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Moduli Spaces in Algebraic Geometry

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Abstract. The workshop on Moduli Spaces in Algebraic Geometry aimed to bring together researchers working in moduli theory, in order to discuss moduli spaces from different points of view, and to give an overview of methods used in their respective fields.

Mathematics Subject Classification (2010): 14D22.

Introduction by the Organisers

The workshop Moduli Spaces in Algebraic Geometry, organized by Dan Abramovich (Brown), Gavril Farkas (HU Berlin), Lucia Caporaso (Rome) and Stefan Kebekus (Freiburg) was held February 4–8, 2013 and was attended by 25 participants from around the world. The participants ranged from senior leaders in the field to young post-doctoral fellows and one advanced PhD student. The range of expertise covered areas ranging from classical algebraic geometry to mathematics inspired by string theory. Researchers reported on the substantial progress achieved within the last three years, discussed open problems, and exchanged methods and ideas. Most lectures were followed by lively discussions among participants, at times continuing well into the night. For a flavor of the range of subjects covered, a few of the talks are highlighted below.

Stable pairs and knot invariants. Rahul Pandharipande (ETH Zürich) reported on work of Shende, Oblomkov and Maulik concerning Hilbert schemes $\mathrm{Hilb}(C, n)$ of n points on a curve C with an isolated, planar singularity. Building on ideas of Pandharipande-Thomas and Diaconescu, Shende and Oblomkov proposed a relation between the Euler characteristics $\chi\big(\mathrm{Hilb}(C,n)\big)$ and coefficients in the HOMFLY polynomial of the curve singularity link. This was recently established by Maulik.

Higher codimension loci in the moduli space of curves. Nicola Tarasca (Leibniz Universität Hannover) reported on results in his PhD Thesis on the calculation of the cohomology class of the codimension two Brill-Noether locus of curves with a pencil of degree k in the moduli space $\overline{\mathcal{M}}_{2k}$ of stable curves of genus $2k$. Remarkable here is that, while one has a large number of divisor class calculations on the moduli space, it is for the first time that a closed formula for a higher codimension locus on the moduli space is found.

Tautological rings of moduli space of curves. In a very impressive talk, Aaron Pixton (Princeton) proposed a rather amazing conjecture generalizing at the level of the moduli space $\overline{\mathcal{M}}_{g,n}$ the Faber-Zagier relations in the cohomology of the moduli space $\mathcal{M}_g$. The increase in complexity when passing from smooth to singular curves is considerable and it is a major step forward that a concrete prediction has been put forward. The field is facing an interesting change of paradigm, in the sense that the largely accepted Faber Conjectures predicting that the corresponding tautological rings of moduli of curves satisfy Poincare duality, are being replaced by new predictions, according to which the suitable generalizations of Faber-Zagier relations span all relations between tautological classed. It is already clear that in genus 24 the two conjectures rule out each other (whereas for $g < 24$ they are equivalent) and it will be interesting to monitor future developments.

Geometric compactifications of the moduli space of K3 surfaces. A classical unsolved problem of moduli theory asks for a modular compactification of the moduli space of polarized K3 surface. While several compactifications of the moduli space have been discussed, none of them is known to date to support a universal family. Bernd Siebert (Hamburg) reported on joint work Mark Gross (San Diego), Paul Hacking (Amherst) and Sean Keel (Austin) which might lead to a solution of this long-standing problem. Building on work of Gross-Siebert which uses Mirror symmetry to study degenerations of Calabi-Yau manifolds, there is hope to single out one particular toroidal compactification for which a family might exist. While many details still need to be filled in, and a discussion of the geometric and modular properties of the construction is still pending, this is a very exciting project which might eventually solve a classical problem.

The moduli stack of semistable curves. This is a development providing a glimpse of the lively discussions which happened at this very meeting. Jarod Alper (ANU) reported in the most timely manner possible on present joint work with Andrew Kresch (Zürich) on the structure of the moduli stack of semistable curves. One of the main questions one must ask about any stack is whether or not it is a global-quotient stack, or at least if it can be approximated by a global-quotient stack. A central example is the stack $\mathfrak{M}_g^{ss}$ of semistable curves, a keystone in constructing many moduli spaces. Kresch has shown that even the first stage of this stack, $\mathfrak{M}_g^{ss,\leq 1}$, is not a global-quotient stack, but Alper conjectured that $\mathfrak{M}_g^{ss}$ falls in a general class of stacks well-approximated by global-quotient stacks. Alper reported that this was established during this meeting by him and Kresch for $\mathfrak{M}_g^{ss,\leq 1}$, with strong evidence for the result to hold for the full moduli stack of semistable curves.

Moduli of slope-semistable bundles. Daniel Greb (Ruhr-Universität Bochum) reported on joint work with Matei Toma (Nancy), discussing wall-crossing and compactifications for moduli spaces of slope-semistable sheaves on higher-dimensional projective manifolds. Generalizing work of Joseph Le Potier and Jun Li, he constructed projective moduli spaces for slope-semistable sheaves by showing semiampleness of certain equivariant determinant line bundles. While the geometry of the resulting moduli spaces is presently only partially understood, these spaces are likely to shed new light on the question whether Tian's topological compactifications of moduli spaces of slope-semistable vector bundles admit complex or even algebraic structures.

Workshop: Moduli Spaces in Algebraic Geometry

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Abstracts

Stable pairs and knot invariants (after Shende, Oblomkov, and Maulik)

Rahul Pandharipande

My lecture concerned the work of Shende, Oblomkov, Maulik, and others on the connection between the Hilbert schemes of plane curve singularities and the invariants of the associated link.

The main results concern the geometry of the Hilbert scheme of points of plane curve singularities. Let C be a curve with an isolated planar singularity $p \in C$, and let $\mathrm{Hilb}(C, n)$ be the Hilbert scheme of n points. In papers with R. Thomas, we proved the generating series of the Euler characteristics of the Hilbert schemes of points

$$P_C(q) = \sum_{n\geq 0} \chi(\mathrm{Hilb}(C,n))\ q^n$$

is actually a rational function in q of a very constrained form:

$$P_C(q) = \sum_{h=g_{\mathrm{geom}}}^{g_{\mathrm{ar}}} n_{h,C}\ q^{g_{\mathrm{ar}}-h}(1-q)^{2h-2}$$

for integers $n_{h,C}$ where h lies between the geometric genus g_{geom} and the arithmetic genus g_{ar}. We termed the integers $n_{h,C}$ the BPS state counts associated to C for their relationship to 3-fold Donaldson-Thomas theory. We observed in examples $n_{h,C} > 0$.

Shende and Oblomkov have undertaken a systematic study of the integers $n_{h,C}$. Their first discovery is the relationship between $n_{h,C}$ and the knot invariants of the link of the plane curve singularity. Shende and Oblomkov conjecture that the $n_{h,C}$ are coefficients of the Jones polynomial of the link. Since the Jones polynomial occurs as a specialization of the HOMFLY polynomial, a natural question is how to obtain the full HOMFLY from the Hilbert schemes of points. Here, the idea is to consider the filtration on the punctual Hilbert scheme $\mathrm{Hilb}_p(C, n)$ at the singularity $p \in C$ given by minimal number of generators of the ideal. The two variate series of Euler characteristics (indexed by number of points and number of generators) Shende and Oblomkov conjecture to be equal after a simple and universal change of variables to the two variate HOMFLY polynomial. Shende and Oblomkov prove the conjecture for torus knots associated to singularities $X^n - Y^m$ and in a few other examples. The result for torus knot is not trivial — on one side, an exact calculation of the HOMFLY polynomial by Jones is used, on the other side, new techniques of dealing with the Hilbert scheme have to be developed.

Recently, Maulik (using also ideas of Diaconescu and collaborators) was able to prove the orginal conjecture by Shende and Oblomokov. The main ideas are to lift the conjecture to relate certainly stable pairs theories on local $\mathbb{P}^1$ to the colored HOMFLY polynomial. Wall-crossing methods in sheaf counting are then used to prove a blow-up formula. The conifold transition plays a central role.

It is natural to consider the motives associated to the Hilbert scheme to produce extra variables. In work with Oblomkov and Rasmussen at Cambridge, Shende has found a conjecture linking the motivic invariants to modern 3-variate extensions of HOMFLY.

In a second line of work undertaken by Shende by himself, he finds the basic geometric meaning of the integers $n_{h,C}$. In the versal deformation space of C, there are loci V_h which parameterize the closures of deformations of C with geometric genus h. These subvarieties V_h are usually singular at the point $[C] \in V_h$. Shende conjectured in 2009 that $n_{h,C}$ equals the multiplicity of V_h at $[C]$. In the case $h = g_{\mathrm{geom}}$, this conjecture is a consequence of basic results by Göttsche-Fantechi-Van Straten. In the summer of 2010, Shende proved the full conjecture via an improved understanding of tangent spaces to relative Hilbert schemes. Shende's results explains the positivity $n_{h,C} > 0$.

Compact moduli spaces for slope-semistable sheaves on higher-dimensional projective manifolds

Daniel Greb
(joint work with Matei Toma)

My talk focussed on the "variation of semistability"-problem for moduli spaces of sheaves on higher-dimensional varieties. In dimension greater than one, both Gieseker-semistability (which yields projective moduli spaces in arbitrary dimension) and slope-semistability (which is better behaved geometrically, e.g. with respect to tensor products and restrictions) depend on a parameter, classically the class of a line bundle in the ample cone of the underlying variety. As a consequence, it is of great importance to understand how the moduli space of semistable sheaves changes when the semistability parameter varies.

1. Known results on surfaces, and the situation on threefolds

In the case where the underlying variety is of dimension two this problem has been investigated by a number of authors and a rather complete geometric picture has emerged, which can be summarised as follows:

(i) A compact moduli space for slope-semistable sheaves also exists as a projective scheme. It is homeomorphic to the Donaldson-Uhlenbeck compactification, endowing the latter with a complex structure, and admits a natural morphism from the Gieseker compactification. This was proven independently by Joseph Le Potier [LP92] and Jun Li [Li93].

(ii) In the ample cone of the underlying variety there exists a locally finite chamber structure given by linear rational walls, so that the notion of slope/Gieseker-semistability (and hence the moduli space) does not change within the chambers, see [Qin93].

(iii) Moreover, at least when the second Chern class of the sheaves under consideration is sufficiently big, moduli spaces corresponding to two chambers separated by a common wall are birational, and the change in geometry can be understood by studying the moduli space of sheaves that are slope-semistable with respect to the class of an ample bundle lying on the wall, see [HL95].

However, starting in dimension three several fundamental problems appear:

(i) While there are gauge-theoretic generalisations of the Donaldson-Uhlenbeck compactification to higher-dimensional varieties [Tia00], these are not known to possess a complex structure.

(ii) Adapting the notion of "wall" as in [Qin93], one immediately finds examples where these walls are not locally finite inside the ample cone.

(iii) Looking at segments between two integral ample classes in the ample cone instead, Schmitt [Sch00] gave examples of threefolds such that the point on the segment where the moduli space changes is irrational.

2. Stability with respect to movable curves and the main result

In my talk I presented a novel approach to attack the above-mentioned problems, developed in joint work with Matei Toma (Nancy). It is based on the philosophy that the natural "polarisations" to consider when defining slope-semistability on higher dimensional base manifolds are not ample divisors but rather movable curves.

For any n-dimensional smooth projective variety X we consider the open set $P(X)$ of powers of ample divisor classes inside the cone of movable curves and show that it supports a locally finite chamber structure given by linear rational walls such that the notion of slope-(semi)stability is constant within each chamber. Moreover, any chamber (even if it is not open) contains products $H_1H_2...H_{n-1}$ of integer ample divisor classes. We are thus led to the problem of constructing moduli spaces of torsion-free sheaves which are slope-semistable with respect to a *multipolarisation* $(H_1, ..., H_{n-1})$, where $H_1, ..., H_{n-1}$ are integer ample divisor classes on X.

The main result of our preprint [GT13] is the following:

Theorem. *Let X be a smooth projective threefold, $H_1, H_2 \in \mathrm{Pic}(X)$ two ample divisors, $c_1 \in H^2\big(X, \mathbb{Z}\big)$, $c_2 \in H^4\big(X, \mathbb{Z}\big)$, $c_3 \in H^6\big(X, \mathbb{Z}\big)$ three classes, r a positive integer, $c \in K(X)_{num}$ a class with rank r, and Chern classes $c_j(c) = c_j$, and Λ a line bundle on X with $c_1(\Lambda) = c_1 \in H^2(X, \mathbb{Z})$. Denote by $\underline{M}^{\mu ss}$ the functor that associates to each weakly normal variety S the set of isomorphism classes of S-flat families of (H_1, H_2)-semistable torsion-free coherent sheaves of class c and determinant Λ on X. Then, there exists a class $\hat{u}_2 \in K(X)_{num}$, a natural number $N \in \mathbb{N}^{>0}$, a weakly normal projective variety $M^{\mu ss}$ with an ample line bundle $\mathscr{O}_{M^{\mu ss}}(1)$, and a natural transformation*

$$\underline{M}^{\mu ss} \to \underline{Hom}(\cdot, M^{\mu ss})$$

with the following properties:

(1) *For any S-flat family $\mathscr{F}$ of μ-semistable sheaves of class c and determinant Λ with induced classifying morphism $\Phi_{\mathscr{F}} : S \to M^{\mu ss}$ we have*

$$\Phi^*_{\mathscr{F}}(\mathscr{O}_{M^{\mu ss}}(1)) = \lambda_{\mathscr{F}}(\hat{u}_2)^N,$$

where $\lambda_{\mathscr{F}}(\hat{u}_2)$ is the determinant line bundle on S induced by $\mathscr{F}$ and $\hat{u}_2$.

(2) *For any other triple $(M', \mathscr{O}_{M'}(1), N')$ consisting of a projective variety M', an ample line bundle $\mathscr{O}_{M'}(1)$ on M' and a natural number N' fulfilling the conditions spelled out in (1), one has $N|N'$ and there exists a uniquely determined morphism $\psi : M^{\mu ss} \to M'$ such that $\psi^*(\mathscr{O}_{M'}(1)) \cong \mathscr{O}_{M^{\mu ss}}(\frac{N'}{N})$.*

The triple $(M^{\mu ss}, \mathscr{O}_{M^{\mu ss}}(1), N)$ is uniquely determined up to isomorphism by the properties (1) and (2).

In addition, $M^{\mu ss}$ contains the weak normalisation of the moduli space of (isomorphism classes of) (H_1, H_2)-stable reflexive sheaves as a Zariski-open set, answering a particular case of a question raised among others by Teleman [Tel08].

