Mathematisches Forschungsinstitut Oberwolfach

Report No. 11/2013

DOI: 10.4171/OWR/2013/11

Structured Function Systems and Applications

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24 February – 2 March 2013

Abstract. Quite a few independent investigations have been devoted recently to the analysis and construction of structured function systems such as e.g. wavelet frames with compact support, Gabor frames, refinable functions in the context of subdivision and so on. However, difficult open questions about the existence, properties and general efficient construction methods of such structured function systems have been left without satisfactory answers. The goal of the workshop was to bring together experts in approximation theory, real algebraic geometry, complex analysis, frame theory and optimization to address key open questions on the subject in a highly interdisciplinary, unique of its kind, exchange.

Mathematics Subject Classification (2010): 42C15, 44A60 (primary), 42C40, 30E05, 42A70, 90C22, 90C26, 14P10 (secondary).

Introduction by the Organisers

We outline below some of the main themes and open problems which have guided the participants during the workshop.

1. Several fundamental results by two groups of authors I. Daubechies, B. Han, A. Ron, Z. Shen [5] and respectively C. Chui, W. He, J. Stöckler [2, 3] laid the foundations of the theory of tight wavelet frames and address practical aspects of their construction. The resulting characterizations of compactly supported tight wavelet frames establish a novel connection between frame constructions and a challenging algebraic problem of existence of sums of squares representations of non-negative Laurent polynomials [8, 10]. In other words, the existence of the *certificate* of non-negativity, i.e. the existence of a sum of squares representation for a given real Laurent polynomial which is non-negative on the d-dimensional torus, became a *necessary* condition for the existence of such tight wavelet frames. In the multivariate case, the difficult questions about their existence and general efficient construction methods have been left unanswered. Another very promising approach of dealing with non-negative polynomials was derived from techniques of semi-definite programming [1, 13]. One of the aims of the workshop was to reveal these interdisciplinary links to several groups of experts from different fields of mathematics and to set up a new framework of addressing open problems in wavelet frame theory. This was done in a truly open and interactive fashion. Our anticipation of the success of cross-disciplinary cooperation strengthened after three of the organizers (Charina, Putinar, Stöckler) joined by Claus Scheiderer from the University of Konstanz spent three productive weeks in October 2011 in Oberwolfach, as a part of the Research in Pairs program. The partial outcome is a paper accepted for publication in the Journal of "Constructive Approximation" which, e.g., gives an affirmative answer to the long standing open question about the existence of bivariate tight wavelet frames satisfying the Unitary Extension Principle.

2. Another open problem is to obtain the estimates for the number of frame generators in the multivariate case. This problem of finding an upper bound for the number of frame generators is closely related to getting an estimate of the Pythagoras number of the ring of rational functions on a real algebraic subvariety of the torus, [12].

3. Problems of multivariate Gabor frames lead to sampling and interpolation questions for entire functions of several complex variables. The construction of the multivariate interpolating functions requires much more precise growth estimates of generalized Weierstrass products than currently known [6]. Moreover, completely new types of sampling and interpolation problems arise [7].

4. Refinable functions arising in the context of subdivision are the starting point of the multiresolution based wavelet frame constructions. Most properties of such functions are encoded in the coefficients (so-called mask) of the corresponding refinable equations, or, equivalently, in the coefficients of the corresponding Laurent polynomials, or polynomials after an appropriate mask shift. Important approximation properties of refinable functions can be characterized in terms of the structure of the associated polynomial ideals [15]. In the univariate case, the structure of these polynomial ideals also determines the smoothness of the corresponding refinable function [4], i.e. the derived wavelet frame. In the multivariate setting, no such general results are available.

5. From the point of view of duality of locally convex spaces, the above questions can be interpreted as generalized moment problems pertaining to (usually) positive measures supported by basic semi-algebraic sets of the Euclidean space. This apparently innocent change of perspective brings into the focus a novel and powerful combination of operator theoretic methods and effective positivity criteria circulating in real algebra. The recent generalization of the moment technique beyond the classical setting of polynomial bases, see [9, 11, 16], is very encouraging for the specific aims of the workshop. Well studied integral transforms of measures, such as the Fourier-Laplace or Fantappiè transforms [14] naturally appear in this context. The study of these transforms involve refined techniques of bounded interpolation and approximation in spaces of analytic functions of several variables. Particular attention was also paid to the entropy method in solving truncated moment problems arising in continuum mechanics, control theory and ergodic theory of dynamical systems.

6. Optimization and semi-definite programming methods, as developed during the last decade [11], also offer unexpected methods for tackling several of the above mentioned problems. The difficulties in solving these arise either at the early modeling stage, e.g. when the problem is formulated as a generalized moment problem or as a reconstruction or synthesis problem, and/or at the final stage when the problem is formulated as a global optimization problem. In addition, the techniques involved in the construction of structured function systems will be general enough to have important applications to e.g., image processing, stability of differential equations with a delay in the argument and mathematical finance with Lévy driven processes. An important feature in all topics is the necessity of working with systems of non-polynomial functions.

The workshop offered longer expository talks on recent advances in real algebraic geometry, complex analysis, frame theory, moment problems and optimization. There were also shorter specialized talks. It was highly rewarding to be present at the beginning of new cooperations, to experience genuine interest of the participants from different fields in the research of others and their willingness to find a common language. All these together will undoubtedly contribute to further mathematical progress.

The reports below offer a more precise picture of the variety of questions addressed during the workshop. The open problems listed at the end of the report also illustrate the viability and long term impact of this emerging new field of interdisciplinary research.

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Workshop: Structured Function Systems and Applications

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Abstracts

Introduction: trigonometric polynomials and multivariate tight wavelet frames

Maria Charina and Mihai Putinar
(joint work with Claus Scheiderer and Joachim Stöckler)

The major theme of the workshop is derived from a technical question arising in the construction of certain wavelet frames.

1. Motivation from frame theory.

In wavelet frame theory and its applications the constructions of compactly supported tight wavelet frames usually boil down to algebraic manipulations with trigonometric polynomials. This is due to the fact that such multiresolution constructions are build upon refinable functions that in the frequency domain are given by infinite products of factors all obtained by a simple group action from a single trigonometric polynomial defined on the d-dimensional torus.

To be more specific, let $M \in \mathbb{Z}^{d\times d}$ be a general expansive matrix, i.e., all its eigenvalues are strictly larger than 1 in absolute value. The order of the finite abelian group

$$G := 2\pi M^{-T}\mathbb{Z}^d/2\pi\mathbb{Z}^d$$

is $m = |\det M|$. The compactly supported refinable function $\phi \in L_2(\mathbb{R}^d)$ is given and is assumed to satisfy the refinement equation

$$\widehat{\phi}(M^T\omega) = p(\boldsymbol{e}^{-i\omega})\widehat{\phi}(\omega), \quad \boldsymbol{e}^{-i\omega} = (e^{-i\omega_1}, \ldots, e^{-i\omega_d}),$$

written in terms of its Fourier transform $\widehat{\phi}$. The trigonometric polynomial p is given by

$$p(\boldsymbol{e}^{-i\omega}) = \sum_{\alpha\in\mathbb{Z}^d} p(\alpha)e^{-i\alpha\cdot\omega}, \qquad \omega \in \mathbb{R}^d,$$

and has only finitely many nonzero coefficients $p(\alpha) \in \mathbb{C}$. If $p(1, \ldots, 1) = 1$, then

$$\widehat{\phi}(\omega) = \widehat{\phi}(0, \ldots, 0)\prod_{j=1}^{\infty} p(\boldsymbol{e}^{-iM^{-T}\omega}), \quad \omega \in \mathbb{R}^d.$$

It is remarkable that most properties of the refinable function ϕ and the associated wavelet tight frames are encoded in the corresponding trigonometric polynomials. Indeed, the functions $\psi_j \in L_2(\mathbb{R}^d)$, $j = 1, \ldots, N$, that generate a tight wavelet frame are assumed to be of the form

$$\widehat{\psi}_j(M^T\omega) = q_j(\boldsymbol{e}^{-i\omega})\widehat{\phi}(\omega), \quad \omega \in \mathbb{R}^d,$$

where q_j are trigonometric polynomials as well. The trigonometric polynomials q_j are defined to satisfy the so-called Unitary Extension Principle from [12].

Theorem 1. *Assume that the trigonometric polynomial p satisfies $p(1,\dots,1)=1$. If the trigonometric polynomials q_j, $1\le j\le N$, satisfy the identities*

$$\delta_{\sigma,\tau}-\overline{p(\boldsymbol{e}^{-i(\omega+\sigma)})}p(\boldsymbol{e}^{-i(\omega+\tau)})=\sum_{j=1}^{N}\overline{q_j(\boldsymbol{e}^{-i(\omega+\sigma)})}q_j(\boldsymbol{e}^{-i(\omega+\tau)}),\ \sigma,\tau\in G,\ \omega\in\mathbb{R}^d,$$

then the family $\Psi=\{m^{1/2}\psi_j(M^\ell\cdot-\alpha)\ :\ j=1,\dots,N,\ \ell\in\mathbb{Z},\ \alpha\in\mathbb{Z}^d\}$ is a tight wavelet frame of $L_2(\mathbb{R}^d)$.

One of the well-known necessary conditions for the existence of the trigonometric polynomials q_j, $1\le j\le N$, in Theorem 1 requires that

$$f=1-\sum_{\sigma\in G}|p(\boldsymbol{e}^{-i(\omega+\sigma)})|^2\ge 0\quad\text{for all}\quad \omega\in\mathbb{R}^d. \tag{1}$$

Our results in [4] and the result in [8], respectively, provide necessary and sufficient conditions for the existence of q_j, $1\le j\le N$, in Theorem 1 and state that in this case f is a sum of hermitian squares of some trigonometric polynomials h_j, $j=1,\dots,K$, i.e.

$$f=\sum_{j=1}^{K}|h_j(\boldsymbol{e}^{-i(\omega)})|^2,\quad K\in\mathbb{N}.$$

This establishes a link between the recent advances in (real) algebraic geometry and wavelet frame theory. In particular, in the 2−dimensional case, we show that the result in [14] yields an affirmative answer to the long standing open problem of existence of multivariate tight wavelet frames. In dimension $d=3$, we construct a class of counterexamples showing that, in general, the condition (1) is not sufficient for the existence of tight wavelet frames. Our construction, on the one hand, relies on the properties of the so-called Motzkin polynomial. On the other hand, we make use of the following local-global result from algebraic geometry [11]: if the Taylor expansion of f at one of its roots, in local coordinates, has a homogeneous part of lowest degree which is the Motzkin polynomial, then f is not a sum of hermitian squares. In dimension $d\ge 3$, we use the results in [13] to derive stronger sufficient conditions for determining the existence of tight wavelet frames.

The polynomial identities and polynomial inequalities in Theorem 1 and in (1) can be recast in terms of positive semi-definite matrices. This allows us to use the techniques of semi-definite programming, see [9, 10], to provide, for every d, efficient numerical methods for checking the existence of multivariate tight wavelet frames and for their construction.

It is still not known, if analogous existence results hold for the tight wavelet frames constructed using the so-called Oblique Extension Principle in [5, 6].

Theorem 2. *Assume that the trigonometric polynomial p satisfies $p(1,\dots,1)=1$. If the trigonometric polynomials q_j, $1\le j\le N$, and strictly positive trigonometric polynomial s, $s(1,\dots,1)=1$, satisfy the identities*

$$\delta_{\sigma,\tau}s(\boldsymbol{e}^{-i(\omega+\tau)})-s(\boldsymbol{e}^{-iM^T\omega})\overline{p(\boldsymbol{e}^{-i(\omega+\sigma)})}p(\boldsymbol{e}^{-i(\omega+\tau)})=\sum_{j=1}^{N}\overline{q_j(\boldsymbol{e}^{-i(\omega+\sigma)})}q_j(\boldsymbol{e}^{-i(\omega+\tau)})$$

for all $\sigma, \tau \in G$ *and* $\omega \in \mathbb{R}^d$, *then the family* $\Psi = \{m^{1/2}\psi_j(M^\ell \cdot -\alpha) \; : \; j = 1, \ldots, N, \; \ell \in \mathbb{Z}, \; \alpha \in \mathbb{Z}^d\}$ *is a tight wavelet frame of* $L_2(\mathbb{R}^d)$.

