.Toms and W.Winter. We come back later to the properties mentioned in (3) and (4).

Note that if A is not stably finite, then $T(A) = \varnothing$ and (2) implies that A is purely infinite. Since A is nuclear it follows $A \cong A \otimes \mathcal{O}_\infty \cong A \otimes \mathcal{Z}$.

The condition that $\partial_e T(A)$ is finite had been weekend recently by independent approaches.

The isomorphism $A \cong A \otimes \mathcal{Z}$ is necessary for the verification of the (old original) Elliott conjecture: The (non-elementary) counterexamples to the Elliott conjecture are all C*-algebras A that do not absorb the Jiang-Su algebra (e.g. those given by Villardsen, Rørdam, Toms, ... – compare the talk of G. Gong). For any of this unital C*-algebras, $\mathcal{Z}$ does not embed unitally into $A_\omega \cap A'$, but for the stably finite ones among them there a surjection $A_\omega \cap A' \to \mathcal{R}^\omega \cap \mathcal{R}'$, so $A_\omega \cap A'$ is not small (or abelian), where $\mathcal{R}$ denotes the hyperfinite II_1-factor with separable predual.

It was known for separable C*-algebras A that $A \cong A \otimes \mathcal{Z}$ if and only if the unital C*-algebra $F(A) := (A' \cap A_\omega)/\mathrm{Ann}(A, A_\omega)$ contains a copy of $\mathcal{Z}$ unitally. Here $\omega \in \gamma(\mathbb{N}) := \beta(\mathbb{N}) \setminus \mathbb{N}$ denotes a free ultrafilter,

$$A_\omega := \ell^\infty(A)/c_\omega(A), \quad c_\omega(A) := \big\{(x_n) \in \ell^\infty(A) \mid \lim_\omega \|x_n\| = 0\big\},$$

and $\quad \mathrm{Ann}(A, A_\omega) := \{b \in A_\omega \mid bA = 0 = Ab\}$. For fixed $\omega \in \gamma(\mathbb{N})$ the algebra $F(A)$ is in a natural way an invariant of Morita equivalence. In particular the natural map from $F(A \otimes \mathcal{K})$ to $F(A)$ is an isomorphism, and – in case of simple A – one can pass to a hereditary sub-C*-algebra B of A with the property that every lower semi-continuous semi-finite 2-quasi-trace on B_+ is bounded. The system of isomorphism classes of unital separable C*-algebras of $F(A)$ and several (later defined) σ-ideals of $F(A)$ are independent from the chosen free ultrafilter ω. Combination of results of M. Dadarlat and A. Toms [1] and E.K. [3] show that $A \cong A \otimes \mathcal{Z}$ for separable A is equivalent to the existence of a sub-homogenous C*-subalgebra $B \subset F(A)$ without characters and $1_{F(A)} \in B$.

Mathui and Sato [5] have introduced two interesting new properties on central sequence algebras: Property (SI) and "Excision in small central sequences". They allow to apply similar technics as in case of the study of strongly purely infinite nuclear C*-algebras.

We outline how this leads (among others) to a proof of the Toms-Winter conjecture in case of non-elementary, separable, simple, and nuclear A with the property that the extremal rays of the cone C_T of lower semi-continuous semi-finite (additive) traces $\tau\colon A_+ \to [0, \infty]$ have a compact and finite-dimensional generating subset $K \subset C_T$: $\partial C_T = \mathbb{R}_+ \cdot K$.

Similar results have been also obtained in [8], [9], [2] and [6] for the case of nuclear algebras. Here I present my view on this topics.

Conjecture 2. The invariant $F(A)$ contains all technical informations that decide if A can be classified, and some information on the class of A.

It seems that classification of the stable $\mathcal{Z}$-absorbing amenable A follows sometimes from a complete understanding of the invariant

$$F_{\mathrm{cont}}(A) := \big(A' \cap \mathrm{C}_b(\mathbb{R}_+, A)\big)/\big(\mathrm{Ann}(A, \mathrm{C}_b(\mathbb{R}_+, A)\big)$$

K-groups of $F_{\mathrm{cont}}(A)$ should define the KK-equivalence class of A.