The proof of the main result follows ideas of Le Potier [LP92] and Jun Li [Li93] in the two-dimensional case: first, using boundedness we parametrise slope-semistable sheaves by a locally closed subscheme $R^{\mu ss}$ of a suitable Quot-scheme. Isomorphism classes of semistable sheaves correspond to orbits of a special linear group G in $R^{\mu ss}$. We then consider a certain determinant line bundle $\mathscr{L}_2$ on $R^{\mu ss}$ and aim to show that it is generated by G-invariant global sections. Le Potier mentions in [LP92] that in the case when $H_1 = ... = H_{n-1} =: H$ his proof of this fact in the two-dimensional case could be extended to higher dimensions if a restriction theorem of Mehta-Ramanathan type were available for Gieseker-H-semistable sheaves. Indeed, such a result would be needed if one proceeded by restrictions to hyperplane sections on X. We avoid this Gieseker-semistability issue and instead restrict our families directly to the corresponding complete intersection curves, where slope-semistability and Gieseker-semistability coincide. The price to pay is some loss of flatness for the restricted families. In order to overcome this difficulty we pass to weak normalisations for our family bases and show that sections in $\mathscr{L}_2$ extend continuously, and owing to weak normality hence holomorphically, over the non-flat locus. The moduli space $M^{\mu ss}$ then arises as the Proj-scheme of a ring of G-invariant sections in powers of $\mathscr{L}_2$ over the weak normalisation of $R^{\mu ss}$.

Our construction works for base manifolds of any dimension $n \geq 3$ and will be explitly carried out in future versions of our paper.

3. Outlook

Based on example computations and partial results, it is natural to expect that the moduli space $M^{\mu ss}$ realises the following equivalence relation on the set of isomorphism classes of slope-semistable torsion-free sheaves: Two slope-semistable sheaves F_1 and F_2 give rise to the same point in the moduli space $M^{\mu ss}$ if and only if the graded sheaf associated with Jordan-Hölder filtrations of F_1 and F_2, respectively, as well as naturally associated 2-codimensional cycles coincide. Comparing with the description of the geometry of the known topological compactifications of the moduli space of slope-stable vector bundles constructed by Tian [Tia00], we expect that the moduli spaces $M^{\mu ss}$ provide new insight concerning the question whether these higher-dimensional analogues of the Donaldson-Uhlenbeck compactificationspaces admit natural complex or even algebraic structures.

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Cycle Classes of a Stratification on the Moduli of K3 surfaces in Positive Characteristic

Gerard van der Geer

(joint work with Torsten Ekedahl)

This talk is on joint work with Torsten Ekedahl who died in November 2011. His sharp intellect and strong and generous personality will be deeply missed.

Moduli spaces in positive characteristic possess stratifications for which we do not know characteristic zero analogues. These stratifications are very helpful in understanding these moduli spaces. Here we deal with the moduli of polarized K3 surfaces in characteristic $p > 0$, actually $p > 2$. For a K3 surface in positive characteristic there is a special invariant, the height of the formal Brauer group, introduced by Artin and Mazur in the 1970s. The formal Brauer group of a K3 surface is a 1-dimensional formal group and 1-dimensional formal groups over an algebraically closed field k of characteristic $p > 0$ are characterized by their height. If t is a local parameter then multiplication by p can be written as

$$[p] \cdot t = a\, t^{p^h} + \text{higher order terms}$$

with $a \neq 0$. This defines the height h. If $h = \infty$ then we are dealing with $\hat{\mathbb{G}}_a$, the formal additive group. If $h < \infty$ we have a p-divisible formal group. Artin and Mazur deduced a consequence for the geometry of a K3 surface:

$$\text{if} \quad h \neq \infty \quad \text{then} \quad \rho \leq 22 - 2h$$

with ρ the rank of the Neron-Severi group. It follows that if $\rho = 22$ then $h = \infty$. This case occurs, for example, the Fermat surface of degree 4 in characteristic $p \equiv 3(\text{mod} 4)$ has $\rho = 22$. In general, if $h < \infty$ then $1 \leq h \leq 10$.

The case $h = 1$ is the generic case and $h = \infty$ is called the supersingular case.

Artin conjectured that if $h = \infty$ then $\rho = 22$. This has now been proved by Maulik, Charles and Madapusi Pera for $p > 2$, see [3, 1, 2].

Using the height we get strata on the moduli space $\mathcal{F}_g$ of polarized K3 surfaces over an algebraically closed field k of characteristic p. Let V_h be the locus of K3 surfaces with height $\geq h$. Since $1 \leq h \leq 10$ or $h = \infty$ we get 11 strata and we know that $\operatorname{codim} V_h \leq h - 1$ for finite h. But for supersingular K3 surfaces there is another invariant, given by

$$\operatorname{disc}(\mathrm{NS}(X)) = -p^{2\sigma_0},$$

and σ_0 is called the Artin invariant. The idea is that though $\rho = 22$ stays fixed for supersingular K3 surfaces, divisor classes in the limit might become divisible by p, thus changing σ_0. So besides the height loci V_h we have loci V_{σ_0} where the Artin invariant is $\leq \sigma_0$. Here $\sigma_0 = 11$ is the generic supersingular K3, while $\sigma_0 = 1$ is the superspecial case, the most degenerate situation.

In joint work with Katsura ([5]) we determined the cycle classes of the height strata:

$$[V_h] = (p-1)(p^2-1)\cdots(p^{h-1}-1)\,\lambda^{h-1},$$

where $\lambda = c_1(\pi_*\Omega^2_{\mathcal{X}/\mathcal{F}_g})$ is the first Chern class of the Hodge bundle of the universal K3 surface $\pi : \mathcal{X} \to \mathcal{F}_g$. The remaining classes of the Artin invariant strata turned out to be very elusive, but were finally determined in joint work with Torsten Ekedahl, see [4].

We also gave a uniform approach to all strata. This is done by looking at (almost) complete flags on the cohomology $H^2_{\mathrm{dR}}(X)$.

We consider K3 surfaces with an isometric embedding $N \to \mathrm{NS}(X)$ of non-degenerate lattices, where we assume that N contains a semi-ample line bundle. The corresponding moduli space is denoted $\mathcal{F}_N$. Let $N^\perp$ the primitive cohomology in H^2_{dR}. It has a Hodge filtration

$$0 = U_{-1} \subset U_0 \subset U_1 \subset U_2 = N^\perp$$

of dimension (say) $0, 1, n-1, n$. In positive characteristic we then get another filtration, the conjugate filtration

$$0 = U^c_{-1} \subset U^c_0 \subset U^c_1 \subset U^c_2 = N^\perp$$

that comes from relative Frobenius $F : X \to X^{(p)}$ and the associated spectral sequence with $E_2^{ij} = H^i(X^{(p)}, \Omega^j_{X^{(p)}/k})$ converging to $H^2_{\mathrm{dR}}(X/k)$. The inverse Cartier operator induces an isomorphism $F^*(U_i/U_{i-1}) \cong U^c_{2-i}/U^c_{1-i}$. We thus have two flags. We refine the flag on the conjugate filtration and use Cartier to transfer it to the Frobenius pull back of the Hodge filtration. We thus get two (almost complete) flags on $N^\perp$ and the relative position of these two flags can be given by an element of a Weyl group (of an orthogonal group). We show that in this way one recovers the invariants h and σ_0.

There are some very subtle issues related to $SO(n)$ versus $O(n)$ in case n is even, involving a discriminant on the middle part of the flag. By working on the flag space (parametrizing flags on $N^\perp$) and using a Pieri formula we were able to calculate the cycle classes. I state the result only in case n is odd. The general result can be found in [4]. The strata on $\mathcal{F}_g$ correspond to the case $n = 2m+1 = 21$.

Theorem 1. *There are $2m$ strata $\mathcal{V}_k$ on $\mathcal{F}_N$ with $k = 1, \ldots, 2m$. For $k = 1, \ldots, m$ we have the finite height strata, the stratum $\mathcal{V}_{m+1}$ is the supersingular locus, while $\mathcal{V}_{m+k}$ for $k = 2, \ldots, m$ give the Artin invariant strata. Their classes are given by the following formulae.*
The finite height case (with $1 \le k \le m$):

$$\mathcal{V}_k = (p-1)(p^2-1)\cdots(p^{k-1}-1)\,\lambda^{k-1}$$

The supersingular case:

$$\mathcal{V}_{m+1} = \frac{1}{2}(p-1)(p^2-1)\cdots(p^m-1)\,\lambda^m$$

The Artin invariant case (with $2 \le k \le m$):

$$\mathcal{V}_{m+k} = \frac{1}{2}\,\frac{(p^{2k}-1)(p^{2k+2}-1)\cdots(p^{2m}-1)}{(p+1)(p^2+1)\cdots(p^{m-k+1}+1)}\,\lambda^{m+k-1}$$

The results of the paper can also be applied to the moduli of higher-dimensional varieties, like hyperkähler varieties.

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Stabilization of discriminants in the Grothendieck ring

Ravi Vakil
(joint work with Melanie Matchett Wood)

This talk is a report on the results of [1]. We study the classes of discriminants (loci in a moduli space of objects with specified singularities) and their complements in the Grothendieck ring of varieties, focusing on the cases of moduli of hypersurfaces and configuration spaces of points. The main contributions of this paper are two theorems and one conjecture ("motivic stabilization of symmetric powers").

I. *(the limiting motive of the space of hypersurfaces with a given number of singularities)* If $\mathcal{L}$ is an ample line bundle on a smooth variety X, we show that the motive of the subset of the linear system $|\mathcal{L}^{\otimes j}|$ consisting of divisors with precisely s singularities (normalized by $|\mathcal{L}^{\otimes j}|$), tends to a limit as $j \to \infty$ (in the completion of the localization of the Grothendieck ring at $\mathbb{L} := [\mathbb{A}^1]$), given explicitly in terms of the motivic zeta function of X.

II. *(motivic stabilization of symmetric powers)* We conjecture that if X is geometrically irreducible, then the ratio $[\mathrm{Sym}^n X]/\mathbb{L}^{n \dim X}$ tends to a limit. This is an algebraic version of the Dold-Thom theorem, and is also motivated by the Weil conjectures. There are a number of reasons for considering this conjecture, see [1, §4].

III. *(the limiting motive of discriminants in configuration spaces)* We show that if X is geometrically irreducible and satisfies motivic stabilization (**II**, e.g. if X is stably rational), then the motive of strata (and their closure) of configurations of points with given "discriminant" (clumping of points) tends to a limit as the number of points $n \to \infty$, and (more important) we describe the limit in terms of motivic zeta values. In the case of s multiple points, the result is the same as that of **I**, except the expression in terms of motivic zeta functions is evaluated at a different value. The reliance on the motivic stabilization conjecture can be removed by specializing to Hodge structures, where the analogous conjecture holds, or by working with generating series.

These results are motivated by a number of results in number theory and topology (including, notably, stability/stabilization theorems), and they generalize analogues of many of these statements. (An elementary motivation is an analogue of both **I** and **III** for $X = \mathrm{Spec}\,\mathbb{Z}$: the probability of an integer being square free is $1/\zeta(2)$. One has to first make sense of the word "probability" as a limit, then show that the limit is a zeta value. These features will be visible in our arguments as well.) Our results also support Denef and Loeser's motto [2, l. 1-2]: "rational generating series occurring in arithmetic geometry are motivic in nature".

Our results suggest a number of new conjectures in arithmetic, algebraic geometry, and topology that may be tractable by other means.

For more detail and context, see [1, §1].

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Tautological relations on $\overline{M}_{g,n}$

Aaron Pixton
(joint work with Rahul Pandharipande and Dimitri Zvonkine)

The tautological ring of the moduli space of stable curves $\overline{M}_{g,n}$ is a subring $R^*(\overline{M}_{g,n})$ of the Chow ring $A^*(\overline{M}_{g,n})$ consisting of the cycles that arise most naturally in geometry. The tautological rings can be collectively defined (see [3]) for all g, n as the smallest subrings that are closed under pushforward by the following morphisms:

- the maps $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$ forgetting a marked point;
- the maps $\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g_1+g_2,n_1+n_2}$ gluing two curves together at marked points;
- the maps $\overline{M}_{g,n+2} \to \overline{M}_{g+1,n}$ gluing two marked points together on a single curve.

The tautological rings of subspaces of $\overline{M}_{g,n}$, such as the moduli space of smooth curves $M_{g,n}$ or the moduli space of curves of compact type $M^c_{g,n}$, can then be defined by restriction. In the case of M_g, the tautological ring $R^*(M_g)$ is simply the ring of polynomials in the Arbarello-Cornalba [1] kappa classes $\kappa_1, \kappa_2, \ldots$. A *tautological relation* on M_g is an element of the kernel of the surjection from the ring of formal kappa polynomials $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ to $R^*(M_g)$.

All known tautological relations on M_g are linear combinations of the Faber-Zagier (FZ) relations, a large family of explicit kappa polynomials that were proven to be tautological relations in [5] using the moduli space of stable quotients. If the FZ relations give a complete description of the tautological relations, then this would contradict Faber's celebrated Gorenstein conjecture [2] for $R^*(M_g)$ when $g \geq 24$.

I will discuss the analogous situation for $R^*(\overline{M}_{g,n})$. Here the kappa classes are not enough to generate the tautological ring, and the ring of formal kappa polynomials $\mathbb{Q}[\kappa_1, \kappa_2, \ldots]$ must be replaced by a more complicated combinatorial object, the *strata algebra* $\mathcal{S}_{g,n}$. The strata algebra is additively defined as a $\mathbb{Q}$-vector space with basis elements corresponding to the additive generators of the tautological ring $R^*(\overline{M}_{g,n})$ described by Graber and Pandharipande [4]: pick a dual graph Γ and take the pushforward of an arbitrary monomial in the kappa and psi classes along the associated gluing map

$$\xi_\Gamma : \prod_v \overline{M}_{g_v,n_v} \to \overline{M}_{g,n}.$$

Multiplication in the strata algebra is defined using the rules for multiplying these additive generators described in [4]. Then a tautological relation on $\overline{M}_{g,n}$ is an element of the kernel of the natural surjection $\mathcal{S}_{g,n} \to R^*(\overline{M}_{g,n})$.

In [8], the author described a large ideal $\mathcal{R}$ in $\mathcal{S}_{g,n}$. This ideal can be interpreted as the ideal generated by pullbacks and pushforwards of special elements $R(g,n,r)$ that are defined as sums over dual graphs: $R(g,n,r)$ is the degree r part of

$$\sum_\Gamma \frac{1}{|\mathrm{Aut}(\Gamma)|} \xi_{\Gamma *} \prod_{v \text{ vertex of } \Gamma} A_v \prod_{e \text{ edge of } \Gamma} B_e \prod_{l \text{ leg of } \Gamma} C_l$$

for certain local contributions A_v, B_e, C_l from the vertices (irreducible components), edges (nodes), and legs (marked points) of the dual graph.

Conjecture ([8]). *$\mathcal{R}$ is the ideal of tautological relations on $\overline{M}_{g,n}$.*

These conjectural relations can be restricted to $M^c_{g,n}$ and $M^{rt}_{g,n}$ to give analogous conjectures. In the case of M_g, the FZ relations are recovered. In each case, as with the FZ relations, all known relations are linear combinations of these conjectured relations.

These conjectures would have numerous implications for the structure of the tautological rings. Currently, the only known counterexample to Faber's Gorenstein conjectures is in the moduli of stable curves: Petersen and Tommasi [7] proved that $R^*(\overline{M}_{2,n})$ is not Gorenstein for some $n \leq 20$. Computing the ranks of the quotients by the (restrictions of the) conjectural relations $\mathcal{R}$, we see that the tautological rings of $M_6^c, M_{5,2}^c, M_{24}, M_{20,1}^{rt}, M_{17,2}^{rt}, M_{14,3}^{rt}, M_{11,4}^{rt}, M_{10,5}^{rt}$, and $M_{9,6}^{rt}$ are also not Gorenstein if $\mathcal{R}$ gives all the tautological relations. (The case M_{24} is of course just the FZ relation prediction.)