The requirement that s is a trigonometric polynomial can be relaxed and one could look for a rational function s with the above mentioned properties. A necessary condition for the existence of the trigonometric polynomials q_j and s can be derived similarly to (1) by computing the determinant of the matrix version of the identities in (2). It is not clear how to extend the sufficient conditions in [8] to the case of Oblique Extension Principle to guarantee the existence of the trigonometric polynomials q_j in Theorem 2.

2. Positive trigonometric polynomials.

The Unitary Extension Principle has a straightforward generalization to the vector valued case. Thus, the heart of the matter is a decomposition of a matrix valued, non-negative trigonometric polynomial on the torus, namely

$$I - \sum_{\sigma \in G} P^{\sigma *}(\boldsymbol{e}^{-i\omega}) P^{\sigma}(\boldsymbol{e}^{-i\omega}) \tag{2}$$

into the modulus square (of an analytic, rectangular, matrix valued trigonometric polynomial). The good news is that this is a century old, well studied question, but the bad news is that in higher dimensions ($d \geq 2$) much remains to be done.

It all starts with an observation that $\cos\theta \leq 1$ for all $\theta \in [0, 2\pi)$. Everybody has an explanation for this inequality. Remarkably, the trigonometric identity $1 - \cos\theta = \frac{1}{2}(1 - \cos\theta)^2 + \frac{1}{2}\sin^2\theta$ is the leading tune of this workshop.

A celebrated Theorem of Riesz and Fejér asserts that every univariate trigonometric polynomial $p(\cos\theta, \sin\theta)$ which is non-negative on the unit torus, can be factored as

$$p(\cos\theta, \sin\theta) = |h(e^{-i\theta})|^2, \quad \theta \in \mathbb{R},$$

where $h \in \mathbb{C}[z]$ is a complex polynomial in the complex variable z. The appearance of the complex variable z is not at all accidental: it brings into the structure of positive polynomials on the torus the whole machinery of function theory and operator theory. To give only a glimpse into this area, we remark that our positive polynomial can be decomposed into a Fourier series in z and $\overline{z}$, with the not less remarkable result (solution of the Dirichlet problem):

$$p(\cos\theta, \sin\theta) = f(z) + \overline{f(z)}, \qquad z = e^{-i\theta}$$

and $f \in \mathbb{C}[z]$.

Thus, we are led to consider non-negative harmonic polynomials on the unit disk, instead of its boundary. This is very fortunate indeed, as their structure is unveiled by Riesz and Herglotz Theorem:

An analytic function f *in the unit disk* $|z| < 1$ *has non-negative real part, if and only if*

$$f(z) = i\Im f(0) + \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma(\theta),$$

where σ is any positive Borel measure on the torus.

One step further, one can refer to the multiplication operator U by z, on $L^2(\sigma)$, and obtain the matricial representation

$$f(z) = i\Im f(0) + \langle (U+z)(U-z)^{-1}\mathbf{1}, \mathbf{1}\rangle = f(0) + 2z\langle (U-z)^{-1}\mathbf{1}, \mathbf{1}\rangle,$$

where $\mathbf{1}$ is the constant function. Again, the argument can be reversed, with the result of allowing *any* unitary transformation U in the above formula.

The interplay alluded above between positive polynomials, analytic functions, structured matrices and positive measures is amply reflected by the quasi-totality of abstracts below. Without being exhaustive, we only hint in the next section on some specific topics that will be discussed in the remaining part of the report.

3. Construction techniques of tight wavelet frames.

Returning to tight wavelet frames and the Unitary Extension Principle, we encounter at least three convergent methods of tackling the question of finding the factorization of the matrix valued polynomial in (2). Of course the higher dimensional case $d \geq 2$ is more subtle and, hence, more interesting for future research. After all, we seek flexible and effective constructions of positive polynomials on tori, in arbitrary dimension.

3.1. Realization of non-negative pluriharmonic functions as transfer functions of conservative, multivariate linear systems. This is a preferred territory of control theory experts, and we only cite some recent references [1, 2, 3, 7] and promise to give more details in a forthcoming continuation of [4].

3.2. Positivstellensätze in real algebraic geometry. Riesz-Fejér Theorem has far reaching generalizations, obtained by algebraic and mathematical logic techniques. They are revealed in the abstracts of Scheiderer, Plaumann, Schweighofer and Netzer.

3.3 Duality and moment problems. Checking the non-negativity of a function f defined on a topological support X amounts at verifying

$$\int_X f d\mu \geq 0$$

for every positive Borel measure defined on X, so that $f \in L^1(\mu)$. This naive observation has deep consequences, moving the positivity of f question to the space of (generalized) moments of the measures μ.

Moment problems of various sorts, including interpolation problems in spaces of analytic functions with bounds, appear in the abstracts of Lyubarskii, Gröchenig, Ambrozie, Junk, Budisic, Schmüdgen.

3.4. Semi-definite programming via Lasserre relaxations. The numerical implementation of optimization in the space of moments was advocated and carried out by Lasserre, Henrion and their collaborators. In short, they propose a sequence of semi-definite programming approximations to any polynomial minimization problem, constrained also by polynomial inequalities. Their two abstracts explain the idea and show its universality on some surprising applications.

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MRA based tight wavelet frame and applications

Zuowei Shen

Since the publication of [25] on the multiresolution analysis and the publication of [16] on construction of the compactly supported orthonormal wavelet generated by the multiresolution analysis (MRA), the wavelet analysis and its applications lead the area of applied and computational harmonic analysis over the last two decades and the MRA based wavelet methods become a powerful tool in various applications in image and signal analysis and processing. One of the well known examples is to compress image by using orthonormal or bi-orthogonal wavelet bases generated by the MRA as given in [14, 16]. Another successful example is noise removal by using redundant wavelet system by [15, 18].

The publication of the unitary extension principle of [26] generates a wide range of interests in tight wavelet systems derived by multiresolution analysis. For example, the oblique extension principle was presented in [13, 17]. Having tight wavelet frames with a multiresolution structure is very important in order to make any use of them in applications, since this guarantees the existence of the fast decomposition and reconstruction algorithms. Recently, tight wavelet frames derived by the multiresolution analysis are used to open a new area of applications of frames. The application of tight wavelet frames in image restorations is one of them that includes image inpainting, image denoising, image deblurring and blind deburring, and image decompositions (see e.g. [3, 2, 1, 7, 8, 10, 11, 6, 4, 9, 12]). In particular, the unitary extension principle is used in [3, 10, 11, 9, 12] to design a tight wavelet frame system adaptive to the real life problems in hand. Frame based algorithms for image and surface segmentation, $3D$ surface reconstruction is give in [20, 21] and CT image reconstruction in [19].

In this talk, we start with a brief survey of the theory of tight wavelet frames. A characterization of the tight wavelet frame of [22, 24, 26] is given. We then focus on the tight wavelet frames and their constructions via the multiresolution analysis (MRA). In particular, the unitary extension principle of [26] and the construction of tight wavelet frame from it is given. We also give a short overview of the generalizations of the unitary extension principle. The second part of this talk focuses on the recent applications of tight wavelet frames in image restorations. In particular, the balanced approach of [3, 2, 1, 10, 11, 4, 9, 12] and the corresponding algorithms for image denoising, deblurring, inpainting and decomposition is discussed in details. The link of the frame based image restoration to the total variational based image restoration is also discussed. Indeed, it is shown in [5] that, by choosing parameters properly, a special case of the wavelet tight frame approach can be seen as sophisticated discretization of minimizations involving the TV regularization or their generalizations.

The interested reader should consult the survey article [27] and the lecture note [20] for details of the theory and applications of MRA based tight wavelet frames.

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Refinable functions and algebraic aspects of subdivision

Kurt Jetter
(joint work with Maria Charina, Costanza Conti, Georg Zimmermann)

Refinable function systems are at the foundation of multiresolution methods. Such functions ϕ, say, can be represented by dilated and translated versions of itself. In order to avoid technical details, we concentrate here on translations with respect to the multi-integer grid $\mathbb{Z}^d$, and to dilations with dilation matrix $2I$. In this situation, the refinability of ϕ may be expressed as

$$\phi(x) = \sum_{\beta \in \mathbb{Z}^d} m_\beta \, \phi(2x - \beta)$$

with a finitely supported sequence $m = (m_\beta)_{\beta\in\mathbb{Z}^d}$, the so-called *mask* of the refinable function and of its related (binary) subdivision operator

$$S_m \;:\; \ell^\infty(\mathbb{Z}^d) \to \ell^\infty(\mathbb{Z}^d)\,,\; c \mapsto S_m c\,, \quad \text{with} \quad (S_m c)_\alpha = \sum_{\beta\in\mathbb{Z}^d} m_{\alpha-2\beta}\, c_\beta\;.$$

The z-transform of the mask, viz. the Laurent polynomial

$$m(z) = \sum_{\beta\in\mathbb{Z}^d} m_\beta\; z^\beta$$

and its restriction to the d-dimensional torus, the trigonometric polynomial

$$h(\xi) = \frac{1}{2^d} \sum_{\beta\in\mathbb{Z}^d} m_\beta\; e^{-i\beta\cdot\xi}\;,$$

are usually employed in order to describe algebraic properties of subdivision operators.

Examples of refinable functions abound in the wavelet literature, and have appeared earlier in the spline literature, with univariate B-splines and multivariate box (or cube) splines being the most prominent examples. In particular, the symbols of bivariate box splines based on the three-directional mesh appear in a recent proper parametrization of bivariate subdivision schemes: they provide a basis of the ideal of symbols representing any convergent subdivision scheme. Other ideal bases have been described earlier in the literature, in particular by H. M. Möller and T. Sauer in [2, 3], but the basis of box spline symbols used in [1] is within the scope of refinable functions systems, since the basis itself consists of symbols of refinable functions.

One sample result along these lines is the following

Theorem. *Any bivariate subdivision scheme satisfying the sum rules of order k has a symbol $m(z)$ in the ideal I^k, which can be represented using the normalized box spline symbols*

$$N^{\#}_{\beta,\beta,\alpha}, N^{\#}_{\beta,\alpha,\beta}, N^{\#}_{\alpha,\beta,\beta}\;, \quad \alpha = 0, 1, \ldots, \lfloor\frac{k}{2}\rfloor\;, \quad \beta = k-\alpha\;,$$

as the generators of this ideal, within the ring of Laurent polynomials. Here,

$$N^{\#}_{n_1,n_2,n_3}(z_1,z_2) = \left(\frac{1+z_1}{2}\right)^{n_1} \left(\frac{1+z_2}{2}\right)^{n_2} \left(\frac{1+z_1 z_2}{2}\right)^{n_3}\;.$$

In the multivariate situation, an analogous results is still missing, and would be highly desirable in the three-variable case. The basic ideal I, referring to the sum rule of order 1, is here generated by normalized box spline symbols which refer to d-variate box splines with corresponding directional matrix

$$\Theta = (\theta_1\; \cdots\; \theta_d)$$

a $(d\times d)$ integer matrix whose columns $\theta_i \in \mathbb{Z}^d$, $i = 1, \ldots, d$, consist of any possible choices among

– the d-variate canonical unit vectors $e_1, \ldots, e_d$, and

– the combinations $e_i + e_j$, for $i \neq j$,

subject that the matrix Θ is unimodular.

Details and many more results along these lines can be found in the paper [1].