Some facts about the invariant $F(A)$ of separable A:

(i) $F(A) = \mathbb{C}$, if and only if, A is **elementary** (i.e. $A = M_n$ or $A = \mathcal{K}$).
(ii) $F(A)$ simple and $[1]_0 = 0$ in $K_0(F(A))$, if and only if, $A = \mathcal{O}_2$ or $A = \mathcal{O}_2 \otimes \mathcal{K}$.
(iii) Any separable nuclear C*-algebra B is in the UCT-class, if and only if, $[1]_0 = 0$ in $F(A)$ for every A with $K_*(A) = 0$ and simple $F(A)$.
(iv) $F(A) = \big(A' \cap \mathcal{M}(A)_\omega\big)/\big(\mathrm{Ann}(A, \mathcal{M}(A)_\omega\big)$

Theorem 3 (K.1994, opposite direction 2004)**.** A (non-zero) separable C*-algebra A is simple, purely infinite **and nuclear**, if and only if, $F(A)$ is simple and $F(A) \neq \mathbb{C}$.

If $F(A) \neq \mathbb{C}$ is simple then $F(A)$ is purely infinite.

In particular, $\mathcal{O}_\infty \hookrightarrow F(A)$ unitally, which entails that $A \cong A \otimes \mathcal{O}_\infty$.

The conclusion $A \cong A \otimes \mathcal{O}_\infty$ follows from the more general **fact:**
Let D is any separable unital tensorially self-absorbing C*-algebra then $A \cong$

$A \otimes D \iff \exists$ unital *-homomorphism $D \to F(A)$.
(Which shows e.g. that $A \cong A \otimes \mathcal{Z}$ for all unital and approximately divisible separable A.)

The important argument for the easy direction (from simple nuclear A to simple purely infinite $F(A)$) is:

If B is simple and purely infinite, $A \subset B$ separable, $V\colon A \to B$ nuclear, then there is $s \in B_\omega$ with $s^*as = V(a)$ for all $a \in A$. ("Excision" property of V with respect to $1 \in \mathcal{M}(B)$.)

D. McDuff: If M is a separable II_1 von Neumann factor, ω is a free ultrafilter, and if $M^\omega \cap M'$ is not abelian, then $\mathcal{R} \hookrightarrow M^\omega \cap M'$, where $\mathcal{R}$ denotes the hyperfinite II_1 factor. We denote by M^ω the ultrapower von Neumann algebra with respect to the norm $\|a\|_2$, and adopt unusual notation A_ω for the (operator) norm-ultrapower.

If the latter holds, then M is said to be a *McDuff factor.*

Proposition 4 (Strengthened version of a theorem of Sato)**.** *Let A be a separable C*-algebra with a bounded trace τ, $M := \pi_\tau(A)''$, and let ω be a free ultrafilter on $\mathbb{N}$. The natural maps*

$$A_\omega \to M^\omega \qquad \textit{and} \qquad A_\omega \cap A' \to M^\omega \cap M'$$

are surjective and map the annihilator $\mathrm{Ann}(A, A_\omega)$ *to zero.*

In particular, if M is a McDuff factor then some quotient of the unital algebra $F(A)$ contains a subalgebra isomorphic to $\mathcal{R}$ unitally.

The argument (and similar others later considered) use implicitly that the kernel of $A_\omega \mapsto M^\omega$ is a σ-ideal:

Definition 5. Let J a closed ideal of a C*-algebra B. J is called σ-ideal (of B) if for every $a \in J_+$ and every separable sub-C*-algebra $C \subset B$ there exists a positive contraction $e \in C' \cap J$ with $ea = a$.

More generally we have that $c_\omega(A)$ is a σ-ideal of $\ell_\infty(A)$, or that $T_\omega\colon A_\omega \to C_\omega$ defines a σ-ideal J_T of A_ω by $J_T := \{a \in A_\omega \; : \; \|a\|_{T,\omega} = 0\}$, where $\|\pi_\omega(a_1, a_2, \ldots)\|_T := \lim_\omega \|T(a_n^* a_n)\|^{1/2}$ for a central cp contraction $T\colon A \to C$, with C commutative.

The prove is straight-forward if one uses the ε-test indicated in [3, Lemma A.1] for suitable sets X_n (of morphisms) and non-negative functions $f_n^{(k)}$ on X_n.