In ongoing joint work with R. Pandharipande and D. Zvonkine [6], we have constructed these relations in cohomology.

Theorem ([6]). *$\mathcal{R}$ is contained in the kernel of the composition*

$$\mathcal{S}_{g,n} \to R^*(\overline{M}_{g,n}) \to H^*(\overline{M}_{g,n}).$$

The proof uses the purity of Witten's class on the moduli space of 3-spin curves together with Teleman's classification of semisimple cohomological field theories [9]. This also gives a new proof of the FZ relations in cohomology.

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Stable cohomology of compactifications of $\mathcal{A}_g$

Klaus Hulek
(joint work with Samuel Grushevsky and Orsola Tommasi)

1. Introduction

Let $\mathcal{A}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathbb{H}_g$ be the moduli space of principally polarized abelian varieties of dimension g (over $\mathbb{C}$). It is a well known result of Borel [Bor74] that $\mathcal{A}_g$ has stable cohomology. To describe this let $\mathbb{E}$ be the Hodge bundle on $\mathcal{A}_g$ and denote its Chern classes by $\lambda_i = c_i(\mathbb{E}) \in H^{2i}(\mathcal{A}_g,\mathbb{Z})$.

Theorem 1 (Borel)**.** *The cohomology of $\mathcal{A}_g$ stabilizes and the stable cohomology is freely generated by the classes $\lambda_1, \lambda_3, \ldots$. More precisely, for all $k < g$ we have*

$$H^k(\mathcal{A}_g) = \mathbb{Q}^k[\lambda_1, \lambda_3, \ldots]$$

where the degree of λ_i is $2i$.

This result can be generalized to local systems. Recall that the irreducible local systems $\mathbb{V}_{\underline{\mu}}$ on $\mathcal{A}_g$ are enumerated by Young diagrams $\underline{\mu}$. It turns out that the only local system with non-trivial stable cohomology is the trivial local system.

Theorem 2 (Borel, Hain)**.** *For a fixed Young diagram $\underline{\mu}$, and for all $k < g$ we have*

$$H^k(\mathcal{A}_g, V_{\underline{\mu}}) = \begin{cases} \mathbb{Q}^k[\lambda_1, \lambda_3, \ldots] & \text{if } \underline{\mu} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. Compactifications

It is natural to ask whether stabilization results can also be obtained for compactifications of $\mathcal{A}_g$. This has been answered positively for the Satake compactification. Recall that this is set-theoretically given by

$$\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \ldots \sqcup \mathcal{A}_0.$$

Theorem 3 (Charney, Lee)**.** *The Satake compactification $\mathcal{A}_g^{\mathrm{Sat}}$ has stable cohomology in degrees $k < g$. This is freely generated by the classes $\lambda_1, \lambda_3, \ldots$ and by classes $\alpha_3, \alpha_5, \ldots$ where the degree of α_j is $2j$.*

The next step is to look at toroidal compactifications $\mathcal{A}_g^{\mathrm{tor}}$. The two toroidal compactifications which have been studied the most are the second Voronoi compactification $\mathcal{A}_g^{\mathrm{Vor}}$ and the perfect cone or first Voronoi compactifcation $\mathcal{A}_g^{\mathrm{Perf}}$. The first is known to have a good modular interpretation due to the work of Alexeev and Olsson, whereas the second has good properties from the point of view of the minimal model program: if $g \geq 12$, then $\mathcal{A}_g^{\mathrm{Perf}}$ is a canonical model of $\mathcal{A}_g$, in particular its canonical bundle is ample, as was shown by Shepherd-Barron.

One cannot expect that the second Voronoi compactification has stable cohomology: if we denote by $l(g)$ the number of 1-dimensonal orbits of the second Voronoi decomposition, then it is known that $l(2) = l(3) = 1$, $l(4) = 2$, $l(5) = 9$

and $l(6) \geq 20000$. It also follows from work of Baranovskii and Grishukhin that at least $l(g) \geq g-3$, in other words the number of boundary divisors (and thus of irreducible boundary components of $\mathcal{A}_g^{\mathrm{Vor}}$) grows with g. In contrast, the boundary of $\mathcal{A}_g^{\mathrm{Perf}}$ is an irreducible divisor for all values of g. The main purpose of this talk was to show that one has indeed a stabilization result for the cohomology of $\mathcal{A}_g^{\mathrm{Perf}}$.

3. Universal families and Mumford's partial compactifiction

Let $\mathcal{X}_g \to \mathcal{A}_g$ be the universal family. We denote its n-fold cartesian product by $\mathcal{X}_g^{\times n} \to \mathcal{A}_g$. On this family we have natural divisor classes. For this let T be (the class of) the theta divisor (trivialized over the 0-section) on $\mathcal{X}_g$ and let P be the (class of) the Poincaré bundle on $\mathcal{X}_g^{\times n}$ (again trivialized along the 0-section). Using the projections p_i and $p_{i,j}$ onto the i-th and (i,j)-th factor respectively we obtain via pullback classes T_i, $1 \leq i \leq n$ and $P_{i,j}$, $1 \leq i < j \leq n$ on $\mathcal{X}_g^{\times n}$.

Theorem 4. *The universal family $\mathcal{X}_g^{\times n} \to \mathcal{A}_g$ has stable cohomology in degree $k < g$. The stable cohomology is generated freely as an algebra over the stable cohomology of $\mathcal{A}_g$ by the classes T_i and $P_{i,j}$.*

To prove the theorem one uses the Leray spectral sequence with E_2-term $E_2^{p,q} := H^p(\mathcal{A}_g, R^q\pi_*^{\times n}\mathbb{Q})$ for the projection $\pi^{\times n} : \mathcal{X}_g^{\times n} \to \mathcal{A}_g$. Since this is a projective fibration the Leray spectral sequence degenerates at E_2-level. The main point is then to compute the number of trivial local systems in $R^q\pi_*^{\times n}(\mathbb{Q})$, which can be done by representation theory, and to compare this number to the number of polynomials of given degree in the classes λ_i, T_i and $P_{i,j}$. The result then follows since by [GZ12] the classes T_i and $P_{i,j}$ do not fulfill non-trivial relations in degree $\leq g$.

As an easy corollary of this result one obtains stable cohomology for Mumford's partial toroidal compactification $\mathcal{A}'_g = \mathcal{A}_g \cup \mathcal{X}_{g-1}$, parametrizing ppav together with torus rank 1 degenerations. This partial compactification of $\mathcal{A}_g$ is contained in all toroidal copactifications $\mathcal{A}_g^{\mathrm{tor}}$ as the part which, under the projection to $\mathcal{A}_g^{\mathrm{Sat}}$, lives over $\mathcal{A}_g \cup \mathcal{A}_{g-1}$.

Proposition 5. *The partial compactificaton $\mathcal{A}'_g$ has stable cohomology. More precisely, for $k < g$ one has*

$$H^k(\mathcal{A}'_g, \mathbb{Q}) = \mathbb{Q}^k[D, \lambda_1, \lambda_3, \ldots]$$

where D is the (class of) the boundary and has degree 2.

The proof of this consists of an application of the Gysin exact sequence for the pair $(\mathcal{A}'_g, \mathcal{X}_{g-1})$ for cohomology with compact support. Since $\mathcal{A}'_g$ and $\mathcal{X}_{g-1}$ are smooth (as stacks) one can then dualize to cohomology.

4. Stable cohomology of $\mathcal{A}_g^{\mathrm{Perf}}$

The projection $p : \mathcal{A}_g^{\mathrm{Perf}} \to \mathcal{A}_g^{\mathrm{Sat}}$ defines a stratification of $\mathcal{A}_g^{\mathrm{Perf}}$ into strata $\beta_i = p^{-1}(\mathcal{A}_{g-i})$. Each stratum β_i is itself stratified into strata $\beta_i = \sqcup\beta_i(\sigma)$ where

σ runs through all cones in the perfect cone decomposition of $\mathrm{Sym}^2_{\geq 0}(\mathbb{Q}^i)$ whose general element is a rank i matrix. The strata $\beta_i(\sigma)$ are finite quotients of torus bundle $\mathcal{T}_i(\sigma)$ over i-fold products $\mathcal{X}^{[i]}_{g-i} \to \mathcal{A}_{g-i}$ by a group $G(\sigma)$, which is the stabilizer of σ in $\mathrm{GL}(i,\mathbb{Z})$ (for details see [HT11]). The strategy is then to compute the $G(\sigma)$-invariant stable cohomology of $\mathcal{T}_i(\sigma)$ and thus the stable cohomology of $\beta_i(\sigma)$. One can then use the Gysin spectral sequence for cohomology with compact support to obtain information on the stable cohomology with compact support for the strata β_i and, after another use of the Gysin spectral sequence, for $\mathcal{A}_g^{\mathrm{Perf}}$ itself. We note that one cannot translate this back into cohomology as $\mathcal{A}_g^{\mathrm{Perf}}$ is a singular space. One finally obtains

Theorem 6. *The cohomology groups $H^{\mathrm{top}-k}(\mathcal{A}_g^{\mathrm{Perf}},\mathbb{Q})$ stabilize for $k < g-1$ (where $\mathrm{top} = g(g+1)$ is the real dimension of $\mathcal{A}_g^{\mathrm{Perf}}$).*

We finally remark that J. Giansiracusa and G. K. Sankaran have independently from us obtained stabilization results for $H^k(\mathcal{A}_g^{\mathrm{matr}},\mathbb{Q})$ where $\mathcal{A}_g^{\mathrm{matr}}$ is the partial compactification of $\mathcal{A}_g$ given by the matroidal locus.

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Classical vs Tropical Brill-Noether Theory

Margarida Melo

(joint work with Lucia Caporaso)

1. Introduction

Classical Brill-Noether theory is the study of linear series on smooth curves. For given degree d, genus g and rank r, one expects the space of linear series of degree d and rank r on a curve C of genus g, denoted by $W^r_d(C)$, to be either empty or to have a certain dimension, given by the so-called Brill-Noether number $\rho = g-(r+1)(g-d+r)$. The Brill-Noether theorem, first proved by Griffiths and Harris in [5] ensures that the expectation holds for general curves of given genus. Classical proofs of the Brill-Noether theorem use degeneration and semicontinuity arguments. Still, Brill-Noether varieties of singular, specially reducible, curves are hard to deal with and in several aspects it is not even clear what to expect.

Let X be a nodal curve and let $f : \mathcal{X} \to B$ be a regular one parameter smoothing of X over a smooth curve B, i.e., $\mathcal{X}_{b_0} \cong X$ and, for $b \neq b_0$, $\mathcal{X}_b$ is a smooth curve. Let also $\mathcal{L}$ be a line bundle of a certain degree d over $\mathcal{X}$. Then, given an irreducible component $C \subset X$, $\mathcal{O}_{\mathcal{X}}(-C)$ is a line bundle, a so called twister, and $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(-C)_{|\mathcal{X}_b} \cong \mathcal{L}_{\mathcal{X}_b}, \forall b \neq b_0$. This phenomena immediately shows that $\mathcal{L}_{|\mathcal{X}_b}$ can specialize to elements in $W^d_r(X)$ for all $r \geq 0$. In fact, by twisting enough times, one gets that the degree of $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(-nC)$ in C gets as big as we want, and along with it the rank of $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(-nC)_{|\mathcal{X}_{b_0}}$ gets to infinity.

2. Combinatorial rank

For any nodal curve X, we can associate to it the dual (weighted) graph of G of X, whose set of vertices corresponds to the irreducible components of X and such that edges connecting two vertices (who might be the same) correspond to nodes lying in the correspondent components. Likely, for any divisor D in Div X, let $\underline{d} \in$ Div G be the divisor in G associated to the multidegree of D on X. There is a well-established theory of linear series on graphs, most notably due to the pioneering work of Baker and Norine in [2] for graphs with no loops nor weights and to Amini-Caporaso in [1] in the general case.

Given a graph G, its divisor group Div G is the free abelian group generated by $V(G)$, the set of vertices of G. So, an element in Div G has the form $\underline{d} = \sum_{v \in V(G)} \underline{d}(v)v$. The degree of a divisor $\underline{d}$ is defined as $\sum_{d \in V(G)} d(v)$ and the set of effective divisors is $\mathrm{Div}_+(G) := \{\underline{d} \in$ Div $G : \underline{d}(v) \geq 0, \forall v \in V(G)\}$. We write $\underline{d} \geq 0$ if $\underline{d} \in \mathrm{Div}_+ G$. The group of principal divisors Prin G can be defined as the group of multidegrees of twisters of a curve with G as dual graph. Then we have

Definition. *Let $\underline{d}$ and $\underline{d}'$ be two divisors on G. Then $\underline{d}$ and $\underline{d}'$ are said to be linearly equivalent, and we write $d \sim d'$ if $\exists \in$ Prin G such that $\underline{d} - \underline{d}' = t$.*
We then define Pic $G :=$ Div $G/\sim$.

Given a graph with no loops nor weights, Baker and Norine defined in [2] the combinatorial rank of a divisor $\underline{d} \in$ Div G, $r_G(\underline{d})$, as follows:

$$r_G(\underline{d}) = \max\{r : \forall \underline{e} \geq 0, |\underline{e}| = r, \exists \underline{t} \in \text{ Prin } G : \underline{d} - \underline{e} + \underline{t} \geq 0\}.$$

In the case when G has loops or weights, the combinatorial rank is defined according to Amini and Caporaso in [1] using an auxiliary graph $\widehat{G}$, obtaining by adding $w(v)$(=weight of v) loops on each vertex v and then by inserting a vertex in each loop of this new graph. The divisor $\underline{d}$ clearly extends to a divisor $\underline{\hat{d}}$ on G by putting $d(v) = 0$ on all vertices $v \in V(\widehat{G}) \setminus V(G)$ and one then defines $r_G(\underline{d}) := r_{\widehat{G}}(\underline{\hat{d}})$.

The combinatorial rank $r_G(\underline{d})$ of a divisor $\underline{d} \in$ Div G is clearly independent of the representative of the linear equivalence class of $\underline{d}$, so it makes sense to write $r_G(\delta)$ for the combinatorial rank of a divisor class $\delta \in$ Pic G as well.

Given, as before, a one parameter smoothing $f : \mathcal{X} \to B$ of a curve X with dual graph equal to G and a line bundle $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}_{|\mathcal{X}_{b_0}}$ has multidegree $\underline{d}$, Baker's specialization lemma states that $r_G(\underline{d})$ is bigger or equal than $r(\mathcal{X}_b, \mathcal{L}_{|\mathcal{X}_b})$ for $b \neq b_0$ varying in a certain open neighborhood of b_0. Notice that this result was a very important tool in the recent proof by Cools, Draisma, Payne and Roveba in [4] of the Brill-Noether theorem using linear series on graphs.

3. Algebraic rank

Given a graph G, denote by $\mathrm{M}^{\mathrm{alg}}\mathrm{G}$ the set of nodal curves X whose dual graph is equal to G. Recently, in [3], L. Caporaso defined the following notion of *algebraic rank* of a divisor on G, using linear series on nodal curves $X \in \mathrm{M}^{\mathrm{alg}}\mathrm{G}$.