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Gabor Frames and Complex Analysis

Karlheinz Gröchenig

(joint work with Yura Lyubarskii)

1. Given a point $z = (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$, the time-frequency shift $\pi(z)$ acting on a function g is defined as

$$\pi(z)g(t) = e^{2\pi i \xi \cdot t} g(t - x) \qquad x, \xi, t \in \mathbb{R}^d .$$

Gabor analysis deals with structured function systems that are generated by time-frequency shifts of a single function g. To impose more structure, one assumes that the time-frequency shifts a generated by a lattice $\Lambda = A\mathbb{Z}^{2d}$, where $A \in \mathrm{GL}(2d, \mathbb{R})$. Then $\mathcal{G}(g, \Lambda) = \{\pi(\lambda)g : \lambda \in \Lambda\}$ is a so-called Gabor family. For the separable lattice $\Lambda = \alpha\mathbb{Z}^d \times \beta\mathbb{Z}^d$ this is just the set of functions $\{e^{2\pi i \beta l \cdot t} g(t - \alpha k) : k, l \in \mathbb{Z}^d\}$. See [1] for an introduction.

The fundamental questions concern the spanning properties of $\mathcal{G}(g, \Lambda)$ in $L^2(\mathbb{R}^d)$. When is $\mathcal{G}(g, \Lambda)$ a Riesz basis for the generated subspace? Precisely, $\mathcal{G}(g, \Lambda)$ is frame (a *Gabor frame*), if there exist constants $A, B > 0$, such that

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d) ,$$

and $\mathcal{G}(g, \Lambda)$ is a Riesz sequence, if there exist constants $A', B' > 0$, such that

$$A'\|c\|_2^2 \leq \|\sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g\|_2^2 \leq B'\|c\|_2^2 \qquad \forall c \in \ell^2(\Lambda) .$$

The frame property and the Riesz sequence property are not independent of each other. Let

$$\Lambda^\circ = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} (A^T)^{-1}\mathbb{Z}^{2d}$$

be the *adjoint* lattice of Λ. Then the fundamental duality principle of Janssen, Ron and Shen [4] states that $\mathcal{G}(g, \Lambda)$ *is a frame for* $L^2(\mathbb{R}^d)$, *if and only if* $\mathcal{G}(g, \Lambda^\circ)$ *is a Riesz sequence.*

From the duality theory two important consequences can be derived:

(i) The biorthogonality condition: The Riesz sequence property can be restated by saying that there exists a dual window $\gamma \in L^2(\mathbb{R}^d)$, such that

$$\langle \gamma, \pi(\mu) g \rangle = \delta_{\mu,0} \qquad \forall \mu \in \Lambda^\circ \,. \tag{1}$$

(In addition, $\mathcal{G}(\gamma, \Lambda^\circ)$ must also be a Bessel sequence).

(ii) The density theorem: If $\mathcal{G}(g, \Lambda)$ is a frame, then $\mathrm{vol}(\Lambda) = |\det A| \leq 1$.

2. Transition to complex analysis. If g is chosen to be the Gaussian $g(t) = 2^{d/4} e^{-\pi t \cdot t/2}$, then the problem of Gabor analysis is equivalent to an interpolating problem in complex analysis.

A small computation reveals that

$$\pi(x, -\xi) g(t) = e^{\pi i x \cdot \xi} e^{\pi z^2/2} \int_{\mathbb{R}^d} f(t) e^{-\pi t^2} e^{2\pi t \cdot z} dt \, e^{-\pi |z|^2/2} \,.$$

The expression

$$Bf(z) = e^{\pi z^2/2} \int_{\mathbb{R}^d} f(t) e^{-\pi t^2} e^{2\pi t \cdot z} dt$$

is an entire function on $\mathbb{C}^d$ and is called the *Bargmann transform* of f. As a consequence of Plancherel's theorem one obtains the isometry property

$$\|f\|_2^2 = \|Bf\|_{\mathcal{F}}^2 := \int_{\mathbb{C}^d} |Bf(z)|^2 e^{-\pi |z|^2} \, dz \,.$$

Then the biorthogonality condition (1) is equivalent to the following statement: $\mathcal{G}(g, \Lambda)$ *is a Gabor frame, if and only if there exists an interpolating function* $\Gamma \in \mathcal{F}$*, such that* $\gamma(\bar{\mu}) = \delta_{\mu,0}, \forall \mu \in \Lambda^\circ$.

3. In **dimension** $d = 1$ this interpolation problem is solvable with mathematics from the 19th century. Let σ be the Weierstrass sigma function defined by

$$\sigma_{\Lambda^\circ}(z) = z \prod_{\mu \in \Lambda^\circ \setminus \{0\}} (1 - \frac{z}{\mu}) e^{\frac{z}{\mu} + \frac{z^2}{2\mu^2}} \,.$$

Then the function $\Gamma(z) = \frac{\sigma_{\Lambda^\circ}(z)}{z}$ is interpolating on Λ°. Using the invariance properties of the sigma function and standard tools from complex analysis, the growth of Γ can be estimated by

$$|\Gamma(z)| \leq C \exp\left(\tfrac{\pi}{2\mathrm{vol}(\Lambda^\circ)} |z|^2\right)$$

and $|\Gamma(z)| \geq c \exp\left(\tfrac{\pi}{2\mathrm{vol}(\Lambda^\circ)} |z|^2\right)$ outside a neighborhood of the zeros of Γ. This means that $\Gamma \in \mathcal{F}$, if and only if $\mathrm{vol}(\Lambda^\circ) > 1$. As a consequence one obtains a complete characterization of all Gabor frames with Gaussian window in dimension $d = 1$. Thus $\mathcal{G}(g, \Lambda)$ *is a Gabor frame, if and only if* $\mathrm{vol}(\Lambda) < 1$. This result is special case of the fundamental work of Lyubarskii [3] and Seip [5].

4. In **higher dimensions** the situation is completely different and very little is known. The problem is much more difficult, because the zero set of an entire function of several complex variable is no longer discrete, but is an analytic manifold.

To be able to use complex variable techniques, we restrict ourselves to complex lattices, i.e., we identify a time-frequency pair $(x, \xi) \in \mathbb{R}^2$ with the complex variable $z = x + i\xi \in \mathbb{C}$ and assume that the lattice is of the form $\Lambda = A(\mathbb{Z} + i\mathbb{Z})^d$ for some $A \in \mathrm{GL}(d, \mathbb{C})$. Then the adjoint lattice is $\Lambda^\circ = (A^*)^{-1}(\mathbb{Z}+i\mathbb{Z})^d$. It is easy to build an interpolating function for Λ°. Abbreviating the sigma function on $\mathbb{Z}+i\mathbb{Z}$ by σ, a natural interpolating function on the product lattice $(\mathbb{Z} + i\mathbb{Z})^d$ is then $\tau(z) = \prod_{j=1}^d \frac{\sigma(z_j)}{z_j}$. By a coordinate transformation, the function $\Gamma(z) = \tau(A^*z)$ is is an interpolating function on Λ°, but its growth depends on A, i.e., on the choice of a basis for Λ°, rather than on the lattice itself. In general this construction yields an interpolating function that grows too fast and is not in Fock space. To decide the frame property of $\mathcal{G}(g, \Lambda)$ requires more subtle constructions.

5. Let us now focus on dimension $d = 2$. After choosing a reduced basis consisting of the shortest vectors in Λ and rotating the lattice, we may assume without loss of generality that the matrix generating Λ has the form

$$A = \begin{pmatrix} \gamma_1 & \beta \\ 0 & \gamma_2 \end{pmatrix} \qquad \text{with } \gamma_1, \gamma_2 > 0, |\mathrm{Re}\beta|, |\mathrm{Im}\beta| \le \gamma_1/2, \gamma_1^2 \le |\beta|^2 + \gamma_2^2\,.$$

In this parametrization we have the following results [2].

Theorem. Let g be the normalized Gaussian and $\Lambda = A(\mathbb{Z} + i\mathbb{Z})^2$ a complex lattice with a reduced basis.
(i) If $\gamma_1 < 1$ and $\gamma_2 < 1$, then $\mathcal{G}(g, \Lambda)$ is a frame.
(ii) If $\gamma_2 \ge 1$, then $\mathcal{G}(g, \Lambda)$ is not a frame.

Hardly anything is known about the remaining cases. Since we are in a nice situation with a maximum amount of structure, we would expect that, as in dimension $d = 1$, every lattice with $\mathrm{vol}(\Lambda) < 1$ generates a Gabor frame.

However, consider the lattices $\Lambda_j = A_j(\mathbb{Z} + i\mathbb{Z})^2$ with

$$A_1 = \begin{pmatrix} 1 & \frac{1}{5} \\ 0 & \frac{\sqrt{24}}{5} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & \frac{\sqrt{24}}{5} \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & \frac{2}{5} \\ 0 & \frac{\sqrt{21}}{5} \end{pmatrix}.$$

For all three lattices the necessary density condition is satisfied and $\mathrm{vol}(\Lambda_j) < 1$. However, neither $\mathcal{G}(g, \Lambda_1)$ nor $\mathcal{G}(g, \Lambda_2)$ constitute a frame. For $\mathcal{G}(g, \Lambda_3)$ we conjecture that it is a frame, but we have no reliable proof of this statement.

A general negative result is the following.

Theorem. Let $A = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{q} \\ 0 & \gamma \end{pmatrix}$ with $\delta_1, \delta_2 \ge 1, q \in \mathbb{Z} + i\mathbb{Z}, |q| \ge 2$ and $|q|^{-2} + \gamma^2 = 1$. Then $\mathcal{G}(g, A(\mathbb{Z} + i\mathbb{Z})^2)$ is *not* a frame for $L^2(\mathbb{R}^2)$.

Based on the known classes of examples, we envision that the generic lattice Λ with $\mathrm{vol}(\Lambda) < 1$ generates a frame. The non-generic examples as in the above theorem constitute "rare" cases. It remains a big challenge to prove a rigorous result supporting this vision.

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Efficient characterizations of nonnegativity on closed sets

Jean B. Lasserre

Tractable characterizations of polynomials (and even semi-algebraic functions) which are nonnegative on a set, is a topic of independent interest in Mathematics but is also of primary importance in many important applications, and notably in global optimization.

We will review two kinds of *tractable* characterizations of polynomials which are nonnegative on a basic closed semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$. Remarkably, both characterizations are through *Linear Matrix Inequalities* and can be checked by solving a hierarchy of semidefinite programs or generalized eigenvalue problems.

The first type of characterization is when knowledge on $\mathbf{K}$ is through its defining polynomials, i.e., $\mathbf{K} = \{\mathbf{x} : g_j(\mathbf{x}) \geq 0, \, j = 1, \ldots, m\}$, in which case some powerful certificates of positivity can be stated in terms of some sums of squares (SOS)-weighted representation. For instance, in global optimization this allows to define a hierarchy of semidefinite relaxations which yields a monotone sequence of *lower bounds* converging to the global optimum (and in fact, finite convergence is generic). Another (dual) way of looking at nonnegativity is when knowledge on $\mathbf{K}$ is through *moments* of a measure whose support is $\mathbf{K}$. In this case, checking whether a polynomial is nonnegative on $\mathbf{K}$ reduces to solving a sequence of *generalized eigenvalue* problems associated with a countable (nested) family of real symmetric matrices of increasing size. When applied in global optimization over $\mathbf{K}$, this results in a monotone sequence of *upper bounds* converging to the global minimum, which complements the previous sequence of lower bounds. These two (dual) characterizations provide convex *inner* (resp. *outer*) approximations (by spectrahedra) of the convex cone of polynomials nonnegative on $\mathbf{K}$.

Research Interests. I am mainly interested in optimization in a broad sense and in particular, using some results and tools from algebraic geometry for solving global optimization problems defined through polynomials (and even semi-algebraic functions). Concerning applications, we are especially interested in optimal control problems, and some inverse problems from moments.

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Positive Polynomials

Daniel Plaumann

This is a short survey about modern results on representations of positive polynomials in real algebraic geometry. As in the talk, statements will be kept as untechnical as possible, often at the expense of scope and precision. For a more complete presentation, the reader might consult the books of Marshall [3], Prestel-Delzell [5] and Lasserre [2], as well as Scheiderer's survey [7]. All results here that are not explicitly cited can be found in Marshall's book.