Proposition 6 (E.K. [3])**.** *Let A separable and $B_1, B_2, \ldots$ unital separable C*-algebras, $\omega, \omega' \in \gamma(\mathbb{N})$, and $h_n\colon B_n \to F(A)_{\omega'}$ injective unital *-morphisms. Then there exists an – on the tensor factors injective – unital *-morphism*

$$h\colon B_1 \otimes_{\max} B_2 \otimes_{\max} \cdots \;\to F(A)\,.$$

We consider the universal C*-algebras $D_n := C^*(c_1, \ldots, c_n, d_1, \ldots, d_n; R)$ with relations

$$(R):\; c_k^* d_k = 0,\; c_k^* c_k = d_k^* d_k,\; \text{and} \; \sum_k c_k^* c_k = 1\,.$$

Clearly D_n has no character, and is weakly semi-projective.

Lemma 7. If a unital C*-algebra B has no character, then $\exists\ n \in \mathbb{N}$ and a unital *-homomorphism $D_n \to B$.

Corollary 8. *If A is separable and $F(A)$ has no character, then $\exists\ n \in \mathbb{N}$ and a unital *-homomorphism $D_n \otimes_{\max} D_n \otimes_{\max} \cdots \to F(A)$.*

Theorem 9 ([1])**.** Let D be a unital C*-algebra. If $\bigotimes_{k=1}^{\infty} D$ contains a unital subhomogeneous C*-algebra without characters, then $\mathcal{Z} \hookrightarrow \bigotimes_{k=1}^{\infty} D$.

Hence: $A \cong A \otimes \mathcal{Z}$ if and only if $F(A)$ contains a unital subhomogeneous C*-algebra without characters.

Question 10. *Suppose that A is a separable C*-algebra such that $F(A)$ has no characters. Does it follow that $F(A)$ contains a unital copy of $\mathcal{Z}$ (so that $A \cong A \otimes \mathcal{Z}$)?*

Question 11 (Dadarlat–Toms)**.** *Does $\mathcal{Z}$ embed unitally into $\bigotimes_{n=1}^{\infty} D$ whenever D is a unital C*-algebra without characters?*

The two questions above are equivalent!

Results of L.Robert and M.Rørdam about divisibility properties for C*-algebras [7] imply the existence of non-elementary, unital, simple, separable, nuclear C*-algebras A such that $F(A)$ has a character (but has also a sub-quotient $\cong \mathcal{R}$).

Let A separable with $T(A) \neq \varnothing$ (traces of norm = 1). Define

$$\|a\|_{2,\tau} := \tau(a^* a)^{1/2}, \qquad \|a\|_2 := \sup_{\tau \in T(A)} \|a\|_{2,\tau}, \quad a \in A.$$

Define $\|\cdot\|_2$ on A_ω by $\|\pi_\omega(a_1, a_2, a_3, \dots)\|_2 := \lim_\omega \|a_n\|_2$, where $\pi_\omega : \ell^\infty(A) \to A_\omega$ is the quotient map. Set $J_A := \{x \in A_\omega : \|a\|_2 = 0\} \vartriangleleft A_\omega$.

It is not difficult to see that $\mathrm{Ann}(A, A_\omega) \subset J_A$. Hence, J_A defines a quotient

$$F_t(A) := (A' \cap A_\omega)/(A' \cap J_A) \subset A_\omega / J_A$$

of $F(A)$, and on $F_t(A) \subset A_\omega/J_A$ we get a norm $\|a\|_2 := \|\mathcal{T}_\omega(a^* a)\|^{1/2} \le \|a\|$ for $a \in A_\omega/J_A$, where $\mathcal{T}_\omega : A_\omega \to \mathrm{C}_b(\partial T(A))_\omega$ is the ultrapower of the map $\mathcal{T} : A \to \mathrm{C}_b(\partial T(A))$ given by $\mathcal{T}(a)(\tau) := \tau(a)$.

If $\omega' \in \gamma(\mathbb{N})$ (not necessarily $\omega' = \omega$) then we can repeat this construction with $F_t(A)$ and $\|\cdot\|_2$ on $F_t(A)$ (in place of A and $\|\cdot\|_2$ on A). Get an ideal $J_{F_t(A)}$ of $F_t(A)_{\omega'}$.