Let then $\delta \in \mathrm{Pic}\ G$ be a divisor class. For any $\underline{d} \in \delta$ and $X \in \mathrm{M}^{\mathrm{alg}}\mathrm{G}$, we set

$$\mathrm{r}^{\max}(\mathrm{X}, \underline{\mathrm{d}}) := \max\{\mathrm{r}(\mathrm{X}, \mathrm{L}),\ \forall \mathrm{L} \in \mathrm{Pic}^{\underline{\mathrm{d}}}(\mathrm{X})\} = \max\{\mathrm{r} : \mathrm{W}^{\mathrm{r}}_{\underline{\mathrm{d}}}(\mathrm{X}) \neq \emptyset\}.$$

Next, we set

$$\mathrm{r}(\mathrm{X}, \delta) := \min\{\mathrm{r}^{\max}(\mathrm{X}, \underline{\mathrm{d}}),\ \forall \underline{\mathrm{d}} \in \delta\}.$$

Finally set

$$\mathrm{r}^{\mathrm{alg}}(\mathrm{G}, \delta) := \max\{\mathrm{r}(\mathrm{X}, \delta),\ \forall \mathrm{X} \in \mathrm{M}^{\mathrm{alg}}(\mathrm{G})\}.$$

Then the following is Conjecture 1 in [3]

Conjecture 1. *Let G be a graph and $\delta \in \mathrm{Pic}(\mathrm{G})$. Then*

$$\mathrm{r}^{\mathrm{alg}}(\mathrm{G}, \delta) = \mathrm{r}_{\mathrm{G}}(\delta).$$

Conjecture 1 is shown in [3] to hold in the following cases:

(1) $g \leq 1$;
(2) $d \leq 0$ and $d \geq 2g - 2$;
(3) $|V(G)| = 1$;
(4) G is a stable graph of genus 2.

Moreover, in loc. cit. it is also proved that if Conjecture 1 holds for semistable graphs, i.e., graphs G such that all vertices of G of weight zero have at least two incident half-edges (in other words, dual graphs of Deligne-Mumford semistable curves), then it holds generally for all nodal curves.

4. Our results

In a joint work on progress with Lucia Caporaso we show furthermore that the algebraic rank satisfies nice properties as the Riemann-Roch formula and that Conjecture 1 holds in the following cases.

Theorem. *Let G be any graph and $\delta \in$ Pic G a divisor class. Then*

(1) $\mathrm{r}^{\mathrm{alg}}(\mathrm{G}, \delta) \leq \mathrm{r}_{\mathrm{G}}(\delta)$;
(2) *if moreover G is either loopless and weightless or if $r_G(\delta) = -1, 0$, we have that $r^{alg}(G, \delta) = r_G(\delta)$.*

We are currently working towards proving that Conjecture 1 is true in general for any graph and any divisor class.

Finally, we would like to mention that it is interesting to exploit which consequences can our results have for the study of linear series on nodal curves. For instance, Clifford's inequality trivially fails for reducible curves for the reasons we explained above. However, we have the following:

Proposition (Clifford)**.** *Let $\delta \in$ Pic G with degree d satisfying* $0 \leq d \leq 2g-2$. *Then*

$$\mathrm{r}^{\mathrm{alg}}(\mathrm{G}, \delta) \leq \left\lfloor \frac{\mathrm{d}}{2} \right\rfloor .$$

The above result implies that given any nodal curve X and $0 \leq d \leq 2g-2$, there is a certain multidegree $\underline{d}$ with total degree d such that for every $L \in \mathrm{Pic}^{\underline{d}}(X)$, $r(X, L) \leq \lfloor \frac{d}{2} \rfloor$.

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Teichmüller modular forms and their relation to 'new' Galois representations in $H^*(\overline{M}_{3,n})$

Carel Faber

Teichmüller modular forms are sections of powers of the determinant of the Hodge bundle $\mathbb{E}$ on $\overline{M}_g$, or, more generally, of the vector bundles obtained by applying a Schur functor for an irreducible representation of $\mathrm{GL}(g)$ to $\mathbb{E}$. Teichmüller modular forms not coming from Siegel modular forms and vanishing on the boundary of M_g are of most interest. Such sections of $\det(\mathbb{E})^{\otimes k}$ were studied in detail by T. Ichikawa in the 1990's [3, 4, 5]. In particular, he proved that $\det(\mathbb{E})^{\otimes 9}$ admits a nonzero section χ_9 on $\overline{M}_3$ vanishing on $\overline{H}_3$, the closure of the locus of hyperelliptic curves, and the boundary divisors Δ_0 and Δ_1.

'Old' Galois representations in $H^*(\overline{M}_{3,n})$ are those that can be expressed in terms of L, the Lefschetz motive, and $S[k]$, $S[j,k]$, and $S[i,j,k]$, Galois representations associated to Siegel modular forms (in general vector valued) of genus 1, resp. 2, resp. 3 (cf. [1]). Let $\lambda = (a, b, c)$ with $a \geq b \geq c \geq 0$ be a weight for Sp_6 and denote by $\mathbb{V}_\lambda$ the local system for GSp_6 corresponding to λ on A_3, or its pull-back to M_3. ($\mathbb{V}_{(1,0,0)} = \mathbb{V} = R^1\pi_*\mathbb{Q}_\ell$ for $\pi\colon U \to A_3$ the universal principally polarized abelian threefold.)

We can prove that 'new' Galois representations, not expressible in the above terms, occur in $H^i_c(M_3, \mathbb{V}_\lambda)$ for $\lambda = (11, 3, 3)$ and $\lambda = (7, 7, 3)$ (probably for $i = 6$). For all other λ with $|\lambda| = a + b + c \leq 17$, it seems that old Galois representations suffice, but non-Tate-twisted terms of motivic weight $|\lambda| + 6$ are found for $\lambda = (5, 5, 5)$, $(8, 4, 4)$, and $(9, 5, 3)$. Teichmüller modular forms corresponding to the pieces of maximal Hodge degree have been constructed on $M_3 - H_3$ and seem to extend. For $(5,5,5)$, this is χ_9; for $(8,4,4)$, it is a vector valued Siegel modular form; the other cases are vector valued Teichmüller modular forms not coming from Siegel modular forms. The recent work of Chenevier and Renard [2] suggests that the two new Galois representations are six-dimensional. (Joint work with Jonas Bergström and Gerard van der Geer.)

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Fibrations in quartic del Pezzo surfaces

Brendan Hassett

(joint work with Yuri Tschinkel)

Let B be a smooth projective curve over an algebraically closed field k with $\mathrm{char}(k) \neq 2$. A *quartic del Pezzo surface fibration* is a flat projective morphism $\pi : \mathcal{X} \to B$ with fibers complete intersections of two quadrics in $\mathbb{P}^4$. We assume $\mathcal{X}$ is smooth and the singular fibers have at worst one ordinary double point; this is equivalent to the discriminant divisor $\Delta \subset B$ being square-free.

The fundamental invariant is the *height*

$$h(\mathcal{X}) = \deg(c_1(\omega_\pi)^3),$$

where ω_π is the relative dualizing sheaf. We may compute

$$h(\mathcal{X}) = -2\deg(\pi_*\omega_\pi^{-1}) = \deg(\Delta)/2,$$

as the Picard group of the moduli space is isomorphic to $\mathbb{Z}$. Indeed, the moduli space is isomorphic to $\mathbb{P}(1,2,3) \setminus s$, where s is a point where the minimal degree invariant is nonzero.

From now on, we assume that $B = \mathbb{P}^1$. In this case

$$\chi(\Omega^1_{\mathcal{X}}) = h^2(\Omega^1_{\mathcal{X}}) - h^1(\Omega^1_{\mathcal{X}}) = h(\mathcal{X}) - 7,$$

so that if $\mathrm{Pic}(\mathcal{X}) \simeq \mathbb{Z}^2$ then $h^2(\Omega^1_{\mathcal{X}}) = h(\mathcal{X}) - 5$. The expected number of parameters for $\mathcal{X}$ is $-\chi(T_{\mathcal{X}}) = \frac{3}{2}h(\mathcal{X}) - 1$.

The cohomology of $\mathcal{X}$ has a natural Prym construction: If $X \subset \mathbb{P}^4$ is a smooth quartic del Pezzo surface then $X = \{Q_0 = Q_1 = 0\}$ where Q_0 and Q_1 are homogeneous quadratic forms. Let

$$D = \{[t_0, t_1] : \mathrm{rank}(t_0Q_0 + t_1Q_1) < 5\} \subset \mathbb{P}^1,$$

which has degree five and parametrizes quadric cones containing X. Each quadric cone has two rulings, each of which traces out a conic fibration on X. Given a fibration $\mathcal{X} \to \mathbb{P}^1$ as above with generic fiber X, the conic fibrations and quadric cones are parametrized by finite morphisms

$$\tilde{\mathcal{D}} \to \mathcal{D} \to \mathbb{P}^1,$$

where the first is étale of degree two and the second has degree five. The Galois group of $\widetilde{\mathcal{D}} \to \mathbb{P}^1$ is a subgroup of the Weyl group $W(D_5)$; we say π has *maximal monodromy* if it is the full Weyl group. Kanev [5] has shown that the intermediate Jacobian of $\mathcal{X}$ may be expressed

$$\mathrm{IJ}(\mathcal{X}) = \mathrm{Prym}(\widetilde{\mathcal{D}}/\mathcal{D}).$$

Some natural questions include

(1) Are there fibrations with square-free discriminant of height two? With height six and maximal monodromy?
(2) Are the fibrations with square-free discriminant and maximal monodromy of a given height irreducible?
(3) How do we characterize the Prym varieties that arise from del Pezzo fibrations?

Our main interest is in sections $\sigma : \mathbb{P}^1 \to \mathcal{X}$ of π. The *height* of a section is

$$h_{\omega_\pi^{-1}}(\sigma) = \deg(\sigma^*\omega_\pi^{-1}) = \deg(N_\sigma),$$

the degree of its normal bundle. The deformation space of σ has dimension at least $h_{\omega_\pi^{-1}}(\sigma) + 2$. We expect there exists an $h \in \mathbb{N}$ and a canonically defined irreducible component of

$$\mathrm{Sect}(\pi, h) = \{\text{sections } \sigma : \mathbb{P}^1 \to \mathcal{X} : h_{\omega_\pi^{-1}}(\sigma) = h\}$$

that is birational to $\mathrm{Prym}(\widetilde{\mathcal{D}}/\mathcal{D})$, or perhaps a rationally connected fibration over this variety.

This has arithmetic significance: If everything is defined over a finite field $\mathbb{F}_q$ then our expectation would imply $\mathcal{X} \to \mathbb{P}^1$ has a section defined over $\mathbb{F}_q$. This follows from results of Lang [6] (that principal homogeneous spaces over abelian varieties split over finite fields) and Esnault [2] (that rationally connected varieties over finite fields admit rational points). This would imply long-standing conjectures [1] on the existence of rational points on del Pezzo surfaces over function fields.

We turn to a concrete example, where $\mathcal{X} \to \mathbb{P}^1$ has height twelve. Note the natural inclusion

$$\begin{array}{ccccc} \mathcal{X} & \subset & \mathbb{P}(\pi_*\omega_\pi^{-1}) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1)^4 \oplus \mathcal{O}_{\mathbb{P}^1}) & \subset & \mathbb{P}^1 \times \mathbb{P}^8 \\ \downarrow\pi & & & & \\ \mathbb{P}^1 & & & & \end{array}$$

where $\mathcal{X}$ contains the canonical section $\sigma_0 : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^8$ associated with the summand $\mathcal{O}_{\mathbb{P}^1}$. Are there any other sections?

We consider two natural contractions: Projection onto the second factor induces

$$\mathcal{X} \to \mathcal{Y} := \pi_2(\mathcal{X}) \subset \mathbb{P}^8,$$

a small contraction of the canonical section; the image is a nodal Fano threefold of degree 12. Fiberwise projection from σ_0 gives

$$\mathbb{P}^1 \times \ell \subset \widetilde{\mathcal{X}} := \mathrm{Bl}_{\sigma_0(\mathbb{P}^1)}(\mathcal{X}) \subset \mathbb{P}^1 \times \mathbb{P}^3,$$

where ℓ is the exceptional line and $\widetilde{\mathcal{X}}$ is a pencil of cubic surfaces $\{t_0F_0+t_1F_1 = 0\}$ with base locus

$$\{F_0 = F_1 = 0\} = \ell \cup C \subset \mathbb{P}^3.$$

The curve $C \subset \mathbb{P}^3$ is a tetragonal curve of genus seven, embedded in $\mathbb{P}^3$ via the adjoint $K_C - g^1_4$, and ℓ is its four-secant line. Here the intermediate Jacobian $\mathrm{IJ}(\mathcal{X}) \simeq \mathrm{J}(C)$, the Jacobian of C.

Theorem 1 ([3, 4])**.** *There exist a distinguished irreducible component of* $\mathrm{Sect}(\pi, 5)$ *birational to* $\mathrm{J}(C)$.

These have an explicit interpretation: Given a generic divisor class $A \in \mathrm{Pic}^{17}(C)$, there exists a unique sextic rational curve $R \subset \mathbb{P}^3$ such that $R \cap C \in |A|$. The proper transform of this curve in $\mathcal{X}$ yields the desired section.

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Conformal blocks divisors and the birational geometry of $\overline{M}_{g,n}$

Angela Gibney

1. Vector bundles of conformal blocks

There is a seemingly endless supply of vector bundles $\mathbb{V} = \mathbb{V}(\mathfrak{g}, \ell, \underline{\lambda})$ on the moduli stack $\overline{\mathcal{M}}_{g,n}$, of stable n pointed curves of genus g, constructed from the data of a simple Lie algebra $\mathfrak{g}$, a positive integer ℓ, and an n-tuple $\underline{\lambda}$ of dominant weights for $\mathfrak{g}$ of level ℓ. We refer to $\mathbb{V}$ as a vector bundle of conformal blocks, since over a closed point $X = (C, \underline{p})$ in $\mathcal{M}_{g,n}$ corresponding to a *smooth curve* C, the vector space $\mathbb{V}|_{[X]}$ can be identified with a conformal block, a basic object in the WZW model of rational conformal field theory [Bea96, TUY89]. One could also call $\mathbb{V}$ a vector bundle of generalized theta functions. For example, in case $\mathfrak{g} = \mathfrak{sl}_{r+1}$, one also has the beautiful description that over a closed point $X = (C, \underline{p})$ in the interior of $\overline{\mathcal{M}}_{g,n}$, the conformal block $\mathbb{V}|_{[X]}$ is canonically isomorphic to the vector space of generalized parabolic theta functions $H^0(SU_C(r+1, \underline{\lambda}), \mathcal{L})$, where $SU_C(r+1, \underline{\lambda})$ is the moduli space of semi-stable vector bundles of rank $r+1$ with trivial determinant bundle, and with parabolic structure of type $\underline{\lambda}$ on the marked curve X, and $\mathcal{L}$ is a canonical element of its Picard group [Pau96, LS97].

There is no such interpretation of the vector space $\mathbb{V}|_{[X]}$ as a conformal block or as a space of generalized theta functions in case $X = (C, \underline{p}) \in \overline{\mathcal{M}}_{g,n}$ is on the boundary – in other words, when the curve C has a node. Nevertheless, these bundles have been constructed by Tsuchiya, Ueno and Yamada on the entire stack $\overline{\mathcal{M}}_{g,n}$, including at points on the boundary [TUY89, Uen08].

