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Graph Theory

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Abstract. This was a workshop on graph theory, with a comprehensive approach. Highlights included the emerging theories of sparse graph limits and of infinite matroids, new techniques for colouring graphs on surfaces, and extensions of graph minor theory to directed graphs and to immersions.

Mathematics Subject Classification (2010): 05Cxx.

Introduction by the Organisers

The aim of this workshop was to offer an exchange forum for graph theory and related fields in pure mathematics. We had eight 'main' longer talks, 22 shorter talks, seven informal workshops on topics suggested by the participants, and plenty of further informal interaction.

Particular emphasis was given to fields that have seen particularly exciting recent developments:

- Graph limits, either dense or sparse;
- Infinite matroids;
- Colouring graphs on surfaces;
- Graph minors, graph immersions, and tree-structure.

The theory of *graph limits*, initiated a few years ago by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi, has been taken up and developed further by a number of leading researchers. The original idea was to describe properties of dense graphs by continuous objects, *graphons*, that occur as limit objects of sequences of such graphs, so that analytical methods could be brought to bear on the study of such properties. In an independent development, Razborov proposed a theory of *flag algebras*, whose applications to extremal graph homomorphism and induced subgraph density problems have turned out to be interchangeable with those of graph limits.

For (very sparse) graphs of bounded degree, such as Cayley graphs of finitely generated groups, Benjamini and Schramm had earlier developed another limit theory, also with probabilistic ingredients, but whose limits were essentially still graphs (albeit infinite). Extending these ideas, Elek recently proposed a more general notion of *graphings* as limit objects of sparse graphs.

Several of our talks were from this area, including the main talks by László Lovász on *Borel graphs, graphings, and limits of bounded-degree graphs*, and by Christian Reiher with a proof of *The clique density theorem.* This had been conjectured by Lovász and Simonovits in the 1970s.

Following the recent axiomatization of infinite matroids with duality by Bruhn, Diestel, Kriesell, Pendavingh and Wollan, infinite matroid theory has seen a surge of activity that has produced some deep results and conjectures, unifying some of its major open problems, and relating it to both infinite graph theory and logic.

Nathan Bowler gave a main talk about these developments. He put an emphasis on newly emerging bonds between infinite matroids and graphs topologized with their ends; these can be described elegantly in terms of the determinacy of infinite games. A highlight of the talk was a new packing/covering conjecture for infinite matroids, a central conjecture that unifies some of the main classical open conjectures about infinite matroids, such as matroid intersection and union.

The area of colouring graphs on surfaces is a classical one dating back to Heawood's formula from 1890. One chapter of its development was completed by the map color theorem of Ringel and Youngs in the 1960s. A modern approach to the subject and a research programme was initiated by Thomassen in the 1990s.

We had a main talk by Luke Postle on *Linear isoperimetric bounds for graph colouring*, a new technique in the area based on the recent discovery that many colouring results are a direct consequence of the fact that the corresponding 'critical' graphs (those that are minimally uncolourable) satisfy a certain isoperimetric inequality.

The theory of graph minors, initiated by Robertson and Seymour in the 1980s, is both maturing and expanding. It is maturing in that its main structural results are increasingly well understood: their assertions and proofs have been simplified and strengthened in a long process of identifying the essentials, converging to a leaner and more powerful theory that is now taking shape. It is also expanding: in the direction of directed graphs (in particular, of tournaments), of matroids, and of graph orderings stronger than that of minors, such as topological minors or immersions. There is now a unifying theory establishing the existence of canonical tree-decompositions of graphs and matroids that can distinguish their dense parts, such as higher-order blocks or tangles. Finally, graph minor theory is increasingly used in computer science, both for concrete algorithms and abstract complexity theory issues such as fixed-parameter tractability.

We had several talks on these topics, including main talks by Dániel Marx on *The k-disjoint paths problem in directed planar graphs*, and by Paul Seymour on *Grid immersion and multicommodity flows*.

Further main talks were given by János Pach on *Geometric graph theory* and Lex Schrijver on *The edge colouring model*.

Finally, we had a number of informal workshops focused on a particular topic, often initiated by the participants but open to all. Their topics were:

- Graph limits and flag algebras (convenor: Král)
- Ramsey goodness (convenor: Conlon)
- Infinite matroids (convenors: Bowler and Carmesin
- Sparse regularity (convenors: Conlon and Fox)
- Flows (convenor: Šámal)
- Colin de Verdière - type parameters (convenor: van der Holst)
- The Lovász theta function (convenor: Šámal)

Reinhard Diestel
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Workshop: Graph Theory

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Abstracts

Borel graphs, graphings, and limits of bounded-degree graphs

László Lovász

We describe objects that can serve as limit objects of bounded degree graphs. We fix a positive integer D, and consider graphs with all degrees bounded by D.

Let $\mathbf{G}$ be a graph with node set $V(\mathbf{G}) = [0,1]$. We call $\mathbf{G}$ a *Borel graph*, if its edge-set is a Borel set in $[0,1] \times [0,1]$. As an example, consider the graph $\mathbf{C}_a$ on $[0,1)$ in which a node x is connected to $x+a \pmod 1$ and $x-a \pmod 1$. If a is irrational, we get a graph that consists of two-way infinite paths; if a is rational the graph will consist of cycles.

There is a rather rich theory of Borel graphs (see e.g. Kechris and Miller [6]). In the talk we state and prove only a few results. For example, Kechris, Solecki and Todorcevic [7] proved an extension of Brooks' Theorem to Borel graphs: *Every Borel graph has a node coloring with $D+1$ colors in which nodes with any given color form a Borel set.*

Now we take the Lebesgue measure on $[0,1]$ into account. We say that a Borel graph $\mathbf{G}$ on $[0,1]$ is a *graphing*, if it is Borel and for any two measurable sets A and B, we have

$$\int_A \deg_B(x)\, d\lambda(x) = \int_B \deg_A(x)\, d\lambda(x). \tag{1}$$

This condition makes it possible to do "double counting" arguments in graphings.

It is not quite trivial to prove that *every Borel subgraph of a graphing is a graphing* [8]. From this it is easy to derive that the union and intersection of two graphings are graphings.

Let Γ be a finitely generated group of measure preserving transformations acting on $[0,1]$. Connect every $x \in [0,1]$ to its images under the generators. The resulting graph is a graphing. Conversely, every graphing arises this way.

As defined by Benjamini and Schramm [2], a sequence of graphs G_n with $|V(G_n)| \to \infty$ is called *locally convergent* if the distribution of the r-neighborhood of a random node of G_n converges for every r. For every convergent sequence there is a graphing in which the distribution of r-neighborhoods is the limit of these distributions (Aldous-Lyons [1], Elek [4]).

It was proved by Hatami, Lovász and Szegedy [5] (see also [8]) that graphings can represent the limit objects for a stronger notion of convergence, the "local-global convergence" introduced by Bollobás and Riordan [3].

References

[1] D. Aldous and R. Lyons: Processes on Unimodular Random Networks, *Electron. J. Probab.* **12**, Paper 54 (2007), 1454–1508.

[2] I. Benjamini and O. Schramm: Recurrence of Distributional Limits of Finite Planar Graphs, *Electronic J. Probab.* **6** (2001), paper no. 23, 1–13.

[3] B. Bollobás and O. Riordan: Metrics for sparse graphs, in: *Surveys in Combinatorics 2009*, Cambridge Univ. Press, Cambridge (2009) 211–287.
[4] G. Elek: On limits of finite graphs, *Combinatorica* **27** (2007), 503–507.
[5] H. Hatami, L. Lovász and B. Szegedy: Limits of local-global convergent graph sequences `http://arxiv.org/abs/1205.4356`
[6] A. Kechris and B.D. Miller: *Topics in orbit equivalence theory*, Lecture Notes in Mathematics **1852**. Springer-Verlag, Berlin, 2004.
[7] A. Kechris, S. Solecki and S. Todorcevic: Borel chromatic numbers, *Advances in Math.* **141** (1999) 1–44.
[8] L. Lovász: *Large networks and graph limits*, Amer. Math. Soc., Providence, R.I. (2012).

The edge coloring model

Alexander Schrijver

Following de la Harpe and Jones [2], a *vertex model*, also called *edge coloring model* (with n *colors* or *states*), is any linear function $h : \mathbb{F}[x_1, \dots, x_n] \to \mathbb{F}$, where $\mathbb{F}$ is a field. The *partition function* $p(h)$ of h is defined by, for any undirected graph $G = (V, E)$,

$$p(h)(G) := \sum_{\phi: E \to [n]} \prod_{v \in V} h\Big(\prod_{e \in \delta(v)} x_{\phi(e)} \Big).$$

Here $\delta(v)$ is the set of edges incident with v. So p is a linear function, with

$$p : \mathbb{F}[x_1, \dots, x_n]^* \to \mathbb{F}^{\mathcal{G}},$$

where $\mathcal{G}$ denotes the set of undirected graphs, including the 'vertexless loop' O (so $p(h)(O) = n$).

The edge coloring model is dual to the "vertex coloring model" (called spin model by de la Harpe and Jones), where the roles of vertices and edges are interchanged.

Recently, several studies of the vertex coloring model have been made, in particular characterizations of their partition functions where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and studies of the limit behaviour of these functions, where$\mathbb{F} = \mathbb{C}$. In this talk we consider similar questions for the edge coloring model.

The basic characterization, and a basic technique involving invariant theory, was given by Szegedy [6], who characterized partition functions of edge coloring models over the reals. It is the counterpart of a characterization of Freedman, Lovász, and Schrijver [1] for partition functions of the vertex coloring model — however, quite different techniques were needed.

We characterize such edge coloring partition functions for the case of the complex numbers, by means of an exponential upper bound on the rank of the corresponding 'connection matrices'.

Define a *k-fragment* to be an undirected graph, with k of its vertices being labeled $1, \dots, k$, each having degree 1. Let $\mathcal{G}_k$ denote the collection of k-fragments. If $G, H \in \mathcal{G}_k$, let $G \cdot H$ be the graph obtained from the disjoint union of G and H by identifying the two vertices labeled i, and joining the two edges incident with it to one edge, while deleting the vertex (for $i = 1, \ldots, k$). (In this way, the vertexless loop can arise.)

If $f : \mathcal{G} \to \mathbb{F}$, the *k-th connection matrix* $C_{f,k}$ of f is the $\mathcal{G}_k \times \mathcal{G}_k$ matrix with

$$(C_{f,k})_{G,H} := f(G \cdot H)$$

for $G, H \in \mathcal{G}_k$. Szegedy [6] proved that any function $f : \mathcal{G} \to \mathbb{R}$ is the partition function of some real edge coloring model if and only if $f(\emptyset) = 1$, $\text{rank}(C_{f,0}) = 1$, and $C_{f,k}$ is positive semidefinite for each k. (Here $\emptyset$ is the graph with no vertices and no edges.)

As a counterpart, we show in [5]:

Theorem 1. *Let $f : \mathcal{G} \to \mathbb{C}$ and $n \in \mathbb{N}$. Then f is the partition function of some n-color edge coloring model over $\mathbb{C}$ if and only if $f(\emptyset) = 1$, $f(O) = n$, and*

$$\mathit{rank}(C_{f,k}) \leq n^k$$

for each $k \in \mathbb{N}$.

Theorem 1 thus characterizes the image of p. Note that p is invariant under the action of the complex orthogonal group $O(n, \mathbb{C})$ on $\mathbb{F}[x_1, \ldots, x_n]^*$. Let, as usual ,

$$\mathbb{C}[x_1, \ldots, x_n]^* // O(n, \mathbb{C})$$

denote the corresponding closed orbit space (the variety made by the Zariski-closed orbits of the action of $O(n, \mathbb{C})$ on $\mathbb{C}[x_1, \ldots, x_n]^*$). It can be proved that this variety is isomorphic to the variety of partition functions of n-color edge coloring models:

$$\mathbb{C}[x_1, \ldots, x_n]^* // O(n, \mathbb{C}) \approx \{f \in \mathbb{C}^{\mathcal{G}} \mid \exists h \in \mathbb{C}[x_1, \ldots, x_n]^* : f = p(h)\}.$$

So for each fixed number n of colors, the partition functions form an affine variety.

Moreover, we show, in joint work with Guus Regts [4], that limits of (real) edge coloring models do exist, which answers a question of Lovász [3]. To describe it, consider the set $\mathbb{R}[x_1, x_2, \ldots]$ of polynomials in infinitely many variables (each polynomial uses only finitely many variables). Let B be the set of elements of $h \in \mathbb{R}[x_1, x_2, \ldots]^*$ with

$$\sum_{\psi \in \mathbb{N}^d} h(x_{\psi(1)} \cdots x_{\psi(d)})^2 \leq 1$$

for each d. As before, the *partition function* $p(h)$ of $h \in B$ is defined by, for any undirected graph $G = (V, E)$,

$$p(h)(G) := \sum_{\phi: E \to \mathbb{N}} \prod_{v \in V} h(\prod_{e \in \delta(v)} x_{\phi(e)}).$$

Theorem 2. *Let $h_1, h_2, \ldots \in B$ be such that for each simple graph G, $p(h_i)(G)$ converges as $i \to \infty$. Then there exists $h \in B$ such that for each simple graph G,*

$$\lim_{i \to \infty} p(h_i)(G) = p(h)(G).$$

This theorem is proved in the framework of Hilbert spaces, and is based on the Alaoglu theorem of the weak compactness of the unit ball of Hilbert spaces. The theorem also gives rise to approximating multi-variate homogeneous polynomials by low-rank polynomials — the rank only depending on the degree of approximation required, and not on the number of variables in the polynomial.

References

[1] M.H. Freedman, L. Lovász, A. Schrijver, *Reflection positivity, rank connectivity, and homomorphisms of graphs*, Journal of the American Mathematical Society **20** (2007), 37–51.

[2] P. de la Harpe, V.F.R. Jones, *Graph invariants related to statistical mechanical models: examples and problems*, Journal of Combinatorial Theory, Series B **57** (1993), 207–227.

[3] L. Lovász, *Graph homomorphisms: Open problems*, 2008, `http://www.cs.elte.hu/ lovasz/problems.pdf`

[4] G. Regts, A. Schrijver, *Compact orbit spaces in Hilbert spaces and limits of edge-colouring models*, ArXiv `http://arxiv.org/abs/1210.2204`

[5] A. Schrijver, *Characterizing partition functions of the vertex model by rank growth*, ArXiv `http://arxiv.org/abs/1210.2204`

[6] B. Szegedy, *Edge coloring models and reflection positivity*, Journal of the American Mathematical Society **20** (2007), 969–988.

The clique density theorem

CHRISTIAN REIHER

Suppose you are confronted with a graph on n vertices having at least $\gamma \cdot n^2$ edges (where γ denotes a real number from the half open interval $[0, 1/2)$) and based on these data you want to give a lower bound on the number of its triangles as large as possible. Here we consider γ as fixed whilst n is thought of as being very large or even as tending to infinity. By the easy direction of TURÁN'S classical theorem from [8] we know that no triangle needs to exist in case $\gamma < 1/4$. If $\gamma > 1/4$, however, then an easy triangle removal argument shows that the number of triangles is proportional to n^3. So let us denote the largest number δ such that any graph on n vertices having at least $\gamma \cdot n^2$ edges contains at least $\delta \cdot n^3$ triangles by $F_3(\gamma)$. It can be shown that this maximum really exists and that it is does not "depend on small graphs". Thus it is an interesting question to determine this function F_3.

Repeating the well known proof of TURÁN'S in which one compares the sum of the squares of the degrees of the vertices with the square of the sum of the degrees of the vertices by means of the Cauchy Schwarz Inequality, GOODMAN proved $F_3(\gamma) \geq \gamma(4\gamma - 1)/3$ in [1]. A cursory inspection of this argument, however, reveals that equality only seems possible if γ belongs to the discrete set $\{0, 1/4, 1/3, 3/8, 2/5, \ldots, (s-1)/2s, \ldots\}$, and that in these cases equality holds exactly at TURÁN graphs. This makes it plausible that for general values of γ GOODMAN'S bound may be improved and e.g. that for $\gamma \in [1/4, 1/3]$ a graph where the true value of $F_3(\gamma)$ is achieved may be constructed by "interpolating" between the complete bipartite and the complete tripartite TURÁN graph. Doing some calculations one finds that writing $\gamma = (1 - \alpha^2)/3$ with $\alpha \in [0, 1/2]$ and imagining a complete tripartite graph the two bigger of whose vertex classes have size $(1+\alpha)n/3$ whilst the remaining smaller vertex class contains the remaining $(1-2\alpha)n/3$ vertices has $\gamma \cdot n^2$ edges and $(1+\alpha)^2(1-2\alpha)n^3/27$ triangles. Thus $(1+\alpha)^2(1-2\alpha)/27$ is a reasonable candidate for the true value of $F_3(\gamma)$ in this range. The easiest known proof of this fact is described in the fifth section of [5] as an application of Razbrov's "differential calculus for flag algebra homomorphisms".

For larger values of γ the picture looks in some respects similar: To predict the value of $F_3(\gamma)$ one commences by locating γ in an interval of the form $[\frac{s-1}{2s}, \frac{s}{2(s+1)}]$, where $s \geq 2$ is integral, visualizes a complete $(s+1)$–partite graph with $\gamma \cdot n^2$ edges all of whose vertex classes except for one possibly smaller one are of equal size, and counts the number of triangles that it contains. It has been conjectured by Lovász and Simonovits in the 1970s that this procedure does indeed give rise to the true value of $F_3(\gamma)$. Recently this has been proved by Razborov in [6] using again his flag algebraic devices. It should be pointed out that the general proof is by far more difficult than the corresponding argument in the special case $\gamma \in [1/4, 1/3]$. A reasons for this is that after some calculations one reaches a point where it would be helpful to know something about the least number of 4–cliques such a graph can contain. This number is plainly zero if γ is smaller than 1/3, and so the corresponding term can simply be thrown away in this case, whereas in the general case some delicate analysis seems to be necessary to handle this term.

Now actually Lovász and Simonovits conjectured much more in [2], namely that the same strategy leads to a true conjecture about the function F_r defined similarly but using r–cliques instead of triangles, the minimum number of which is, of course proportional to n^r. The case $r = 4$ of this conjecture has recently been proved by Nikiforov in [3], who also gave an alternative treatment of the case $r = 3$. His proofs are easier to read than [6] for people unfamiliar with [5], for he uses a finitary analytical language similar to the one occurring in the alternative proof of Turán's Theorem described by Motzkin and Strauss in [4] rather than Razborov's infinitary flag algebraic language.

The talk uses this elementary language of weighted graphs and sketches the proof of the general version of Lovász' and Simonovits' clique density conjecture obtained in November 2010, [7]. The argument is much closer in spirit to Razborov's original treatment of triangles than to Nikiforov's approach, and both languages could have been with equal ease. An additional difficulty encountered in the argument is that when thinking about the number of r–cliques one arrives at an inequality involving the numbers of triangles, r–cliques, and $(r+1)$–cliques that is difficult to analyze further.