The permanence properties for the family of the separable sub-C*-algebras of $F_t(A)$ are better than in case of $F(A)$:

Theorem 12. The natural *-morphism $A \odot F(A) \to A_\omega$ defines a *-morphism

$$A \otimes_{\min} F_t(A) \to A_\omega / J_A .$$

It is injective if A is simple.

Let $\omega, \omega' \in \gamma(\mathbb{N})$, $B_1, B_2, \dots$ unital separable sub-C*-algebras of $F_t(A)_{\omega'}/J_{F_t(A)}$ then there exists a unital *-morphism

$$B_1 \otimes_{\min} B_2 \otimes_{\min} \cdots \to F_t(A) .$$

If $F_t(A)$ contains a sub-homogenous algebra without character unitally, then $\mathcal{R}$ is unitally contained in $F_t(A)$.

Question 13. *Let B a separable unital C*-algebra without character and separating $T(B)$. Does $B \otimes_{\min} B \otimes_{\min} \cdots$ contain a sub-homogenous C*-algebra without character?*

A positive answer gives that $\mathcal{R} \subset F_t(A)$. If there is a positive answer then the following questions remain:
When $F_t(A)$ has not a character?
When we can conclude from $M_2 \subset F_t(A)$ that $\mathcal{Z} \subset F(A)$ (i.e. that $A \otimes \mathcal{Z} \cong A$)?

Definition 14 (Matui–Sato, reformulated)**.** A separable simple C*-algebra A is said to have *property (SI)* if for all positive contractions $e, f \in F(A)$ such that

$$e \in J_A, \qquad \sup_k \|1 - f^k\|_2 < 1,$$

there is $s \in F(A)$ with $fs = s$ and $s^*s = e$.

Proposition 15. *Let A be a separable, simple, unital, stably finite C*-algebra with property (SI). TFAE:*

(i) $A \cong A \otimes \mathcal{Z}$.
(ii) $\exists$ *unital *-homomorphism* $\mathcal{R} \to F_t(A)$.
(iii) $\exists$ *unital *-homomorphism* $M_2 \to F_t(A)$.
(iv) $\exists$ *unital *-homomorphism* $I(2,3) \to F(A)$.

Proposition 16. *If A is a non-elementary, unital, simple, separable, stably finite C*-algebra with $T(A) = QT(A)$ such that*

(i) $\pi_\tau(A)''$ *is McDuff factor for all* $\tau \in \partial_e T(A)$.
(ii) $\partial_e T(A)$ *is (weak *) closed in* $T(A)$ *(i.e.,* $T(A)$ *is a Bauer simplex).*
(iii) $\partial_e T(A)$ *has finite covering dimension.*

*Then there is a unital *-homomorphism $M_2 \to F_t(A)$.*

The last proposition and the following theorem have also a version for *stably projection-less* (hence non-unital) A:
Then one has to require in place of (2) and (3) that the extremal rays of the cone $\mathbb{R}_{>0} \cdot T(A)$ of non-zero traces on A contain a finite-dimensional compact subset M with $\mathbb{R}_+ \cdot M = \mathbb{R}_+ \cdot \partial_e T(A)$.

Results similar to the ones above and below have been obtained independently by Y.Sato, and in a paper of A. Toms, S. White and W. Winter in case of amenable A.

Theorem 17. Let A be a non-elementary, unital, simple, separable, stably finite C*-algebra such that

(i) $\pi_\tau(A)''$ is McDuff factor for all $\tau \in \partial_e T(A)$.
(ii) $\partial_e T(A)$ is weak * closed in $T(A)$ (i.e., $T(A)$ is a Bauer simplex).
(iii) $\partial_e T(A)$ has finite covering dimension.
(iv) A has property (SI).

Then $A \cong A \otimes \mathcal{Z}$.

- Note that $A \cong A \otimes \mathcal{Z}$ implies (1), but not (2) and (3).
- It is not known if $A \cong A \otimes \mathcal{Z}$ implies (4).