The basic setup is as follows. Let

$$S = \{a \in \mathbb{R}^n \mid g_1(a) \geq 0, \ldots, g_k(a) \geq 0\}$$

be a basic closed semi-algebraic set defined by real polynomials $g_1, \ldots, g_k \in \mathbb{R}[\underline{x}]$ in n variables $\underline{x} = (x_1, \ldots, x_n)$. Let always

$$\Sigma = \{h_1^2 + \cdots + h_l^2 \mid h_1, \ldots, h_l \in \mathbb{R}[\underline{x}], l \in \mathbb{N}\}$$

denote the cone of sums of squares of polynomials in $\mathbb{R}[\underline{x}]$.
The full cone of positive polynomials

$$\mathcal{P}(S) = \{f \in \mathbb{R}[\underline{x}] \mid f \geq 0 \text{ on } S\}$$

on S is a notoriously difficult, inaccessible object. On the other hand, it clearly contains the cone

$$M = M(g_1, \ldots, g_k) = \{s_0 + s_1 g_1 + \cdots + s_k g_k \mid s_i \in \Sigma\}$$

called the *quadratic module* generated by $g_1, \ldots, g_k$. The principal question is how close M is to $\mathcal{P}(S)$. There are two fundamental results for the compact case:

Schmüdgen 1991. If S is compact and $M \cdot M \subset M$, then

$$f|_S > 0 \quad \Longrightarrow \quad f \in M.$$

Note that, explicitly, $M \cdot M \subset M$ means $M = \{\sum_{i \in \{0,1\}^k} s_i g_1^{i_1} \cdots g_k^{i_k} \mid s_i \in \Sigma\}$.

Putinar 1993. If there is $g \in M$ with $\{a \in \mathbb{R}^n \mid g(a) \geq 0\}$ compact, then

$$f|_S > 0 \quad \Longrightarrow \quad f \in M.$$

It is customary to refer to the hypothesis in Putinar's theorem as the *archimedean property*. It follows from the theorem that the archimedean property is equivalent to the existence of $R > 0$ such that $R - \sum_{i=1}^n x_i^2 \in M$.

Natural questions that arise next are the following.

- What if S is not compact?
- What about computations and complexity?
- What if f has zeros on S?

The first question played a role in the later talk of Schmüdgen and the second in those of Netzer and Scheiderer, at this same workshop. The remainder of this note focuses on the third question.

Subsets of the line. A non-negative polynomial $f \in \mathbb{R}[x]$ in one variable is a sum of two squares. For such f must be of even degree, with positive leading coefficient, and real zeros must be of even multiplicity. Hence it can be written as $f = c^2 \cdot \prod_{j=1}^d (x-\alpha_j)\overline{(x-\alpha_j)} = \mathrm{Re}(p)^2 + \mathrm{Im}(p)^2$, where $p = c \cdot \prod_{j=1}^d (x - \alpha_j)$.

If $g = 1 - x^2$, then every polynomial $f \in \mathbb{R}[x]$ with $f|_{[-1,1]} \geq 0$ has a representation $f = s_0 + s_1 g$ with $s_0, s_1 \in \Sigma$. This is easy to prove by induction. By contrast, $\widetilde{g} = (1-x^2)^3$ defines the same interval, but there does not exist an identity $x = s_0 + s_1 \widetilde{g}$ with $s_0, s_1 \in \Sigma$.

In general, if S is any semialgebraic subset of $\mathbb{R}$, the cone $\mathcal{P}(S)$ is a finitely generated quadratic module, and one can precisely identify the generators. (A complete analysis was first given by Kuhlmann, Marshall and Schwartz; see [3]).

The higher dimensional case. It is not too surprising that what really governs the behaviour of $\mathcal{P}(S)$ is not so much the number of variables but rather the dimension of the semialgebraic set S. If $\dim(S) \geq 3$, then $\mathcal{P}(S)$ is never finitely generated, i.e. there do not exist $g_1', \ldots, g_l'$ describing S with $\mathcal{P}(S) = M(g_1', \ldots, g_l')$. With a localization argument, this essentially follows from the existence of a positive definite homogeneous polynomial in three variables (of degree at least 6) that is not a sum of squares (Hilbert - Motzkin).

On the other hand, any non-negative polynomial can be represented as a sum of squares of rational functions, i.e. with denominators. This was Hilbert's 17th problem, resolved by Artin in 1927. It generalizes to any preorder (multiplicative quadratic module) without further assumptions.

Positivstellensatz (Krivine 1964, Stengle 1974). If $M \cdot M \subset M$, then for every $f \in \mathcal{P}(S)$ there exist $s, t \in M$ and an integer $N \geq 0$ such that $sf = f^{2N} + t$.

If S is compact and $M \cdot M \subset M$ (or M archimedean), a polynomial in $\mathcal{P}(S) \setminus M$ must necessarily have zeros on S. On the positive side, there exist local sufficient conditions for membership in M in this case, such as the following.

Marshall 2006. Assume that M is archimedean, i.e. there exists $R > 0$ such that $R - \sum_{i=1}^n x_i^2 \in M$. Then M contains all $f \in \mathcal{P}(S)$ satisfying

(1) If $a \in \mathrm{int}(S)$ with $f(a) = 0$, then the Hessian $D^2 f(a)$ is positive definite;

(2) If $a \in \partial S$ with $f(a) = 0$, then f satisfies a suitable *boundary Hessian condition* in a.

For more refined versions of such results, see also [1].

Nie 2012 [4]**.** The conditions above hold for generic f (in a suitable sense).

Scheiderer 2003-2006 [8, 10]**.** If S is compact of dimension at most 2 and sufficiently regular, then $\mathcal{P}(S)$ is finitely generated.

Here, "sufficiently regular" amounts to about the following: If $\dim(S) = 1$, then S should be contained in a smooth algebraic curve. If $\dim(S) = 2$, then S should be contained in a smooth algebraic surface V and the relative boundary of S in V must satisfy a regularity condition (e.g. smooth boundary curves intersecting with independent tangents).

Examples. 1) If $h = x^2 + y^2 - 1$ and $\mathbb{T} = \{a \in \mathbb{R}^2 \mid h(a) = 0\}$ is the unit circle, then every $f \in \mathcal{P}(\mathbb{T})$ has a representation $f = s + rh$ with $s \in \Sigma$ and $r \in \mathbb{R}[x, y]$. (Note here that, according to our definitions, we should take $M = M(h, -h)$, which is in fact equal to $\Sigma + (h)$, where (h) is the ideal generated by h.) This can also be verified directly, for example by working in the complex plane or by taking a stereographic projection onto a line.

2) Likewise, if $h = y^2 + (2-x)(1-x)(1+x)(2+x)$, then $S = \{a \in \mathbb{R}^2 \,|\, h(a) = 0\}$ consists of two smooth ovals in the plane and every $f \in \mathcal{P}(S)$ has a representation $f = s + rh$ with $s \in \Sigma$ and $r \in \mathbb{R}[x, y]$. But this cannot be shown as easily by elementary means, since the curve here does not admit any parametrization by rational functions. (In fact, it has a node at infinity and its geometric genus is 1). The non-rationality precludes any direct analogue of the splitting of positive polynomials into complex-conjugate linear factors.

3) Every polynomial $f \in \mathbb{R}[x, y]$ that is non-negative on the unit disc in $\mathbb{R}^2$ possesses a representation $f = s_0 + s_1 \cdot (1 - x^2 - y^2)$ with $s_0, s_1 \in \Sigma$.

4) Every non-negative polynomial function f on the two-dimensional torus $\mathbb{T} \times \mathbb{T}$ is a sum of squares in the ring $\mathbb{R}[\mathbb{T} \times \mathbb{T}]$ of polynomial functions.

Remark. When looking for representations of the form

$$f = s_0 + s_1 g_1 + \cdots + s_k g_k$$

in M, one cannot in general bound the degrees of the sums of squares s_i in terms of the degree of f. Suppose for example that $M \cdot M \subset M$ and that S is compact with non-empty interior. If $f \in \mathcal{P}(S)$, then $f + \epsilon$ is strictly positive on S for any $\epsilon > 0$. Thus, by Schmüdgen's theorem, we can obtain representations

$$f + \epsilon = s_0^{(\epsilon)} + s_1^{(\epsilon)} g_1 + \cdots + s_k^{(\epsilon)} g_k$$

where $s_0^{(\epsilon)}, \ldots, s_k^{(\epsilon)} \in \Sigma$ depend on ϵ. If the degrees of $s_i^{(\epsilon)}$ were uniformly bounded (i.e. independent of ϵ), one could make an argument for the existence of a convergent subsequence and conclude $f \in M$. But we know that is not always possible if $\dim(S) \geq 3$. However, for $\dim(S) \leq 2$ the situation is less clear. It turns out that uniform degree bounds usually exist if $\dim(S) = 1$ (see [11] and the talk of Scheiderer), but cannot exist if S is compact and $\dim(S) = 2$ by [9].

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A generalized Prony method for sparse approximation

Gerlind Plonka

(joint work with Thomas Peter)

In many situations (e.g. in system theory or nonlinear approximation) one is faced with the problem of determining parameters for a certain signal structure from the measured data. One well-known method to tackle this problem is the Prony method that indeed serves as the basic concept for a series of reconstruction methods as e.g. MUSIC, ESPRIT, the matrix pencil method, the annihilating filter method etc. We want to provide a *universalized approach* to the Prony method that applies to a very general underlying signal structure.

Original Prony method. Let us first consider an exponential sum of the form

$$f(\omega) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{\omega T_j}$$

with (unknown) complex parameters c_j and T_j, $j = 1, \ldots, M$, and assume that $-\pi < \operatorname{Im} T_1 < \ldots < \operatorname{Im} T_M < \pi$. We aim to reconstruct c_j and T_j from a given small amount of (possibly noisy) measurement values $f(\ell)$. Using Prony's method or one of its stabilized variants, we are able to reconstruct f with only

$2M$ function values $f(\ell)$, $\ell = 0, \ldots, 2M-1$, see e.g. [2, 3, 4, 7] and references therein. The solution of this problem involves the determination of a so-called **Prony polynomial**

$$P(z) := \prod_{j=1}^{M}(z - \lambda_j)$$

with $\lambda_j := \mathrm{e}^{T_j}$. Assuming that $P(z)$ has the monomial representation $P(z) = \sum_{k=0}^{M} p_k z^k$, and using the structure of f, a short computation yields for $m = 0, \ldots, M-1$,

$$\begin{aligned}\sum_{k=0}^{M} p_k f(k+m) &= \sum_{k=0}^{M} p_k \sum_{j=1}^{M} c_j \mathrm{e}^{(k+m)T_j} = \sum_{j=1}^{M} c_j \lambda_j^m \left(\sum_{k=0}^{M} p_k \lambda_j^k\right) \\ &= \sum_{j=1}^{M} c_j \lambda_j^m P(\lambda_j) = 0.\end{aligned}$$

With $p_M = 1$ we obtain the linear Hankel system

$$\sum_{k=0}^{M-1} p_k\, f(k+m) = -f(M+m), \quad m = 0, \ldots, M-1,$$

providing the coefficients p_k of the Prony polynomial $P(z)$. Now, the unknown parameters T_j can be extracted from the zeros $\lambda_j = \mathrm{e}^{T_j}$ of $P(z)$. Afterwards, the coefficients c_j are obtained by solving the overdetermined linear system

$$f(\ell) = \sum_{j=1}^{M} c_j \mathrm{e}^{\mathrm{i}\ell T_j}, \quad \ell = 0, \ldots, 2M-1.$$

Generalized Prony method. Based on the above Prony approach, we want to present a new very general approach for the reconstruction of sparse expansions of eigenfunctions of suitable linear operators. In particular, we will show that the Prony-like reconstruction methods for exponentials [3] and polynomials [1] known so far, can be seen as special cases of this approach. Moreover, the new insight into Prony-like methods enables us to derive reconstruction algorithms for orthogonal polynomial expansions including Jacobi, Laguerre, and Hermite polynomials, [5, 6]. The approach also applies to finite dimensional vector spaces, and we derive a deterministic reconstruction method for M-sparse vectors from only $2M$ measurements. Particularly, a stable numerical method based on matrix pencils can also be transferred to this general Prony approach.