To conclude this abstract, we would like to state the Clique Density Theorem: If $r \geq 3$ and $\gamma \in \left[0, \frac{1}{2}\right)$, then every graph on n vertices with at least $\gamma \cdot n^2$ edges contains at least

$$\frac{1}{(s+1)^r}\binom{s+1}{r}(1+\alpha)^{r-1}\left(1-(r-1)\alpha\right) \cdot n^r$$

cliques of size r, where $s \geq 1$ is an integer for which $\gamma \in [\frac{s-1}{2s}, \frac{s}{2(s+1)}]$, and $\alpha \in [0, \frac{1}{s}]$ is implicitly defined by $\gamma = \frac{s}{2(s+1)}(1 - \alpha^2)$.

References

[1] A. W. Goodman: *On sets of acquaintances and strangers at any party*, American Mathematical Monthly **69** (1959), 778–783.

[2] László Lovász and Miklós Simonovits: *On the number of complete subgraphs in a graph II*, Studies in pure mathematics, pages 459–495, Birkhäuser, 1983.
Also available from Lovász' homepage
`http://www.cs.elte.hu/ lovasz/scans/no-comp-sub.pdf`,
or alternatively from Simonovits' homepage
`http://www.renyi.hu/ miki/download.html`.

[3] Vlado Nikiforov: *The number of cliques in graphs of given order and size*, Transactions of the American Mathematical Society **363** (2011), 1599–1618.

[4] T. S. Motzkin and E. G. Strauss: *Maxima of graphs and a new proof of a theorem of Turán*, Canadian Journal of Mathematics **17** (1965), 533–540.

[5] Alexander A. Razborov: *Flag algebras*, Journal of Symbolic logic **72** (2007), 1239–1282.

[6] Alexander A. Razborov: *On the minimal density of triangles in graphs*, Combinatorics, Probability and Computing **17** (2008), 603–618.

[7] Christian Reiher: *The Clique Density Theorem*, submitted.

[8] Pál Turán: *Egy gráfelméleti szélsöertékfeladatról*, Matematikai és Fizikai Lapok **48** (1941), 436–451.

Infinite matroids

Nathan Bowler

The results presented below are joint work with Hadi Afzali, Johannes Carmesin and Robin Christian, in various constellations.

The talk begins with an overview of the recent rapid progress in the development of the theory of infinite matroids. Although there was some early progress in this field in the '60s and '70s [15], and the correct class of objects (called at the time B-Matroids, and due to Higgs [13]) was known, it was only after the introduction by Bruhn *et al.* of simpler axiomatisations for this class [10] that it became practical to work with these objects. Developments since then include the extension to infinitary matroids of many familiar concepts and theorems from finite matroid theory, including connectivity and the linking theorem [11], tree decompositions over 2-separations [4], representability [1, 6], graphic matroids [9, 7], and matroid union [3] and intersection [5].

Next, I discuss a rich class of examples of infinite matroids with a close relationship to infinite graphs. Let G be a locally finite graph. We call the new limit points in the Freudenthal compactification $|G|$ of G the *ends* of G. For a given set Ψ of ends of G, we consider the subsets of the ground set given by edge sets of topological circles in $|G|$ that don't use any ends except those in Ψ. If Ψ is Borel, these sets give the circuits of a matroid $M_\Psi(G)$ [8]. The proof relies on Borel determinacy [14], and the question of when such constructions give matroids is closely linked to the question of which games are determined. This class of examples is rich enough to show that there are as many countably infinite matroids as there could possibly be $(2^{2^{\aleph_0}})$ and that the class of infinite planar matroids is not wellquasiordered. We may also deduce that any connected subspace of $|G|$ meeting the ends in a Borel set is path-connected. It was previously known that in general $|G|$ may have subspaces that are connected but not path-connected [12].

Finally, I discuss the most important open problem in infinite matroid theory - the packing-covering conjecture [5]. A *packing* for a family of matroids on the same ground set consists of a disjoint family of bases, one from each matroid. Similarly, a *covering* for such a family of matroids consists of a family of bases, one from each matroid, whose union is the whole ground set. The packing-covering conjecture states that for any family of matroids there is a partition of the ground set into two parts, the *packing side* and the *covering side*, such that the family obtained by contracting the matroids onto the packing side has a packing and the family obtained by restricting the matroids to the covering side has a covering. I explain how this conjecture provides a common unifying generalisation of several key results of finite matroid theory: the matroid intersection theorem, the base packing and covering theorems and the rank formula in the matroid union theorem. It also would imply the Aharoni-Berger theorem [2], a deep result generalising Menger's theorem to infinite graphs.

References

[1] H. Afzali and N. Bowler, *Thin sums matroids and duality*, arXiv1204.6294 (2012).

[2] R. Aharoni and E. Berger, *Menger's theorem for infinite graphs* Invent. math.. **176** (2009), 1-62.

[3] E. Aigner-Horev, J. Carmesin and J. Fr ohlich, *Infinite matroid union*, arXiv:1111.0602 (2011)

[4] E. Aigner-Horev, R. Diestel and L. Postle, *The structure of 2-separations of infinite matroids*, arXiv:1201.1135 (2012).

[5] N. Bowler and J. Carmesin, *Matroid intersection, base packing and base covering for infinite matroids*, arXiv:1202.3409 (2012).

[6] N. Bowler and J. Carmesin, *An excluded minors method for infinite matroids*, arXiv:1212.3939 (2012).

[7] N. Bowler and J. Carmesin, *Graph-like spaces and infinite matroids*, in preparation.

[8] N. Bowler and J. Carmesin, *Infinite matroids and determinacy of games*, in preparation.

[9] H. Bruhn and R. Diestel, *Infinite matroids in graphs*, arXiv:1011.4749 (2010).

[10] H. Bruhn, R. Diestel, M. Kriesell, R. Pendavingh and P. Wollan, *Axioms for infinite matroids*, arXiv:1003.3919 (2010).

[11] H. Bruhn and P. Wollan, *Finite connectivity in infinite matroids* Eur. J. Comb. **33**, 8 (2012), 1900-1912.

[12] A. Georgakopoulos, *Connected but not path-connected subspaces of infinite graphs*, Combinatorica **27**, 6 (2007), 317-328.

[13] D.A. Higgs, *Matroids and duality*, Colloq. Math. **20** (1969), 215–220.

[14] D.A. Martin, *A purely inductive proof of Borel Determinacy*, Proc. Sympos. Pure Math. **42**, 303–308 (1985).

[15] J.G. Oxley, *Infinite Matroids*, Proc. London Math. Soc. **37** (1978), no. 3, 259–272.

Linear isoperimetric bound for coloring graphs on surfaces

Luke Postle

The new results presented below are joint work with Robin Thomas.

1. Coloring Graphs on Surfaces

Mathematicians have long been interested in coloring maps. A natural question is to ask what is the fewest number of colors so that the regions or countries of a map that touch one another have different colors. For planar maps, it was long conjectured that four colors suffices. The Four-Color Theorem, proved in the 1970s, settled this conjecture in the affirmative.

Let X be a nonempty set. We say that a function $\phi : V(G) \to X$ is a *coloring* of G if for all $e = uv \in E(G)$, $\phi(u) \neq \phi(v)$. We say that a coloring $\phi : V(G) \to X$ is a *k-coloring* if $|X| = k$. We say that a graph G is *k-colorable* if there exists a k-coloring of G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum k such that G is k-colorable.

Mathematicians have wondered what generalizations of the Four-Color Theorem might be true. A natural class of graphs to determine the coloring properties for is graphs embedded in a surface. A fundamental question in topological graph theory is as follows: Given a surface Σ and an integer $t > 0$, which graphs embedded in Σ are t-colorable? Heawood proved that if Σ is not the sphere, then every graph in Σ is t-colorable as long as $t \geq H(\Sigma) := \lfloor (7 + \sqrt{24g + 1})/2 \rfloor$, where g is the Euler genus of Σ.

Ringel and Youngs [9] proved that the bound is best possible for all surfaces except the Klein bottle. Dirac [5] and Albertson and Hutchinson [2] improved Heawood's result by showing that every graph in Σ is actually $(H(\Sigma)-1)$-colorable, unless it has a subgraph isomorphic to the complete graph on $H(\Sigma)$ vertices.

Thus the maximum chromatic number for graphs embeddable in a surface has been found for every surface. Yet the modern view argues that most graphs embeddable in a surface have small chromatic number. To formalize this notion, we need a definition. We say that a graph G is *t-critical* if it is not $(t-1)$-colorable, but every proper subgraph of G is $(t-1)$-colorable. Using Euler's formula, Dirac [6] proved that for every $t \geq 8$ and every surface Σ there are only finitely many t-critical graphs that embed in Σ. By a result of Gallai [8], this can be extended to $t = 7$. Indeed, this was extended to $t = 6$ by Thomassen 1.

Theorem 1. *For every surface Σ, there are finitely many 6-critical graphs that embed in Σ.*

Furthermore, Theorem 1 yields an algorithm for deciding whether a graph on a fixed surface is 5-colorable.

Corollary 1. *There exists a linear-time algorithm for deciding 5-colorability of graphs on a fixed surface.*

This follows from a result of Eppstein which gives a linear-time algorithm for testing subgraph isomorphism on a fixed surface. Hence if the list of 6-critical graphs embeddable on a surface is known, one need merely test whether a graph contains one of the graphs on the list. The list is known only for the projective plane, torus, and Klein bottle.

2. List-Coloring Graphs on Surfaces

There exists a generalization of coloring where the vertices do not have to be colored from the same palette of colors. We say that L is a *list-assignment* for a graph G if $L(v)$ is a set of colors for every vertex v. We say L is a *k-list-assignment* if $|L(v)| = k$ for all $v \in V(G)$. We say that a graph G has an *L-coloring* if there exists a coloring ϕ such that $\phi(v) \in L(v)$ for all $v \in V(G)$. We say that a graph G is *k-choosable*, also called *k-list-colorable*, if for every k-list-assignment L for G, G has an L-coloring. The *list chromatic number* of G, denoted by $ch(G)$, is the minimum k such that G is k-list-colorable.

Note that $\chi(G) \leq ch(G)$ as a k-coloring is a k-list-coloring where all the lists are the same. In fact, Dirac's Theorem[5] has been generalized to list-coloring by Bohme, Mohar and Stiebitz for most surfaces; the missing case, $g(\Sigma) = 3$, was completed by Kral and Skrekovski. Nevertheless, list-coloring differs from regular coloring. One notable example of this is that the Four Color Theorem does not generalize to list-coloring. Indeed Voigt constructed a planar graph that is not 4-choosable.

Yet the list chromatic number of planar graphs is now well understood, thanks to Thomassen [10]. He was able to prove the following remarkable theorem with an outstandingly short proof.

Theorem 2. *Every planar graph is* 5*-choosable.*

If L is list assignment for a graph G, then we say that G is *L-critical* if G does not have an L-coloring but every proper subgraph of G does. Similarly, we say that G is *k-list-critical* if G is not $(k-1)$-list-colorable but every proper subgraph of G is. Thomassen [12] gave a simple proof that there are only finitely many 7-list-critical graphs on a fixed surface. Indeed, Thomassen proved the following stronger theorem. Naturally then, Thomassen conjectured (see Problem 5 of [12]) that Theorem 1 generalizes to list-coloring, which we proved:

Theorem 3. *[Postle and Thomas] If G is a* 6*-list-critical graph embedded on a surface Σ of genus g, then $|V(G)| = O(g)$. Hence for every surface, there exist only finitely many* 6*-list-critical graphs that embed in Σ.*

An immediate corollary of Theorem 3 is that we are now able to decide 5-list-colorablity on a fixed surface in linear-time. Our proof also gives a new proof of Theorem 1 as his techniques do not apply for list-coloring. In addition, we use Theorem 7 to give an independent proof of a version of the following theorem of DeVos, Kawarabayashi and Mohar while improving the bound on the necessary edge-width from exponential in genus to logarithmic in genus.

Theorem 4. *If G is* 2*-cell embedded in a surface Σ and $ew(G) \geq \Omega(\log g(\Sigma))$, then G is* 5*-list-colorable.*

3. Extending Precolored Subgraphs

An important technique in the proofs of Thomassen is to ask what colorings of a graph are possible when a certain subgraph has already been precolored. To that end if H is a subgraph of G and ϕ is a coloring of H and ϕ' is a coloring of G, we say that ϕ *extends* to ϕ' if $\phi'(v) = \phi(v)$ for all $v \in V(H)$. We proved the following.

Theorem 5. *Let G be a 2-connected plane graph with outer cycle C and L a 5-list-assignment for G. Then G contains a connected subgraph H with at most $29|C|$ vertices such that for every L-coloring ϕ of C either*

(i) *ϕ cannot be extended to an L-coloring of H, or,*
(ii) *ϕ can be extended to an L-coloring of G.*

This settles in the affirmative a conjecture of Dvorak et al. [7] who had improved Thomassen's version of this theorem with exponential bound to quadratic. The fact that the bound is linear is crucial to proving many of the main results. Indeed we prove the main results outline in a general setting about families of graphs satisfying a more abstract version of Theorem 5.

Thomassen extended his version of Theorem 1 to the case when the precolored subgraph has more than one component. He proved the following stronger version of Theorem 1.

Theorem 6. *For all $g, q \geq 0$, there exists a function $f(g, q)$ such the following holds: Let G be a graph embedded in a surface Σ of Euler genus g and let S be a set of at most q vertices in G. If ϕ is a 5-coloring of S, then ϕ extends to a 5-coloring of G unless there is a graph H with at most $f(g, q)$ vertices such that $S \subseteq H \subseteq G$ and the 5-coloring of S does not be extend to a 5-coloring of H.*

We generalize Theorem 6 to list-coloring. Indeed, we prove that f is linear, which is best possible up to a multiplicative constant:

Theorem 7. *Let G be a connected graph 2-cell embedded in a surface Σ, $S \subseteq V(G)$ and L a 5-list-assignment of G. Then there exists a subgraph H with $|V(H)| = O(|S| + g(\Sigma))$ such that for every L-coloring ϕ of S either*

(1) *ϕ does not extend to an L-coloring of H, or*
(2) *ϕ extends to an L-coloring of G.*

Furthermore, Thomassen wondered though whether the dependence of f on the number of components in Theorem 6 could be dropped if certain conditions were satisfied. Specifically, Thomassen conjectured [12] that if all the components of S were just isolated vertices whose pairwise distance in the graph was large, then any precoloring of S always extends. Albertson [1] proved this in 1997. He then conjectured that this generalizes to list-coloring.

Conjecture 1. *There exists D such that the following holds: If G is a plane graph with a 5-list assignment L and $X \subset V(G)$ such that $d(u, v) \geq D$ for all $u \neq v \in X$, then any L-coloring of X extends to an L-coloring of G.*

Dvorak, Lidicky, Mohar, and Postle [7] recently announced a proof of Albertson's conjecture. In addition, Albertson and Hutchinson [3] have generalized Albertson's result to locally planar graphs on surfaces. We prove the following common generalization about list-coloring locally planar graphs on surfaces:

Theorem 8. *Let G be 2-cell embedded in a surface Σ, $ew(G) \geq \Omega(\log g)$ and L be a 5-list-assignment for G. If $X \subset V(G)$ such that $d(u,v) \geq \Omega(\log g(\Sigma))$ for all $u \neq v \in X$, then every L-coloring of X extends to an L-coloring of G.*

In fact, we generalize the theorem to cycles of size at most four, which in turn implies the following theorem:

Theorem 9. *Let G be drawn in a surface Σ with a set of crossings X and L be a 5-list-assignment for G. Let G_X be the graph obtained by adding a vertex v_x at every crossing $x \in X$. If $ew(G_X) \geq \Omega(\log g(\Sigma))$ and $d(v_x, v_{x'}) \geq \Omega(\log g(\Sigma))$ for all $v_x \neq v_{x'} \in V(G_X) \setminus V(G)$, then G is L-colorable.*

4. Exponentially Many Colorings

Thomassen wondered whether a planar graph has many 5-list-colorings. He [13] proved the following.

Theorem 10. *If G is a planar graph and L is a 5-list assignment for G, then G has $2^{|V(G)|/9}$ L-colorings.*

Thomassen [11, 13] then conjectured that Theorem 10 may be generalized to other surfaces. Of course not every graph on other surfaces is 5-list-colorable. Hence, Thomassen conjectured the following.

Conjecture 2. *Let G be a graph embedded in a surface Σ and L is a 5-list-assignment for G. If G is L-colorable, then G has $2^{c|V(G)|}$ L-colorings where c is a constant depending only on g, the genus of Σ.*

Indeed, we show that precoloring a subset of the vertices still allows exponentially many 5-list-colorings where the constant depends only on the genus and the number of precolored vertices. In fact, we show that the dependence on genus and the number of precolored vertices can be removed from the exponent.

Theorem 11. *For every surface Σ there exists a constant $c > 0$ such that following holds: Let G be a graph embedded in Σ and L a 5-list-assignment for G. If G has an L-coloring, then G has at least $2^{c|V(G)|}$ L-colorings of G.*

References

[1] M. Albertson, You can't paint yourself into a corner, J. Combin. Theory Ser. B, 73 (1998), pp. 189–194.

[2] M. Albertson and J. Hutchinson, The three excluded cases of Dirac's map-color theorem, Second International Conference on Combinatorial Mathematics (New York, 1978), pp. 7–17, Ann. New York Acad. Sci. 319, New York Acad. Sci., New York, 1979.

[3] M. Albertson and J. Hutchinson, Extending colorings of locally planar graphs, J. Graph Theory 36 (2001), no. 2, 105–116.

[4] M. DeVos, K. Kawarabayashi, B. Mohar, Locally planar graphs are 5-choosable, J. Combin. Theory Ser. B 98 (2008) 1215–1232.
[5] G. A. Dirac, Map color theorems, Canad. J. Math. 4 (1952) 480–490.
[6] G. A. Dirac, The coloring of maps, J. London Math. Soc. 28 (1953) 476–480.
[7] Z. Dvorak, B. Lidicky, B. Mohar, L. Postle, 5-list-coloring graphs with precolored vertices far apart, manuscript.
[8] T. Gallai, Kritische Graphen I, II, Publ. Math. Inst. Hungar. Acad. Sci. 8 (1963) 165–192 and 373–395.
[9] G. Ringel and J.W.T. Youngs, Solution of the Heawood map-coloring problem, Proc. Nat. Acad. Sci. USA 60 (1968), 438–445.
[10] C. Thomassen, Every planar graph is 5-choosable, J. Combin. Theory Ser. B 62 (1994), 180–181.
[11] C. Thomassen, The number of k-colorings of a graph on a fixed surface, Discrete Math. 306 (2006), 3145–3253.
[12] C. Thomassen, Color-critical graphs on a fixed surface, J. Combin. Theory Ser. B 70 (1997), 67–100.
[13] C. Thomassen, Exponentially many 5-list-colorings of planar graphs, J. Combin. Theory Ser. B 97 (2007), 571–583.

The k-disjoint paths problem in directed planar graphs

Dániel Marx

The new results presented below are joint work with Marek Cygan, Marcin Pilipczuk and Michał Pilipczuk.

A classical problem of combinatorial optimization is finding disjoint paths with specified endpoints:

k-Disjoint Paths
Input: A graph G and k pairs of vertices (s_1, t_1), ..., (s_k, t_k).
Question: Do there exist k pairwise vertex-disjoint paths P_1, ..., P_k such that P_i goes from s_i to t_i?