Definition 18 (Matui–Sato, reformulated)**.** A cp map $\varphi: A \to A \subseteq A_\omega$ can be *excised in small central sequences* if for all positive contractions $e, f \in A_\omega \cap A'$ with

$$e \in J_A, \qquad \sup_k \|1 - f^k\|_2 < 1,$$

there exists $s \in A_\omega$ such that

$$fs = s, \qquad s^* a s = \varphi(a) e, \qquad a \in A.$$

Proposition 19 (Matui–Sato)**.** *Let A be a unital simple C*-algebra.*

(i) *If* $\mathrm{id}_A: A \to A$ *can be excised in small central sequences, then A has property (SI).*
(ii) *If A is simple, separable, unital and nuclear, and if A has strict comparison, then* id_A *can be excised in small central sequences.*

Definition 20. Let A be a unital, simple, stably finite C*-algebra. Then A has *local weak comparison* if there exists a constant $\gamma = \gamma(A)$ such that for all positive element $a, b \in A$:

$$\gamma \cdot \sup_{\tau \in QT(A)} d_\tau(a) < \inf_{\tau \in QT(A)} d_\tau(b) \implies a \precsim b.$$

▷
A has strict comparison $\iff$ $\mathrm{Cu}(A)$ is almost unperforated $\implies$ $\mathrm{Cu}(A)$ has strong tracial m-comparison for some $m < \infty$ (in the sense of Winter) $\implies$ A has local weak comparison.

Proposition 21. *Let A be a unital, simple, stably finite C*-algebra.*

(i) *If A has local weak comparison, then every nuclear cp $\varphi: A \to A$ can be excised in small central sequences.*
(ii) *(Matui-Sato) If* id_A *can be excised in small central sequences, then A has property (SI).*

Corollary 22. *Let A be a non-elementary, stably finite, simple, separable, unital and nuclear C*-algebra. Suppose that $\partial_e T(A)$ is closed in $T(A)$ and that $\partial_e T(A)$ has finite covering dimension. Then the following are equivalent:*

(i) $A \cong A \otimes \mathcal{Z}$,
(ii) *A has local weak comparison,*
(iii) *A has strict comparison ($\iff$ $\mathrm{Cu}(A)$ is almost unperforated).*

Question 23. *Are (1), (2) and (3) above equivalent for* all *non-elementary, stably finite, simple, separable, unital and nuclear C*-algebra?*

Are (2) and (3) above equivalent for all *non-elementary, stably finite, simple, separable, unital and nuclear C*-algebra!?*

Let $bT(A)$ denote the closure of $\partial_e T(A)$ in A^*, $\mathcal{T}_\omega\colon A_\omega \to \mathrm{C}(bT(A))_\omega$ the ultrapower of the cp evaluation map $\mathcal{T}(a)(\tau) := \tau(a)$. The multiplicative domain of $\mathcal{T}_\omega$ is denoted by $\mathrm{Mult}(\mathcal{T}_\omega)$. Since $\|\mathcal{T}_\omega(a^*a)\| = \|a\|_2^2$ it follows that the ideal J_A is contained in the kernel of $\mathcal{T}_\omega$, and that every positive element $a \in A_\omega$ with $\mathcal{T}_\omega(a) = 0$ is in J_A. Thus $\mathrm{Mult}(\mathcal{T}_\omega)$ defines a "*faithful*" completely positive map $\Psi\colon A_\omega/J_A \to \mathrm{C}(bT(A))_\omega$. *This does* not *mean that* Ψ *is faithful as a linear map!*

We identify naturally $\mathrm{Aff}_c(T(A))$ with a unital subspace of $\mathrm{C}(bT(A),\mathbb{R}) = \mathrm{C}(bT(A))_{\mathrm{sa}}$, and identify $\mathbb{C}\text{-}\mathrm{Aff}_c(T(A))$ with a unital subspace of $\mathrm{C}(bT(A))$.

The following proposition shows the limitation of the method for the proof of the main theorem: One has approximately to imitate inside A constructions (e.g. decompositions of the unit of $bT(A)$) by positive elements of A in a way that $\mathcal{T}\colon A \to \mathrm{C}(bT(A))$ becomes on those elements of A almost multiplikative.

Proposition 24. *Let A a unital and separable C^*-algebra with $T(A) \neq \varnothing$. The following properties of A are equivalent:*

(i) *For each $\tau_0 \in bT(A)$ and every non-empty compact set $K \subseteq \partial_e T(A) \setminus \{\tau_0\}$ there exists a positive contraction $a \in A$ with*

$$\tau_0(a) > 1/2, \qquad \sup_{\tau\in K} \tau(a) < 1/4.$$

(ii) $\partial_e T(A)$ *is closed in* $T(A)$.