2. Conformal blocks divisors and morphisms

An important feature of vector bundles of conformal blocks is that when $g = 0$ they give rise to morphisms from the fine moduli spaces $\overline{\mathrm{M}}_{0,n}$ to other projective *varieties.* This comes from the fact, proved by Fakhruddin in [Fak12], that every vector bundle $\mathbb{V}$ of conformal blocks on $\overline{\mathcal{M}}_{0,n}$ is globally generated, and hence its first Chern class $c_1(\mathbb{V})$ is a semi-ample divisor. As Fakhruddin shows, this is not always true for $g > 0$. While some vector bundles of conformal blocks, like the Hodge bundle on $\overline{\mathcal{M}}_g$, are generated by their global sections, others are not.

Fakhruddin has given a recursive formula for the first Chern classes $c_1(\mathbb{V})$. We have learned that these *Conformal blocks divisors* often occur naturally in families having interesting properties. For example, the nonzero divisors $\{c_1\mathbb{V}(\mathfrak{sl}_2, 1, \underline{\lambda}) : \underline{\lambda}\}$ forms a basis for the Picard group of $\overline{\mathcal{M}}_{0,n}$ [Fak12], and $\{c_1\mathbb{V}(\mathfrak{sl}_n, 1, \underline{\lambda}) : S_n \text{ invariant } \underline{\lambda}\}$ forms a basis for $\overline{\mathcal{M}}_{0,n}/S_n$ [AGSS12]. All but one of the divisors $c_1\mathbb{V}(\mathfrak{sl}_n, 1, \underline{\lambda})$ lie on extremal faces of the nef cone, $\mathrm{Nef}(\overline{\mathcal{M}}_{0,n})$, and as we show in [AGSS12], the divisors $c_1\mathbb{V}(\mathfrak{sl}_n, 1, \underline{\lambda})$ for S_n- invariant $\underline{\lambda}$ span extremal rays of $\mathrm{Nef}(\overline{\mathcal{M}}_{0,n}/S_n)$. Similar statements can be made about the family of $\mathfrak{sl}_2$ conformal blocks divisors with weights $\lambda_i = \omega_1$, and varying level, studied in [AGS10].

We have identified a number of morphisms associated to conformal blocks divisors on $\overline{\mathcal{M}}_{0,n}$, including those which have already figured prominently in the literature [Fak12, AGS10, AGSS12, Gib12], as well as new maps [Gia10, GG12, GJMS12, Fed11]. For example, there are natural birational models of $\overline{\mathrm{M}}_{0,n}$ obtained via Geometric Invariant Theory which are moduli spaces of pointed rational normal curves of a fixed degree d, where the curves and the marked points are weighted by non-negative rational numbers $(\gamma, A) = (\gamma, (a_1, \cdots, a_n))$ [Gia10, GS11, GJM11]. These so-called Veronese Quotients specialize to nearly every known compactification of $\mathrm{M}_{0,n}$ [GJM11]. There are birational morphisms from $\overline{\mathrm{M}}_{0,n}$ to these GIT quotients, and these maps have been shown to be given by conformal blocks divisors in many special cases [Gia10, GG12, GJMS12]. I believe this always true.

3. Conformal blocks divisors and the Mori Dream Space Conjecture

The Mori Dream Space Conjecture of Hu and Keel [HK00] says that the Cox Ring of $\overline{\mathrm{M}}_{0,n}$ is finitely generated, and so is a "dream space" from the point of view of Mori Theory. For example, if this conjecture were true, then the cone of nef divisors would be the convex hull of a finite number of extremal rays. A second implication would be that every element of $\mathrm{Nef}(\overline{\mathrm{M}}_{0,n})$ would be a semi-ample divisor. In other words, if $\overline{\mathrm{M}}_{0,n}$ is a Mori Dream Space, then one could hope to explicitly describe all the nef divisors, and all the morphisms from $\overline{\mathcal{M}}_{0,n}$ to any projective variety.

One could regard the conformal blocks divisors as both support for and evidence against the Mori Dream Space Conjecture. On the one hand, the fact that there are so many – a potential infinitude – of conformal blocks divisors, lends support to the implication that every nef divisor on $\overline{\mathrm{M}}_{0,n}$ might be semi-ample. If the cone generated by conformal blocks divisors lies properly inside the cone of net divisors, then while it may be interesting if it is not polyhedral, it won't have an impact on the Mori Dream Space Conjecture.

On the other hand, especially taking into account the fact that most of the conformal blocks divisors that we have studied lie on extremal faces of the nef cone, it is natural to ask whether these cones are distinct. In the fantastic event that every nef divisor were a conformal blocks divisor, then every nef divisor would of course be semi-ample. But then if the cone of conformal blocks divisors is not finitely generated, $\overline{\mathrm{M}}_{0,n}$ would not be a Mori Dream Space. Giansiracusa and I considered a special case of this question, studying the subcone of $\mathrm{Nef}(\overline{\mathcal{M}}_{0,n})$ generated by level one divisors $\{c_1\mathbb{V}(\mathfrak{sl}_k, 1, \underline{\lambda}) : \underline{\lambda}\}$, and we show that this cone is indeed finitely generated [GG12].

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Toward a geometric compactification of the moduli space of polarized K3 surfaces

Bernd Siebert
(joint work with Mark Gross, Paul Hacking, and Sean Keel)

The deformation type of a polarized K3 surface is determined by a single integer $h \geq 4$, the degree of a general hyperplane section. Period theory provides a description of the corresponding analytic moduli stack F_h as a quotient $\mathbb{D}_h/\Gamma_h$ of a bounded symmetric domain $\mathbb{D}_h$ by an arithmetic group Γ_h. The underlying coarse moduli space is a quasiprojective scheme. It has a Baily-Borel compactification F_h^{BB} which only adds strata of dimension zero and one, called 0- and 1-cusps. This compactification, however, is too small to support an extension of the universal family of K3 surfaces.

At the other extreme are various toroidal compactifications [1][15], which add divisors to arrive at a projective scheme with toroidal singularities. This compactification depends on the choice of a compatible collection Σ of infinite fans, one for each cuspidal point, leading to a partial resolution $F_h^{\Sigma} \to F_h^{\mathrm{BB}}$. However, no proposal has been made for a toroidal compactification that supports a family of K3 surfaces.

Morrison pointed out that mirror symmetry sometimes provides canonical choices of Σ by the Mori fan of a mirror degeneration [12]. The mirror family for F_h^{BB} is a one-dimensional family of lattice polarized $K3$-surfaces [2][3]. This project started by the observation of Paul Hacking and Sean Keel that my joint construction with Mark Gross of degenerating families of Calabi-Yau varieties [8] may provide a canonical family of degenerating K3 surfaces over this toroidal compactification. One basic problem is that the singularities of such a degeneration can not only be of the type treated in [8]. This problem has been solved in an affine situation in [4] by combining the construction of [8] away from codimension two with the results of [6] and the construction of global functions. The point of using [6] is that Gromov-Witten theory on the mirror variety provides the coefficients of a consistent scattering diagram at the singular point. Scattering diagrams are the bookkeeping device in [8] for the gluing data of standard affine patches, much as in the construction of cluster varieties. In the global situation the ring of global functions is replaced by the homogeneous coordinate ring, using the fact that our construction comes with canonical sections of the polarizing line bundle [5].

We then have the following central technical result.

Theorem 1. *Let $\mathcal{Y} \to S$ be a semistable model (normal crossings central fibre) of a cusp of the one-dimensional mirror family. Then for each embedding of the cone of effective curves* $\mathrm{NE}(\mathcal{Y})$ *of $\mathcal{Y}$ into a sharp toric monoid P, there is a canonical degeneration*

$$\mathfrak{X} \longrightarrow \mathrm{Spec}\big(\mathbb{C}[[P]]\big)$$

of h-polarized K3 surfaces. The central fibre $X_0 \subset \mathcal{X}$ is a union of $\mathbb{P}^2$'s, one for each zero-stratum of the central fibre $Y_0 \subset \mathcal{Y}$.

Recall that a sharp toric monoid is the submonoid of some $\mathbb{Z}^r$ of integral points of a strictly convex, rational polyhedral cone. In particular, the interior integral points form an ideal of P, and $\mathbb{C}[[P]]$ denotes the completion with respect to this ideal. Thus $\mathrm{Spec}\,\big(\mathbb{C}[[P]]\big)$ is the completion of an affine toric variety with a zero stratum along the toric boundary.

The construction comes with many interesting features, notably reflecting the birational geometry of $\mathcal{Y}$ in terms of the deformation theory of X_0. For example, one can see that the general points of the (type III) toric boundary of $\mathrm{Spec}\,\big(\mathbb{C}[[P]]\big)$ correspond to log K3 surfaces with the kind of singularities arising in our mirror symmetry program [7][8]. Implicit in the construction is also the existence of a torus worth of

To make the connection to toroidal compactifications of F_h one notes that the support of the fan Σ at a 0-cusp is naturally the nef cone of the generic fibre $\mathcal{Y}_{\mathrm{gen}}$. Thus if $\mathrm{Pic}\,(\mathcal{Y}) \to \mathrm{Pic}\,(\mathcal{Y}_{\mathrm{gen}})$ were an isomorphism each maximal cone in Σ would give a choice of P, provided Σ refines the Mori fan. This is not quite true, because for a semistable model, $\mathrm{Pic}\,(\mathcal{Y}) \to \mathrm{Pic}\,(\mathcal{Y}_{\mathrm{gen}})$ is only an epimorphism with fibres of rank $g = 2h - 2$. The overcount of g gets reflected in the action of a g-dimensional algebraic torus on $\mathcal{X}$. The way out is to restrict to a canonical slice of the action that curiously is suggested by those models of the one-dimensional mirror family which contract all but one component of a semistable model.

At this point we have a family of K3 surfaces over the formal completion of a toroidal compactification F_h^{Σ} along the boundary divisor, provided the fans Σ refine the Mori fan of the one-dimensional mirror family at each maximal degeneration point. To patch to the existing family over F_h we intend to use an explicit calculation of period integrals along with a descent result due to Moret-Bailly [11]. The relevant period integrals turn out to be proportional to $\log(z^p)$ for some $p \in P$. This is a manifestation of the fact that our construction produces families in the canonical coordinates suggested by mirror symmetry.

Of course, several details remain to be filled in, but the established results clearly point to the existence of distinguished toroidal compactifications of F_h that support a family of $K3$ surfaces. Concerning a modular meaning of these compactifications, I would like to point out that our family is locally trivial as a family of polarized schemes over large parts of the compactifying divisor. Thus our families are not versal in the scheme-theoretic sense. Our compactification neither appears to be embeddable into stable pairs moduli. My personal opinion is that one should rather add the data of a certain log structure to the degenerate K3 surfaces, modifying Olsson's moduli stack of semistable log K3 surfaces [13]. An approach along these lines, however, would require a better understanding of the singularities of the log structure at the zero-dimensional strata of X_0.

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The moduli stack of semistable curves

Jarod Alper
(joint work with Andrew Kresch)

A fundamental question in the theory of moduli spaces is to characterize which algebraic stacks are global quotient stacks. Recall that an algebraic stack $\mathcal{X}$ is a *global quotient stack* if $\mathcal{X} \cong [X/\operatorname{GL}_n]$ where X is an algebraic space. For a noetherian algebraic stack $\mathcal{X}$ with affine diagonal, there are the following characterizations:

$\mathcal{X} \cong [\text{algebraic space}/\operatorname{GL}_n]$	$\iff$	there exists a vector bundle on $\mathcal{X}$ such that the stabilizers act faithfully on the fibers
$\mathcal{X} \cong [\text{quasi-affine}/\operatorname{GL}_n]$	$\iff$	$\mathcal{X}$ satisfies the resolution property; i.e., every coh. sheaf is the quotient of a vector bundle
$\mathcal{X} \cong [\text{affine}/\operatorname{GL}_n]$	$\underset{\text{char}\,=\,0}{\iff}$	for every coherent sheaf $\mathcal{F}$, $\mathrm{H}^i(X, \mathcal{F}) = 0$ for $i > 0$, and $\mathcal{X}$ satisfies the resolution property

The first characterization follows from the usual relationship between principal GL_n-bundles and vector bundles. The second characterization is due to Totaro [11] and generalized by Gross [6]. To summarize the known general results concerning global quotient stacks, we have:

- Every smooth, separated Deligne-Mumford stack with generically trivial stabilizer is a global quotient stack [5].
- In characteristic 0, every separated Deligne-Mumford stack with quasi-projective coarse moduli space is a global quotient stack [10].
- For a banded $\mathbb{G}_m$-gerbe $\mathcal{X} \to X$ which corresponds to $\alpha \in \mathrm{H}^2(X, \mathbb{G}_m)$, $\mathcal{X}$ is a global quotient stack if and only if α is in the image of the Brauer map $\operatorname{Br}(X) \to \mathrm{H}^2(X, \mathbb{G}_m)$ [5].

In particular, [5] uses the third result to produce the first examples of non-quotients stacks. The general question regarding which algebraic stacks are quotient stacks appears to be quite difficult. It is not known for instance whether every separated Deligne-Mumford stack is a global quotient stack. Indeed, as the above results indicate, the question is related to both global geometric properties as well as arithmetic properties. Instead, we turn our attention to the local structure of algebraic stacks. It is natural to conjecture as in [2] that algebraic stacks are étale locally quotient stacks. Precisely,

Conjecture 1. *Let $\mathcal{X}$ be an algebraic stack with separated and quasi-compact diagonal of finite type over an algebraically closed field k. Suppose that all points have affine stabilizer groups. Let $x \in \mathcal{X}(k)$ be a point with linearly reductive stabilizer. Then there exists an étale, representable morphism*

$$f\colon [\operatorname{Spec}(A)/G_x] \to \mathcal{X}$$

and a point w above x such that f induces an isomorphism of stabilizer groups.

There are various variants of Conjecture 1 where one can replace the affine scheme $\mathrm{Spec}(A)$ with an algebraic space, or remove the condition that f is stabilizer preserving at w. However, the above is the most desirable conjecture that one could hope is true. For instance, if $\mathcal{X}$ admits a coarse moduli space (or good moduli space) $\phi\colon \mathcal{X} \to X$, then there is a morphism f as above which is the base change of an étale morphism $\mathrm{Spec}(A^{G_x}) \to X$. The condition that *all* points have affine stabilizers is necessary–if $\mathcal{E} \to C$ is a family of smooth elliptic curves degenerating to a nodal cubic over a smooth curve C, then $B(\mathrm{Aut}(\mathcal{E}))$ does not satisfy Conjecture 1 around the point with $\mathbb{G}_m$-stabilizer. Similarly, the linearly reductive hypothesis is necessary as non-linearly reductive groups are not rigid. Indeed, $\mathbb{G}_a$ to deforms to $\mathbb{G}_m$ in a family $\mathcal{G} \to \mathbb{A}^1$ and the corresponding stack $B(\mathcal{G})$ cannot satisfy Conjecture 1. We have the following evidence for the conjecture:

- It is true for Deligne-Mumford stacks essentially by the Keel-Mori theorem [9].
- It is true for tame Artin stacks [1].
- It is true for gerbes over Deligne-Mumford stacks [8].
- It is true for quotient stacks $[X/G]$ with X a separated, normal scheme and G a connected algebraic group by an application of Sumihiro's theorem and a Luna-slice argument [2].