This problem is NP-hard even on undirected planar graphs if the number k of paths is part of the input. However, for every fixed k, Robertson and Seymour showed that there is a cubic-time algorithm for the problem in general graphs [4]. Their proof uses the structure theory of graphs excluding a fixed minor. Obtaining polynomial running time for fixed k is significantly simpler in the special case of planar graphs; see for example the self-contained presentation of Adler et al. [1].

The problem becomes dramatically harder for directed graphs: it is NP-hard even for $k = 2$ in general directed graphs [3]. Therefore, we cannot expect an analogue of the undirected result of Robertson and Seymour [4] saying that the problem is polynomial-time solvable for fixed k. For directed planar graphs, however, Schrijver gave an algorithm with polynomial running time for fixed k:

Theorem 12 (Schrijver [5])**.** *The k-Disjoint Paths problem on directed planar graphs can be solved in time $n^{O(k)}$.*

The algorithm of Schrijver is based on enumerating all possible homology types of the solution and checking in polynomial time whether there is a solution for a fixed type. Therefore, the running time is mainly dominated by the number $n^{O(k)}$ of homology types. Our main result is improving the running time by removing k from the exponent of n:

Theorem 13. *The k-Disjoint Paths problem on directed planar graphs can be solved in time $f(k) \cdot n^{O(1)}$ for some computable function f.*

In other words, we show that the k-Disjoint Paths problem is fixed-parameter tractable on directed planar graphs.

Our first step is analogous to many undirected algorithms (e.g., of Adler et al. [1]): we try to identify an *irrelevant vertex* whose deletion provably does not change the problem. In particular, we show that if a vertex is surrounded by a sequence of $f(k)$ directed cycles of alternating orientation that do not enclose any terminals, then the vertex can be safely removed. Next we show that if no irrelevant vertex can be removed this way, then the graph has a decomposition into a bounded number of "discs" and "rings" such that the edges connecting these components can be grouped into a bounded number of unidirectional sets. We show that given such a decomposition, we need to consider only a set of homology types whose number is bounded by a function of $f(k)$. This bound requires, among other things, a careful understanding of routing directed paths between the inside and the outside of a ring. Fortunately, this is in very close connection to the existence of disjoint directed cycles of specified homotopy, which is well understood by the results of Ding, Schrijver, and Seymour [2].

References

[1] I. Adler, S. G. Kolliopoulos, P. K. Krause, D. Lokshtanov, S. Saurabh, and D. M. Thilikos. Tight bounds for linkages in planar graphs. In *ICALP (1)*, pages 110–121, 2011.

[2] G. Ding, A. Schrijver, and P. D. Seymour. Disjoint cycles in directed graphs on the torus and the klein bottle. *J. Comb. Theory, Ser. B*, 58(1):40–45, 1993.

[3] S. Fortune, J. Hopcroft, and J. Wyllie. The directed subgraph homeomorphism problem. *Theoret. Comput. Sci.*, 10(2):111–121, 1980.

[4] N. Robertson and P. D. Seymour. Graph minors. XIII. The disjoint paths problem. *J. Combin. Theory Ser. B*, 63(1):65–110, 1995.

[5] A. Schrijver. Finding k disjoint paths in a directed planar graph. *SIAM J. Comput.*, 23(4):780–788, 1994.

Geometric graph theory

János Pach

A graph drawn in the plane with possibly crossing straight-line edges is called a *geometric graph.* If the edges are not necessarily straight, but are represented by any continuous Jordan arcs, then the drawing is called a *topological graph.* It is usually assumed for simplicity that (1) no edge of a topological graph passes through a vertex different from its endpoints, (2) no two edges touch each other (i.e., have precisely one interior point in common, and at this point the two curves do not cross properly), and (3) no three edges share an interior point.

Given a set of n points in the plane with maximum distance 1, connect two of them by a segment if their distance is precisely 1. The resulting geometric graph is called the *diameter graph* associated with the point set. Following Hopf and Pannwitz [3], Paul Erdős observed that the maximum number of edges that the diameter graph of a set of $n \geq 3$ points in the plane can have is n. A beautiful possible generalization of this statement was conjectured by Schur [8].

Conjecture 1. (Schur) *For every $n \geq d+1$, the maximum number of $(d-1)$-dimensional simplices in the graph of diameters of a set of n points in $\mathbf{R}^d$ is n.*

Schur, Perles, Martini, and Kupitz [8] have proved this conjecture for $d = 3$, but it is open for all other values.

Theorem 2. (P.-Morić [5]) *If every pair of $(d-1)$-dimensional simplices in the graph of diameters induced by n points in $\mathbf{R}^d$ share at least $d-2$ vertices, then Schur's conjecture is true.*

It follows from a result of Dolnikov [2] that the condition in Theorem 2 holds for $d = 3$, and it is possible that it is also true for every finite point set in d-dimensional space and for every $d \geq 3$. We cannot even prove the following weaker

Conjecture 3. [5] *For any $d \geq 4$, every pair of $(d-1)$-dimensional simplices in a graph of diameters in $\mathbf{R}^d$ share at least one vertex.*

Erdős noticed that in the plane his theorem mentioned above can be strengthened as follows: Every geometric graph with n vertices, which contains no 2 disjoint edges (that is, no 2 edges that do not even share an endpoint), has at most n edges. Of course, this statement does not generalize to topological graphs, because it is easy to draw a complete graph of n vertices in the plane so that no pair of its edges are disjoint. However, the statement may remain true for topological graphs satisfying the condition that every pair of edges have at most one point in common. A topological graph with this property is said to be *simple*.

Conjecture 4. (Conway's thrackle conjecture [11]) *Every simple topological graph with n vertices, which contains no 2 disjoint edges, has at most n edges.*

Problem 5. *Is it true that every simple topological graph with n vertices, which contains no k disjoint edges, has at most $O(kn)$ edges?*

The answer is affirmative for *convex* geometric graphs, that is, for geometric graphs whose vertices form the vertex set of a convex n-gon; see [4]. It was proved by G. Tóth [10] that the maximum number of edges that a geometric graph of n vertices can have without containing k disjoint edges is $O(k^2 n)$. See [1] and [6] for the first bounds linear in n.

Theorem 6. (P.-Tóth [7]) *Every simple topological graph with n vertices, which contains no k disjoint edges, has at most $O(n \log^{4k} n)$ edges.*

It is an easy consequence of Theorem 6 that every complete simple topological graph with n vertices has at least (a positive) constant times $\log n / \log \log n$ pairwise disjoint edges. This was improved by A. Suk [9].

Theorem 7. (Suk) *Every complete simple topological graph with n vertices has at least constant times $n^{1/3}$ pairwise disjoint edges.*

If the answer to Problem 5 is yes, the bound in Theorem 7 can be improved to constant times n. A. Ruiz Vargas and R. Fulek have recently come up with an alternative proof of Theorem 7. Their method offers some hope that the bound in Theorem 7 can be improved to at least $\Omega(n^{1/2})$. It would be sufficient to prove that the answer to Problem 5 is yes for simple topological graphs that can be drawn on the surface of a circular cylinder with the property that every straight line parallel to the axis of the cylinder intersects every edge at most once.

References

[1] N. Alon and P. Erdős, *Disjoint edges in geometric graphs*, Discrete Comput. Geom. **4** (1989), 287–290.

[2] V. L. Dolnikov, *Some properties of graphs of diameters*, Discrete Comput. Geom. **24** (2000), 293–299.

[3] H. Hopf and E. Pannwitz, *Aufgabe Nr. 167*, Jahresbericht d. Deutsch. Math.-Verein. **43** (1934), 114.

[4] Y. S. Kupitz, Extremal Problems of Combinatorial Geometry, *Lecture Notes Series* **53**, Aarhus University, Denmark, 1979.

[5] F. Morić and J. Pach, *Remarks on Schur's conjecture*, submitted.

[6] J. Pach and J. Törőcsik, *Some geometric applications of Dilworth's theorem*, Discrete Comput. Geom. **12** (1994), 1–7.

[7] J. Pach and G. Tóth, *Disjoint edges in topological graphs*, J. Comb. **1** (2010), 335–344.

[8] Z. Schur, M. A. Perles, H. Martini, and Y. S. Kupitz, *On the number of maximal regular simplices determined by n points in $\mathbb{R}^d$*, in: Discrete and Computational Geometry, The Goodman-Pollack Festschrift (Aronov et al., eds.), Algorithms Combin. **25**, Springer, Berlin, 2003, 767–787.

[9] A. Suk, *Disjoint edges in complete topological graphs*, in: Proc. 28th Symposium on Computational Geometry (SoCG'12), ACM Press, New York, 2012, 383–386.

[10] G. Tóth, *A note on geometric graphs*, Journal of Combinatorial Theory, Ser. A **89** (2000), 126–132.

[11] D. R. Woodall, *Thrackles and deadlock*, in: Combinatorics, Proc. Conf. Comb. Math. (D. Welsh, ed.), Academic Press, London, 1971, 335–347.

Grid immersion and multicommodity flows

Paul Seymour

The multicommodity flow problem is, we are given k pairs (s_i, t_i) of vertices of a graph G, and we wish to determine whether there are k flows in G, where the ith flow is between s_i and t_i and has total value 1, and for every edge the sum of the absolute values of the flows through the edge is at most 1. For fixed integer p, we may also restrict the flow values on each edge to be rational with denominator p. A problem is *critical* if it is not feasible, but contracting any edge makes it feasible. We are concerned with whether (for fixed k, p) all critical instances have bounded size.

For $(p, k) = (1, 2)$ there is no bound. It turns out that for all fixed k, p with $p > 1$ there is a bound. (This is a consequence of a theorem about immersing a grid in G, due to M. Chudnovsky, Z. Dvorak, T. Klimosova and the speaker.) Since the talk, this has been improved; now I think I can prove that for all fixed k, there is a bound for all $p > 1$ independent of p, and also the same bound holds when no value of p is specified (ie we permit arbitrary values).

Groups and graph limits

Miklós Abért

Benjamini and Schramm introduced a convergence notion for sequences of finite networks with an absolute degree bound. This notion turned out to be useful not only in graph theory, but in group theory and geometry as well. Surprisingly, it turned out that energy flows both ways, and group theory comes into the picture when one aims to understand very large sparse networks. In the talk I introduced the basic notions, surveyed the known results and discussed some of the major open problems. In particular, I discussed how the Lück Approximation Theorem follows from the fact that the normalized rank of the adjacency matrix is continuous in the Benjamini-Schramm topology. A recent result of myself with Hubai about the distribution of the roots of the chromatic polynomial, and its very recent generalization by Csíkvári and Frenkel were also discussed.

Canonical tree decomposition into highly connected pieces

Johannes Carmesin
(joint work with R. Diestel, F. Hundertmark, M. Stein)

Considering systems of separations in a graph that separate every pair of a given set of vertex sets that are themselves not separated by these separations, we determine conditions under which such a separation system contains a nested subsystem that still separates those sets and is invariant under the automorphisms of the graph.

As an application, we show that the k-blocks – the maximal vertex sets that cannot be separated by at most k vertices – of a graph G live in distinct parts of a suitable tree decomposition of G of adhesion at most k, whose decomposition tree is invariant under the automorphisms of G. This extends recent work of Dunwoody and Krön and, like theirs, generalizes a similar theorem of Tutte for $k = 2$.

Under mild additional assumptions, which are necessary, our decompositions can be combined into one overall tree decomposition that distinguishes, for all k simultaneously, all the k-blocks of a finite graph.

We think that it is worth investigating how the existence of a k-block is related to other properties of the graph. In particular, we want to know which average degree d is needed to force a k-block. We can show that $2k - 1 \leq d \leq 3k$ but we do not know what the precise value of d is.

Excluding pairs of graphs

Maria Chudnovsky
(joint work with Alex Scott, Paul Seymour)

Let G be a graph. The *complement* G^c of G is the graph with vertex set $V(G)$, and such that two vertices are adjacent in G if and only if they are non-adjacent in G^c. For a graph G and a set of graphs $\mathcal{H}$, we say that G is *$\mathcal{H}$-free* if no induced subgraph of G is isomorphic to a member of $\mathcal{H}$. Given an integer $P > 0$, a graph G, and a set of graphs $\mathcal{F}$, we say that G *admits an $(\mathcal{F}, P)$-partition* if the vertex set of G can be partitioned into P subsets $X_1, \ldots, X_P$, so that for every $i \in \{1, \ldots, P\}$, either $|X_i| = 1$, or the subgraph of G induced by X_i is $\{F\}$-free for some $F \in \mathcal{F}$.

The main result of [2] is the following:

Theorem 14. *For every pair (H, J) of graphs such that H is the disjoint union of two non-null graphs H_1 and H_2, and J^c is the disjoint union of two non-null graphs J_1^c and J_2^c, there exists an integer $P > 0$ such that every $\{H, J\}$-free graph has an $(\{H_1, H_2, J_1, J_2\}, P)$-partition.*

A *clique* in a graph is a set of pairwise adjacent vertices, and a stable set is a set of pairwise non-adjacent vertices. 14 implies a number of previously known theorems. One of the consequences is a famous theorem of Ramsey, that states that every graph with no large clique and no large stable set has bounded size. Another one is the main result of [3], which we describe next.

Let $k > 0$ be an integer, and say a graph is *k-split* if its vertex set is the union of two sets A, B, where A contains no clique of size $k+1$, and B contains no stable set of size $k+1$.

Theorem 15. [3] *For every graph H_1 that is a disjoint union of cliques, and every complete multipartite graph H_2, there exists k such that every $\{H_1, H_2\}$-free graph is k-split.*

A *cograph* is a graph obtained from single vertices by repeatedly taking disjoint unions and disjoint unions in the complement. For every cograph there is a parameter measuring its complexity, called its *height*. Given a graph G and a pair of graphs H_1, H_2, we say that G is *$\{H_1, H_2\}$-split* if $V(G) = X_1 \cup X_2$, where the subgraph of G induced by X_i is $\{H_i\}$-free for every $i \in \{1, 2\}$. Another consequence of 14 is that for every integer $k > 0$ and pair $\{H, J\}$ of cographs each of height $k+1$, where neither of H, J^c is connected, there exists a pair of cographs $(\tilde{H}, \tilde{J})$, each of height k, where neither of $\tilde{H}^c, \tilde{J}$ is connected, such that every $\{H, J\}$-free graph is $\{\tilde{H}, \tilde{J}\}$-split.

Let us now go back to 15. The interest in that stems from the attempt to understand the so-called "heroic sets" in graphs. The *co-chromatic number* of a graph G is the smallest number of cliques and stable sets with union $V(G)$. Let us say a set of graphs $\mathcal{H}$ is *heroic* if there exists k such that every $\mathcal{H}$-free graph has cochromatic number at most k. It is not difficult to see (and it is also a result of [3]) that every finite heroic set of graphs contains a compelte multipartite graph, a graph which is the disjoint union of cliques, a forest, and the complement of a forest. Thus 15 is a step on the way to proving the converse, which is a conjecture in [3]:

Theorem 16. Conjecture *A finite set of graphs is heroic if and only if it contains a clique partition graph, a complete multipartite graph, a forest, and the complement of a forest.*

We remark that using 15 it is easy to see that 16 is equivalent to a well-known old conjecture independently proposed by Gyárfás [4] and Sumner [5]) (which we state here in the language of heroic sets):

Theorem 17. Conjecture: *For every complete graph K and every tree T, the set $\{K, T\}$ is heroic.*

Finally, let us mention another application of 14. A *tournament* is a complete graph with directions on edges. A set $X \subseteq V(G)$ is *transitive* if $G|X$ has no directed cycles. The *chromatic number* of G is the smallest integer k for which $V(G)$ can be partitioned into k transitive subsets. Given tournaments H_1 and H_2 with disjoint vertex sets, we write $H_1 \Rightarrow H_2$ to mean the tournament H with $V(H) = V(H_1) \cup V(H_2)$, and such that $H|V(H_i) = H_i$ for $i = 1, 2$, and every vertex of $V(H_1)$ is adjacent to (rather than from) every vertex of $V(H_2)$. Similarly to graphs, a tournament H is a *hero* if there exists c (depending on H) such that every H-free tournament has chromatic number at most c. One of the results of [1] is a complete characterization of all heroes. An important and the most difficult step toward that is the following:

Theorem 18. *If H_1 and H_2 are heroes, then so is $H_1 \Rightarrow H_2$.*

It turns out that translating the proof of 14 into the language of tournaments gives a proof of 18 that is much simpler than the one in [1].

References

[1] E. Berger, K. Choromanski, M. Chudnovsky, J. Fox, M. Loebl, A. Scott, P. Seymour and S. Thomassé, *Tournaments and colouring*, Journal of Combinatorial Theory, Series B, **103** (2013), 1-20.

[2] M. Chudnovsky, A. Scott and P. Seymour *Excluding pairs of graphs*, manuscript.

[3] M. Chudnovsky and P. Seymour, *Extending the Gyárfás-Sumner conjecture*, submitted for publication.

[4] A. Gyárfás, *On Ramsey covering-numbers*, *Coll. Math. Soc. János Bolyai*, in *Infinite and Finite Sets*, North Holland/American Elsevier, New York (1975), **10**.

[5] D.P. Sumner, *Subtrees of a graph and chromatic number*, in *The Theory and Applications of Graphs*, (G. Chartrand, ed.), John Wiley & Sons, New York (1981), 557–576.

Ramsey numbers of cubes versus cliques

David Conlon
(joint work with Jacob Fox, Choongbum Lee, Benny Sudakov)

For graphs G and H, the *Ramsey number* $r(G,H)$ is defined to be the smallest natural number N such that every red/blue edge-coloring of the complete graph K_N on N vertices contains a red copy of G or a blue copy of H.

One obvious construction, noted by Chvátal and Harary [11], which gives a lower bound for these numbers is to take $\chi(H)-1$ disjoint red cliques of size $|G|-1$ and to connect every pair of vertices which are in different cliques by a blue edge. If G is connected, the resulting graph contains neither a red copy of G nor a blue copy of H, so that $r(G,H) \geq (|G|-1)(\chi(H)-1)+1$. Burr [5] strengthened this bound by noting that if $\sigma(H)$ is the smallest color class in any $\chi(H)$-coloring of the vertices of H, we may add a further red clique of size $\sigma(H)-1$, obtaining

$$r(G,H) \geq (|G|-1)(\chi(H)-1)+\sigma(H).$$

Following Burr and Erdős [5, 7], we say that a graph G is *H-good* if the Ramsey number $r(G,H)$ is equal to this bound. If $\mathcal{G}$ is a family of graphs, we say that $\mathcal{G}$ is H-good if all sufficiently large graphs in $\mathcal{G}$ are H-good. When $H = K_s$, where $\sigma(K_s) = 1$, we simply say that G or $\mathcal{G}$ is *s-good*.

The classical result on Ramsey goodness, which predates the definition, is the theorem of Chvátal [10] showing that all trees are s-good for any s. On the other hand, the family of trees is not H-good for every graph H. For example [9], a construction of $K_{2,2}$-free graphs due to Brown [4] allows one to show that there is a constant $c < \frac{1}{2}$ such that

$$r(K_{1,t}, K_{2,2}) \geq t + \sqrt{t} - t^c$$

for t sufficiently large. This is clearly larger than $(|K_{1,t}|-1)(\chi(K_{2,2})-1)+\sigma(K_{2,2}) = t+2$.