(iii) $\mathrm{Aff}_c(T(A)) = \mathrm{C}(bT(A))_{\mathrm{sa}}$.

(iv) $\mathbb{C}\text{-}\mathrm{Aff}_c(T(A)) = \mathrm{C}(\partial_e T(A))$.

(v) $\mathcal{T}_\omega$ *maps the closed unit ball of* A_ω *onto the closed unit ball of* $\mathrm{C}(bT(A))_\omega$.

(vi) $\mathrm{Mult}(\Psi) \subseteq \mathrm{center}(A_\omega/J_A)$ *and* $\Psi\colon \mathrm{Mult}(\Psi) \to \mathrm{C}(bT(A))_\omega$ *is a surjective* **-isomorphism.*

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Dimension reduction and Jiang-Su stability

Aaron Tikuisis
(joint work with Wilhelm Winter)

Recent developments in the study of classification of C^*-algebras have suggested an important role of new regularity conditions (more stringent than amenability). These developments arise in response to an example of Villadsen [7], which was built on by Rørdam [3] (cf. also [5]), to disprove the Elliott conjecture. The general idea is captured by the following conjecture:

Conjecture. For a simple, separable, unital, nonelementary, nuclear C^*-algebra A in the UCT class, the following are equivalent:

(i) A is $\mathcal{Z}$-stable;
(ii) A has finite nuclear dimension;
(iii) A has strict comparison of positive elements;
(iv) A is an inductive limit of nice building blocks (2-NCCW complexes, direct sums of $M_n \otimes \mathcal{O}_m \otimes C(\mathbb{T})$).

Moreover, the algebras satisfying (i)-(iv) are classifiable.

(Closely related is the Toms-Winter conjecture, stating that (i),(ii), and (iii) are equivalent even without assuming the UCT.) It should be noted that the conjecture is known to hold for the examples of Villadsen [6].

This talk focused on the relationship between properties (i) and (ii), although it is important to view their relationship in context of the other two properties. A C^*-algebra is said to be $\mathcal{Z}$-stable if it is isomorphic to its tensor product with $\mathcal{Z}$. Nuclear dimension is a non-commutative generalization of topological dimension, building on the idea that the completely positive approximation property is a noncommutative version of (arbitrarily fine) partitions of unity [9].

Among the many partial verifications of the conjecture, we note that (ii) $\Rightarrow$ (i) has been shown by Winter in full generality [8]. On the other hand, (i) $\Rightarrow$ (ii) is perhaps the least-understood implication of the conjecture, and earlier verifications of this implication have always relied on classification (i.e. factored through (iv)).

The following result is, we hope, the beginning of a new approach to establishing and understanding (i) $\Rightarrow$ (ii):

Theorem. *(T-Winter [4]) The decomposition rank of $C(X, \mathcal{Z})$ is at most 2, independent of X.*

This result, and (i) $\Rightarrow$ (ii) in general, amounts to dimension reduction: showing that tensoring with the Jiang-Su algebra has the effect of lowering the dimension (at least, when the dimension is sufficiently high beforehand). Some notable earlier results about dimension reduction are the following: Villadsen's example A (mentioned above) has infinite nuclear dimension, but by classification, $A \otimes \mathcal{Z}$ has decomposition rank at most 2. Gong's reduction theorem [1] states that, if A is a simple AH algebra with very slow dimension growth then it is a limit of algebras with topological dimension at most three. Finally, Kirchberg and Rørdam [2] showed that for any space X, $C_0(X, \mathbb{C}\cdot 1_{\mathcal{O}_2}) \subset C(X,\mathcal{O}_2)$ factors as

$$C_0(X) \to C_0(Y) \to C(X,\mathcal{O}_2),$$

where $\dim Y \le 1$. The latter result highly relies on $K_*(\mathcal{O}_2) = 0$ (and little else), and is used in the proof of the result of myself and Winter mentioned above.