General approach. Let V be a normed vector space over $\mathbb{C}$, and let $\mathcal{A} : V \to V$ be a linear operator. Let $\Lambda := \{\lambda_j : j \in I\}$ be a (sub)set of pairwise distinct eigenvalues of $\mathcal{A}$, where I is a suitable index set. We consider the eigenspaces $\mathcal{V}_j = \{v : \mathcal{A}v = \lambda_j v\}$ to the eigenvalues λ_j, and for each $j \in I$, we predetermine a one-dimensional subspace $\tilde{\mathcal{V}}_j$ of $\mathcal{V}_j$ that is spanned by the normalized eigenfunction v_j. In particular, we assume that there is a unique correspondence between λ_j and v_j, $j \in I$.

An expansion f of eigenfunctions of the operator $\mathcal{A}$ is called *M-sparse* if its representation consists of only M non-vanishing terms, i.e. if

(1) $$f = \sum_{j\in J} c_j v_j, \quad \text{with } J \subset I \text{ and } |J| = M.$$

Due to the linearity of the operator $\mathcal{A}$, the k-fold application of $\mathcal{A}$ to f yields

$$\mathcal{A}^k f = \sum_{j\in J} c_j \mathcal{A}^k v_j = \sum_{j\in J} c_j \lambda_j^k v_j.$$

Further, let $F : V \to \mathbb{C}$ be a linear functional with $Fv_j \neq 0$ for all $j \in I$.

Theorem [5]**.** *With the above assumptions, the expansion f in* (1) *of eigenfunctions $v_j \in \tilde{\mathcal{V}}_j$, $j \in J \subset I$, of the linear operator $\mathcal{A}$, with $c_j \neq 0$ for all $j \in J$, can be uniquely reconstructed from the values $F(\mathcal{A}^k f)$, $k = 0, \ldots, 2M-1$, i.e., the "active" eigenfunctions v_j as well as the coefficients $c_j \in \mathbb{C}$, $j \in J$, in* (1) *can be determined uniquely.*

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A constructive approach to Putinar's and Schmüdgen's Positivstellensätze with applications to degree bounds and matrix polynomials

Markus Schweighofer

In the year 1900, Hilbert presented a list of 23 very influential mathematical problems [33]. In the 17th of these problems he mainly asked whether each (globally) nonnegative (real) polynomial (in several variables) could be written as a sum of squares of rational functions (already knowing that sums of squares of polynomials are not enough [34, page 347], i.e. denominators are needed). Artin solved this problem to the affirmative [31, Satz 6][8, 1.4.1][18, Thm. 2.1.12]. To do this, he had to develop together with Schreier the theory of ordered fields [32]. In particular, he had to introduce the notion of a real closed field [32, p. 87]. Real closed fields relate to ordered fields very much like algebraically closed fields relate to fields. The real numbers form the prototype of a real closed field just like the complex numbers build the prototype of an algebraically closed field. Real closed fields seem to be an indispensable tool for answering Hilbert's 17th problem. Moreover, the answer to Hilbert's 17th problem remains positive over an arbitrary real closed field (i.e. when one allows for coefficients from a real closed field instead of the real numbers). By general arguments from model theory, this implies that the degree of the numerator and denominator of the rational functions in a sums of squares representation of a given nonnegative polynomial can be bounded in terms of the number of variables and the degree of this polynomial. To get a concrete bound however is extremely tedious (and the known bounds are horribly bad) since Artin's proof is highly non-constructive [23, 24].

At about the same time when Artin solved Hilbert's 17th problem, Pólya proved another theorem on positive polynomials of a totally different flavor. He characterized (real) homogeneous polynomials (strictly) positive on an open orthant. Namely, he showed that these can be written as a quotient of an homogeneous polynomial with only (strictly) positive coefficients and a power of the sum of the variables [30]. For the case of two variables this is easily seen to be equivalent to the fact that a univariate polynomial positive on a given interval has only positive coefficients when expressed in the Bernstein basis of the vector space of polynomials of sufficiently high degree (associated to the interval). In sharp contrast to Artin's theorem, in Pólya's theorem it is self-evident how to compute the guaranteed representation: One simply multiplies the polynomial repeatedly with the sum of the variables until all coefficients get positive. Powers and Reznick proved an upper bound for the number of repetitions needed [17]. This bound unfortunately depends on a measure of how close the polynomial gets to zero (loosely speaking the size of the coefficients divided by the minimum on the standard simplex). Pólya's theorem does not hold over any real closed field.

Just a few years after the discovery of Artin's and Pólya's theorems, Tarski invented the method of real quantifier elimination. This was published only about 20 years later [29]. Then another thirteen years passed before this tool which is omnipresent in modern real algebra showed its impact on the further developments around Hilbert's 17th problem.

Namely, the next major step in the theory of positive polynomials was Krivine's seminal paper [28] where he introduces amongst others preordered rings and their maximal real spectrum. Thus he introduced basic notions and indispensable tools of modern real algebraic geometry in his very first scientific article. This brilliant work came too early for people to understand what is going on and has been neglected for a long time [18, Section 4.7]. Combining this newly developed theory with Tarski's real quantifier elimination, Krivine proves in the same article both the Positivstellensatz [28, Thm. 7][18, 4.2.10][21, 4.4.2] and the archimedean Positivstellensatz [18, Lemma 5.2.7].

The Positivstellensatz is a refinement and generalization of Hilbert's 17th problem which might at first glance look like a technical improvement but actually it is a very crucial and important enhancement of Artin's theorem. The archimedean Positivstellensatz at first sight looks like a variant of the Positivstellensatz but actually it is more of a Pólya-like nature: Under the additional assumption that the given preorder is archimedean, it provides a concrete denominator (namely a natural number which in many rings can be assumed to be 1) and it does not hold over any real closed field. However, Krivine uses the Positivstellensatz to prove the archimedean Positivstellensatz (combine [28, Thm. 11] with [28, Thm. 7]). Note however, that in the archimedean setting one can easily avoid the use of Tarski's theorem to prove the Positivstellensatz. Krivine's proof of the archimedean Positivstellensatz is completely constructive up to the starting point of the proof where he applies the Positivstellensatz to the element one wants to represent. Much later the author of this note gives a different and completely constructive proof of the archimedean Positivstellensatz by reducing it to Pólya's Theorem [16] (see also [1] for a recent exposition). We will come back to this later.

The content of Krivine's work was disremembered for about 35 years (though the work has occasionally been cited even in [21, page 95]) until Prestel took notice of this. Even now it continues to be ignored by many authors. Therefore the Positivstellensatz is often attributed to Stengle who rediscovered it ten years later [27]. Independently, Prestel rediscovered at about the same time the Positivstellensatz [26, Thm. 5.10] and gave the modern standard proof.

Unexpectedly, the next major breakthrough in the theory of positive polynomials came from functional analysis. In 1991, Schmüdgen used the Positivstellensatz to prove that multiplication operators arising in a GNS construction are bounded and used the spectral theorem and separating techniques for convex sets to prove what is now the celebrated Schmüdgen's Positivstellensatz [25, Cor. 3][18, Thm. 5.2.9][8, Cor. 6.1.2]. It is a denominator-free version of the Positivstellensatz over compact semialgebraic sets. It took more than seven years until people from real algebraic geometry could find an algebraic proof for Schmüdgen's Positivstellensatz. Namely, Wörmann found in his thesis (see [19]) an amazingly short but ingenious algebraic argument that allows to deduce Schmüdgen's Positivstellensatz from the Positivstellensatz and the archimedean Positivstellensatz. Using the Positivstellensatz Wörmann could show that the preorder involved in Schmüdgen's Positivstellensatz is archimedean and fulfills therefore the hypotheses of the archimedean Positivstellensatz. In hindsight, Schmüdgen's theorem is thus a characterization of finitely generated archimedean preorderings in the real polynomial ring [18, Thm. 5.1.17][8, Thm. 6.1.1] rather than a theorem about positive polynomials. But the original proof worked very differently. In the original proof there is a gap reported by Marshall in [8, pages x,88,89 and 98]. This gap has been found by Prestel and shortly after it has been bridged by Schmüdgen in an unpublished erratum which was apparently not known to Marshall.

Just two years later, in 1993, Putinar proved also with functional-analytic methods a sharpening of the archimedean Positivstellensatz which is now known as Putinar's Positivstellensatz [22][18, Thm. 5.3.8]. He uses quadratic modules instead of preorderings. The sums of squares representation is therefore weighted only by the defining polynomials of the semialgebraic set instead of all their exponentially many products. It is a common misperception that Putinar's Positivstellensatz is a strengthening of Schmüdgen's Positivstellensatz. In fact, it is a strengthening of the archimedean Positivstellensatz although one could formulate it in a way that it would generalize at the same Schmüdgen's Positivstellensatz (by imposing condition (i) from [12, Thm. 1] instead of the archimedean condition as a hypothesis). But any such phrasing of Putinar's Positivstellensatz just borrows from Schmüdgen's characterization of archimedean preorderings which is much deeper than Putinar's theorem. In fact, the innovative aspect of Putinar's article was mostly something different and his Positivstellensatz was "just" a by-product. Nevertheless it took again more than seven years until people from real algebraic geometry could find an algebraic proof for Putinar's Positivstellensatz. It was Jacobi who found a very technical and long algebraic argument [18, Lemma 5.3.7]. Another seven years later Marshall found an ingenious argument that radically shortened Jacobi's proof [8, Thm. 5.4.4].

The author's constructive approach. In 2002, the author found a new proof of the archimedean Positivstellensatz which is completely constructive [13]. It uses Pólya's theorem instead of the Positivstellensatz. It is therefore also an algorithmic approach to the Positivstellensatz up to Schmüdgen's characterization of archimedean preorderings. The latter is still not constructive at all since it uses the Positivstellensatz and Tarski's real quantifier elimination (note that we said above that Tarski could be avoided in the Positivstellensatz in the presence of the archimedean condition, however this does not help since it is used at a point in the proof before the archimedean condition is established).

In 2005, the author found a similar approach to Putinar's Positivstellensatz. The constructions involved are much more "dirty" than for the archimedean Positivstellensatz in the sense that there is an additional step with a polynomial of potentially very large degree appearing even before Pólya's procedure is applied.

The main advantages of these constructive approaches are the following:

(1) Computation of sums of squares representations. One can actually try to compute the sums of squares representation in the archimedean Positivstellensatz or in Putinar's Positivstellensatz. Once a Positivstellensatz certificate for the archimedean property is known, this then applies also to Schmüdgen's Theorem. Such a certificate can be found in many cases, and one gets it for free by adding a redundant inequality defining a big ball to the description of the semialgebraic set (if a ball containing the set is known).

(2) Complexity analysis. By taking much more care in the constructions, one can take track of the degree complexity of the sums of squares representations in the archimedean Positivstellensatz and in Schmüdgen's Positivstellensatz [13]. One of the main ingredients is the upper bound on the exponent needed in Pólya's theorem proved by Powers and Reznick [17]. Therefore it is not surprising that again the bound depends on a measure of how close the polynomial gets to zero on the semialgebraic set (roughly speaking again the size of the coefficients divided by the minimum on the semialgebraic set).

The same is true for Putinar's Positivstellensatz [9]. However, the bound is considerably worse. It seems that the price one has to pay for avoiding the exponentially many products of the defining inequalities is an exponential in the degree bound (though it is not known if the bounds are sharp).

(3) Parameterized families of sums of squares representations. The constructions, if performed carefully enough, can often be done uniformly for parameterized families of polynomials to represent. For the final stage of the procedure, namely the repeated multiplication step in Pólya's theorem, to terminate, it is often advantageous if the parameters come from a compact space.

Applications. The applications of the author's procedure seem to be numerous and are by far not exhausted. We give here just a few examples.

A. Computing minima of polynomials on compact semialgebraic sets. One can try to get a sums of squares representation of a polynomial minus an unknown lower bound of the polynomial. After each multiplication step in Pólya's procedure, one solves a linear program in only two (!) variables with the objective of maximizing the unknown lower bound. The second variable in the linear program comes from a parameter introduced in the author's constructions. This is an example of (3) with a linearly parameterized family of polynomials, the parameter ranging over an interval of the real line (namely the set of strict lower bounds of the polynomial on the given semialgebraic set). This procedure was implemented by Datta [15].