In an effort to determine what properties contribute to being Ramsey good, Burr and Erdős [6, 7] conjectured that if Δ was fixed then the family of graphs with bounded maximum degree Δ should be s-good for any s (and perhaps even H-good for all H). This conjecture holds for bipartite graphs H [8] but is false in general, as shown by Brandt [3]. He proved that for $\Delta \geq \Delta_0$ almost every Δ-regular graph on a sufficiently large number of vertices is not even 3-good. His result (and a similar result in [12]) actually prove something stronger, namely that if a graph G has strong expansion properties then it cannot be 3-good.

On the other hand, it has been shown if a family of graphs exhibits poor expansion properties then it will tend to be good [1, 12]. To state the relevant results, we define the *bandwidth* of a graph G to be the smallest number ℓ for which there exists an ordering $v_1, \dots, v_n$ of the vertices of G such that every edge v_iv_j satisfies $|i-j| \leq \ell$. This parameter is known to be intimately linked to the expansion properties of the graph. In particular, any bounded-degree graph with poor expansion properties will have sublinear bandwidth [2].

The first such result, shown by Burr and Erdős [7], states that for any fixed ℓ the family of connected graphs with bandwidth at most ℓ is s-good for any s. This result was recently extended by Allen, Brightwell and Skokan [1], who showed that the set of connected graphs with bandwidth at most ℓ is H-good for every H. Their result even allows the bandwidth ℓ to grow at a reasonable rate with the size of the graph G. If G is known to have bounded maximum degree, their results are particularly strong, saying that for any Δ and any fixed graph H there exists a constant c such that if G is a graph on n vertices with maximum degree Δ and bandwidth at most cn then G is H-good.

Many of the original problems of Burr and Erdős [7] have now been resolved [12], but one that remains open is to determine whether the family of hypercubes is s-good for every s. The *hypercube* Q_n is the graph on vertex set $\{0,1\}^n$ where two vertices are connected by an edge if and only if they differ in exactly one coordinate. This family of graphs has sublinear bandwidth but does not have bounded degree, so the result of Allen, Brightwell and Skokan does not apply.

To get a first bound for $r(Q_n, K_3)$, note that a simple greedy embedding implies that any graph with maximum degree d and at least $dn + 2^n$ vertices has a copy of Q_n in its complement. Suppose now that the edges of a complete graph have been 2-colored in red and blue and there is neither a blue triangle nor a red copy of Q_n. Then, since the blue neighborhood of any vertex forms a red clique, the maximum degree in blue is at most $2^n - 1$. Hence, the graph must have at most $(2^n - 1)n + 2^n < 2^n(n + 1)$ vertices. We may therefore conclude that $r(Q_n, K_3) \leq 2^n(n + 1)$.

It is not hard to extend this argument to show that for any s there exists a constant c_s such that $r(Q_n, K_s) \leq c_s 2^n n^{s-2}$. This is essentially the best known bound. We improve this bound, obtaining the first upper bound which is within a constant factor of the lower bound.

Theorem 19. *For any natural number $s \geq 3$, there exists a constant c_s such that*

$$r(Q_n, K_s) \leq c_s 2^n.$$

The original question of Burr and Erdős [7] relates to s-goodness but it is natural to also ask whether the family of cubes is H-good for any H. For bipartite H, this follows directly from a result of Burr, Erdős, Faudree, Rousseau and Schelp [8]. Our result clearly implies that for any H, there is a constant c_H such that $r(Q_n, H) \leq c_H 2^n$.

References

[1] P. Allen, G. Brightwell and J. Skokan, Ramsey-goodness – and otherwise, *Combinatorica*, to appear.

[2] J. Böttcher, K. Pruessman, A. Taraz and A. Würfl, Bandwidth, expansion, treewidth, separators, and universality for bounded degree graphs, *European J. Comb.* **31** (2010), 1217–1227.

[3] S. Brandt, Expanding graphs and Ramsey numbers, available at Freie Universität, Berlin preprint server, ftp://ftp.math.fu-berlin.de/pub/math/publ/pre/1996/pr-a-96-24.ps (1996).

[4] W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **9** (1966), 281–285.

[5] S.A. Burr, Ramsey numbers involving graphs with long suspended paths, *J. London Math. Soc. (2)* **24** (1981), 405–413.
[6] S. A. Burr, What can we hope to accomplish in generalized Ramsey theory?, *Discrete Math.* **67** (1987), 215–225.
[7] S.A. Burr and P. Erdős, Generalizations of a Ramsey-theoretic result of Chvátal, *J. Graph Theory* **7** (1983), 39–51.
[8] S. A. Burr, P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, The Ramsey number for the pair complete bipartite graph–graph of limited degree, in: Graph theory with applications to algorithms and computer science (Kalamazoo, Mich., 1984), 163–174, Wiley, New York, 1985.
[9] S. A. Burr, P. Erdős, R. J. Faudree, C. C. Rousseau and R. H. Schelp, Some complete bipartite graph–tree Ramsey numbers, *Ann. Discrete Math.* **41** (1989), 79–90.
[10] V. Chvátal, Tree-complete graph Ramsey numbers, *J. Graph Theory* **1** (1977), 93.
[11] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs, III. Small off-diagonal numbers, *Pacific J. Math.* **41** (1972), 335–345.
[12] V. Nikiforov and C. C. Rousseau, Ramsey goodness and beyond, *Combinatorica* **29** (2009), 227–262.

Sampling limits of finite trees

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(joint work with Gábor Tardos)

1. Convergence of finite graphs

Let G be a finite, simple graph and $\phi : [n] \to V(G)$ be a map. Then $V(G_\phi) = [n]$ and $(i,j) \in E(G_\phi)$ if $(\phi(i), \phi(j)) \in E(G)$. Hence, we have a probability distribution μ^n_G on $\mathcal{G}^n$, the set of finite graphs with vertex set $[n]$. We say that $\{G_k\}^\infty_{k=1}$ is a convergent sequence, if for any $n \geq 1$, $\{\mu^n_{G_k}\}$ converges in the weak topology. Let $W = [0,1]^2 \to [0,1]$ be a symmetric function (a graphon) (see [2]) Let $\phi : [n] \to [0,1]$. Then the random graph G^n_W is constructed the following way. The vertex set is $[n]$ and we draw the edge (i,j) with probability $W(\phi(i), \phi(j))$. Hence we define a probability measure μ^n_W on $\mathcal{G}^n$. We say that $\{G_k\}^\infty_{k=1}$ converges to W, if for any $n \geq 1$, $\lim_{k\to\infty} \mu^n_{G_k} = \mu^n_W$ weakly.

2. Metrics on function spaces

Let us consider the space (up to zero measure perturbation) of bounded measurable functions $L^\infty([0,1]^2, \lambda^2)$ taking values in $[0,1]$. We define three metrics on this space.

- $\Box_1(f,g)$ is defined as the supremal ϵ such that there exists $T_\epsilon \subset [0,1]$, $\lambda(T_\epsilon) \leq \epsilon$, satisfying
$$|f(x,y) - g(x,y)| \leq \epsilon$$
almost everywhere, if $(x,y) \in ([0,1]\backslash T_\epsilon) \times ([0,1]\backslash T_\epsilon)$.
- $d_1(f,g) = \int_0^1 \int_0^1 |f-g| dxdy$.
- $d_\Box(f,g) = \sup_{S,T\subset[0,1]} |\int_S \int_T (f-g) dxdy|$

Clearly, $d_\Box(f,g) \leq d_1(f,g) \leq 3\Box_1(f,g)$.

3. Distances of mm-spaces

By an mm-space (metric measure space) we mean a Polish (complete, separable) space of diameter at most 1, equipped with a probability measure. We denote the set of such spaces (up to isometry) by χ. We can define three metric structures on χ using the distances of the previous section. Let $(X, \mu_X, d_X), (Y, \mu_Y, d_Y) \in \chi$. We denote by $MPM(X)$ the set of all measure preserving transformations from $([0,1], \lambda)$ to X. Then

- $\underline{\square}_1(X, Y) = \inf_{\Psi \in MPM(X), \Phi \in MPM(Y)} \square_1(\Psi^{-1}(d_X), \Phi^{-1}(d_Y))$
- $\delta_1(X, Y) = \inf_{\Psi \in MPM(X), \Phi \in MPM(Y)} d_1(\Psi^{-1}(d_X), \Phi^{-1}(d_Y))$
- $\delta_\square(X, Y) = \inf_{\Psi \in MPM(X), \Phi \in MPM(Y)} d_\square(\Psi^{-1}(d_X), \Phi^{-1}(d_Y))$

Note that we never used the fact that d is a distance function, so the same way we can define the distances of arbitrary $[0,1]$-valued measurable functions on a probability measure space. Since any graph can be viewed as $[0,1]$-valued functions on a finite probability measure space, the graph distances

$$\underline{\square}_1(G, H), \delta_1(G, H), \delta_\square(G, H)$$

and even the distances from a graphon

$$\underline{\square}_1(G, W), \delta_1(G, W), \delta_\square(G, W)$$

are well-defined.

Theorem 20. [2]*[Lovasz-Szegedy]* $\{G_k\}_{k=1}^\infty$ *converges to* W *if and only if* $\lim_{k\to\infty} \delta_\square(G_k, W) = 0$.

Theorem 21. [4]*[Pikhurko] If* $\lim_{k\to\infty} \delta_1(G_k, W) = 0$*, then* W *is random-free, that is, it takes only the values* 0 *and* 1 *(almost everywhere). Also, if* $\{G_k\}_{k=1}^\infty$ *converges to* W*, and* W *is random-free then* $\lim_{k\to\infty} \delta_1(G_k, W) = 0$.

4. Sampling mm-spaces

Let $(X, \mu_X, d_X) \in \chi$. Pick a μ_X-random function $\phi : [n] \to X$. Let $\hat{d}(i,j) = d(\phi(i), \phi(j)$. In this way, we obtain a probability measure μ_X^n on M_n, where M_n is the compact space of pseudo-metrics on $[n]$ with diameter at most 1. Similarly to graphs, we can talk about the (sampling) convergence of mm-spaces. We say that $\{X_k\}$ converges if the weak limit $\lim \mu_{X_k}^n$ exists for any $n \geq 1$. Note that a finite graph G can be regarded as a metric space, where $d_G(x,y) = 1/2$ if x, y is adjacent vertices and $d_G(x,y) = 1$ otherwise.

Theorem 22. [1]*[Gromov]* $\mu_X^n = \mu_Y^n$ *for any* $n \geq 1$ *if and only if* X *and* Y *are isomorphic, which is equivalent to* $\underline{\square}_1(X, Y) = 0$.

5. The Gromov-Prohorov metric

Let μ, ν be probability measures on the mteric space (X, d). The Prohorov distance is given as follows.

$$d_{PR}(\mu, \nu) := \inf(\epsilon > 0 \mid \mu(A) \leq \nu(A^\epsilon) + \epsilon, \text{ for any Borel-set } A)$$

where A^ϵ is the ϵ-neighborhood of the set A. The Gromov-Prohorov metric on χ is defined as

$$d_{GP}(X,Y) := \inf_{f,g} d_{PR}(f_\star(\mu_1), g_\star(\mu_2)),$$

where the infimum is taken for all isometries of X and Y to a common metric space.

Theorem 23. [3] *[Löhr]* $d_{GP} \leq \square_1 \leq 2d_{GP}$

6. Sampling finite trees

6.1. **Metric sampling.** We consider the finite tree T as a connected metric space with normalized diameter and the natural uniform measure.

Result 1. $\{T_n\}_{n=1}^{\infty}$ *is convergent in sampling if and only if it converges in the* δ_1*-metric.*

Result 2. *The limit objects of metric samplings are Polish trees with decorations. That is*

- *A Polish tree* T.
- *The direct product* $T \times (0,1]$ *(hanging lines for each point of* T*).*
- *A probability measure on the set* $(T \times [0,1])$.

References

[1] M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics **152** Birkhauser, 1999

[2] L. Lovász and B. Szegedy, *Limits of dense graph sequences*, J.Combin.Theory B **96** (2006), 933–957.

[3] W. Lohr, *Equivalence of Gromov-Prohorov- and Gromov's Box-Metric on the Space of Metric Measure Spaces* (preprint)

[4] O. Pikhurko, *An analytic approach to stability*, Discrete Math. **310** (2010), 2951–2964.

Chromatic number, clique subdivisions, and the conjectures of Hajós and Erdős-Fajtlowicz

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(joint work with Choongbum Lee, Benny Sudakov)

A *subdivision* of a graph H is any graph formed by replacing edges of H by internally vertex disjoint paths. This is an important notion in graph theory, e.g., the celebrated theorem of Kuratowski uses it to characterize planar graphs. For a graph G, we let $\sigma(G)$ denote the largest integer p such that G contains a subdivision of a complete graph of order p. Clique subdivisions in graphs have been extensively studied and there are many results which give sufficient conditions for a graph G to have large $\sigma(G)$. For example, Bollobás and Thomason [5], and Komlós and Szemerédi [10] independently proved that every graph of average degree at least d has $\sigma(G) \geq cd^{1/2}$ for some absolute constant c. Motivated by a conjecture of Erdős, in [2] the authors further showed that when $d = \Omega(n)$ in the above subdivision one can choose all paths to have length two. Similar result for subdivisions of general graphs with $O(n)$ edges (a clique of order $O(\sqrt{n})$ clearly satisfies this) was obtained in [9].

For a given graph G, let $\chi(G)$ denote its chromatic number. A famous conjecture made by Hajós in 1961 states that $\sigma(G) \geq \chi(G)$. Dirac [7] proved that this conjecture is true for all $\chi(G) \leq 4$, but in 1979, Catlin [6] disproved the conjecture for all $\chi(G) \geq 7$. Subsequently, several researchers further studied this problem. On the negative side, by considering random graphs, Erdős and Fajtlowicz [8] in 1981 showed that the conjecture actually fails for almost all graphs. On the positive side, recently Kühn and Osthus [11] proved that all graphs of girth at least 186 satisfy Hajós' conjecture. Thomassen [12] studied the relation of Hajós' conjecture to several other problems of graph theory such as Ramsey theory, maximum cut problem, etc., and discovered many interesting connections.

In this paper, we revisit Hajós' conjecture and study to what extent the chromatic number of a graph can exceed the order of its largest clique subdivision. Let $H(n)$ denote the maximum of $\chi(G)/\sigma(G)$ over all n-vertex graphs G. The example of graphs given by Erdős and Fajtlowicz which disprove Hajós' conjecture in fact has $\sigma(G) = \Theta(n^{1/2})$ and $\chi(G) = \Theta(n/\log n)$. Thus it implies that $H(n) = \Omega(n^{1/2}/\log n)$. In [8], Erdős and Fajtlowicz conjectured that this bound is tight up to a constant factor so that $H(n) = O(n^{1/2}/\log n)$. Our first theorem verifies this conjecture.

Theorem 24. *There exists an absolute constant C such that $H(n) \leq Cn^{1/2}/\log n$ for $n \geq 2$.*

The proof shows that we may take $C = 10^{120}$, although we do not try to optimize this constant. For the random graph $G = G(n,p)$ with $0 < p < 1$ fixed, Bollobás and Catlin [4] determined $\sigma(G)$ asymptotically almost surely and later Bollobás [3] determined $\chi(G)$ asymptotically almost surely. These results imply, by picking the optimal choice $p = 1 - e^{-2}$, the lower bound $H(n) \geq (\frac{1}{e\sqrt{2}} - o(1))n^{1/2}/\log n$.

For a graph G, let $\alpha(G)$ denote its independence number. The theorem above actually follows from the study of the relation between $\sigma(G)$ and $\alpha(G)$, which might be of independent interest. Let $f(n, \alpha)$ be the minimum of $\sigma(G)$ over all graphs G on n vertices with $\alpha(G) \leq \alpha$.

Theorem 25. *There exist absolute positive constants c_1 and c_2 such that the following holds.*

(1) *If $\alpha < 2\log n$, then $f(n,\alpha) \geq c_1 n^{\frac{\alpha}{2\alpha-1}}$, and*
(2) *if $\alpha = a \log n$ for some $a \geq 2$, then $f(n,\alpha) \geq c_2\sqrt{\frac{n}{a\log a}}$.*

Note that for $\alpha = 2\log n$, both bounds from the first and second part gives $f(n,\alpha) \geq \Omega(\sqrt{n})$. Moreover, both parts of this theorem establish the correct order of magnitude of $f(n,\alpha)$ for some range of α. For $\alpha = 2$, it can be shown that in the triangle-free graph constructed by Alon [1], every set of size at least $37n^{2/3}$ contains at least n edges. This implies that the complement of this graph has independence number 2 and the largest clique subdivision of size $t < 37n^{2/3}$.

Indeed, if there is a clique subdivision of order $t \geq 37n^{2/3}$, then between each of the at least n pairs of nonadjacent vertices among the t vertices of the subdivided clique, there is at least one additional vertex along the path between them in the subdivision. However, this would require at least $t + n$ vertices in the n-vertex graph, a contradiction. On the other hand, for $\alpha = \Theta(\log n)$, by considering $G(n, p)$ with constant $0 < p < 1$, one can see that the second part of the above theorem is tight up to the constant factor. Even for $\alpha = o(\log n)$, by considering the complement of $G(n, p)$ for suitable $p \ll 1$, one can easily verify that there exists an absolute constant c' such that $f(n, \alpha) \leq O(n^{\frac{1}{2}+\frac{c'}{\alpha}})$.

Our second theorem can also be viewed as a Ramsey-type theorem which establishes an upper bound on the Ramsey number of a clique subdivision versus an independent set.

References

[1] N. Alon, Explicit Ramsey graphs and orthonormal labelings, *Electron. J. Combin.* **1** (1994), R12.

[2] N. Alon, M. Krivelevich and B. Sudakov, Turan numbers of bipartite graphs and related Ramsey-type questions, *Combinatorics, Probability and Computing* **12** (2003), 477–494.

[3] B. Bollobás, The chromatic number of random graphs, *Combinatorica* **8** (1988), 49–55.

[4] B. Bollobás and P. A. Catlin, Topological cliques of random graphs, *J. Combin. Theory Ser. B* **30** (1981), 224–227.

[5] B. Bollobás and A. Thomason, Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs, *European J. Combin.* **19** (1998), 883–887.

[6] P. Catlin, Hajós' graph-coloring conjecture: variations and counterexamples, *J. Combin. Theory Ser. B* **26** (1979), 268–274.

[7] G. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.* **27** (1952), 85–92.

[8] P. Erdős and S. Fajtlowicz, On the conjecture of Hajós, *Combinatorica* **1** (1981), 141–143.

[9] J. Fox and B. Sudakov, Density theorems for bipartite graphs and related Ramsey-type results, *Combinatorica* 29 (2009), 153–196.

[10] J. Komlós and E. Szemerédi, Topological cliques in graphs. II, *Combin. Probab. Comput.* **5** (1996), 79–90.