Part of this talk concerned explaining some key ideas from the proof of the result of myself and Winter. A key point of this proof is establishing the following:

Lemma. Let $X = [0,1]^d$. Then $C(X, \mathbb{C}\cdot 1_{n^\infty}) \subset C(X, M_{n^\infty})$ can be approximately factorized as

$$C(X) \xrightarrow{\psi} C_0(Y, \mathbb{C}\cdot 1_{\mathcal{O}_2}) \oplus F \subset C_0(Y,\mathcal{O}_2) \oplus F \xrightarrow{\varphi} C(X, M_{n^\infty}),$$

where ψ, φ are c.p.c. and φ is order zero when restricted to $C_0(Y,\mathcal{O}_2)$ or F.

In fact, the result follows (at least with M_{n^∞} in place of $\mathcal{Z}$) from this and Kirchberg-Rørdam's result for $C_0(Y) \subset C_0(Y,\mathcal{O}_2)$. The lemma is proven somewhat explicitly for the case $d = 1$ (using as input a c.p.c. approximate embedding of $C_0((0,1],\mathcal{O}_2)$ to M_{n^k}, given by quasidiagonality of the cone over $\mathcal{O}_2$), and then for general d roughly by taking products.

The result on dimension reduction opens many questions, including the following: Can we say more about the structure of $C(X) \subset C(X,\mathcal{Z})$; does it (approx.) factorize through subhomogeneous algebras with low topological dimension? Is $\dim_{\mathrm{nuc}}(A \otimes \mathcal{Z}) < \infty$ for every nuclear C^*-algebra A? What is the decomposition rank of $C(X) \subset (X, M_n)$; is it $< \dim X$, or does this dimension drop only occur when we put a UHF algebra in for M_n?

Slides from the talk may be found on my website: http://www.math.uni-muenster.de/u/aaron.tikuisis

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The p-shift endomorphism and purely infinite C^*-algebras

Eduard Ortega
(joint work with Enric Pardo)

We talked about the crossed product C^*-algebra associated to an injective endomorphism, but we showed that this turns out to be equivalent to study the crossed product C^*-algebra associated to the dilated autormorphism. In particular, given a C^*-algebra A with a non-trivial projection p, we study the so called *Bernoulli p-shift* endomorphism $\Delta_p : A^{\otimes\infty} \longrightarrow A^{\otimes\infty}$ defined by $x \longmapsto p \otimes x$ for every $x \in A^{\otimes\infty}$. We showed that the dilation of the Bernoulli p-shift endomorphism is topologically free, and this provides of a natural way to twist any endomorphism of a $\mathcal{D}$-absorbing C^*-algebra, where $\mathcal{D}$ is a strongly self-absorbing C^*-algebra with non-trivial a projection, into one that its dilated automorphism is essentially free and have the same K-theory map than the original one. Finally we gave conditions on the endomorphism and in the C^*-algebra to guarantee that the associated crossed product C^*-algebra is purely infinite. Therefore, combining both results we constructed purely infinite C^*-algebras with diverse ideal structure and various ideal related K-theory.

On the invariant translation approximation property

Joachim Zacharias

The purpose of the lecture was to present some recent developments on the invariant translation approximation property (ITAP) for groups. This approximation property has been introduced by John Roe around 2000 in connection with the coarse Baum-Connes conjecture. In his 2003 lectures on coarse geometry ([9]) he proved that amenable and finitely generated free groups have the ITAP. It is still an open problem whether all groups have the ITAP. There is also a stronger version (SITAP) of the ITAP involving coefficients which was introduced by the author in 2006 in [10]. Until recently it was not known whether all discrete groups verify both conditions, but thanks to the work of De la Salle and Lafforgue ([2]) we can show now that the version with coefficients is not verified for the group $\Gamma = SL(3,\mathbb{Z})$ (and other groups [4]). There has also been some very recent work by Katsura and Uuye [6], who resolved a technical question form [10]. The new results we describe are partly in collaboration with Jacek Brodzki and Issan Patri.