B. Positive polynomials on cylinders with compact cross section. Powers had the idea to consider a polynomial on a cylinder with compact cross section as a parameterized family of polynomials on the same compact semialgebraic set (namely the cross section). In this way she found mild and reasonable geometric conditions that guarantee the existence of sums of squares representations of polynomials positive on such cylinders [14]. This is again an example of (3) with a linearly parameterized family of polynomials, the parameter ranging over the whole real line.

C. Positive matrix polynomials. A symmetric matrix polynomials in several variables can be interpreted as a polynomial in the same variables with coefficients which are quadratic forms in new variables (one for each row or column). Since quadratic forms are given by their values on the unit sphere, one can therefore think of symmetric matrix polynomials as parameterized polynomials with parameters in the unit sphere which is a compact space. The ideas in (3) above therefore apply. This was carried out by Hol and Scherer in order to prove a version of Putinar's theorem for matrix polynomials [10] (see also [13]).

D. Semidefinite representations. In two seminal articles, Helton and Nie proved that many convex semialgebraic sets are semidefinitely representable [6, 5] (attention: the two articles appeared in the wrong order). To prove this they need sums of squares representation of bounded degree complexity for linear polynomials nonnegative on the given semialgebraic set. The main focus lies on the linear polynomials whose kernel is a supporting hyperplane of the convex set. Therefore neither Schmüdgen's nor Putinar's Positivstellensatz is applicable since the linear form is not strictly positive on the set. Although there are meanwhile a lot of theorems generalizing these theorems by allowing for a certain kind of zeros [11, 7, 2], there are no general complexity bounds available (perhaps one could try to generalize the author's constructive approach by using versions of Pólya's theorems that allow for zeros [3] but this seems a long way to go). Helton and Nie found a truly ingenious way to control the degree complexity by using Karush-Kuhn-Tucker conditions (i.e., “Lagrange-multipliers” for inequalities) and a sums of squares representation of the Hessian. The Hessian is a symmetric matrix polynomial which can very roughly speaking be assumed to be positive with some additional arguments given by Helton and Nie. This created the need for a matrix version of Schmüdgen’s and Putinar’s Positivstellensatz with control on the degree complexity. But with the observation made in the last point that matrix polynomials fall under the general idea (3) above, the arguments in [13, 9] go through almost literally as Helton and Nie observed.

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Introduction to Zonotopal Algebra

Amos Ron

General. The most common methodology for constructing multivariate splines is via their definition as volume functions. One begins with a linear surjection

$$X : \mathbb{R}^N \to \mathbb{R}^n,$$

and restricts this map to a polyhedron $\mathbb{Z} \subset \mathbb{R}^N$. In the theory of **box splines**, $\mathbb{Z} = [0,1]^N$. Two geometries underscore box spline theory: that of *zonotopes*, and the dual geometry of *hyperplane arrangements*. The geometries are associated with dual algebraic structures, and results in a seamless cohesion of the geometry, the algebra, the spline function and combinatorial properties of X.

Attempts to extend the aforementioned constructions beyond the original setup of box spline theory began in the mid 90's and reached their successful completion in [HR] where a three-layer theory that was coined there *zonotopal algebra* is introduced. Box spline theory is the middle *central* layer and two novel constructions, external and internal, over the same pair of dual geometries were newly introduced.

This report is devoted to the external theory within zonotopal algebra. In each layer, one starts with a pair of homogeneous polynomial spaces; the first is a "$\mathcal{P}$-space", and the second is a "$\mathcal{D}$-space" and is defined as the joint kernel of an ideal of differential operators known as a "$\mathcal{J}$-ideal".

Zonotopal algebra, central. Each column $x \in X$ induces a form $p_x : \mathbb{R}^n \to \mathbb{R} \ : \ t \mapsto x \cdot t$, and a differential operator $p_x(D)$ i.e., a directional derivative D_x. For $Z \subset X$, $p_Z := \prod_{x \in Z} p_x$. The central zonotopal algebra partitions 2^X into the long subsets $L(X) := \{Z \subset X \mid \operatorname{rank}(X \backslash Z) < n\}$, and the short subsets: $S(X) := 2^X \backslash L(X)$. The central $\mathcal{P}$-space $\mathcal{P}(X)$ is: $\mathcal{P}(X) := \operatorname{span}\{p_Z : \ Z \in S(X)\}$, while the long polynomials p_Z, $Z \in L(X)$ generate the ideal $\mathcal{J}(X)$. The $\mathcal{D}$-space is then the kernel of $\mathcal{J}(X)$:

$$\mathcal{D}(X) := \{f \in \Pi \mid p(D)f = 0, \forall p \in \mathcal{J}(X)\} = \{f \in \Pi \mid p(D)f(0) = 0, \forall p \in \mathcal{J}(X)\},$$

with Π the space of all polynomials in n variables. It is known, [DR], that the pairing

$$\langle \cdot, \cdot \rangle : \Pi \times \Pi \ : \ (p, q) \mapsto \langle p, q \rangle := p(D)q(0) \tag{1}$$

induces a linear bijection between $\mathcal{P}(X)$ and $\mathcal{D}(X)'$, and that, [DM], [DR],

$$\dim \mathcal{P}(X) = \dim \mathcal{D}(X) = \#\mathbb{B}(X),$$

with $\mathbb{B}(X)$ the set of **bases** of X, i.e., subsets of X that form a basis for $\mathbb{R}^n$, and $\mathbb{I}(X)$ the set of all **independent subsets** of X.

Connection with geometry and the least map. We discuss here the connection of $\mathcal{D}(X)$ with hyperplane arrangements; cf. [BDR] and [HR] for connections of $\mathcal{P}(X)$ and related spaces to zonotopes. One starts, [DR], by associating each $x \in X$ with $q_x := p_x - \lambda_x$, $\lambda_x \in \mathbb{R}$. Each $B \in \mathbb{B}(X)$ defines a *vertex* $b(B) \in \mathbb{R}^n$, viz, the common zero of the polynomials $(q_x)_{x \in B}$. Assume that the map $b : \mathbb{B}(X) \to \mathbb{R}^n$ is injective. The set $b(\mathbb{B}(X))$ is then the vertex set of the hyperplane arrangement $\mathcal{H}(X)$ generated by the zero sets H_x of q_x, $x \in X$.

We apply then to the vertex set $b(\mathbb{B}(X))$ the *least map* of [BR90]. The least map associates each finite $\Theta \subset \mathbb{R}^n$ with a polynomial space $\Pi(\Theta)$ such that the restriction of functions defined on $\mathbb{R}^n$ to the set Θ induces a bijection between $\Pi(\Theta)$ and $\mathbb{C}^\Theta$ (so, in particular, $\dim \Pi(\Theta) = \#\Theta$). It is proved in [BR91] that $\Pi(b(\mathbb{B}(X))) = \mathcal{D}(X)$.

External zonotopal algebra. One first complements X with an ordered $Y \subset \mathbb{R}^n$. In [HR] and [HRX], Y is an arbitrary ordered basis of $\mathbb{R}^n$. In [LR], Y is a (sufficiently long) sequence in general position in $X \cup Y$. Given Y, one selects a subset $\mathbb{B}' \subset \mathbb{B}(X \cup Y)$. The corresponding $\mathcal{J}$-ideal is then defined as

$$\mathcal{J}_{\mathbb{B}'} := \text{Ideal}\{p_Z \mid Z \subset X \cup Y,\ Z \cap B \neq \emptyset, \forall B \in \mathbb{B}'\}. \tag{2}$$

The corresponding $\mathcal{D}$-space $\mathcal{D}_{\mathbb{B}'}$ is the *kernel* of $\mathcal{J}_{\mathbb{B}'}$. The selection is **external** whenever $\mathbb{B}(X) \subset \mathbb{B}'$. While we are interested in particular, structured, choices of $\mathbb{B}'$, we have the following unqualified estimate on $\dim \mathcal{D}_{\mathbb{B}'}$, from [BR91]:

$$\dim \mathcal{D}_{\mathbb{B}'} \geq \#\mathbb{B}'.$$

We say that the external selection $\mathbb{B}(X) \subset \mathbb{B}' \subset \mathbb{B}(X \cup Y)$ is **coherent** if $\dim \mathcal{D}_{\mathbb{B}'} = \#\mathbb{B}'$. Thus, the central selection $\mathbb{B}' = \mathbb{B}(X)$ is coherent. [HR] was the first to consider an external setup. It chose Y to be a basis for $\mathbb{R}^n$, and defined a set injection

$$\text{ex} : \mathbb{I}(X) \to \mathbb{B}(X \cup Y),$$

via a greedy extension of each independent set to a basis using the elements of Y. The corresponding $\mathcal{D}$-space is then denoted there as $\mathcal{D}_+(X)$ and its corresponding ideal $\mathcal{J}_+(X)$. It is indeed proved in [HR] that $\mathbb{B}' := \text{ex}(\mathbb{I}(X))$ is coherent:

$$\dim \mathcal{D}_+(X) = \#\mathbb{I}(X).$$

$\mathcal{P}$-spaces. The original external version $\mathcal{P}_+(X)$ was introduced independently in [PS] and [HR]. It is defined as

$$\mathcal{P}_+(X) := \text{span}\{p_Z \mid \ Z \subset X\}.$$

It is proved in [HR] that $\mathcal{P}_+(X)$ and $\mathcal{D}_+(X)$ are dual, i.e., that $\mathcal{J}_+(X) \oplus \mathcal{P}_+(X) = \Pi$. It follows that $\dim \mathcal{P}_+(X) = \dim \mathcal{D}_+(X)$, hence $\dim \mathcal{P}_+(X) = \#\mathbb{I}(X)$.

Homogeneous bases for $\mathcal{P}(X)$. There are no known explicit constructions of bases for $\mathcal{D}$-type spaces. In contrast, there are such basis constructions for $\mathcal{P}$-spaces, all of which follow the construction of bases for $\mathcal{P}(X)$ in [DR]: one fixes an arbitrary order $\prec$ on the elements of X. Then, given $B \in \mathbb{B}(X)$, one defines

$$X(B) := \{x \in X \backslash B \mid x \notin \ \text{span}\{b \in B \mid b \prec x\}\}. \tag{3}$$

The cardinality of $X(B)$ is intimately connected to the *external activity* of B, [B].

Theorem [DR]. The polynomials $p_{X(B)}$, $B \in \mathbb{B}(X)$, form a basis for $\mathcal{P}(X)$.

The construction of homogeneous bases for external $\mathcal{P}$-spaces is derived from the above. Suppose that we have defined a $\mathcal{D}$-space $\mathcal{D}_{\mathbb{B}'}$, corresponding to the basis set $\mathbb{B}' \subset \mathbb{B}(X \cup Y)$, and a related $\mathcal{P}_{\mathbb{B}'}$ and proved a duality between the $\mathcal{D}$- and the $\mathcal{P}$- space. Now, necessarily,

$$\mathcal{P}_{\mathbb{B}'} \subset \mathcal{P}(X \cup Y).$$

Thus, we construct a homogeneous basis for $\mathcal{P}(X \cup Y)$ as above, and select the basis polynomials that correspond to $B \in \mathbb{B}'$. These polynomials are automatically linearly independent. Assuming that $\mathbb{B}'$ is coherent, we combine this coherence together with the assumed duality between $\mathcal{P}_{\mathbb{B}'}$ and $\mathcal{D}_{\mathbb{B}'}$ to conclude that

$$\dim \mathcal{P}_{\mathbb{B}'} = \dim \mathcal{D}_{\mathbb{B}'} = \#\mathbb{B}'.$$

Thus, the polynomials selected above will form a basis for $\mathcal{P}_{\mathbb{B}'}$ once we show that each of them actually lies in $\mathcal{P}_{\mathbb{B}'}$.

General external setup. The setup in [LR] provides a general unified theory and analysis that captures all previous efforts as special cases. We begin with an assignment $\kappa : 2^X \to \mathbb{N}$.