The motivation for Conjecture 1 comes from several sources. First, as algebraic stacks are ubiquitous in algebraic geometry, it is natural to try to understand their local structure. Similar to how affine schemes are the building blocks for schemes, it would be desirable to know in what sense the stacks $[\mathrm{Spec}(A)/\mathrm{GL}_n]$ are the building blocks for algebraic stacks. Since quotient stacks and, in particular, stacks of the form $[\mathrm{Spec}(A)/\mathrm{GL}_n]$ are particularly simple, this conjecture would imply that many properties of general algebraic stacks can be reduced to quotient stacks. Finally, this conjecture arises naturally in the context of developing an intrinsic and systematic procedure to construct projective moduli spaces parameterizing objects with infinite automorphisms–see [4]. In particular, we have:

Theorem. [4] *Let $\mathcal{X}$ be an algebraic stack of finite type over k. Suppose that:*

(1) *For every closed point $x \in \mathcal{X}$, there exists an affine, étale neighborhood $f\colon [\mathrm{Spec(A)/G_x}] \to \mathcal{X}$ of x such that f is stabilizer preserving at closed points of $[\mathrm{Spec(A)/G_x}]$ and f sends closed points to closed points.*
(2) *For any $x \in \mathcal{X}(k)$, the closed substack $\overline{\{x\}}$ admits a good moduli space.*

Then $\mathcal{X}$ admits a good moduli space.

While the above theorem is not particularly hard to prove, it is interesting because although conditions (1) and (2) may seem very technical, they can be in fact verified in practice. The above theorem can be used to generalize [7] to construct the second flip of $\overline{\mathcal{M}}_g$:

Theorem. (–, Fedorchuk, Smyth) *For $\alpha > \frac{2}{3} - \epsilon$, there are moduli interpretations of the log-canonical models*

$$\overline{M}_g(\alpha) = Proj \bigoplus_d \Gamma(\mathcal{M}_g, (K + \alpha\delta)^{\otimes d}).$$

The above theorem was the topic of the Oberwolfach report [3]. In this report, we prefer to focus only on Conjecture 1 and, in particular, its validity in a particular interesting example.

Let $\mathfrak{M}_g^{\text{ss}}$ be the algebraic stack parameterizing Deligne-Mumford semistable curves where one allows rational components to meet the curve in only two points. Also consider the substack $\mathfrak{M}_g^{\text{ss},\leq 1} \subseteq \mathfrak{M}_g^{\text{ss}}$ parameterizing semistable curves with at most one exceptional component. These algebraic stacks have some striking properties. First, there is a stabilization morphism $\text{st}\colon \mathfrak{M}_g^{\text{ss},\leq 1} \to \overline{\mathcal{M}}_g$ which is an isomorphism over the open substack $\overline{\mathcal{M}}_g \subseteq \mathfrak{M}_g^{\text{ss},\leq 1}$, whose complement has codimension 2. Moreover,

- $\mathfrak{M}_g^{\text{ss},\leq 1}$ is not a quotient stack. Indeed, if $\mathcal{V}$ is any vector bundle on $\mathfrak{M}_g^{\text{ss},\leq 1}$, then $\text{st}^*(\mathcal{V})|_{\overline{\mathcal{M}}_g} \cong \mathcal{V}$ since they agree in codimension 2. It follows that $\mathcal{V}$ cannot have a faithful action of the stabilizer at a strictly semistable curve.
- $\mathfrak{M}_g^{\text{ss},\leq 1}$ does not have a quasi-affine diagonal.
- The fiber of a general curve with one node and smooth normalization under the stabilization morphism is isomorphic to $[T/\mathbb{G}_m]$ where T is the nodal cubic (rather than $[\mathbb{A}^1/\mathbb{G}_m]$).
- There is no Zariski-open neighborhood of a strictly semistable curve which admits a good moduli space.

For the above reasons, $\mathfrak{M}_g^{\text{ss}}$ is a natural candidate to test the validity of Conjecture 1. Moreover, semistable curves appear in many contexts such as in admissible covers or the compactification of the universal Jacobian; therefore, it is useful to understand the local structure of the algebraic stack. Our main result is:

Theorem. (–, Kresch) $\mathfrak{M}_g^{\text{ss},\leq 1}$ *satisfies Conjecture 1.*

This work is still in progress and we expect to prove more generally that $\mathfrak{M}_g^{\text{ss}}$ satisfies Conjecture 1 using the machinery of log structures.

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New properties of $\mathcal{A}_5$ via the Prym map

Alessandro Verra
(joint work with Gavril Farkas, Sam Grushevskih, and Riccardo Salvati Manni)

This is a report on some results of [FGSMV11], a joint paper with G. Farkas, S. Grushevskih and R. Salvati Manni. We will revisit the structure of the Prym map in genus six, $P : \mathcal{R}_6 \to \mathcal{A}_5$, and introduce further properties. Then some new results on the moduli space $\mathcal{A}_5$, of principally polarized abelian varieties of dimension 5, will be deduced. The subjects of this report can be summarized as follows:

(1) *Precisions on the ramification and antiramification divisors of the Prym map P and new proof of their irreducibility.*
(2) *Characterization in $\mathcal{A}_5$ of the loci parametrizing ppav's (A, Θ) such that Sing Θ contains a non ordinary double point.*

Let us put (1) and (2) in their own perspectives:

(1) Let $\mathcal{D} \subset \mathcal{A}_5$ and $\mathcal{Q} \subset \mathcal{R}_6$ be the branch divisor and the ramification divisor of P. Let $\mathcal{U} := P^{-1}(\mathcal{D}) - \mathcal{Q}$ be the antiramification divisor. The irreducibility of $\mathcal{Q}$ was first remarked and proved by Donagi [D92].
As the general fibre of P has the configuration of 27 lines in a smooth cubic surface, the special one F, over a general point of $\mathcal{D}$, is biregular to the Hilbert scheme of lines of a cubic surface S such that *Sing* S consists of a node o. The six lines through o correspond to $F \cap \mathcal{Q}$. The study of the monodromy of $P/\mathcal{Q}$ implies the irreducibility of $\mathcal{Q}$. In turn this implies the irreducibility of $\mathcal{U}$.
Notice that $\mathcal{Q}$ parametrizes Prym curves (C, η) whose Prym canonical model is contained in a quadric, that is, the multiplication map $Sym^2 H^(\omega_C \otimes \eta) \to H^0(\omega_C^{\otimes 2})$ is not an isomorphism.
In the talk new divisorial conditions on $\mathcal{R}_6$ are introduced, in particular:
(M) *Mukai condition.* The Prym curve (C, η) satisfies $h^0(E \otimes \eta) \geq 1$, where E is the Mukai bundle of C.
(N) *Nikulin condition.* The Prym curve (C, η) satisfies $C \subset W \subset G(2, 5)$. Here $G(2, 5)$ is the Grassmannian of lines of $\mathbf{P}^4$ and W is the family of lines which are tangent to a smooth quadric $Q \subset \mathbf{P}^4$.

One can show that conditions (M) and (N) define divisors $\mathcal{Q}_M$ and $\mathcal{Q}_N$, such that $\mathcal{Q}_M \subseteq \mathcal{Q}_N \subset \mathcal{Q} \subset \mathcal{R}_6$. In the talk it is shown that $\mathcal{Q}_N$ is irreducible and that

$$\mathcal{Q}_M = \mathcal{Q}_N = \mathcal{Q}.$$

This follows from class computations in $\mathcal{R}_6^*$, a standard partial compactification of $\mathcal{R}_6$. Indeed, denoting by X^* the closure of $X \subset \mathcal{R}_6$ in $\mathcal{R}_6^*$, one shows that

$$[\mathcal{Q}_M^*]\hat{E} = [\mathcal{Q}_N^*]\hat{E} = [\mathcal{Q}^*].$$

The previous geometric characterizations of $\mathcal{Q}$ are useful in the next step.

(2) The second step is related to Andreotti-Mayer loci $N_i \subset \mathcal{A}_g$. Recall that N_i is the moduli space of pairs (A, Θ) such that $dim\ Sing\ \Theta \geq i$. As is well known $N_0 = \theta_{null} \cup N_0'$, θ_{null} and N_0' being integral Cartier divisors. A general point of N_0 is defined by a pair (A, Θ) such that $Sing\ \Theta$ consists of ordinary double points, that is, the quadratic tangent cone has maximal rank g. Moreover one has $Sing\ \Theta = \{x, -x\}$ generically in N_0' and $Sing\ \Theta = \{x\}, 2x = 0$ generically in θ_{null}. A natural and well motivated question, see [HF06], is the following:

Describe the loci $N_0^{g-1} \subset N_0$ such that the quadratic tangent cone Q_x has rank $\leq g-1$ for some quadratic singularity $x \in Sing\ \Theta$.

Previous results on this problem have been obtained by Grushevskih and Salvati Manni [GSM07], [GSM08], [GSM11]. For every g they show that $N_0^{g-1} \cap \theta_{null}$ is also contained in N_0'. Furthermore they have shown that

Theorem 1. *$N_0^{g-1} = \theta_{null} \cap N_0'$ for $g = 4$.*

In the talk the next case, where the dimension is 5, is described, [FGSMV11]:

Theorem 2. *N_0^4 is the union of two irreducible, unirational components θ_{null}^4 and $N_0^{'4}$ of codimension two. Both of them are contained in $\mathcal{U} \cap \mathcal{Q}$. Moreover:*

- *A general point of $N_0^{'4}$ is the image by $P : \mathcal{R}_6 \to \mathcal{A}_5$ of a Prym curve (C, η) such that $h^0(L \otimes \eta) \geq 1$, where L is a line bundle of degree 4 and $dim\ |L| = 1$.*
- *A general point of θ_{null}^4 is the image by $P : \mathcal{R}_6 \to \mathcal{A}_5$ of a Prym curve (C, η) such that $\eta \cong \theta_1 \otimes \theta_2^{-1}$, where θ_1, θ_2 are theta nulls on C.*

The new result includes class computations for these loci. Let $\overline{\mathcal{A}}_g$ be the perfect cone compactification of $\mathcal{A}_g$ so that $CH^1(\overline{\mathcal{A}}_g)$ is $\mathbb{Z}\lambda \oplus \mathbb{Z}\delta$, where δ is the boundary class. As a further application, see also [GSM11], one has:

Theorem 3. *Let s be the slope of $\overline{\mathcal{A}}_5$, then $s = \frac{54}{7}$.*

To widen the picture of the geometry of $\mathcal{Q}, \mathcal{U}$ and $N_0^{'4}$ consider the forgetful map

$$f : \mathcal{R}_6 \to \mathcal{M}_6.$$

The Gieseker Petri divisor $\mathcal{GP}_6$ of $\mathcal{M}_6$ is split in two irreducible components, namely $\mathcal{GP}_6 = \mathcal{G}_{4,6}^1 + \mathcal{GP}_{5,6}^1$. Here a general point of $\mathcal{GP}_{4,6}^1$ is defined by a curve having a line bundle L of degree 4 such that $h^0(L) = 2$ and $h^1(L^{\otimes 2}) = 1$. On the other hand a general point of $\mathcal{GP}^1_{5,6}$ is defined by a curve having a theta null. Keeping the previous notations it turns out that:

$$f^*\mathcal{GP}^1_4 = \mathcal{U}\ ,\ f^*\mathcal{GP}^1_5 = P^*\theta_{null}\ ,\ \mathcal{D} = N'_0\ ,\ P^*\mathcal{D} = 2\mathcal{Q} + \mathcal{U}.$$

We end this report by a picture of the family of Prym curves parametrized by ${N'_0}^4$, since they have very special geometric properties.
Let (C, η) be a Prym curve defining a *general* point $p \in {N'_0}^4$. Then $p \in \mathcal{Q} \cap \mathcal{U}$. Since $p \in \mathcal{Q}$, the Prym canonical model $C \subset \mathbf{P}^4$ of (C, η) is contained in a quadric Q. Let (A, Θ) be the Prym of (C, η), $\{x, -x\} = Sing\ \Theta$. In particular $\mathbf{P}^4$ is naturally identified to the projectivized tangent space to A at x and Q is the projectivized quadratic tangent cone to Θ at x.
Since $p \in \mathcal{U}$, Q has rank four. Moreover one of the two rulings of planes of Q cuts on C a pencil $|L|$ of divisors of degree 4. Furthermore the line bundle L is Petri special, that is, $h^1(L^{\otimes 2}) = 1$. In this situation one can show that there exists a degree four effective divisor $t \subset C$ contained in a 4-secant line $< t >$ to C. We have also that $t \in |L \otimes \eta|$, so that $h^0(L \otimes \eta) \geq 1$ as indicated in theorem 0.2.
The image Γ of C under the linear projection $\nu : C \to \mathbf{P}^2$ of center $< t >$ is a very special plane sextic. It has three collinear nodes n_1, n_2, n_3 and three totally tangent conics. Let o be the fourth node of Γ. Then the strict transform by ν of the pencil of lines through o is $|L|$. Finally there exists an integral plane cubic F, which is nodal at o, contains n_1, n_2, n_3 and is tangent to Γ along $\nu_* t$.
Let $\mathcal{G}$ be the family of plane sextics with the previous properties. One can show that $\mathcal{G}$ is irreducible, unirational and dominates ${N'_0}^4$. This implies the same for ${N'_0}^4$, so we have summarized part of the proof of theorem 0.2.

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Fourier-Mukai partners of K3 surfaces in positive characteristic.

Martin Olsson

(joint work with Max Lieblich)

Let k be a field. Two smooth projective varieties X and Y over k are called *Fourier-Mukai partners* if there exists an equivalence of triangulated categories $D(X) \simeq D(Y)$ between their bounded derived categories of coherent sheaves. In this work we extend to positive characteristic many of the now classical results in characteristic 0, due Mukai, Oguiso, Orlov, and Yau (see [Or] and [HLOY]), on Fourier-Mukai partners of K3 surfaces.

Let X be a K3 surface. For a complex $E \in D(X)$ (and in particular for a coherent sheaf on X) define its *Mukai vector*, denoted $v(E)$, to be

$$(\mathrm{rank}(E), c_1(E), \mathrm{rank}(E) + c_1(E)^2/2 - c_2(E)) \in CH^*(X) \otimes \mathbb{Q}.$$

For a fixed vector $v \in CH^*(X) \otimes \mathbb{Q}$ and polarization h on X (suppressed from the notation), let $M_X(v)$ denote the moduli space of semistable sheaves on X with Mukai vector v. For suitable choices of v it is known that $M_X(v)$ is a smooth projective variety, there exists a universal family $\mathcal{E}$ on $X \times M_X(v)$, and the resulting functor

$$\Phi^{\mathcal{E}} : D(X) \to D(M_X(v)), \quad K \mapsto R\mathrm{pr}_{2*}(L\mathrm{pr}_1^* K \otimes^{\mathbb{L}} \mathcal{E})$$

is an equivalence of triangulated categories. In particular, for suitable choices of v the varieties X and $M_X(v)$ are Fourier-Mukai partners. Our main result is the following:

Theorem 1. *Assume that the characteristic of k is not 2 and that X/k is a K3 surface.*

(i) Any Fourier-Mukai partner of X is of the form $M_X(v)$ for suitable vector $v \in CH^*(X) \otimes \mathbb{Q}$.