[11] D. Kühn and D. Osthus, Topological minors in graphs of large girth, *J. Combin. Theory Ser. B* **86** (2002), 364–380.

[12] C. Thomassen, Some remarks on Hajós' conjecture, *J. Combin. Theory Ser. B* **93** (2005), 95–105.

Hyperbolic graphs

Matthias Hamann

Ever since Gromov's paper [4] on hyperbolic groups appeared, the interactions between hyperbolic graphs and trees, the simplest class of hyperbolic graphs, are studies. Before we turn our attention to these connections, we define hyperbolic graphs.

Let G be a graph. A *geodesic triangle* is a subgraph of G consisting of three vertices and three geodesics, i.e. shortest paths, one between each two of the vertices. The geodesics are the *sides* of the geodesic triangle. The graph G is called *hyperbolic* if there is a $\delta \geq 0$ such that for all triangles any point on one of its sides has distance at most δ to some point of the union of the other two sides.

A ray R is called *geodesic* if $d_R(x, y) = d_G(x, y)$ for all vertices x, y on R. We call two geodesic rays $x_1x_2\ldots$ and $y_1y_2\ldots$ *equivalent* if there is an $m \in \mathbb{N}$ such that for infinitely many $i \in \mathbb{N}$ there is a $j \in \mathbb{N}$ with $d(x_i, y_j) \leq M$. Equivalence of geodesic rays is an equivalence relation in hyperbolic graphs and the set of its equivalence classes forms the hyperbolic boundary.

Let us return to the connections between arbitrary hyperbolic graphs and trees: One such connection was investigated by Buyalo et al. [2] and deals with quasi-isometric embeddings of hyperbolic groups (and more generally of visual hyperbolic spaces) into the product of binary metric trees, where the number of trees you need for the product depends on the topological dimension of the hyperbolic boundary.

If we want to use trees to capture various informations about the hyperbolic graph and its boundary, then it is (usually) more suitable to look at only one tree and not on a product of trees. There are results in which the local structure of the tree resembles the local structure of the hyperbolic graph. Here, we mention a result of Gromov [4] who constructed for a finite subset of the completion of a δ-hyperbolic graph a tree in the graph whose completion contains the given set and all whose geodesics between elements of the finite set are quasi-geodesics in the hyperbolic graph. Another result in the same direction is due to Benjamini and Schramm [1]: they show that if a locally finite hyperbolic graph has exponential growth then the graph has a subtree with exponential growth such that the embeddingof the tree is bilipschitz.

There are also constructions of trees that try to capture the boundary of a given hyperbolic space in a good way. For locally finite graphs, several ideas for constructions of such trees can already be found in Gromov's article [4]. They have been elaborated on in [3]. The constructed trees capture the boundary of the hyperbolic graph in that there is a continuous map from their own boundary onto that of the graph. For hyperbolic graphs of bounded degree, these maps are finite-to-one. However, the constructed trees are abstract trees, that is, they are not necessarily subtrees of the hyperbolic graph.

We combine these last two approaches [5, 6] in that we construct in every locally finite hyperbolic graphs whose hyperbolic boundary has finite Assouad dimension a rooted spanning tree that gives us a good picture of the graph and also of its hyperbolic boundary: the rays in the tree starting at the root are (γ, c)-quasi-geodesics in the graph and every geodesic ray of the graph lies eventually in a Δ-neighbourhood of some ray of the tree for some $\Delta \in \mathbb{N}$. Furthermore, the embedding of the tree extends continuously to the boundaries such that every boundary point of the graph has at least one but a bounded number of inverse images. For both aspects of the tree – representing the graph and representing the hyperbolic boundary in a good way – the constants we obtain depend only on the hyperbolicity constant δ and on the Assouad dimension of the hyperbolic boundary

In addition, we show that the condition on at least some dimension of the hyperbolic boundary is necessary: if the hyperbolic boundary of a locally finite hyperbolic graph G has topological dimension at least n, then for every spanning tree T of G there is a boundary point of G that has at least $n+1$ inverse images under the canonical extension of the embedding of T into G on the boundaries.

References

[1] I. Benjamini and O. Schramm, *Every graph with a positive Cheeger constant contains a tree with a positive Cheeger constant*, Geom. Funct. Anal. **7** (1997), no. 3, 403–419.

[2] S. Buyalo, A. Dranishnikov, and V. Schroeder, *Embedding of hyperbolic groups into products of binary trees*, Invent. Math. **169** (2007), no. 1, 153–192.

[3] M. Coornaert and A. Papadopoulos, *Symbolic dynamics and hyperbolic groups*, Springer Lecture Notes, vol. 1539, Springer-Verlag, 1993.

[4] M. Gromov, Hyperbolic Groups, in *Essays in group theory* (ed. S. M. Gersten), MSRI series vol. 8, pages 75–263. Springer, New York, 1987.

[5] M. Hamann, *Spanning trees in hyperbolic graphs*, arXiv:0910.5605.

[6] M. Hamann, *On the tree-likeness of hyperbolic spaces*, arXiv:1105.3925.

The entropy of random-free graphons and properties

Hamed Hatami

(joint work with Sergey Norin)

1. Introduction

In recent years a theory of convergent sequences of dense graphs has been developed. One can construct a limit object for such a sequence in the form of certain symmetric measurable functions called graphons. Every graphon defines a random graph on any given number of vertices. In [HJS] several facts about the asymptotics of the entropies of these random variables are established. These results provide a good understanding of the situation when the graphon is not "random-free". However in the case of the random-free graphons they completely trivialize. The purpose of this article is to study these entropies in the case of the random-free graphons.

For every natural number n, denote $[n] := \{1, \dots, n\}$. Let $\mathcal{U}$ denote set of all graphs up to an isomorphism. Moreover, for $n \geq 0$, let $\mathcal{U}_n \subset \mathcal{U}$ denote the set of all graphs in $\mathcal{U}$ with exactly n vertices. We will usually work with labeled graphs. For every $n \geq 1$, denote by $\mathcal{L}_n$ the set of all graphs with vertex set $[n]$.

The *homomorphism density* of a graph H in a graph G, denoted by $t(H;G)$, is the probability that a random mapping $\phi : V(H) \to V(G)$ preserves adjacencies, i.e. $uv \in E(H)$ implies $\phi(u)\phi(v) \in E(G)$. The *induced density* of a graph H in a graph G, denoted by $p(H;G)$, is the probability that a random *embedding* of the vertices of H in the vertices of G is an embedding of H in G.

We call a sequence of finite graphs $(G_n)_{n=1}^{\infty}$ *convergent* if for every finite graph H, the sequence $\{p(H;G_n)\}_{n=1}^{\infty}$ converges. It is not difficult to construct convergent sequences $(G_n)_{n=1}^{\infty}$ such that their limits cannot be recognized as graphs, i.e. there is no graph G, with $\lim_{n\to\infty} p(H;G_n) = p(H;G)$ for every H. Thus naturally one considers $\overline{\mathcal{U}}$, the completion of $\mathcal{U}$ under this notion of convergence. It is not hard to see that $\overline{\mathcal{U}}$ is a compact metrizable space which contains $\mathcal{U}$ as a dense subset. The elements of the complement $\mathcal{U}^\infty := \overline{\mathcal{U}} \setminus \mathcal{U}$ are called *graph limits*. Note that a sequence of graphs $(G_n)_{n=1}^\infty$ converges to a graph limit Γ if and only if $|V(G_n)| \to \infty$ and $p(H;G_n) \to p(H;\Gamma)$ for every graph H. Moreover, a graph limit is uniquely determined by the numbers $p(H;\Gamma)$ for all $H \in \mathcal{U}$.

It is shown in [LS06] that every graph limit Γ can be represented by a *graphon*, which is a symmetric measurable function $W : [0,1]^2 \to [0,1]$. The set of all graphons are denoted by $\mathcal{W}_0$. Given a graph G with vertex set $[n]$, we define the corresponding graphon $W_G : [0,1]^2 \to \{0,1\}$ as follows. Let $W_G(x,y) := A_G(\lceil xn\rceil, \lceil yn\rceil)$ if $x,y \in (0,1]$, and if $x = 0$ or $y = 0$, set W_G to 0. It is easy to see that if $(G_n)_{n=1}^\infty$ is a graph sequence that converges to a graph limit Γ, then for every graph H,

$$p(H;\Gamma) = \lim_{n\to\infty} \mathbb{E}\left[\prod_{uv\in E(H)} W_{G_n}(x_u,x_v) \prod_{uv\in E(H)^c} (1 - W_{G_n}(x_u,x_v))\right],$$

where $\{x_u\}_{u\in V(H)}$ are independent random variables taking values in $[0,1]$ uniformly, and $E(H)^c = \{uv : u \neq v, uv \notin E(H)\}$. Lovász and Szegedy [LS06] showed that for every graph limit Γ, there exists a graphon W such that for every graph H, we have $p(H;\Gamma) = p(H;W)$ where

$$p(H;W) := \mathbb{E}\left[\prod_{uv\in E(H)} W(x_u,x_v) \prod_{uv\in E(H)^c} (1 - W(x_u,x_v))\right].$$

Furthermore, this graphon is unique in the following sense: If W_1 and W_2 are two different graphons representing the same graph limit, then there exists a measure-preserving map $\sigma : [0,1] \to [0,1]$ such that

$$W_1(x,y) = W_2(\sigma(x),\sigma(y)), \tag{1}$$

almost everywhere [BCL10]. With these considerations, sometimes we shall not distinguish between the graph limits and their corresponding graphons. We define the δ_1 distance of two graphons W_1 and W_2 as

$$\delta_1(W_1,W_2) = \inf \|W_1 - W_2 \circ \sigma\|_1,$$

where the infimum is over all measure-preserving maps $\sigma : [0,1] \to [0,1]$.

A graphon W is called a *stepfunction*, if there is a partition of $[0,1]$ into a finite number of measurable sets $S_1,\dots,S_n$ so that W is constant on every $S_i \times S_j$. The partition classes will be called the *steps* of W.

Let W be a graphon and $x_1,\dots,x_n \in [0,1]$. The random graph $G(x_1,\dots,x_n,W) \in \mathcal{L}_n$ is obtained by including the edge ij with probability $W(x_i,x_j)$, independently for all pairs (i,j) with $1 \le i < j \le n$. By picking $x_1,\dots,x_n$ independently and uniformly at random from $[0,1]$, we obtain the random graph $G(n,W) \in \mathcal{L}_n$. Note that that for every $H \in \mathcal{L}_n$,

$$\Pr[G(n, W) = H] = p(H; W).$$

1.1. **Graph properties and Entropy.** A subset of the set $\mathcal{U}$ is called a *graph class*. Similarly a *graph property* is a property of graphs that is invariant under graph isomorphisms. There is an obvious one-to-one correspondence between graph classes and graph properties and we will not distinguish between a graph property and the corresponding class. Let $\mathcal{Q} \subseteq \mathcal{U}$ be a graph class. For every $n > 1$, we denote by $\mathcal{Q}_n$ the set of graphs in $\mathcal{Q}$ with exactly n vertices. We let $\overline{\mathcal{Q}} \subseteq \overline{\mathcal{U}}$ be the closure of $\mathcal{Q}$ in $\overline{\mathcal{U}}$.

Define the *binary entropy* function $h : [0, 1] \mapsto \mathbb{R}_+$ as $h(x) = -x \log(x) - (1 - x) \log(1 - x)$ for $x \in (0, 1)$ and $h(0) = h(1) = 0$ so that h is continuous on $[0, 1]$ where here and throughout the paper $\log(\cdot)$ denotes the logarithm to the base 2. The *entropy* of a graphon W is defined as

$$\mathrm{Ent}(W) := \int_0^1 \int_0^1 h(W(x, y)) dx dy.$$

Note that it follows from the uniqueness result (1) that entropy is a function of the underling graph limit, and it does not depend on the choice of the graphon representing it. It is shown in [Ald85] and [Jan, Theorem D.5] that

$$\lim_{n\to\infty} \frac{\mathrm{Ent}(G(n, W))}{\binom{n}{2}} = \mathrm{Ent}(W). \tag{2}$$

A graphon is called *random-free* if it is $\{0, 1\}$-valued almost everywhere. Note that a graphon W is random-free if and only if $\mathrm{Ent}(W) = 0$, which by (2) is equivalent to $\mathrm{Ent}(G(n, W)) = o(n^2)$. Our first theorem shows that this is sharp in the sense that the growth of $\mathrm{Ent}(G(n, W))$ for random-free graphons W can be arbitrarily close to quadratic.

Theorem 26. *Let $\alpha : \mathbb{N} \to \mathbb{R}_+$ be a function with $\lim_{n\to\infty} \alpha(n) = 0$. Then there exists a random-free graphon W such that $\mathrm{Ent}(G(n, W)) = \Omega(\alpha(n)n^2)$.*

A graph property $\mathcal{Q}$ is called *random-free* if every $W \in \overline{\mathcal{Q}}$ is random-free. Our next theorem shows that in contrast to Theorem 26, when a graphon W is the limit of a sequence of graphs with a random-free property, then $\mathrm{Ent}(G(n, W))$ cannot grow faster than $O(n \log n)$.

Theorem 27. *Let $\mathcal{Q}$ be a random-free property, and let W be the limit of a sequence of graphs in $\mathcal{Q}$. Then $\mathrm{Ent}(G(n, W)) = O(n \log n)$.*

Remark 1. *We defined $G(n, W)$ as a* labeled *graph in $\mathcal{L}_n$. Both Theorems 26 and 27 remain valid if we consider the random variable $G_u(n, W)$ taking values in $\mathcal{U}_n$ obtained from $G(n, W)$ by forgetting the labels. Indeed,* $\mathrm{Ent}(G_u(n, W)) = \mathrm{Ent}(G(n, W)) - \mathrm{Ent}(G(n, W) \mid G_u(n, W))$ *and* $\mathrm{Ent}(G(n, W) \mid G_u(n, W) = H) = O(n \log n)$ *for every $H \in \mathcal{U}_n$. It follows that*

$$\mathrm{Ent}(G(n, W)) - O(n \log n) \leq \mathrm{Ent}(G_u(n, W)) \leq \mathrm{Ent}(G(n, W)).$$

References

[Ald85] David J. Aldous. Exchangeability and related topics. In *École d'été de probabilités de Saint-Flour, XIII—1983*, volume 1117 of *Lecture Notes in Math.*, pages 1–198. Springer, Berlin, 1985.

[BCL10] Christian Borgs, Jennifer Chayes, and László Lovász. Moments of two-variable functions and the uniqueness of graph limits. *Geom. Funct. Anal.*, 19(6):1597–1619, 2010.

[HJS] Hamed Hatami, Svante Janson, and Balázs Szegedy. in preparation.

[Jan] Svante Janson. Graphons, cut norm and distance, couplings and rearrangements. ArXiv:1009.2376.

[LS06] László Lovász and Balázs Szegedy. Limits of dense graph sequences. *J. Combin. Theory Ser. B*, 96(6):933–957, 2006.

Partitioning edge-connectivity

Tomáš Kaiser

We discuss the following problem posed by Matt DeVos (who also used the phrase 'partitioning edge-connectivity' to describe it):

Problem 1. *Is it true that for $a_1, a_2 \geq 1$, every $(a_1 + a_2 + 2)$-edge-connected graph G has edge-disjoint spanning subgraphs G_1 and G_2 such that each G_j is a_j-edge-connected $(j = 1, 2)$?*

The only case of Problem 1 where the answer is known to be affirmative is $(a_1, a_2) = (1, 1)$, by the following well-known corollary of Tutte and Nash-Williams' tree-packing theorem [2, 3]:

Proposition 1. *Every $2k$-edge-connected graph has k edge-disjoint spanning trees.*

In the general case, Proposition 1 only implies that the desired spanning subgraphs exist if G is $2(a_1 + a_2)$-edge-connected. In this talk, we outline a proof of the first open case of Problem 1, namely $(a_1, a_2) = (1, 2)$:

Theorem 28. *Every 5-edge-connected graph G admits a spanning tree whose complement is a spanning 2-edge-connected subgraph of G.*

The proof is based on a method similar to one used in [1] for the tree-packing theorem, although the current setting is somewhat more complicated. Roughly speaking, the main steps are as follows:

(1) Given a spanning tree T, an associated sequence of partitions of the vertex set of G is defined. Each member of the sequence is coarser than or equal to its successor, so there is a 'limit' partition $\mathcal{P}$.

(2) If $|\mathcal{P}| \geq 2$, then the assumption that G is 5-edge-connected implies a lower bound on the density of the graph $\overline{T}/\mathcal{P}$, obtained from the complement of T by contracting each class of $\mathcal{P}$. The density bound guarantees that $\overline{T}/\mathcal{P}$ contains a subgraph that can be obtained by taking '2-edge-joins' of 1-fold subdivisions of 3-edge-connected graphs.

(3) Any such subgraph can be used for an exchange step, resulting in an improvement of the spanning tree T in terms of a suitably defined ordering.

(4) After finitely many iterations, we will have $|\mathcal{P}| = 1$, which implies that T is the desired spanning tree with a 2-edge-connected complement.

In fact, step (3) is considerably more involved due to a technical problem mentioned in the talk (and successfully resolved since then). The complete proof of Theorem 28 is being prepared for publication.

References

[1] T. Kaiser, *A short proof of the tree-packing theorem*, Discrete Math. **312** (2012), 1689–1691.
[2] C. St. J. A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, J. London Math. Soc. **36** (1961), 445–450.
[3] W. T. Tutte, *On the problem of decomposing a graph into n connected factors*, J. London Math. Soc. **36** (1961), 221–230.

Getting to the point of Reed's Conjecture: How local can you go?

Andrew D. King
(joint work with Katherine Edwards)

1. Reed's Conjecture, evidence, and tightness

Reed's conjecture, now published 15 years ago, proposes the strongest possible bound on the chromatic number χ as a convex combination of the size of the largest closed neighbourhood $(\Delta + 1)$ and the size of the largest clique (ω), which represent a trivial upper and lower bound on χ, respectively.

Conjecture 3 (Reed '98 [9])**.** *Every graph satisfies*

$$\chi \leq \left\lceil \tfrac{1}{2}(\Delta + 1 + \omega) \right\rceil$$

It is easy enough to see that this is best possible. One example showing this is the uniform expansion of C_5. Another is a graph with no stable set of size 3, and with clique number $o(|V|)$, whose existence is a fundamental result in Ramsey theory.

So why should we believe it? First, the bound holds for the fractional chromatic number:

Theorem 29 (Reed '00 [8])**.** *Every graph satisfies*

$$\chi_f \leq \tfrac{1}{2}(\Delta + 1 + \omega)$$

Second, the bound holds for claw-free graphs [6], which are a natural generalization of quasi-line graphs – quasi-line graphs, in turn, are a class for which χ_f and χ agree asymptotically [5]. So given the previous result, they should be easy victims.

Third, we can at least bound χ as a nontrivial convex combination of the other two bounds:

Theorem 30 (Reed '98 [9]). *There is an $\epsilon > 0$ such that every graph satisfies*

$$\chi \leq \lceil (1-\epsilon)(\Delta+1) + \epsilon\omega \rceil$$

King and Reed recently provided a much simpler proof of this result [7], using an extremely useful existence condition that, for this problem, characterizes a minimum counterexample:

Theorem 31 (King '11 [4]). *Any graph satisfying $\omega > \frac{2}{3}(\Delta+1)$ contains a stable set meeting every maximum clique.*

2. The local and superlocal strengthening

Reed's Conjecture posits a bound on χ based on two global invariants. Can we replace these with a local invariant based on how hard the neighbourhood of a given vertex is to colour?