Definition. An assignment κ as above is **solid** if, given $Z, Z' \subset X$, we have

$$\text{span}Z \subset \text{span}Z' \implies \kappa(Z) \le \kappa(Z').$$

κ is **incremental** if, for every $Z \subset X$ and $x \in X$,

$$\kappa(Z \cup x) \le \kappa(Z) + 1.$$

We define the $\mathcal{P}$-space as

$$\mathcal{P}_\kappa := \sum_{Z \subset X} p_{X \setminus Z} \Pi_{\kappa(Z)},$$

with Π_k the space of polynomials of total degree k. The associated basis set $\mathbb{B}' := \mathbb{B}_\kappa \subset \mathbb{B}(X \cup Y)$ is defined as follows:

$$\mathbb{B}_\kappa := \{B \in \mathbb{B}(X \cup Y) \mid \ B \cap Y \subset Y_{m(B \cap X)}\}, \tag{4}$$

where, for an independent $I \in \mathbb{I}(X)$, $m(I) := \kappa(I) + n - \#I$, while $Y_i := \{y_1, \ldots, y_i\}$. It follows that each independent $I \subset X$ can be extended in $\binom{m(I)}{\kappa(I)}$ different ways to a basis in $\mathbb{B}_\kappa$, hence that $\#\mathbb{B}_\kappa = \sum_{I \in \mathbb{I}(X)} \binom{m(I)}{\kappa(I)}$.

The $\mathcal{D}$-space $\mathcal{D}_\kappa$ is defined as

$$\mathcal{D}_\kappa := \mathcal{D}_{\mathbb{B}_\kappa} = \ker \mathcal{J}_{\mathbb{B}_\kappa},$$

where $\mathcal{J}_{\mathbb{B}_\kappa}$ is defined in (2) with respect to the choice $\mathbb{B}' = \mathbb{B}_\kappa$. As before, we associate each $z \in X \cup Y$ with a constant λ_z and assume the assignment to be generic. Every $B \in \mathbb{B}(X \cup Y)$ then corresponds to $b(B) :=$ the common zero of the polynomials $(q_z)_{z \in B}$, and, by assumption, the map

$$b : \mathbb{B}(X \cup Y) \to \mathbb{R}^n \quad : \quad B \mapsto b(B)$$

is injective. We denote

$$V_\kappa := b(\mathbb{B}_\kappa).$$

Theorem [LR]. Let κ be a solid assignment. Then:

- $\mathbb{B}_\kappa$ is coherent. Furthermore, $\Pi(V_\kappa) = \mathcal{D}_\kappa$.
- $\mathcal{P}_\kappa + \mathcal{J}_\kappa = \Pi$.
- $\mathcal{P}_\kappa$ contains a Lagrange basis for V_κ: for each $v \in V_\kappa$ there exists $L_v \in \mathcal{P}_\kappa$, such that L_v vanishes on $V_\kappa \setminus v$, but not at v.

Assume further that κ is incremental. Set $X' := X \cup Y$, and, for $I \in \mathbb{I}(X)$, $X'_I := X \cup Y_{m(I)}$. Then

- The polynomials $q_{(X'_{B \cap X}) \setminus B}$, $B \in \mathbb{B}_\kappa$, form an inhomogeneous basis for $\mathcal{P}_\kappa$, hence $\dim \mathcal{P}_\kappa = \#\mathbb{B}_\kappa$.
- The polynomials $p_{X'(B)}$, $B \in \mathbb{B}_\kappa$, form a homogeneous basis for $\mathcal{P}_\kappa$.

It follows that $\mathcal{J}_\kappa \oplus \mathcal{P}_\kappa = \Pi$, or in other words that $\mathcal{P}_\kappa$ and $\mathcal{D}_\kappa$ are dual to each other.

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On the solvability of maximum entropy moment problems

Michael Junk

A variety of technical and physical problems leads to so called *reduced moment problems* where one tries to reconstruct a function $f \geq 0$ from a finite number of weighted integral averages

$$\int a_i(x) f(x) \rho(dx) = b_i, \qquad i = 1, \ldots, m. \tag{1}$$

Since the reconstruction of f from the values b_i is not unique, (1) can be viewed as an ill–posed inverse problem. A common way to regularize the problem is the *maximum entropy* approach [4] in which a solution of (1) is singled out as maximizer of the strictly concave entropy functional

$$H(f) = -\int f \ln f - f \rho(dx)$$

subject to the constraints (1). In particular, the original formulation (1) is embedded in the well developed theory of optimization problems which offers a variety of tools both theoretical and numerical [1, 2].

For example, it turns out to be quite useful to consider the dual formulation

$$\min_{\lambda \in \mathbb{R}^m} z(\lambda), \qquad z(\lambda) = \int \exp(\lambda \cdot a(x)) \rho(dx) - \lambda \cdot b$$

which is a finite dimensional convex optimization problem. Assuming that z is differentiable, we see that its critical points $\hat{\lambda}$ give rise to solutions of (1) in the form of strictly positive exponential functions $\exp(\hat{\lambda} \cdot a(x))$

$$0 = \partial_i z(\hat{\lambda}) = \int a_i(x) \exp(\hat{\lambda} \cdot a(x)) \rho(dx) - b_i.$$

Hence, standard minimization algorithms applied to z can be used to solve (1) provided z has at least one critical point.

The crucial question of necessary and sufficient conditions for the existence of critical points of z will be discussed in the following. For simplicity, we restrict our considerations to the case where the measure ρ in (1) is finite and where the weight functions a_i are measurable and bounded.

An obvious necessary condition concerns the data b in (1). We only have a chance to solve (1) with an exponential density if the problem admits at least one solution $f \geq 0$ which is not everywhere zero, i.e. if

$$(2) \qquad b \in \mathcal{M} = \left\{ \int a(x) f(x) \, \rho(dx) \; : \; \|a\| f \in \mathbb{L}^1(\rho), \; f \geq 0, \; \rho(f > 0) > 0 \right\}.$$

To illustrate a second necessary condition, assume that there exists some $\xi \in \mathbb{R}^m$ such that $g(x) = \xi \cdot a(x)$ is a non-positive function which vanishes on a set V of positive measure such that V^c also has positive measure. Then the characteristic function $f(x) = \mathbf{1}_V(x)$ of V is a bounded, non-negative density which does not vanish ρ-a.e. Due to boundedness of the weight functions a_i, we also have $\|a\| f \in \mathbb{L}^1(\rho)$ and hence $b_0 = \int a f \rho \in \mathcal{M}$ with

$$\xi \cdot b_0 = \xi \cdot \int a(x) f(x) \rho(dx) \int g(x) f(x) \rho(dx) = 0.$$

However, z based on b_0 cannot have a critical point $\hat{\lambda}$ because otherwise

$$0 = \xi \cdot b_0 = \xi \cdot \int a(x) \exp(\hat{\lambda} \cdot a(x)) \rho(dx) = \int g(x) \exp(\hat{\lambda} \cdot a(x)) \rho(dx) < 0$$

which is a contradiction. We have thus found a second necessary condition

$$(3) \qquad \xi \cdot a \leq 0 \, \rho\text{-a.e.} \quad \Rightarrow \quad \rho(\xi \cdot a < 0) \rho(\xi \cdot a = 0) = 0.$$

We now show that the two necessary conditions are also sufficient.

Theorem 1. *Assume ρ is a finite measure and a_i are measurable and bounded. Then the function z has a critical point if and only if* (2) *and* (3) *are satisfied.*

Proof. The necessity of the conditions has already been shown. Conversely, if (2) and (3) are satisfied, then the function z is strictly convex and coercive on the subspace $U^\perp \subset \mathbb{R}^m$ with

$$U = \{\xi \in \mathbb{R}^m \; : \; \xi \cdot a = 0 \; \rho\text{-a.e.}\}$$

(up to the trivial case $U^\perp = \{0\}$ where $a = 0$ ρ-a.e. so that z is constant and has critical points everywhere). To see this, choose $0 \neq \xi \in U^\perp$, and consider $Z(t) = z(t\xi)$ for $t > 0$. In the case $\xi \cdot b < 0$, the values $Z(t)$ grow at least linearly for $t \to \infty$. For $\xi \cdot b \geq 0$, the linear combination $\xi \cdot a$ must be positive on a set of positive measure because, otherwise, condition (3) applies and yields $\rho(\xi \cdot a < 0) = 0$ or $\rho(\xi \cdot a = 0) = 0$. Using (2), the second case contradicts $\xi \cdot b \geq 0$ while the first implies $\xi \in U \cap U^\perp = \{0\}$ which contradicts our assumption $\xi \neq 0$. But if $\xi \cdot a > 0$ on a set P with positive measure, there exists $\epsilon > 0$ such that $\xi \cdot a > \epsilon$ on $\tilde{P} \subset P$ with $\rho(\tilde{P}) > 0$. Thus

$$Z(t) \geq \int_{\tilde{P}} \exp(t\xi \cdot a(x))\rho(dx) > \rho(\tilde{P}) \exp(t\epsilon)$$

which also diverges for $t \to \infty$. The strict convexity follows from the strict positive definiteness of the Hessian of z. In fact,

$$\xi \cdot (\nabla^2 z(\lambda)\xi) = \int (\xi \cdot a(x))^2 \exp(\lambda \cdot a(x))\rho(dx) > 0$$

because $(\xi \cdot a)^2 \geq 0$ cannot vanish ρ-a.e. if $0 \neq \xi \in U^\perp$. As a consequence, z has a unique minimizer $\hat{\lambda}$ in $U^\perp$ which means that $\nabla z(\hat{\lambda})$ is perpendicular to $U^\perp$. Since for $\xi \in U$, we have $\xi \cdot b = 0$ and

$$\xi \cdot \nabla z(\lambda) = \int \xi \cdot a(x) \exp(\lambda \cdot a(x))\rho(dx) - \xi \cdot b = 0,$$

$\nabla z(\hat{\lambda})$ is also perpendicular to U and hence $\hat{\lambda}$ is a critical point of z. □

Since (3) is difficult to check in practice, as the non-positive linear combinations of the weight functions are generally difficult to characterize, a simpler sufficient condition is often used which is called *pseudo-Haar* property

$$\rho(\xi \cdot a = 0) > 0 \quad \Rightarrow \quad \xi = 0. \tag{4}$$

To see that (4) implies (3), we assume $\xi \cdot a \leq 0$ and $\rho(\xi \cdot a = 0) > 0$. Then the pseudo-Haar property implies $\xi \cdot a = 0$ ρ-a.e. so that $\rho(\xi \cdot a < 0) = 0$ and (3) follows.

In [3] the solvability of the maximum entropy moment problem is shown in the case where ρ is a regular Borel measure having full support on a compact Hausdorff space with continuous weight functions a_i satisfying the pseudo-Haar property.

Also in the case of non-compact domains, the pseudo-Haar condition is a typical assumption while the solvability of the maximum entropy problem is more complicated [5, 7].

Finally, we mention [6] where the pseudo-Haar condition is shown for the important case where ρ is the Haar-measure on a locally compact group and a_i are natural representation functions.

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Conditioning of truncated moment sequences of singular measures for input into entropy optimization

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(joint work with Mihai Putinar)

We explain the main ideas behind our paper [4], which concerns truncated moment problems for singular measures supported in subsets of $\mathbb{R}^D$. The paper shows how the original moment sequence can be adapted to be a suitable input for optimization of the entropy functional, which is used to obtain a measure that approximates the measure generating the input moment sequence. We convey the core concepts, tracing back to M. Krein [7, 8], by considering the special case of dimension $D = 1$. In the end we describe what exists and what is presently missing for extensions to any finite $D > 1$ and present a few future research direction.

Assume the case of the truncated moment problem: given is the subset of power moments $\gamma_\mu(n) = \int_0^1 x^n d\mu(x)$, $n = 0, \ldots, N < \infty$ of a positive finite-mass measure supported in $[0, 1] \subset \mathbb{R}$. The problem is to find a measure $\tilde{\mu}$ whose first $N + 1$ moments match the given set, i.e., $\int x^n \tilde{\mu}(x) = \gamma_\mu(n)$, $\forall n \in \{0, 1, \ldots, N\}$. Since we only have access to finite data, there exists a whole family of compatible measures whose moments match the set we are given. The entropy optimization is a deterministic procedure resulting in a *positive smooth* measure $\tilde{\mu}$ satisfying moment constraints.