(ii) X has only finitely many Fourier-Mukai partners.

(iii) If the Picard number of X is at least 11 then X has no nontrivial Fourier-Mukai partners.

Remark 2. *Presumably the theorem remains valid in characteristic 2 as well.*

By studying the *Mukai motive* of Fourier-Mukai partners, we also prove the following two results:

Theorem 3. *Let X be a K3 surface over a finite field $\mathbb{F}_q$ of characteristic $\neq 2$. If Y is a Fourier-Mukai partner of X then X and Y have the same zeta function.*

Theorem 4. *Let k be an algebraically closed field of characteristic $\neq 2$, and let W denote the Witt vectors of k. Let X and Y be K3 surfaces over k with lifts $\mathcal{X}/W$ and $\mathcal{Y}/W$ to W giving rise to a Hodge filtration on the F-isocrystal $H^4_{\mathrm{cris}}(X \times Y/K)$. Suppose $Z \subset X \times Y$ is a correspondence coming from a Fourier-Mukai kernel. If the fundamental class of Z lies in $\mathrm{Fil}^2 H^4_{\mathrm{cris}}(X \times Y/K)$ then Z is the specialization of a cycle on $\mathcal{X} \times \mathcal{Y}$.*

Remark 5. *Theorem 3 answers a question of Mustaţă and Huybrechts, while theorem 4 establishes the truth of the variational crystalline Hodge conjecture in a special case. Huybrechts also independently found a proof of 3.*

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Brill-Noether loci in codimension two

Nicola Tarasca

The classical Brill-Noether theory is a powerful tool for investigating subvarieties of moduli spaces of curves. While a general curve admits only linear series with non-negative Brill-Noether number, the locus $\mathcal{M}^r_{g,d}$ of curves of genus g admitting a $\mathfrak{g}^r_d$ with negative Brill-Noether number $\rho(g,r,d) := g-(r+1)(g-d+r) < 0$ is a proper subvariety of $\mathcal{M}_g$.

Such a locus can be realized as a degeneracy locus of a map of vector bundles over $\mathcal{M}_g$ so that one knows that the codimension of $\mathcal{M}^r_{g,d}$ is less than or equal to $-\rho(g,r,d)$ ([8]). When $\rho(g,r,d) \in \{-1,-2,-3\}$ the opposite inequality also holds (see [5] and [3]), hence the locus $\mathcal{M}^r_{g,d}$ is pure of codimension $-\rho(g,r,d)$. Moreover, the equality is classically known to hold also when $r=1$ and for any $\rho(g,1,d) < 0$: B. Segre first showed that the dimension of $\mathcal{M}^1_{g,d}$ is $2g+2d-5$, that is, $\mathcal{M}^1_{g,d}$ has codimension exactly $-\rho(g,1,d)$ for every $\rho(g,1,d) < 0$ (see for instance [1]).

Harris, Mumford and Eisenbud have extensively studied the case $\rho(g,r,d) = -1$ when $\mathcal{M}^r_{g,d}$ is a divisor in $\mathcal{M}_g$ ([7], [4]). They computed the class of its closure in $\overline{\mathcal{M}}_g$ and found that it has slope $6+12/(g+1)$. Since for $g \geq 24$ this is less than $13/2$ the slope of the canonical bundle, it follows that $\overline{\mathcal{M}}_g$ is of general type for g composite and greater than or equal to 24.

While the class of the Brill-Noether divisor has served to reveal many important aspects of the geometry of $\overline{\mathcal{M}}_g$, very little is known about Brill-Noether loci of higher codimension. The main result presented in the talk is a closed formula for the class of the closure of the locus $\mathcal{M}^1_{2k,k} \subset \mathcal{M}_{2k}$ of curves of genus $2k$ admitting a pencil of degree k. Since $\rho(2k,1,k) = -2$, such a locus has codimension two. As an example, consider the hyperelliptic locus $\mathcal{M}^1_{4,2}$ in $\mathcal{M}_4$.

Faber and Pandharipande have shown that Hurwitz loci, in particular loci of type $\mathcal{M}^1_{g,d}$, are tautological in $\overline{\mathcal{M}}_g$ ([6]). When $g \geq 6$, Edidin has found a basis for the space $R^2(\overline{\mathcal{M}}_g,\mathbb{Q}) \subset A^2(\overline{\mathcal{M}}_g,\mathbb{Q})$ of codimension-two tautological classes of the moduli space of stable curves ([2]). It consists of the classes κ_1^2 and κ_2; the following products of classes from $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$: $\lambda\delta_0, \lambda\delta_1, \lambda\delta_2, \delta_0^2$ and δ_1^2; the following push-forwards $\lambda^{(i)}, \lambda^{(g-i)}, \omega^{(i)}$ and $\omega^{(g-i)}$ of the classes λ and $\omega = \psi$ respectively from $\overline{\mathcal{M}}_{i,1}$ and $\overline{\mathcal{M}}_{g-i,1}$ to $\Delta_i \subset \overline{\mathcal{M}}_g$: $\lambda^{(3)}, \ldots, \lambda^{(g-3)}$ and $\omega^{(2)}, \ldots, \omega^{(g-2)}$; finally the classes of closures of loci of curves having two nodes: the classes θ_i of the loci having as general element a union of a curve of genus i and a curve of genus $g-i-1$ attached at two points; the class δ_{00} of the locus whose general element is an irreducible curve with two nodes; the classes δ_{0j} of the closures of the loci of irreducible nodal curves of geometric genus $g-j-1$ with a tail of genus j; at last the classes δ_{ij} of the loci with general element a chain of three irreducible curves with the external ones having genus i and j.

Having then a basis for the classes of Brill-Noether codimension-two loci, in order to determine the coefficients I use the method of test surfaces. The idea is the following. Evaluating the intersections of a given a surface in $\overline{\mathcal{M}}_g$ on one hand with the classes in the basis and on the other hand with the Brill-Noether loci, one obtains a linear relation in the coefficients of the Brill-Noether classes. Hence one has to produce several surfaces giving enough independent relations in order to compute all the coefficients of the sought-for classes.

The surfaces used are bases of families of curves with several nodes, hence a good theory of degeneration of linear series is required. For this, the compactification of the Hurwitz scheme by the space of admissible covers introduced by Harris and Mumford comes into play. The intersection problems thus boil down first to counting pencils on the general curve, and then to evaluating the respective multiplicities via a local study of the compactified Hurwitz scheme.

Theorem ([9]). *For $k \geq 3$, the class of the locus $\overline{\mathcal{M}}^1_{2k,k} \subset \overline{\mathcal{M}}_{2k}$ is*

$$\left[\overline{\mathcal{M}}^1_{2k,k}\right]_Q = c\Bigg[A_{\kappa_1^2}\kappa_1^2 + A_{\kappa_2}\kappa_2 + A_{\delta_0^2}\delta_0^2 + A_{\lambda\delta_0}\lambda\delta_0 + A_{\delta_1^2}\delta_1^2 + A_{\lambda\delta_1}\lambda\delta_1$$

$$+ A_{\lambda\delta_2}\lambda\delta_2 + \sum_{i=2}^{2k-2} A_{\omega^{(i)}}\omega^{(i)} + \sum_{i=3}^{2k-3} A_{\lambda^{(i)}}\lambda^{(i)} + \sum_{i,j} A_{\delta_{ij}}\delta_{ij} + \sum_{i=1}^{\lfloor(2k-1)/2\rfloor} A_{\theta_i}\theta_i\Bigg]$$

in $R^2(\overline{\mathcal{M}}_{2k}, \mathbb{Q})$, where

$$c = \frac{2^{k-6}(2k-7)!!}{3(k!)} \qquad A_{\kappa_1^2} = -A_{\delta_0^2} = 3k^2 + 3k + 5$$

$$A_{\kappa_2} = -24k(k+5) \qquad A_{\delta_1^2} = -(3k(9k+41)+5)$$

$$A_{\lambda\delta_0} = -24(3(k-1)k-5) \qquad A_{\lambda\delta_1} = 24\left(-33k^2+39k+65\right)$$

$$A_{\lambda\delta_2} = 24(3(37-23k)k+185) \qquad A_{\delta_{1,1}} = 48\left(19k^2-49k+30\right)$$

$$A_{\delta_{1,2k-2}} = \frac{2}{5}(3k(859k-2453)+2135) \qquad A_{\delta_{00}} = 24k(k-1)$$

$$A_{\delta_{0,2k-2}} = \frac{2}{5}(3k(187k-389)-745) \qquad A_{\delta_{0,2k-1}} = 2(k(31k-49)-65)$$

$$\begin{aligned} A_{\omega^{(i)}} &= -180i^4 + 120i^3(6k+1) - 36i^2\left(20k^2+24k-5\right) \\ &\quad + 24i\left(52k^2-16k-5\right) + 27k^2 + 123k + 5 \\ A_{\lambda^{(i)}} &= 24\left(6i^2(3k+5) - 6i\left(6k^2+23k+5\right) + 159k^2 + 63k + 5\right) \\ A_{\theta^{(i)}} &= -12i\left(5i^3 + i^2(10-20k) + i\left(20k^2-8k-5\right) - 24k^2 + 32k - 10\right) \end{aligned}$$

and for $i \geq 1$ and $2 \leq j \leq 2k-3$

$$A_{\delta_{ij}} = 2\left(3k^2(144ij-1) - 3k(72ij(i+j+4)+1) + 180i(i+1)j(j+1) - 5\right)$$

while

$$A_{\delta_{0j}} = 2\left(-3\left(12j^2+36j+1\right)k + (72j-3)k^2 - 5\right)$$

for $1 \leq j \leq 2k-3$.

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Deforming rational curves in $\overline{\mathrm{M}}_g$

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The results presented in this talk are contained in the preprint [2]. We work over $\mathbb{C}$. The moduli space $\overline{\mathrm{M}}_g$ of stable curves of genus g is uniruled if and only if a general curve of genus g can be embedded in a projective algebraic surface Y, not ruled irrational, so that $\dim(|C|) > 0$. Consider the fibration $f : X \longrightarrow \mathbb{P}^1$ obtained from a general pencil $\Lambda \subset |C|$ after blowing up its base points. The deformation theory of f is controlled by the sheaf $Ext^1_f(\Omega_{X/\mathbb{P}^1}, \mathcal{O}_X)$, whose H^0 and H^1 are respectively the tangent space and an obstruction space for the functor Def_f. The condition that $\overline{\mathrm{M}}_g$ is uniruled then translates into the condition that there exists a non-isotrivial fibration $f : X \longrightarrow \mathbb{P}^1$ (with general fibre a nonsingular curve) of genus g such that the sheaf $Ext^1_f(\Omega_{X/\mathbb{P}^1}, \mathcal{O}_X)$ is globally generated. We call such a fibration *free*. The first result we prove is the following:

Theorem 1. *Assume that C is a nonsingular curve of genus g in a projective nonsingular surface Y such that $\dim(|C|) = r \geq 1$. Assume that $\Lambda \subset |C|$ is a pencil such that the fibration $f : X \longrightarrow \mathbb{P}^1$ obtained from it is a free fibration of genus g. Then:*

$$10\chi(\mathcal{O}_Y) - 2K_Y^2 \geq 4(g-1) - C^2 - h^0(K_Y - C) \tag{1}$$

If moreover $\dim(|C|) \geq 2$ or $h^1(\mathcal{O}_C(2C)) = 0$ then $h^0(K_Y - C) = 0$.

Inequality (1) can be applied to prove the following theorem.

Theorem 2. *Assume that C is a nonsingular curve of genus g in a projective nonsingular surface Y such that dim$(|C|) = r \geq 1$. Assume that C is general.*

If $0 \leq \kappa - \dim(Y) \leq 1$ *then*

- $p_g(Y) = 0 \Rightarrow g \leq 6$.
- $p_g(Y) = 1 \Rightarrow g \leq 11$.
- $p_g(Y) \geq 2 \Rightarrow g \leq 16$.

If Y is of general type and $K_Z^2 \geq 3\chi(\mathcal{O}_Z) - 10$, where Z is the minimal model of Y, assume that one of the following holds:

(a) *dim*$(|C|) \geq 2$.
(b) $h^0(K_Y - C) = 0$ *and* $C^2 \geq \frac{g-1}{2}$.
(c) $h^1(\mathcal{O}_C(2C)) = 0$.

Then $g \leq 19$.

The above result shows that the deformation theory of fibrations can be applied to bound the genus g of general curves moving in a nontrivial linear system on certain surfaces. The surfaces that are excluded from this analysis are the rational ones, due to the fact that these methods are not effective on such surfaces. The case of rational surfaces has been studied in the classical literature and there are some partial results [1, 3]. Also several other cases are excluded so far in the general type situation. Nevertheless these results indicate that by these methods it might be possible to prove the existence of a g_0 such that $\overline{\mathrm{M}}_g$ is not uniruled if $g > g_0$. Of course this result is well known with $g_0 = 21$ (see [3] for a survey) but the methods we propose here are conceptually simpler than those used so far and might apply to other cases.

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Categorification of Donaldson-Thomas invariants via perverse sheaves

Jun Li
(joint work with Young-Hoon Kiem)

In mid 1990s, the theory of virtual fundamental class was invented and it enabled us to define enumerative invariants more systematically. During the past two decades, many useful curve counting invariants have been defined in this way, like Gromov-Witten invariants (GW invariants for short) and Donaldson-Thomas invariants (DT invariants for short).

Of particular interest in string theory are curve counting invariants in a Calabi-Yau threefold Y. In this case, for each homology class $\beta \in H_2(Y, \mathbb{Z})$, the number of genus g curves with homology class β is expected to be finite. Unfortunately the

GW invariants are virtual counting of maps, which are rational numbers because of multiple cover contributions and automorphisms.

In 1998, Gopakumar and Vafa argued using Super-String theory the existence of a new invariant $n_g(\beta)$, called the Gopakumar invariant (GV invariant, for short). It is integer-valued and should be defined by an $sl_2 \times sl_2$ action on some cohomology of certain moduli space of sheaves on Y. Moreover, GV invariants should be viewed as virtual counting of curves, and are expected to determine all the GW invariants $N_g(\beta)$.

$$\sum_{g,\beta} N_g(\beta) q^\beta \lambda^{2g-2} = \sum_{k,g,\beta} n_g(\beta) \frac{1}{k} \left(2\sin(\frac{k\lambda}{2}) \right)^{2g-2} q^{k\beta} \tag{1}$$

where $\beta \in H_2(Y, \mathbb{Z})$, $q^\beta = \exp(-2\pi \int_\beta c_1(\mathcal{O}_Y(1)))$.