Conjecture 4 (King '09 [3]). *Every graph satisfies*

$$\chi \leq \max_{v \in V} \lceil \tfrac{1}{2}(d(v) + 1 + \omega(G[\tilde{N}(v)])) \rceil$$

(In this case, $\tilde{N}(v)$ denotes the closed neighbourhood of v). This conjecture is known to hold for the fractional relaxation (McDiarmid '00 [8]) and for quasi-line graphs (Chudnovsky, King, Plumettaz and Seymour, '12 [1]). We would very much like to at least prove an ϵ-relaxation of this conjecture:

Conjecture 5. *There is a universal constant $\epsilon > 0$ such that every graph satisfies*

$$\chi \leq \max_{v \in V} \lceil (1-\epsilon)(d(v)+1) + \epsilon\omega(G[\tilde{N}(v)])) \rceil$$

Using $\gamma_\ell(v)$ to denote $12(d(v) + 1 + \omega(G[\tilde{N}(v)]))$, we are now saying that if χ is high, some $\gamma_\ell(v)$ must be high. But seeing as $\gamma_\ell(v)$ represents the difficulty in colouring the neighbourhood of v, it seems as though these difficult-to-colour vertices should not appear merely as a stable set. Hence we propose the *superlocal strengthening* of Reed's Conjecure.

Conjecture 6 (Edwards, King [2]). *Every graph satisfies*

$$\chi \leq \max_{uv \in E} \lceil \tfrac{1}{2}(\gamma_\ell(u) + \gamma_\ell(v)) \rceil$$

We have managed to prove this for the fractional relaxation and for quasi-line graphs.

3. Fractional colourings and stronger conjectures

We might hope that we can push the "local averaging" condition even further, for example with the following question:

Question 1. *Is there a graph for which*

$$\chi_f > \max_{u \in V} \frac{1}{d(u)+1} \sum_{v \in \tilde{N}(u)} \gamma_\ell(v)$$

The answer to this is *yes*, as we can see by looking at a clique of size k with k pendant neighbours added to every vertex. However, we conjecture something weaker, letting $\mathcal{C}$ denote the set of maximal cliques in G:

Conjecture 7. *Every graph satisfies*

$$\chi_f > \max_{C\in\mathcal{C}} \frac{1}{|C|}\sum_{v\in C}\gamma_\ell(v)$$

To prove the fractional relaxation of the previous bounds we have discussed, the same greedy colouring algorithm suffices. This algorithm, introduced by Reed, can be defined as the limit, for increasing k, of the following randomized process:

(1) Begin with weight 1 on every vertex and weight 0 on every stable set.
(2) Take a maximum stable set S in G uniformly at random, and add weight $1/k$ to it.
(3) For any vertex in S, reduce the weight on the vertex by $1/k$.
(4) Delete any vertex with nonpositive weight remaining and go to step 2.

We believe the clique-averaged conjecture would require a different approach. To this end, we modify this algorithm by simply choosing S to be a maximum weight stable set chosen uniformly at random, rather than a maximum stable set. Yet we have not found an effective way to analyze this process – obviously it never requires more than $\Delta + 1$ colours.

Question 2. *Does the weight-modified Reed's fractional colouring always use at most* $\max_v \gamma_\ell(v)$ *colours, like the original algorithm? Can we prove better bounds?*

We would be extremely interested in the answers to such questions, and suspect that they might even be useful in attempts to prove a better asymptotic bound in the direction of Reed's Conjecture.

References

[1] M. Chudnovsky, A. D. King, M. Plumettaz, and P. Seymour. A local strengthening of Reed's ω, Δ, χ conjecture for quasi-line graphs. *SIAM J. Discrete Math.*, 2012. Accepted.

[2] K. Edwards and A. D. King. A superlocal version of Reed's conjecture. Submitted. Arxiv preprint 1208.5188, 2012.

[3] A. D. King. *Claw-free graphs and two conjectures on ω, Δ, and χ*. PhD thesis, McGill University, October 2009.

[4] A. D. King. Hitting all maximum cliques with a stable set using lopsided independent transversals. *J. Graph Theory*, 67(4):300–305, 2011.

[5] A. D. King and B. Reed. Asymptotics of the chromatic number for quasi-line graphs. *SIAM J. Discrete Math.*, 2012. To appear. arXiv preprint 1102.0805.

[6] A. D. King and B. A. Reed. Claw-free graphs, skeletal graphs, and a stronger conjecture on ω, δ, and χ. Arxiv preprint 1212.3036, 2012.

[7] A. D. King and B. A. Reed. A short proof that χ can be bounded ϵ away from $\delta+1$ towards ω. Arxiv preprint 1211.1410, 2012.

[8] M. Molloy and B. Reed. *Graph Colouring and the Probabilistic Method*. Springer, 2000.

[9] B. A. Reed. ω, Δ, and χ. *J. Graph Theory*, 27:177–212, 1998.

Quasirandomness and limits of combinatorial objects

Daniel Král'
(joint work with Oleg Pikhurko)

A sequence of graphs with increasing orders is pseudorandom if the limit of the densisities of any graph H in the graphs in sequence is the same as the density of H in the random graph $G_{n,1/2}$. This property is equivalent to many other properties a sequence of graphs may have, including the non-existence of unbalanced cuts or seperation among eigenvalues [1]. A surprising fact is that if the limits of densities of K_2 and C_4 in the graphs forming the sequence are the same as the corresponding densities in $G_{n,1/2}$, then the sequence must be pseudorandom.

Cooper [2] studied pseudorandomness of permutations. A sequence of permutation is pseudorandum if the limit of densities of any k-element permutation in the sequence is equal to $k!^{-1}$. A question attributed to Graham asks whether there exists k_0 such that a sequence of permutations must be pseudorandom if the above holds for $k \leq k_0$. We show that this is indeed the case with $k_0 = 4$; this is the best possible value as there exist examples that $k_0 = 3$ would not suffice.

Our proof uses the theory of permutation limits developed by Hoppen et al. [3, 4]. We consider as a limit object a measure on a unit measure with a property so-called unit marginals (this limit object is different from that in [3, 4] but it is not hard to show that the two objects are equivalent). Using this machinery, we show that the limit of any sequence of permutations with densitites of all 4-element permutations converging to 1/24 must be the uniform measure on the unit square. A similar argument can be used to show the analogous statement for graphs: if the densities of all 4-element subgraphs in a sequence of graphs converge to their densities in $G_{n,1/2}$, then the limit of the sequence is the graphon uniformly equal to 1/2 on the unit square.

References

[1] F. R. K. Chung, R. L. Graham, R. M. Wilson, *Quasi-random graphs*, Combinatorica **9** (1989), 345–362.
[2] J. N. Cooper, *Quasirandom permutations*, J. Combin. Theory Ser. A **106** (2004), 123–143.
[3] C. Hoppen, Y. Kohayakawa, C. G. Moreira, R. M. Sampaio, *Testing permutation properties through subpermutations*, Theoretical Computer Science **412** (2011), 3555–3567.
[4] C. Hoppen, Y. Kohayakawa, C. G. Moreira, R. M. Sampaio, *Limits of permutation sequences through permutation regularity*, available as arXiv:1106.1663.

Long geodesics in subgraphs of the cube

Imre Leader
(joint work with Eoin Long)

Given a graph G of average degree d, a classic result of Dirac [3] guarantees that G contains a path of length d. Moreover, for general graphs this is the best possible bound, as can be seen by taking G to be K_{d+1}, the complete graph on $d+1$ vertices.

The hypercube Q_n has vertex set $\{0,1\}^n$ and two vertices $x, y \in Q_n$ are joined by an edge if they differ on a single coordinate. In this talk we consider the analogous question for geodesics in the hypercube. A path in Q_n is a geodesic if no two of its edges have the same direction. Equivalently, a path is a geodesic if it forms a shortest path in Q_n between its endpoints. Given a subgraph G of Q_n of average degree d, how long a geodesic path must G contain?

It is trivial to see that any such graph must contain a geodesic of length $d/2$. Indeed, taking a subgraph G' of G with minimal degree at least $d/2$ and starting from any vertex of G', we can greedily pick a geodesic of length $d/2$ by choosing a new edge direction at each step.

On the other hand the d-dimensional cube Q_d shows that, in general, we cannot find a geodesic of length greater than d in G. Our main result is that this upper bound is sharp.

Theorem 32. *Every subgraph G of Q_n of average degree d contains a geodesic of length at least d.*

Noting that the endpoints of the geodesic in G guaranteed by Theorem 32 are at Hamming distance at least d, we see that Theorem 32 extends the following result of Feder and Subi [4].

Theorem 33 ([4])**.** *Every subgraph G of Q_n of average degree d contains two vertices at Hamming distance d apart.*

We remark that neither Theorem 32 nor Theorem 33 follow from isoperimetric considerations alone. Indeed, if G is a subgraph of Q_n of average degree d, by the edge isoperimetric inequality for the cube ([1], [5], [6], [7]; see [2] for background) we have $|G| \geq 2^d$. However if n is large, a Hamming ball of small radius may have size larger than 2^d without containing a long geodesic.

Finally, Feder and Subi's theorem was motivated by a conjecture of Norine [8] on antipodal colourings of the cube. We discuss Theorem 32 in relation to Norine's conjecture.

References

[1] A.J. Bernstein: Maximally connected arrays on the n-cube, *SIAM J. Appl. Math.* **15**(1967), 1485-1489.

[2] B. Bollobás: *Combinatorics: Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability*, Cambridge University Press, 1st ed, 1986.

[3] G.A. Dirac: Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69-81.

[4] T. Feder, C. Subi: On hypercube labellings and antipodal monochromatic paths, *http://theory.stanford.edu/~tomas/antipod.pdf.*

[5] L.H. Harper: Optimal assignments of numbers to vertices, *SIAM J. Appl. Math.* **12**(1964), 131-135.

[6] S. Hart: A note on edges of the n-cube, *Discrete Math.* **14**(1976), 157-163.

[7] J.H. Lindsey: Assignment of numbers to vertices, *Amer. Math. Monthly* **71**(1964), 508-516.

[8] S. Norine, Open Problem Garden, *http://garden.irmacs.sfu.ca/?q=op/edge_antipodal_colorings_of_cubes.*

Local Graph Theory

Nati Linial

(joint work with Hao Huang, Humberto Naves, Yuval Peled, and Benny Sudakov)

Some major challenges that come from application areas such as bioinformatics raise very interesting problems in graph theory. It is often the case that experimental work gives rise to very large data sets that are best expressed as huge graphs. To be concrete one can think of the protein-protein interaction (=PPI) graph of an organism. The vertex set of a PPI graph is comprised of all proteins of the organism in question, where two vertices are adjacent iff the two corresponding proteins are (experimentally) found to be in interaction. What kind of numeric data should one try to extract from such a huge graph? The biologists who generate the data do not have at present even a good vocabulary to describe the kind of knowledge that they wish to derive from such graphs. Current practices are very simplistic at best, e.g., many practitioners look at invariants such as degree distribution and other very simple concepts. The evolving theory of graph limits suggests a better approach. Namely, one should look at the distribution of small (say k-vertex) subgraphs of the big (say n-vertex) graph G. We call this distribution the k-*local* profile of G. This brings us to our first major problem.

Question: What are the possible k-profiles of large G's ?

To illustrate, here is the case $k = 3$ of this general problem. For $i = 0, 1, 2, 3$, let $p_i(G)$ be the probability that three randomly sampled vertices in G span exactly i edges. Let $p(G) = (p_0(G), \dots, p_3(G))$. Consider the set $S \subset \mathbb{R}^4$ which consists of all limit points of a sequence $p(G_n)$ where G_n is a sequence of graphs with $|V(G_n)| \to \infty$.

Question: Describe the set S.

Though we know quite a few things about this set, a full description is presently unavailable. Among other things, the papers below completely describe (i) The intersection of S with the plane (p_0, p_3) and (ii) Its intersection with the plane $p_3 = 0$. We note that at present the analogous problem for 4-vertex graphs seems out of reach.

References

[1] On the 3-local profiles of graphs, Hao Huang, Nati Linial, Humberto Naves, Yuval Peled, and Benny Sudakov, http://arxiv.org/abs/1211.3106.

[2] On the densities of cliques and independent sets in graphs, Hao Huang, Nati Linial, Humberto Naves, Yuval Peled, and Benny Sudakov, http://arxiv.org/abs/1211.4532.

Hexagon Graphs and the Directed Cycle Double Cover Conjecture

Martin Loebl
(joint work with Andrea Jimenez, Mihyun Kang)

We explore the well-known Jaeger's *directed cycle double cover conjecture* [1]. Jaeger's conjecture is equivalent to the assertion that every cubic bridgeless graph has an embedding on a closed Riemann surface with no dual loop. For each cubic graph G we define a new object, the hexagon graph $H(G)$, whose perfect matchings describe all embeddings of G on closed Riemann surfaces. We initiate the study of the properties of the hexagon graphs related to the embeddings with no dual loop. As a consequence we obtain a cut-type sufficient condition for the validity of the DCDC conjecture in the class of *fork graphs*. This condition is satisfied in the class of *lean fork graphs*, and the lean fork graphs hence satisfy the DCDC conjecture. Fork graphs are graphs obtained from a triangle by sequentially adding special fork-type graphs, and lean fork graphs are fork graphs satisfying an additional connectivity property.

How rich is the class of all the fork graphs? Given a graph H with vertices of degree either two or three, let the number of its vertices of degree two be its *degree of freedom*. When constructing the fork graphs, we can obtain intermediate graphs H with arbitrarily large degree of freedom. This leads us to conjecture that the DCDC conjecture is as hard for the fork graphs as for the general cubic bridgeless graphs.

References

[1] F. Jaeger, *A Survey of the Cycle Double Cover Conjecture*, Annals of Discrete Mathematics **115** (1985), 1–12.

Semi-definite method in extremal combinatorics

Sergey Norin

The *density* of a graph H in a graph G, denoted by $t_{\text{inj}}(H;G)$, is the probability that in a random *embedding* of the vertices of H in the vertices of G, every edge of H is mapped to an edge of G. Many fundamental theorems in extremal graph theory can be expressed as algebraic inequalities between the subgraph densities.

The *homomorphism density* of H in G, denoted by $t(H;G)$, is the probability that in a random mapping (not necessarily injective) from the vertices of H to the vertices of G, every edge of H is mapped to an edge of G. It is frequently possible to replace subgraph densities with homomorphism densities. An easy observation shows that one can convert any algebraic inequality between homomorphism densities to a linear inequality. In recent years a new line of research in the direction of treating these inequalities in a unified way has emerged. An interesting result in this context, proved recently in several different forms [8, 20, 18], says that every such inequality follows from the positive semi-definiteness of a certain infinite matrix. As an immediate consequence, every algebraic inequality between the homomorphism densities follows from an infinite number of certain type of applications of the Cauchy-Schwarz inequality. This explains why many results in extremal graph theory are proved by one or more tricky applications of the Cauchy-Schwarz inequality.

In [20] Razborov introduced flag algebras which provide a powerful formal calculus that captures many standard arguments in extremal combinatorics. He observed that a typical proof of an inequality in extremal graph theory between homomorphism densities of some fixed graphs involves only homomorphism densities of finitely many graphs. He stated that one of the most interesting general open questions about asymptotic extremal graph theory is whether every true linear inequality between homomorphism densities can be proved using a finite amount of manipulation with homomorphism densities of finitely many graphs. In [13] we show that the answer to this question and a related question of Lovász [17, Problem 17] is negative by proving the following theorem.

Theorem 34. *The following problem is undecidable.*

- INSTANCE: *A positive integer k, finite graphs $H_1, \ldots, H_k$, and integers $a_1, \ldots, a_k$.*
- QUESTION: *Does the inequality $a_1 t(H_1; G) + \ldots + a_k t(H_k; G) \geq 0$ hold for every graph G?*

In a paper in preparation, Hatami, Hatami, Lovett and myself show that the above problem is undecidable even if we restrict ourselves to bipartite graphs G. Although Theorem 34 shows that not every algebraic inequality between homomorphism densities is a linear combination of a finite number of semidefiniteness inequalities, the positive semidefinite characterization is still a powerful approach for proving such inequalities. Below we describe several such applications.

In [11] we use this method to prove the following.

Theorem 35. *Fo all sufficiently large positive integers l, the maximum number of copies of the cycle of length 5 in a triangle-free graph with $5\ell + a$ vertices ($0 \leq a \leq 4$) is $\ell^{5-a}(\ell+1)^a$,*

The above has been conjectured by Erdös [7]. The proof of the asymptotic relaxation of this statement is a rather standard Cauchy-Schwarz calculation in Razborov's flag algebras. To derive the exact result from the asymptotic one we develop a new approach which appears to differ substantially from the standard one based on stability and removal lemmas (see e.g. [16, 19]). Instead, we convert finite graphs into certain limit objects and then apply analytic methods to prove uniqueness of the optimum in this setting. This general approach in our case allows us to get the exact result from the asymptotic one and is likely to be applicable to other problems.

A natural question in Ramsey theory is how many monochromatic subgraphs isomorphic to a graph H must be contained in any two-coloring of the edges of the complete graph K_n. Equivalently how many subgraphs isomorphic to a graph H must be contained in a graph and its complement?

Goodman [9] has shown that for $H = K_3$, the optimum solution is essentially obtained by a typical random graph. The graphs H that satisfy this property are called *common*. Erdös [6] conjectured that all complete graphs are common. Later, this conjecture is extended to all graphs by Burr and Rosta [2]. Sidorenko [22] disproved Burr and Rosta's conjecture by showing that a triangle with a pendant edge is not common. It is now known that in fact the common graphs are very rare. For example Jagger, Šťovíček and Thomason [15] showed that every graph that contains K_4 as a subgraph is not common. Until recently, all of the known common graphs are of chromatic number at most 3. In [15] the authors mention that they consider the question whether W_5 [the wheel with 5 spokes] is common as one of the most important questions in the area. In [12] we answer the question in the affirmative.

One of the most intriguing problems in extremal (di)graph theory is the following conjecture due to Caccetta and Häggvist [3].

Conjecture 8. *Every n-vertex digraph with minimum outdegree at least r has a cycle with length at most $\lceil n/r \rceil$.*

The case when $r = n/3$ is of particular interest. It asserts that any digraph on n vertices with minimum outdegree at least $n/3$ contains a directed triangle. In [14] we obtained a new minimum degree bound for this case of the Caccetta-Häggvist conjecture.

Theorem 36. *Every n-vertex digraph with minimum outdegree at least $0.3465n$ contains a triangle.*

This improves previous bounds established by Caccetta and Häggvist [3] ($0.3820n$), Bondy [1] ($0.3798n$), Shen [21] ($0.3543n$) and Hamburger, Haxell, and Kostochka [10] ($0.3532n$). The main ingredients of our proof of Theorem 36 are once again the semidefinite method, an induction argument, generalizing the argument of Shen [21] and a recent result of Chudnovsky, Seymour and Sullivan [4] on eliminating cycles in triangle-free digraphs. Brute force computer search is used to find a way of combining these ingredients which yields the optimum bound.