Let Γ be a discrete group acting on $\ell^2(\Gamma)$ via the left and right regular representation given by

$$\lambda_s e_t = e_{st} \quad \text{and} \quad \rho_s e_t = e_{ts^{-1}}$$

The reduced group C^*-algebra $C^*_r(\Gamma)$ is defined as the C^*-algebra generated by $\{\lambda_s \mid s \in \Gamma\}$. We may identify bounded operators on $\ell^2(\Gamma)$ with matrices $[\alpha_{s,t}]_{s,t\in\Gamma}$ and define the uniform Roe algebra $UC^*_r(\Gamma)$ as the closure of scalar $\Gamma\times\Gamma$-matrices $[\alpha_{s,t}]$ of finite width (i.e. $\{st^{-1} \mid \alpha_{s,t} \neq 0\}$ is finite) with uniformly bounded entries acting on $\ell^2(\Gamma)$. The reduced group C^*-algebra $C^*_r(\Gamma)$ is naturally contained in $UC^*_r(\Gamma)$. Indeed the left translation λ_r on $\ell^2(\Gamma)$ is given by the matrix where

$\alpha_{s,t} = 1$ if $st^{-1} = r$ and $\alpha_{s,t} = 0$ otherwise; so the matrices $[\alpha_{s,t}]$ of finite width such that $\alpha_{sr,tr} = \alpha_{s,t}$ for all $s,t,r \in \Gamma$ form precisely the group ring $\mathbb{C}[\Gamma]$. They may also be characterized as those finite width matrices fixed by all automorphisms of the form $\mathrm{Ad}(\rho_t)$.

Definition 1. We say that Γ has the *invariant translation approximation property* (ITAP) if

$$C_r^*(\Gamma) = UC_r^*(\Gamma)^\Gamma.$$

Equivalently, Γ has the ITAP iff $C_r^*(\Gamma) = UC_r^*(\Gamma) \cap L(\Gamma)$, where $L(\Gamma)$ is the von Neumann algebra generated by the left regular representation.

The stronger version with coefficients is defined as follows: let $S \subseteq B(H)$ be a (concrete) operator space. Consider matrices $[a_{s,t}]_{s,t\in\Gamma}$ with finite width, where $a_{s,t} \in S$ for all $s,t \in \Gamma$ and $\|a_{s,t}\|$ is uniformly bounded. Each such matrix defines a bounded operator on $\ell^2(\Gamma, H)$ and we define the operator space $UC_r^*(\Gamma, S)$ as the closure of the set of such matrices. $UC_r^*(\Gamma, S)$ is an operator space in general and a C^*-algebra if S is a C^*-algebra. Note that $\mathrm{Ad}(\rho_t)$ still acts on $UC_r^*(\Gamma, S)$, for every S.

Definition 2. We say that Γ has the *strong invariant translation approximation property* (SITAP) if

$$C_r^*(\Gamma) \otimes S = UC_r^*(\Gamma, S)^\Gamma,$$

where $\otimes$ denotes the minimal tensor product.

It is well-known that exactness of Γ can be characterised by nuclearity of $UC_r^*(\Gamma)$ ([3] and [8]). Note that $UC_r^*(\Gamma, -)$ may be regarded as a functor on C^*-algebras and we can characterise exactness of Γ in terms of exactness properties of this functor.

An important very weak approximation property which we need is the OAP, which may be regarded as the matricial version of Grothendieck's AP. It was introduced in [7] and studied in [5] for group algebras. See [1] for an excellent exposition. Using this property we can characterise the (SITAP) as follows:

Theorem 3. ([10]) Let Γ be exact. Then Γ has the SITAP iff $C_r^*(\Gamma)$ has the OAP.

It has been a longstanding open problem whether the OAP and exactness of $C_r^*(\Gamma)$ are equivalent. By recent results of De la Salle and Lafforgue this is not he case ([2]), they showed that $C_r^*(SL(3,\mathbb{Z}))$ does not verify the OAP but is known to be exact. Thus this group also fails the SITAP.

We have the following permanence properties of the (S)ITAP:

Theorem 4. (i) (S)ITAP is closed under subgroups, inductive limits and extensions byfinite groups.

(ii) If one of Γ_1 and Γ_2 has (S)ITAP, the other SITAP and is exact then $\Gamma_1 \times \Gamma_2$has (S)ITAP.

(iii) If Γ_1 and Γ_2 verify SITAP and are exact then so does $\Gamma_1 \star \Gamma_2$.

(2) can be extended to certain semidirect products.

One can also develop a very similar theory for locally compact groups. Though we know now that not every group verifies the SITAP the question whether every discrete group verifies the ITAP remains open but we have a strong candidate for a counterexample.

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