In 2005, Behrend discovered that the Donaldson-Thomas invariant of $M_Y(\beta)$ is the Euler number of $M_Y(\beta)$, weighted by an integer-valued constructible function ν, called the Behrend function, i.e.

$$DT(M_Y(\beta)) = \sum_k k \cdot e(\nu^{-1}(k))$$

where e denotes the topological Euler number. Since the ordinary Euler number is the alternating sum of Betti numbers of ordinary cohomology groups, it is reasonable to ask if the DT invariant is in fact the Euler number of some cohomology of $M_Y(\beta)$.It is known that the moduli space is locally the critical locus of a holomorphic function, called a local Chern-Simons functional. Given a holomorphic function f on a complex manifold V, one has the perverse sheaf $\phi_f(\mathbb{Q}[\dim V - 1])$ of vanishing cycles supported on the critical locus and the Euler number of this perverse sheaf at a point x equals the value of the Behrend function $\nu(x)$. Joyce and Song asked if there exists a global perverse sheaf $P^\bullet$ on $M_Y(\beta)$ which is locally isomorphic to the sheaf $\phi_f(\mathbb{Q}[\dim V - 1])$. In [1], the authors answered this question affirmatively, possibly after taking a cyclic Galois étale cover

$$\rho : M^\dagger \longrightarrow M = M^\dagger/G \quad \text{where } M = M_Y(\beta).$$

Further, the perverse sheaf if of geometric origin, thus admits a MHM structure.

The hypercohomology $\mathbb{H}^*(M, P^\bullet)$ is a graded vector space whose Euler number is by construction the DT invariant of $M = M_Y(\beta)$. Furthermore by the theory of perverse sheaves, it can be shown that there is an $sl_2 \times sl_2$ action of $\mathbb{H}^*(M^\dagger, \mathrm{gr}\, P^\bullet)$ where gr is the graded object using MHM structure of $P^\bullet$. Then the authors proposed in [1] that this is the desired cohomology for a mathematical theory of GV invariants. We proved that the genus 0 GV invariant thus defined equals the DT invariant of $M_Y(\beta)$ and checked the equation (1) for primitive fiber class of a K3-fibered CY3.

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Topological methods in moduli theory and Moduli spaces of curves with symmetries.

Fabrizio Catanese
(joint work with Ingrid Bauer, resp. Michael Lönne and Fabio Perroni)

1. Symmetry marked varieties

A symmetry marked projective variety is a triple (X, G, ϕ), where

(1) X is a projective variety,
(2) G is a finite group and
(3) $\phi : G \to Aut(X)$ is an injective homomorphism.

Equivalently, one can give the triple (X, G, α) of an action $\alpha : X \times G \to X$, where α determines the injective homomorphism ϕ.

What is important is the notion of isomorphism of marked varieties:

$$(X, G, \alpha) \cong (X', G', \alpha') \Leftrightarrow \exists f : X \to X', \psi : G \to G'$$

$$f \circ \alpha = \alpha' \circ (f \times \psi)(\Leftrightarrow \phi' \circ \psi = Ad(f) \circ \phi),$$

where f, ψ are isomorphisms.

Now, the group of automorphisms $Aut(G)$ acts on marked varieties by replacing ϕ with $\phi \circ \psi^{-1}$. The group $Inn(G)$ of inner automorphisms does not change the equivalence class of a triple, hence the group $Out(G)$ acts on the set of equivalence classes of marked varieties.

2. Projective $K(\pi, 1)$'s

The easiest examples of projective varieties which are $K(\pi, 1)$'s are

(1) curves of genus $g \geq 2$,
(2) AV : = Abelian varieties,
(3) LSM : = Locally symmetric manifolds, quotients of a bounded symmetric domain $\mathcal{D}$ by a cocompact discrete subgroup Γ acting freely, in particular
(4) VIP : = Varieties isogenous to a product, studied in [4], quotients of products of projective curves of respective genera ≥ 2 by the action of a finite group G acting freely,
(5) Kodaira fibrations $F : S \to B$, where S is a smooth projective surface and all the fibres of F are smooth curves of genus $g \geq 2$.

However, an important role is also played by **Rational** $K(\pi, 1)$**'s**, i.e., quasi projective varieties Z such that

$$Z = \mathcal{D}/\pi,$$

where $\mathcal{D}$ is contractible and the action of π on $\mathcal{D}$ is properly discontinuous but not necessarily free.

While for a $K(\pi, 1)$ we have $H^*(G, \mathbb{Z}) \cong H^*(Z, \mathbb{Z})$, $H_*(G, \mathbb{Z}) \cong H_*(Z, \mathbb{Z})$, for a rational $K(\pi, 1)$ we only have $H^*(G, \mathbb{Q}) \cong H^*(Z, \mathbb{Q})$.

Typical examples of such rational $K(\pi, 1)$'s are the moduli space of curves $\mathcal{M}_g$.

3. Inoue type varieties

Inspired by an example of Inoue, [12], who constructed some surfaces of general type with $K^2 = 7, p_g = 0$ as the quotient by a finite gorup G of some subvarieties in the product of 4 elliptic curves, together with Ingrid Bauer we defined in [3] the notion of Inoue type varieties.

- A projective manifold X of dimension ≥ 2 is said to be an **ITM = Inoue Type Manifold** iff

(1) X is the quotient $X = \hat{X}/G$ of a projective manifold $\hat{X}$ by the free action of a finite group G

(2) $\hat{X}$ is an ample divisor in a $K(\pi, 1)$ projective manifold Z

(3) the action of G on $\hat{X}$ is induced by an action of G on Z

(4) the fundamental group exact sequence

$$1 \to \Gamma = \pi_1(\hat{X}) \cong \pi_1(Z) \to \pi_1(X) \to G \to 1$$

induces an injective homomorphism (by conjugation) $G \to Out(\Gamma)$.

- X is said to be a SITM (special Inoue type Manifold) if Z is a product of curves, Abelian varieties, irreducible locally symmetric manifolds.

Together with Ingrid Bauer, we were able to show that, under some technical conditions which we have no space to reproduce here, if X' is homotopically equivalent to a SITM X, then also X' is a SITM of similar type. The fuller investigation of the moduli spaces of such manifolds has been one more motivation to investigate moduli spaces of marked varieties, in particular curves.

4. Moduli spaces of curves with symmetries

Assume that we have a marked curve (C, G, ϕ). Since C is a $K(\pi, 1)$, the homotopy class of the action is determined by an homomorphism (here $\pi_g = \pi_1(C)$) into the mapping class group

$$\rho : G \to Out^+(\pi_g) = Map_g.$$

The homomorphism is injective, as shown by Lefschetz, and determines the differentiable type of the action, as shown by Nielsen.

We have corresponding moduli spaces of curves with symmetries $\mathcal{M}_{g,G,\rho}$, or their images in the moduli space of curves, which were shown in [4] and [5] to be irreducible and closed subsets.

These yield more examples of rational $K(\pi, 1)$'s.

Question. *How to determine exactly the topological type, i.e. the class of ρ modulo automorphisms of G and conjugation in the mapping class group Map_g ?*

Geometry yields some invariants, for instance the genus g' of the quotient curve $C' := C/G$, and, denoting by $y_1, \dots, y_d$ the branch points of the map $C \to C'$, the Nielsen class ν, which counts the conjugacy classes of the local monodromies in the points y_i.

These invariants suffice for cyclic groups, but, as shown by several authors, one needs at least a homological invariant in $H_2(G, \mathbb{Z})$ in the case where there are no branch points (i.e., when G acts freely). This was shown in [11] to be the only invariant for $g' \gg 0$. We were able to treat the more difficult general case.

4.1. **Genus stabilization Theorem.** ([7, 8]) There exists a refined homological invariant $\epsilon(\rho) \in G_\Gamma$ such that, for $g' \gg 0$, there is a bijection between the set of topological types and the set of admissible classes of invariants ϵ.

4.2. **Branch stabilization Theorem([9]).** If the value of the Nielsen function is sufficiently large for the conjugacy classes which occur as local monodromies, and the group G is generated by local monodromies, then there is a bijection between the set of topological types and the set of admissible classes of invariants ϵ.
In [8] we also make the following

Conjecture. *The cohomology groups of the moduli spaces $\mathcal{M}_{g,G,\rho}$ stabilize for $g' \gg 0$.*

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The zero section of the universal semiabelian variety, and the locus of principal divisors on $\overline{\mathcal{M}}_{g,n}$

Samuel Grushevsky

(joint work with Dmitry Zakharov)

Let $\mathcal{A}_g$ denote the moduli space of complex principally polarized abelian varieties of dimension g (ppav), and let $\pi : \mathcal{X}_g \to \mathcal{A}_g$ denote the universal family of ppav. For a very general ppav $B \in \mathcal{A}_g$ the Picard group $\mathrm{Pic}(B)$ is generated by the class of the polarization divisor; the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{A}_g)$ is generated by the first Chern class of the Hodge vector bundle $\lambda_1 := c_1(\mathbb{E}) := c_1(\pi_* \Omega^1_{\mathcal{X}_g/\mathcal{A}_g})$. It thus follows that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}_g) = \mathbb{Q}\lambda_1 \oplus \mathbb{Q}T$, where T denotes the class of the universal theta (polarization) divisor trivialized along the zero section $z_g : \mathcal{A}_g \to \mathcal{X}_g$. Our interest is in fact in computing the class of the zero section.

It is known (see the work of Mumford, van der Geer, Voisin for different approaches) that the equality $[z_g(\mathcal{A}_g)] = T^g/g!$ holds both in cohomology $H^g(\mathcal{X}, \mathbb{Q})$ and in Chow group $CH^g(\mathcal{X}, \mathbb{Q})$. The question we address is extending this to the partial compactification of the universal family $\pi' : \mathcal{X}'_g \to \mathcal{A}'_g$ over Mumford's partial toroidal compactification $\mathcal{A}'_g$ of $\mathcal{A}_g$, parameterizing semiabelian varieties of torus rank one. Our result is as follows:

Theorem 1 ([1]). *For the class of the closure of the zero section, $z'_g : \mathcal{A}'_g \to \mathcal{X}'_g$ we have the following polynomial expression*

$$[z'_g] = \sum_{a+b+2c=g} \alpha_{a,b,c} (T' - D/8)^a D^b (\Delta - 2T'D)^c \in CH^g(\mathcal{X}'_g),$$

where the positive coefficients $\alpha_{a,b,c}$ are given by

$$\alpha_{a,b,c} = \frac{(-1)^{b+c+1}(2^{-b-c} - 2^{1-3b-3c})(2a+2b+2c-1)!!B_{2b+2c}}{(2a+2c-1)!!(2b+2c-1)!!a!b!c!}.$$

(with B the Bernoulli numbers).

Here T' denotes the class of the extension of the universal polarization divisor trivialized along the zero section (the notation Θ is used in [1] to avoid confusion with other natural classes), D is the class of the boundary divisor $D := [\mathcal{X}'_g \setminus \mathcal{X}_g]$, and to define Δ we recall the geometric description of $\mathcal{X}'_g$ given by Mumford (see also our papers with Lehavi, and Erdenberger and Hulek):

$$\begin{tikzcd}
 & & & & & (\widetilde{Y} = \mathbb{P}(\mathcal{P} \oplus \mathcal{O}))/j \arrow[dll] \arrow[d] \\
\mathcal{X}'_g & = & \mathcal{X}_g \arrow[d] & \sqcup & Y \arrow[d] & \mathcal{X}^{\times 2}_{g-1} \ni (B,z,b) \arrow[dl] \\
\mathcal{A}'_g & = & \mathcal{A}_g & \sqcup & \mathcal{X}_{g-1} \ni (B,b) &
\end{tikzcd}$$

where $\mathcal{X}_{g-1}^{\times 2} := \mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \mathcal{X}_{g-1}$ denotes the fiberwise square of the universal family, $\mathcal{P}$ denotes the universal Poincaré bundle trivialized along the zero section, and the gluing j is given by identifying the 0-section of the $\mathbb{P}^1$ (globally given by $\mathbb{P}(\mathcal{P} \oplus 0)$) over the point (B, z, b) with the ∞-section (globally given by $\mathbb{P}(0 \oplus \mathcal{O})$) over the point $(B, z+b, b)$. Then the codimension 2 class Δ is the class of the non-normality locus of Y, i.e. of the image of the glued 0 and ∞ sections of $\widetilde{Y}$. We notice that the zero section in Y is the section 1 (this is the identity for the group law on $\mathbb{C}^* \subset \mathbb{P}^1$) over $z_{g-1}(\mathcal{A}_{g-1}) \subset \mathcal{X}_{g-1}$.

To prove the theorem, we in fact show that all classes on $\widetilde{Y}$ that are polynomials in the divisor classes there, and which are pullbacks from Y, are polynomial in $D, T' - D/8, \Delta - 2T'D$, which establishes the existence of a polynomial expression — and then proceed to compute the coefficients.

Another application of our result is to computing the class of the double ramification cycle, also know as the locus of principal divisors on pointed curves. Indeed, for $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ with $\sum d_i = 0$ define the map $s_{\underline{d}} : \mathcal{M}_{g,n} \to \mathcal{A}_g$ from the moduli of curves with marked points, by sending a curve to its Jacobian (considered as Pic^0) together with the sum of the Abel-Jacobi images of points, $\sum d_i p_i$. The double ramification locus is the closure in $\overline{\mathcal{M}}_{g,n}$ of the preimage of the zero section. Equivalently, we can think of it as the locus where the divisor $\sum d_i p_i$ is principal on the curve. The question of determining the class of this locus is due to Eliashberg, and is of importance for constructing suitable symplectic field theories. The restriction of this class to $\mathcal{M}_{g,n}^{ct}$ was determined by Hain using Hodge-theoretic methods, while the restriction to $\mathcal{M}_{g,n}^{rt}$ was determined by Cavalieri, Marcus, and Wise using the Gromov-Witten theory. Further work and conjectures on this locus are due to Zvonkine.

While we cannot fully compute the class of the double ramification cycle, by pullback under $s_{\underline{d}}$ our computation allows us to compute the class of the restriction of the double ramification cycle to the locus of stable curves of geometric genus at least $g-1$ with at most two non-separating nodes (otherwise the Abel-Jacobi map may not be defined, or we don't end up in $\mathcal{A}_g'$). The result is as follows:

Theorem 2 ([2]). *The double ramification cycle in $\overline{\mathcal{M}}_{g,n}$ restricted to the locus of curves that have at most two non-separating nodes is given by pulling back the formula in Theorem 1, where for the pullbacks of the classes we have*

$$s_{\underline{d}}^* T' = \frac{1}{2}\sum_{i=1}^n d_i^2 K_i - \frac{1}{2}\sum_{P \subseteq I}\left(d_P^2 - \sum_{i \in P} d_i^2\right)\delta_{0,P} - \frac{1}{2}\sum_{h>0, P\subseteq I} d_P^2 \delta_{h,P},$$

$$s_{\underline{d}}^* D = \delta_{irr}, \quad s_{\underline{d}}^* \Delta = \sum_{i=1}^n |d_i| \xi_i.$$

Here K_i denote the pullback to $\overline{\mathcal{M}}_{g,n}$ of the ψ class on $\overline{\mathcal{M}}_{g,1}$ under the forgetful map forgetting all but the i'th point, $\delta_{0,P}$ and δ_{irr} are the usual boundary divisors, $d_P := \sum_{i \in P} d_i$, and finally the last class ξ_i is the closure of the locus of nodal curves having two irreducible components intersecting in two points (i.e. "banana" curves), where one irreducible component has genus 0, and contains only the marked point p_i, and the other component has genus $g-1$ and contains the remaining $n-1$ marked points.

While it seems plausible that with much more work the results could be extended one step further, to the locus of semiabelic varieties of torus rank two, going deeper into the boundary seems harder as questions of existence of universal family over different toroidal compactifications of $\mathcal{A}_g$ come into play.

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