The Turán's brickyard problem [23] is the problem of determining the minimum number of crossings, which must occur in a planar drawing of the complete bipartite graph $K_{n,n}$. It has been investigated since 1950's and recent success in improving the lower bounds for this number is due to an application of semidefinite method and relies on heavy computations. Recently, jointly with Yori Zwols, I have started considering applications of the flag algebra framework to this setting. Our first results suggest that one can apply semi-definite programming to produce a larger class of constraints on the crossing numbers. We have been able to show that the crossing number is lower bounded by 0.905 of the conjectured optimum, improving on the previously best known bound of 0.8594 due to

De Klerk, Pasechnik and Schrijver [5]. It seems likely that using more general class of semi-definite programs one can significantly improve our current understanding of this tantalizing open problem.

References

[1] J. A. Bondy. Counting subgraphs: a new approach to the Caccetta-Häggkvist conjecture. *Discrete Math.*, 165/166:71–80, 1997.

[2] Stefan A. Burr and Vera Rosta. On the Ramsey multiplicities of graphs—problems and recent results. *J. Graph Theory*, 4(4):347–361, 1980.

[3] L. Caccetta and R. Häggkvist. On minimal digraphs with given girth. In *Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1978)*, Congress. Numer., XXI, pages 181–187, Winnipeg, Man., 1978. Utilitas Math.

[4] M. Chudnovsky, P. Seymour, and B. Sullivan. Cycles in dense digraphs. *Combinatorica*, 28(1):1–18, 2008.

[5] E. de Klerk, D.V. Pasechnik, and A. Schrijver. Reduction of symmetric semidefinite programs using the regular $*$-representation. *Math. Programming*, 109:613–624, 2007.

[6] P. Erdős. On the number of complete subgraphs contained in certain graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 7:459–464, 1962.

[7] P. Erdös. On some problems in graph theory, combinatorial analysis and combinatorial number theory. In *Graph theory and combinatorics (Cambridge 1983)*, pages 1–17, 1984.

[8] Michael Freedman, László Lovász, and Alexander Schrijver. Reflection positivity, rank connectivity, and homomorphism of graphs. *J. Amer. Math. Soc.*, 20(1):37–51 (electronic), 2007.

[9] A. W. Goodman. On sets of acquaintances and strangers at any party. *Amer. Math. Monthly*, 66:778–783, 1959.

[10] P. Hamburger, P. Haxell, and A. Kostochka. On directed triangles in digraphs. *Electron. J. Combin.*, 14(1):Note 19, 9 pp. (electronic), 2007.

[11] Hamed Hatami, Jan Hladký, Daniel Král, Serguei Norine, and Alexander Razborov. On the number of pentagons in triangle-free graphs. *J. Combin. Theory Ser. A., to appear.*

[12] Hamed Hatami, Jan Hladký, Daniel Král, Serguei Norine, and Alexander Razborov. Non-three-colourable common graphs exist. *Combin. Probab. Comput.*, 21(5):734–742, 2012.

[13] Hamed Hatami and Serguei Norine. Undecidability of linear inequalities in graph homomorphism densities. *J. Amer. Math. Soc.*, 24(2):547–565, 2011.

[14] Jan Hladký, Daniel Král, and Serguei Norine. Counting flags in triangle-free digraphs. submitted.

[15] Chris Jagger, Pavel Šťovíček, and Andrew Thomason. Multiplicities of subgraphs. *Combinatorica*, 16(1):123–141, 1996.

[16] P. Keevash and B. Sudakov. The Turán number of the Fano plane. *Combinatorica*, 25:561–574, 2005.

[17] László Lovász. Graph homomorphisms: Open problems. *manuscript*, 2008.

[18] László Lovász and Balázs Szegedy. Random graphons and a weak Positivstellensatz for graphs. *J. Graph Theory*, 70(2):214–225, 2012.

[19] O. Pikhurko. The minimum size of 3-graphs without a 4-set spanning no or exactly three edges. Manuscript, 2009.

[20] Alexander A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.

[21] J. Shen. Directed triangles in digraphs. *J. Combin. Theory Ser. B*, 74(2):405–407, 1998.

[22] A. F. Sidorenko. Cycles in graphs and functional inequalities. *Mat. Zametki*, 46(5):72–79, 104, 1989.

[23] P. Turán. A note of welcome. *J. Graph Theory*, 1:7–9, 1977.

Expansion of random graphs: new proofs, new results

Doron Puder

We present a new approach to showing that random graphs are nearly optimal expanders. This approach is based on deep results from Combinatorial Group Theory. It applies both to regular and irregular random graphs.

Let Γ be a random d-regular graph on n vertices, and let λ be the largest absolute value of a non-trivial eigenvalue of its adjacency matrix. It was conjectured by Alon [2] that for every $\varepsilon > 0$, most d-regular graphs satisfy $\lambda < 2\sqrt{d-1} + \varepsilon$. Friedman famously presented a proof of this conjecture in [4]. Here we suggest a new, substantially simpler proof of a nearly-optimal result: we show that for d even, a random d-regular graph satisfies $\lambda < 2\sqrt{d-1} + 1$ asymptotically almost surely.

A main advantage of our aproach is that it is applicable to a generalized conjecture: A random d-regular graph on n vertices in the permutation model is, in fact, a random n-covering space of a bouquet of $d/2$ loops. More generally, fixing an arbitrary base graph Ω , we study the spectrum of Γ, a random n-covering of Ω. Let λ be the largest absolute value of a non-trivial eigenvalue of Γ. Extending Alon's conjecture to this more general model, Friedman [3] conjectured that for every $\varepsilon > 0$, a.a.s. $\lambda < \rho + \varepsilon$, where ρ is the spectral radius of the universal cover of Γ. When Ω is regular we get the same bound as before, and for an arbitrary Ω, we prove a nearly optimal upper bound of $\sqrt{3}\rho$. This is a substantial improvement upon all known results ([3, 5, 6, 1]). The techniques and results from combinatorial group theory which underlie the proofs come mainly from [8] and [7].

References

[1] L. Addario-Berry & S. Griffiths, *The spectrum of random lifts*, Arxiv preprint arXiv:1012.4097 2010.

[2] N. Alon, *Eigenvalues and Expanders*, Combinatorica **6** (1986), no. 2 83–96.

[3] J. Friedman, *Relative expanders or weakly relatively Ramanujan graphs*, Duke Mathematical Journal **118** 2003 no. 1 19–35

[4] J. Friedman, *A proof of Alon's second eigenvalue conjecture and related problems*, Memoirs of the AMS **195** no. 910, AMs 2008

[5] N. Linial & D. Puder, *Words Maps and Spectra of Random Graph Lifts*, Random Structures and Algorithms **37** 1 2010, 100-135

[6] E. Lubetzky, B. Sudakov & V. Vu, *Spectra of lifted Ramanujan graphs*, Advances in Mathematics **22** 4 2011, 1612–1645

[7] D. Puder, & O. Parzanchevski, *Measure Preserving Words are Primitive*, Arxiv preprint arXiv:1202.3269, 2012

[8] D. Puder, *Primitive Words, Free Factors and Measure Preservation*, Israel J. of Math., to appear. Arxiv preprint: arXiv:1104.3991

[9] D. Puder *Expansion of Random Graphs: New Proofs, New Results*, Arxiv preprint arXiv:1212.5216, 2012

The approximate Loebl-Komlós-Sós conjecture

Maya Stein
(joint work with Jan Hladký, János Komlós, Diana Piguet, Miklós Simonovits, Endre Szemerédi)

The Loebl–Komlós–Sós conjecture first appeared in [3]. It suggests that if a graph fulfills a certain condition on the median degree, then it contains every tree of a given order as a subgraph.

Conjecture 9 (Loebl–Komlós–Sós conjecture 1995)**.** *Let G be an n-vertex graph with at least $n/2$ vertices of degree at least k. Let T be any tree on $k+1$ vertices. Then $T \subseteq G$.*

A related but independent conjecture is the Erdős-Sós Conjecture. Here, the condition on the median degree is replaced with a condition on the average degree of the host graph.

Conjecture 10 (Erdős-Sós Conjecture 1963)**.** *Let G be a graph of order n with average degree more than $k-1$ edges. Let T be any tree on $k+1$ vertices. Then $T \subseteq G$.*

A breakthrough has been announced in the early 1990's by Ajtai, Komlós, Simonovits, and Szemerédi [1], who, with methods similar to the ones presented here, solve Conjecture 10 for sufficiently large values of k.

Both Conjectures 9 and Conjecture 10 have an important application in Ramsey theory. It is easy to see that each of them implies that the Ramsey number of two trees T_{k+1}, $T_{\ell+1}$ on $k+1$ and $\ell+1$ vertices, respectively, is bounded by $R(T_{k+1}, T_{\ell+1}) \leq k + \ell + 1$. Actually more is implied: Any 2-edge-colouring of $K_{k+\ell+1}$ contains either *all* trees on $k+1$ vertices in red, or *all* trees on $\ell+1$ vertices in blue.

Let us now turn back to the Loebl–Komlós-Sós conjecture. Conjecture 9 is dominated by two parameters: one quantifies the number of vertices of 'large' degree, and the other tells us how large this degree should actually be. Strengthening either of these bounds sufficiently, the conjecture becomes trivial. On the other hand, the bound of k for the degree is necessary because of the stars, while the bound on the numbers of large degree vertices might be lowered a bit (for details, see [4]).

Several partial results concerning Conjecture 9 have been obtained by placing either restrictions on the host graph, or on the class of trees to be embedded. Also, the case $k = n/2$ has been treated. For references, see [4].

A more general approach is the attack of the dense case of Conjecture 9, that is, the case when k is linear in n. This has been done using a method that employs the Regularity Lemma. The solution of the dense case was achieved by first establishing an approximate result [6], which was then extended, by adding stability arguments, to the full exact dense case (for large graphs) by Hladký and Piguet [5], and independently Cooley [2]. Let us state the two results here.

Theorem 37 (Piguet-Stein [6]). *For any $q > 0$ and $\alpha > 0$ there exists a number n_0 such that for any $n > n_0$ and $k > qn$ the following holds. If G is an n-vertex graph and has at least $(\frac{1}{2} + \alpha)n$ vertices of degree at least $(1 + \alpha)k$, and T is any tree on $k + 1$ vertices, then $T \subseteq G$.*

Theorem 38 (Hladký-Piguet [5], Cooley [2]). *For any $q > 0$ there exists a number $n_0 = n_0(q)$ such that for any $n > n_0$ and $k > qn$ the following holds. If G is an n-vertex graph and has at least $\frac{1}{2}n$ vertices of degree at least k, and T is any tree on $k + 1$ vertices, then $T \subseteq G$.*

It is left to deal with the sparse case of Conjecture 9, that is, when k is sublinear in n. We have recently been able to establish the following analogue of Theorem 37:

Theorem 39 (Hladký, Komlós, Piguet, Simonovits, Stein, Szemerédi [4]). *For any $\alpha > 0$ there exists a number k_0 such that for any $k > k_0$ the following holds. If G is an n-vertex graph and has at least $(\frac{1}{2} + \alpha)n$ vertices of degree at least $(1 + \alpha)k$, and T is any tree on $k + 1$ vertices, then $T \subseteq G$.*

The trouble in the sparse case is that the Regularity Lemma is no longer useful in sparse graphs. To surmount this shortcoming we use a general decomposition technique which applies also to sparse graphs: each graph can be decomposed into vertices of huge degree, regular pairs (in the sense of the Regularity Lemma), and two different expander-like parts. We call this a *sparse decomposition* of the graph. Such a tool has been used also in [1].

In a dense graph, our sparse decomposition yields basically the Szemerédi Regular Partition. But in a sparse graph, it allows us to explore those parts of the graph that escape from being regularized, and use them for our tree embedding. We indicated in the talk very roughly how the different parts can be useful for the embedding; the actual embedding procedures are quite involved (see [4]).

References

[1] M. Ajtai, J. Komlós, M. Simonovits, and E. Szemerédi, *Erdős-Sós conjecture*, in preparation

[2] O. Cooley, *Proof of the Loebl-Komlós-Sós conjecture for large, dense graphs*, Discrete Math., **309** (2009), 21, 6190–6228,

[3] P. Erdős, Z. Füredi, M. Loebl, and V. T. Sós, *Discrepancy of trees*, Studia Sci. Math. Hungar., **30** (1995), 1-2, 47–57

[4] J. Hladký, J. Komlós, D. Piguet, M. Simonovits, M. Stein, E. Szemerédi, *Approximate Loebl-Komlós-Sós Conjecture*, arXiv:1211.3050

[5] J. Hladký, D. Piguet, *Loebl-Komlós-Sós Conjecture: dense case*, arXiv:0805:4834

[6] D. Piguet, M. Stein, *An approximate version of the Loebl-Komlós-Sós conjecture*, J. Combin. Theory Ser. B,**102** (2012), 1, 102–125

Couplings of probability spaces and Sidorenko's conjecture

Balázs Szegedy

We present a new approach to Sidorenko's conjecture. This approach uses entropy calculations and coupligs of probability spaces.

Let $\{\Omega_i = (X_i, \mathcal{A}_i, \mu_i)\}_{i=1}^3$ be three probability spaces. Assume that $\{\psi_i : X_i \to X_3\}_{i=1,2}$ are measure preserving maps. Then we say that X_3 is a **joint factor** of Ω_1 and Ω_2. In this situation there is a unique probability space Ω_4 which is a coupling of Ω_1 and Ω_2 such that they are conditionally independent over Ω_3. This means that there are measure preserving maps $\{\phi_i : \Omega_4 \to \Omega_i\}_{i=1,2}$ such that $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$ and $\mathbb{E}(\phi_1^{-1}(A)|\phi_2) = \mathbb{E}(\phi_1^{-1}(A)|\psi_2 \circ \phi_2)$ holds for every $A \in \mathcal{A}_1$. We say that Ω_4 is the **conditionally independent coupling** of Ω_1 and Ω_2 over the joint factor Ω_3. Assume that $\{\nu_i\}_{i=1}^4$ are also probability measures on $\{\mathcal{A}_i\}_{i=1}^4$ such that, μ_i is absolutely continuous with respect to ν_i. Assume furthermore that (X_4, ν_4) is the conditionally independent coupling of (X_1, ν_1) and (X_2, ν_2) over (X_3, ν_3) using the above maps $\psi_1, \psi_2, \phi_1, \phi_2$. Then the **relative entropy** function defined by $D(\mu \parallel \nu) = \mathbb{E}_\mu(\log(d\mu/d\nu))$ satisfies the following inclusion-exclusion type formula.

$$D(\mu_4 \parallel \nu_4) = D(\mu_1 \parallel \nu_1) + D(\mu_2 \parallel \nu_2) - D(\mu_3 \parallel \nu_3).$$

We will use the above language for probability distributions on copies of a graph H in another graph G. It is convenient to introduce the notion of graph homomorphisms. A homomorphism from H to G is a map $f : V(H) \to V(G)$ such that the image of every edge in H is an edge in G. Let $\mathrm{Hom}(H, G) \subset V(G)^{V(H)}$ denote the set of homomorphisms from H to G and let $t(H, G)$ denote the probability that a random map $f : V(H) \to V(G)$ is a homomorphism. We interpret $\mathrm{Hom}(H, G)$ as the set of copies of H in G and $t(H, G)$ as the density of H in G. Let $\tau(H, G)$ denote the uniform distribution on $\mathrm{Hom}(H, G)$ and let $\nu(H, G)$ denote the uniform distribution on $V(G)^{V(H)}$. Let $D(\mu) := D(\mu \parallel \nu(H, G))$ for an arbitrary probability distribution μ on $\mathrm{Hom}(H, G)$. The Erdös-Simonovits, Sidorenko conjecture ([1],[6]) is the following.

Conjecture 11 (Erdös-Simonovits, Sidorenko)**.** *Let H be a bipartite graph. Then*

$$t(H, G) \geq t(e, G)^{|E(H)|}$$

for every graph G where e is a single edge. Equivalently:

$$D(\tau(H, G)) \leq |E(H)|\ D(\tau(e, G)).$$

Note that any probability distribution μ on $\mathrm{Hom}(H, G)$ satisfies that

$$D(\tau(H, G)) \leq D(\mu).$$

If μ satisfies $D(\mu) \leq |E(H)| D(\tau(e, G))$ then H satisfies the Sidorenko conjecture in G. For various graphs H one can construct such a measure μ by iterated conditionally independent couplings. The required inequality follows from the inclusion-exclusion formula. This verifies Sidorenko's conjecture for various graphs including many old ([4],[2],[5], [3]) and many new cases. In particular we obtain a very simple proof of the famous result [5] by Conlon, Fox and Sudakov which says that if a vertex in H is complete to the other side then H satisfies the conjecture.

References

[1] A.F Sidorenko, A correlation inequality for bipartite graphs, *Graphs Combin.* **9** (1993), 201-204

[2] H. Hatami, Graph norms and Sidorenko's conjecture, *Israel J. Math.* 175(1), (2010), 125-150

[3] X. Li, B. Szegedy, On the logarithmic calculus and Sidorenko's conjecture, *Combinatorica (to appear)*

[4] G.R. Blakley, P.A. Roy, A Hölder type inequality for symmetric matrices with nonnegative entries, *Proc. Amer. Math. Soc.* **16** (1965) 1244-1245

[5] D. Conlon, J. Fox, B. Sudakov, An approximate version of Sidorenko's conjecture, *GAFA*, Vol. 20 (2010) 1354–1366

[6] M. Simonovits, Extremal graph problems, degenerate extremal problems and supersaturated graphs, in "Progress in Graph Theory (Waterloo, Ont., 1982)", Academic Press, Toronto, ON (1984), 419-437.

Hypergraph containers

Andrew Thomason

(joint work with David Saxton)

Given a uniform hypergraph G, it is sometimes helpful to have a collection $\mathcal{C}$ of *containers* for the collection $\mathcal{I}$ of independent sets, such that $|\mathcal{C}|$ is much smaller than $|\mathcal{I}|$ itself, but, for each $I \in \mathcal{I}$, there is some $C \in \mathcal{C}$ with $I \subset C$, and moreover no $C \in \mathcal{C}$ is large. Here is an illustration.

The *list chromatic number* $\chi_\ell(G)$ is the smallest ℓ such that, whenever we assign to each vertex v a list L_v of ℓ colours, there is a vertex colouring of G in which the colour of v is chosen from L_v. Unlike $\chi(G)$, $\chi_\ell(G)$ must grow with the minimum degree of the graph G. Alon [1], improving on earlier results, showed that $\chi_\ell(G) \geq (1/2 + o(1)) \log_2 d$ holds for any graph G of minimum degree d. One can ask, more generally, about simple hypergraphs (those in which no two edges share more than one vertex). Haxell and Verstraëte [5] proved that $\chi_\ell(G) \geq (\log d / 5 \log\log d)^{1/2}$ for every simple, d-regular r-uniform hypergraph G when d is large and $r = 3$. Alon and Kostochka [2] obtained $\chi_\ell(G) \geq (\log d)^{1/(r-1)}$ for general r.

Simple examples show that one would expect $\chi_\ell(G) = \Omega(\log d)$. To attempt a proof, let us assign random lists L_v of size ℓ, chosen from a palette of k colours (the value of k is not critical — take k around ℓ^2). If there is a choice of colours from these lists that form a proper colouring of G, then there is some k-tuple of independent sets $(I_1, \ldots, I_k)$ such that the vertices of colour j lie inside I_j. We say the lists L_v respect a tuple $(I_1, \ldots, I_k)$ if, for each v, L_v is not a subset of $\{j : v \notin I_j\}$. If the lists allow a colouring then there is some tuple $(I_1, \ldots, I_k)$ that the lists respect. Now in a regular r-uniform hypergraph we have $|I_j| \leq (1 - c)n$ where $c = 1/r$. Thus, on average, $|\{j : v \notin I_j\}| \geq ck$, and so the probability that the lists respect a particular k-tuple is at most $(1 - c^\ell)^n \leq e^{-nc^\ell}$ (this can be made precise). Hence if $|\mathcal{I}|^k e^{-nc^\ell} < 1$ then, with positive probability, the lists fail to respect any k-tuple, showing that $\chi_\ell(G) > \ell$.

Taking $\ell = \Theta(\log d)$ we would obtain the lower bound $\chi_\ell(G) = \Omega(\log d)$ this way provided $|\mathcal{I}| \le e^{n/d}$. But this bound on $|\mathcal{I}|$ is hopelessly optimistic. However, suppose that we were able to find a set $\mathcal{C}$ of containers for $\mathcal{I}$, such that

- $|\mathcal{C}| \le e^{n/d}$
- $|C| \le (1-c)n$ for all $C \in \mathcal{C}$, and
- for all $I \in \mathcal{I}$ there exists $C \in \mathcal{C}$ with $I \subset C$.

Then we could use $\mathcal{C}$ and $(C_1, \ldots, C_k)$ in the argument above rather than $\mathcal{I}$ and $(I_1, \ldots, I_k)$, and so establish that $\chi_\ell(G) = \Omega(\log d)$.

For regular graphs, Sapozhenko [6] showed that such a set of containers exists. In [7] we proved that containers exist for regular simple uniform hypergraphs. Given an independent set I, there are small sets R, S generated randomly, and a third small set T generated deterministically, and a container $C = f(R, S, T)$ with $I \subset C$. Because the containers are determined by small sets, $|\mathcal{C}|$ is not large.

We wish to extend these ideas to non-regular hypergraphs, partly to gain a bound on $\chi_\ell(G)$ in general, but also because we can then *iterate* the container process to obtain smaller containers. Indeed, if C is a container for I in G, then I is independent in the hypergraph $G' = G[C]$, and we can find a smaller container $I \subset C'$ inside G'. However, for non-regular hypergraphs, we cannot require that $|C|$ be bounded, since $|I|$ can be arbitrarily close to n. By using the notion of *degree measure* $\mu(S) = \sum_{v \in S} d(v) / \sum_{v \in G} d(v)$, though, we can aim to find containers with $\mu(C) \le 1 - c$ (this implies $|C| \le (1-c)n$ for regular hypergraphs).

In fact, we can construct such containers for *any* r-uniform hypergraph, not just simple ones [8]. The construction is via a deterministic algorithm, which generates containers from r small sets, so ensuring that $|\mathcal{C}|$ is small. The actual size of $|\mathcal{C}|$ is expressed in terms of a parameter δ which can be readily computed from the co-degrees in G. The construction is surprisingly efficient, insofar as in many cases it can be shown that no smaller set of containers exists.

One consequence of the existence of these containers is that we can show $\chi_\ell(G) \ge \frac{1}{(r-1)^2} \log_r d$ for any simple r-uniform hypergraph G of average degree d (in particular this improves Alon's result for graphs by a factor of 2).

There are other consequences in graph theory. For example, let H be some ℓ-uniform hypergraph. We might be interested in ℓ-uniform H-free hypergraphs of order n. Consider the $e(H)$-uniform hypergraph G on vertex set $[n]^2$, whose edges correspond to copies of H on vertex set $[n]$. An independent set I in G is precisely an H-free graph on $[n]$. By building containers in G we discover that all H-free graphs are contained in just a few almost H-free graphs. This can be applied to prove a variety of facts, such as that there are at most $2^{(\pi(H)+o(1)\binom{n}{\ell})}$ H-free ℓ-uniform hypergraphs of order n. It also implies the recently proved sparse Túran theorems of Conlon-Gowers [4] and Schacht [9], as well as the full KŁR conjecture. Balogh, Morris and Samotij [3] have obtained closely related results here. Corresponding results pertain for other structures, such as the number of solution-free sets for collections of linear equations.

References

[1] N. Alon, Degrees and choice numbers, *Random Structures and Algorithms* **16** (2000), 364–368.
[2] N. Alon and A. Kostochka, Hypergraph list coloring and Euclidean Ramsey theory. (submitted).
[3] J. Balogh, R. Morris and W. Samotij, Independent sets in hypergraphs, (preprint).
[4] D. Conlon and W.T. Gowers, Combinatorial theorems in sparse random sets, submitted.
[5] P. Haxell and J. Verstraëte, List coloring hypergraphs, *Electronic J. Combinatorics* **17** (2010) R129, 12pp.
[6] A.A. Sapozhenko, The Cameron-Erdős conjecture, *Discrete Math.* **308** (2008), 4361–4369.
[7] D. Saxton and A. Thomason, List colourings of regular hypergraphs, *Combinatorics, Probability and Computing* **21** (2012), 315–322.
[8] D. Saxton and A. Thomason, Hypergraph containers (submitted).
[9] M. Schacht, Extremal results for random discrete structures, submitted.

Graph coloring, communication complexity, and the stubborn problem

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(joint work with Nicolas Bousquet, Aurélie Lagoutte)

A classical result of Graham and Pollak asserts that the edge set of the complete graph on n vertices cannot be partitioned into less than $n-1$ complete bipartite graphs. A natural question is then to ask for some properties of graphs G_ℓ which are edge-disjoint unions of ℓ complete bipartite graphs. An attempt in this direction was proposed by Alon, Saks and Seymour, asking if the chromatic number of G_ℓ is at most $\ell+1$. This wild generalization of Graham and Pollak's theorem was however disproved by Huang and Sudakov who provided graphs with chromatic number $\Omega(\ell^{6/5})$. The $O(\ell^{\log \ell})$ upper-bound being routine to prove, this leaves as open question the *polynomial Alon-Saks-Seymour conjecture* asking if an $O(\ell^c)$ coloring exists for some fixed c.

A well-known communication complexity problem introduced by Yannakakis, involves a graph G of size n and the usual suspects Alice and Bob. Alice plays on the stable sets of G and Bob plays on the cliques. Their goal is to exchange the minimum amount of information to decide if Alice's stable set S intersect Bob's clique K. In the nondeterministic version, one asks for the minimum size of a certificate one should give to Alice and Bob to decide whether S intersects K. If indeed S intersects K, the certificate consists in the vertex $x = S \cap K$, hence one just has to describe x, which cost is $\log n$. The problem becomes much harder if one want to certify that $S \cap K = \emptyset$ and this is the core of this problem. A natural question is to ask for a $O(\log n)$ upper bound. Yannakakis observed that this would be equivalent to the following *polynomial clique-stable separation conjecture*: There exists a c such that for any graph G on n vertices, there exists $O(n^c)$ vertex bipartitions of G such that for every disjoint stable set S and clique K, one of the bipartitions separates S from K.

A variant of Feder and Vardi celebrated dichotomy conjecture for Constrait Satisfaction Problems, the List Matrix Partition (LMP) problem asks whether all $(0, 1, *)$ CSP instances are NP-complete or polytime solvable. The LMP was investigated for small matrices, and was completely solved in dimension 4, save for a unique case, known as the *stubborn problem*: Given a complete graph G which edges are labelled by 1,2, or 3, the question is to partition the vertices into three classes V_1, V_2, V_3 so that V_i does not span an edge labelled i. An easy branching majority algorithm computes $O(n^{\log n})$ 2-list-coloring of the vertices such that every solution of the stubborn problem is covered by at least one of these 2-list-coloring. The stubborn problem hence reduces to $O(n^{\log n})$ 2-SAT instances, yielding a pseudo polynomial algorithm. A polynomial algorithm was recently discovered by Cygan et al., but whether the original branching algorithm could be turned into a polynomial algorithm is still open. Precisely one can ask the *polynomial stubborn 2-list cover conjecture* asking if the set of solutions of any instance of the stubborn problem can be covered by $O(n^c)$ instances consisting of lists of size 2.

In this talk, I will show that the polynomial Alon-Saks-Seymour conjecture, the polynomial clique-stable separation conjecture and the polynomial stubborn 2-list cover conjecture are indeed equivalent. One of the implications linking the two first problems was already proved by Alon and Haviv.

Approximating minimum cost k-node-connectivity augmentation via independence-free graphs

László Végh
(joint work with Joseph Cheriyan)

For a set V, let $\binom{V}{2}$ denote the edge set of the complete graph on the node set V. In the *minimum-cost k-connectivity augmentation problem*, we are given a graph $G = (V, E)$ and nonnegative edge costs $c : \binom{V}{2} \to \mathbb{R}_+$, and the task is to find a minimum cost set $F \subseteq \binom{V}{2}$ of edges such that $G + F$ is k-node-connected. Let $\mathrm{opt}(G)$ denote the optimum value.

The problem is NP-hard for $k \geq 2$. In the *asymptotic setting*, we restrict ourselves to instances where the number of nodes is lower bounded by a function of k. Our main result addresses such a setting:

Theorem 40. *Let $G = (V, E)$ be an undirected graph with at least $k^3(k-1)+k$ nodes. There is a polynomial-time algorithm that finds an edge set $F \subseteq \binom{V}{2}$ such that $G + F$ is k-connected and $c(F) \leq 6\mathrm{opt}(G)$.*

This is the first constant factor approximation algorithm even for the asymptotic setting. In [3], an $O(\log k)$ approximation guarantee was given for the asymptotic setting, assuming that $n \geq 6k^2$. Most research efforts subsequent to [3] focused on finding near-logarithmic approximation guarantees for all possible ranges of n and k. The best current approximation ratio is $O(\log k \log \frac{n}{n-k})$ by Nutov [9].

For the analogous problem of edge-connectivity augmentation, the seminal result by Jain [8] gives a 2-approximation algorithm. This works for the more general *survivable network design problem* (SNDP).[1] This result was proved by the novel technique of iterative rounding. The key theorem asserts that every basic feasible solution to the standard linear programming (LP) relaxation has at least one edge of value at least $\frac{1}{2}$. A 2-approximation is obtained by iteratively adding such an edge to the graph and solving the LP relaxation again.

As tempting as it might be to apply iterative rounding for SNDP with node-connectivity requirements, unfortunately the standard LP relaxation for this problem might have basic feasible solutions with small fractional values on every edge. Such an example was presented already in [2]. Recently, [1] improved on previous constructions by exhibiting an example of the min-cost k-connected spanning subgraph problem with a basic feasible solution that has value $O(1/\sqrt{k})$ on every edge. Still, iterative rounding has been applied to problems with node-connectivity requirements: for example, Fleischer, Jain and Williamson [4] gave a 2-approximation for node-connectivity SNDP with maximum requirement 2.

Our new insight is that whereas iterative rounding fails to give $O(1)$-approximations for arbitrary instances, we can isolate a class of graphs where it does work; and moreover, we are able to transform an arbitrary input instance to a new instance from this class.

Frank and Jordán [5] introduced the framework of set-pairs for node-connectivity problems; the LP relaxation is also based on this notion. By a *set-pair*, we mean a pair of nonempty disjoint sets of nodes, not connected by any edge of the graph; the two sets are called *pieces*. If the union of the two pieces has size $> n-k$, then the set-pair is called *deficient*, since it corresponds to the two sides of a node cut of size $< k$. Clearly, a k-connected graph must not contain any deficient set-pairs. A new edge has to cover every deficient set-pair, that is, an edge whose endpoints lie in the two different pieces. Two set-pairs are called *independent*, if they cannot be simultaneously covered by an edge (of the complete graph), It can be seen that the two set-pairs are independent if and only if one of them has a piece disjoint from both pieces of the other set-pair.

A graph is called *independence-free* if any two deficient set-pairs are dependent. This notion of independence-free graphs was introduced by Jackson and Jordán [7] in the context of minimum cardinality k-connectivity augmentation (the special case of our problem where each edge in $\binom{V}{2} \setminus E$ has cost 1).

We observed that bad examples for iterative rounding (such as the one in [1]) always contain independent deficient set-pairs. We show that this is the only possible obstruction: in independence-free graphs, the analog of Jain's theorem holds, that is, every basic feasible solution to the LP relaxation has an edge with value at least $\frac{1}{2}$. The proof is a simple extension of Jain's result on edge-connectivity.

The first phase of our algorithm uses "combinatorial methods" to add a set of edges of cost $\leq 4\text{opt}(G)$ to obtain an independence-free graph. The second phase

[1] In the SNDP, we are given an undirected graph with non-negative costs on the edges, and for every unordered pair of nodes i, j, we are given a number $\rho_{i,j}$; the goal is to find a subgraph of minimum cost that has at least $\rho_{i,j}$ edge-disjoint paths between i and j for every pair of nodes i, j.

applies iterative rounding to add a set of edges of cost $\leq 2\text{opt}(G)$ to obtain an augmented graph that is k-connected.

In the first phase we guarantee a property stronger than independence-freeness. By a *rogue set* we mean a set $U \subseteq V$ with $d(U) < k$ and $|U| < k$. We call a graph *rogue-free* if it does not contain any rogue-sets; it is easy to see that a rogue-free graph must also be independence-free. Our main tool is the Frank-Tardos algorithm [6] for k-outconnectivity augmentation, a standard tool in connectivity-augmentation algorithms. We show that a graph with at least $k^3(k-1)+k$ nodes can be made rogue-free by two applications of this algorithm, with suitably chosen root sets.

References

[1] A. Aazami, J. Cheriyan, and B. Laekhanukit, *A bad example for the iterative rounding method for mincost k-connected spanning subgraphs*, Discrete Optimization, (2012).

[2] J. Cheriyan and S. Vempala, *Edge covers of setpairs and the iterative rounding method*, in IPCO, vol. 2081 of LNCS, 2001, pp. 30–44.

[3] J. Cheriyan, S. Vempala, and A. Vetta, *An approximation algorithm for the minimum-cost k-vertex connected subgraph*, SIAM J. Comput., 32 (2003), pp. 1050–1055.

[4] L. Fleischer, K. Jain, and D.P. Williamson, *Iterative rounding 2-approximation algorithms for minimum-cost vertex connectivity problems*, Journal of Computer and System Sciences, 72 (2006), pp. 838–867.

[5] A. Frank and T. Jordán, *Minimal edge-coverings of pairs of sets*, J. Comb. Theory Ser. B, 65 (1995), pp. 73–110.

[6] A. Frank and É. Tardos, *An application of submodular flows*, Linear Algebra and its Applications, 114 (1989), pp. 329–348.

[7] B. Jackson and T. Jordán, *Independence free graphs and vertex connectivity augmentation*, J. Comb. Theory Ser. B, 94 (2005), pp. 31–77.

[8] K. Jain, *A factor 2 approximation algorithm for the generalized Steiner network problem*, Combinatorica, 21 (2001), pp. 39–60.

[9] Z. Nutov, *Approximating minimum-cost edge-covers of crossing biset families*, Tech. Report, (2009). Preliminary version "An almost $O(\log k)$-approximation for k-connected subgraphs," in SODA, pages 912–921, 2009.

Immersions in highly connected graphs

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(joint work with D. Marx and P. Seymour)

We consider graphs with parallel edges but no loops. A graph G admits an *immersion* of a graph H if there exist functions $\pi_v : V(H) \to V(G)$ and π_e mapping the edges of H to subgraphs of G satisfying the following:

a. the map π_v is an injection;
b. for every edge $f \in E(H)$ with endpoints x and y, $\pi_e(f)$ is a path with endpoints equal to $\pi_v(x)$ and $\pi_v(y)$;
c. for edges $f, f' \in E(H)$, $f \neq f'$, $\pi_e(f)$ and $\pi_e(f')$ have no edge in common.

The vertices $\{\pi_v(x) : x \in V(H)\}$ are the *branch vertices* of the immersion. We will also say that G immerses H or alternatively that G contains H as an immersion.

There is an easy structure theorem for graphs which exclude a fixed H as an immersion [3], [2]. If we fix the graph H and let Δ be the maximum degree of a vertex in H, then one obvious obstruction to a graph G containing H as an immersion is if every vertex of G has degree less than Δ. The structure theorem shows that this is approximately the only obstruction. The structure theorem says that for all $t \geq 1$, any graph which does not have an immersion of K_t can be decomposed into a tree-like structure of pieces with at most t vertices of degree at least t^2.

The structure theorem gives rise to a variant of tree decompositions based on edge cuts instead of vertex cuts, called *tree-cut decompositions.* The minimum width of a tree-cut decomposition is the tree-cut width of a graph. The tree-cut width shares many of the standard properties of tree-width translated into terms of immersions. For example, the tree-cut width of a graph is monotone decreasing under taking immersions. See again [3] for further details.

We consider here the extremal problem of how much edge connectivity is necessary to force a fixed graph H as an immersion. DeVos, Dvorak, Fox, Mcdonald, Mohar, Scheide [1] have shown that in a simple graph, minimum degree 200t suffices to force an immersion of K_t. Moreover, there exist examples of graphs with minimum degree $t - 1$ which do not contain K_t as an immersion. More generally, it is easy to construct graphs which are highly edge connected and still have no K_t immersion. Consider, for example, the graph obtained from taking a path and adding $t^2/4 - t$ parallel edges to each edge. Such a graph is $O(t^2)$ edge connected but still contains no K_t immersion.

The example above of a highly edge connected graph with no K_t immersion has tree-cut width bounded by a function of t. Thus, one might hope that all the highly edge connected graphs which do not admit K_t as an immersion similarly have bounded tree-cut width. This is in fact the case.

Theorem 41. *There exists a function g satisfying the following. Let $k \geq 4, n \geq 1$ be positive integers. Then for all graphs H with maximum degree k on n vertices and for all k-edge connected graphs G, either G admits an immersion of H, or G has tree-cut width at most $g(k, n)$.*

The theorem is not true for $k = 3$. This is because if G and H are 3-regular graphs, then G contains H as an immersion if and only if G contains H as a topological minor. Thus, if H is any 3-regular graph which cannot be embedded in the plane, then any 3-regular planar graph G cannot contain H as an immersion and such graphs can have arbitrarily large tree-width.

References

[1] M. DeVos, Z. Dvorak, J. Fox, J. McDonald, B. Mohar, D. Schiede, *Minimum degree condition forcing complete graph immersion*, preprint.

[2] M. DeVos, J. McDonald, B. Mohar, D. Schiede, *A note on forbidding clique immersions,* `http://arxiv.org/abs/1207.2117`.

[3] P. Wollan, *The structure of graphs not admitting a fixed immersion*, preprint.

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