A popular method [3, 9] for finding an absolutely continuous positive measure $d\tilde{\mu}(x) = f(x)dx$ constrained by $\int x^n f(x)dx = \gamma_\mu(n)$ is by optimizing the entropy functional $S(f) = \int f(x) \ln f(x)dx$. Solving the Lagrange dual problem, one can obtain the ansatz for solutions $f(x) = \exp \sum_{n=0}^N \lambda_n x^n$, where λ_n are Lagrange multipliers. However, not all moment sequences $\gamma_\mu(n)$ can be matched by exponential polynomial densities $e^{P(x)}$. For example, a Dirac-δ measure at $x = 0$, $\int f d\mu(x) = f(0)$, has the moment sequence containing $\gamma_\delta(n) = 0$, $\forall n > 0$, however due to strict positivity of any $\rho(x) = e^{P(x)}$, $\gamma_\rho(n) > 0$, $\forall n > 0$, therefore moments of $\delta(x)dx$ cannot be matched by an entropy-optimizing density. More general conditions are given in [6].

To be able to produce a measure $d\tilde{\mu}$ that in some sense approximates the unknown, possibly singular, measure $d\mu$, we propose to regularize the problem. Specifically, we seek an auxiliary regular measure $\varphi(x)dx$, such that first $N + 1$

moments $\gamma_\varphi(n)$ can be computed from first $N+1$ moments $\gamma_\mu(n)$ (conditioning), and that the measure $d\tilde{\mu} \approx d\mu$ can be evaluated from pointwise knowledge of density φ (inversion). If such a *regularized* density φ exists, then it can be obtained from entropy optimization with moments constrained by $\int \varphi(x)x^n dx = \gamma_\varphi(n), n = 0, \dots, N$.

We show that the regularized density $\varphi(x)$ can be obtained by considering the complex moment-generating function $(\mathcal{C}\mu)(z) := \int (z-x)^{-1} d\mu(x)$. Function $\mathcal{C}\mu$ is analytic in the upper-half plane with positive imaginary part (Nevanlinna class), it can be represented by another Nevanlinna function $\varphi(z)$ through relation $1 + (\mathcal{C}\mu)(z) = \exp \varphi(z)$. The boundary values of $\varphi(z)$ as $z \to \Re z = x \in \mathbb{R}$ produce a function $\varphi(x)$ such that $\varphi(z) = \int (z-x)^{-1}\varphi(x)dx = (\mathcal{C}\varphi)(x)$. The paper [2] gives a list of properties of φ; most importantly $\varphi \in L^1(\mathbb{R})$, $\varphi(x) \in [0, \pi]$ pointwise, and the support of φ is bounded whenever μ is of finite mass. These properties hold regardless of regularity of μ.

To relate moments of μ to moments of φ, one interprets the relation $1 + (\mathcal{C}\mu)(z) = \exp(\mathcal{C}\varphi)(z)$ as a relation between moment generating functions, as $(\mathcal{C}\mu)(z) = -\sum_{n=0}^{\infty} z^{-(k+1)}\gamma_\mu(n)$. If $M_N(z)$ is the Laurent polynomial obtained by generating moment function $\mathcal{C}\mu$ to $N+1$ coefficients, then the analogous truncation P_N for $\mathcal{C}\varphi$ can be obtained by summing $N+1$-truncations of powers of polynomials $(M_N)^k$ for $k = 1, \dots, N$, divided by k, i.e.,

$$\{\mathcal{C}\varphi(z)\}_N = P_N(z) = \textstyle\sum_{k=1}^{N} \frac{1}{k}\{[M_N(z)]^k\}_N,$$

where we denote the truncation by $\{.\}_N$. The moments γ_φ can now be extracted as coefficients of P_N.

Moments γ_φ are a feasible input to entropy optimization, as we saw that φ was a compactly supported, bounded, positive integrable function. We obtain the density $\tilde{\varphi}$ which approximates φ by matching its moments. It is important to notice that $\tilde{\varphi}$ is more regular than even φ, in particular, $\tilde{\varphi} \in C^\infty$ on $(0, 1)$, as it is of form $e^{P(x)}$ for some polynomial P. Now, the relation between two generating functions still holds, $1 + F(z) = \exp(\mathcal{C}\tilde{\varphi})(z)$. The density $f(x)$ that the generating function $F(z) = \int (z-x)^{-1} f(x)dx$ represents, we would need to evaluate the boundary value of $F(z)$ at the real line. The pointwise boundary (non-tangential) limits for smooth densities exist, in particular they exist for $(\mathcal{C}\tilde{\varphi})(z)$, and by analyticity of exp-function, also for $F(z) = \exp(\mathcal{C}\tilde{\varphi})(z) - 1$. Explicit formulas that evaluate the boundary limits are known as Plemelj-Sokhotski formulas [5].

Applying Plemelj-Sokhotski formulas, we obtain the density f as

$$f(x) = \pi^{-1} \exp[-\pi(\mathcal{H}\tilde{\varphi})(x)] \sin[\pi\tilde{\varphi}(x)],$$

where $\mathcal{H}$ denotes the Hilbert transform. Since an analytic expression for $\tilde{\varphi}$ is known, as $\tilde{\varphi}$ is in the form of entropy-optimal ansatz on a compact support, we can evaluate the Hilbert transform numerically from pointwise evaluations using discrete Fourier Transform algorithms, e.g., FFT.

The generalization of the conditioning procedure to supports in subsets of $\mathbb{R}^D$ relies on the Fantappiè transform [1] as the representation of the moment-generating function. Our paper [4] gives the explicit computations for several domains that contain supports of μ, in particular, positive orthant $\mathbb{R}^D_+$ for unbounded supports, and ℓ^2 and ℓ^1 balls in $\mathbb{R}^D$ for bounded supports. Presently, a practical generalization of the inversion procedure is not known, however, we do propose two possible approaches. The first approach would require a practically computable version of the extension of the Plemelj-Sokhotski formulas to multivariate setting, in which the complex field is replaced by a Clifford algebra. The second would be based on disintegration of measures over multidimensional domains to measures along lines. The inversion on each line could be solved by the presented one-dimensional method, followed by a tomographic approach for integrating the one dimensional solutions into a fully supported measure, completing the inversion.

In addition to completing the inversion step for measures supported in $\mathbb{R}^D$, an interesting problem would be a concrete description of the manner in which $f(x)dx \approx d\mu(x)$. From current work, it is clear that $f(x)dx \approx d\mu(x)$ in the sense that moments of their respective phase functions $\tilde{\varphi}$ and φ are equal. It would be clarifying to formulate the manner of approximation in terms of direct properties of $f(x)dx$ and $d\mu$. A related question is the characterization of the density φ, e.g., its differentiability dependent on properties of μ. A more practical direction of research could explore whether approximation performance is improved if the entropy optimization is replaced by optimization of another functional, or by an alternative method of inversion of the truncated moment problem for continuous measures.

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On the truncated problem of moments

Călin Ambrozie

The existence of the Lebesgue integrable solutions $f = f(t) \geq 0$ where $t = (t_1, \ldots, t_n)$ for a problem of moments [1], [4] with truncated data: $\int_T t^i f(t) dt = g_i$ ($i \in I$) is known [2], [3], [5] – [8] to be characterized under certain hypotheses by the existence of a (unique) solution f_* maximizing Boltzmann's entropy $H(f) = -\int_T f \ln f dt$, and hence by the existence of the maximum λ^* of the Lagrangian $L(\lambda) = \sum_{i \in I} g_i \lambda_i - \int_T e^{\sum_{i \in I} \lambda_i t^i} dt$. In the present context $T \subset \mathbb{R}^n$ is a closed subset, $I \subset (\mathbb{Z}_+)^n$ a finite set of multiindices $i = (i_1, \ldots, i_n)$ and g_i are the given data of the problem; as usual, $t^i = t_1^{i_1} \cdots t_n^{i_n}$. We give certain applications of this idea for certain slightly modified versions of the functional H, under various hypotheses on the data. In particular, in the case of two variables $n = 2$, if $T = \mathbb{R}^2$ and for moments up to order four $I = \{i \in \mathbb{Z}_+^2 : |i| \leq 4\}$, we give recurrence relations $\tilde{g}_j = \sum_{|i| \leq 4} r_{ji}(\lambda^*) g_i$ that provide the higher order moments $\tilde{g}_j := \int t^j f_*(t) dt$ ($|j| \geq 5$) in terms of the prescribed data g_i, where r_{ji} are rational functions that we can describe by concrete linear recurrence relations. These could be used to improve the approximation process used in the computation of λ^*, for instance by providing algebraic formulas for the second order derivatives of L ($\sim -\tilde{g}_j$ for various j) in the usual application [6], [9] of Newton's method to the problem $\max L = L(\lambda^*)$.

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Wavelet characterization of some spaces of distributions and its applications

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Besov spaces Let $V_0 \subset V_1 \subset V_2 \subset ...$ be a an r-regular multiresolution approximation (MRA) of $L^2(\mathbf{R}^n)$, and let W_j be the orthogonal complement of V_j in V_{j+1}. We denote by E_j the orthogonal projection of $L^2(\mathbf{R}^n)$ onto V_j and its extension to larger spaces of functions and distributions and let $D_j = E_{j+1} - E_j$. The characterization of functions in Besov spaces $B_q^{s,p}$ in terms of their projections onto W_j is strikingly simple and efficient. A function (or distribution) f belongs to $B_q^{s,p}$ if and only if $\|D_j f\|_p = 2^{-js} a_j,\ \{a_j\} \in l^q$, for a regular MRA with $r > |s|$, [8, ch. 2.9]. It is even tempting to take this description as a definition of the Besov spaces. The problem is that one has to specify an MRA or prove that the definition does not depend on one.

We study the spaces of distributions $B_\infty^{-s,\infty}$ for $s \geq 0$ and its generalizations. These spaces can be considered as spaces of boundary values of harmonic functions in $\mathbf{R}_+^{n+1}$ with some growth restrictions. We assume that all our distributions are of finite order and satisfy some regularity at infinity such that they can be convolved with the Poisson kernel $P_y(x) = y^{-n}P(x/y) = c_n y(y^2 + |x|^2)^{-(n+1)/2}$. Then for $s > 0$ we have $T \in B_\infty^{-s,\infty}$ if and only if $\|P_y * T\|_\infty \leq Cy^{-s}$, $y < 1$. For $s = 0$ the corresponding Besov space $B_\infty^{0,\infty} = \mathcal{B}$ is the Bloch space and $T \in \mathcal{B}$ if and only if $\|\nabla(P_y * T)\|_\infty \leq Cy^{-1}$.

Let ϕ be the father wavelet and $\psi^{(p)}$ be a finite system of generating wavelets with $\psi_{jk}^{(p)}(x) = 2^{nj/2}\psi^{(p)}(2^j x - k)$, $j = 0, 1, 2, ...$, $k \in \mathbf{Z}^n$, $p = 1, ..., q$. The classical Besov spaces can be also characterized in terms of the absolute values of wavelet coefficients, for our case of the spaces $B_\infty^{-s,\infty}$ with $s \geq 0$ this characterization reads like $|(T, \phi(\cdot - k))| \leq C$ and $|(T, \psi_{jm}^{(p)})| \leq C2^{-nj/2}2^{-js}$.

Weighted spaces of distributions and spaces of harmonic functions Let v be a positive decreasing function on $\mathbf{R}_+$, $v(y) = 1$ when $y > 1$, $\lim_{y\to 0} v(y) = \infty$, that satisfies $v(y/2) \leq Dv(y)$ for some D, in this case we say that v is a weight function. Examples include $\max\{1, y^{-s}\} = h_s(y)$ and $\max\{1, (\log e/t)^a\} = l_a(y)$ for $s, a > 0$. For each weight function we define the space of distributions $D_\infty(v) = \{T : |P_y * T| \leq C(T)v(y)\}$. Naturally, those appear as boundary values of harmonic functions in the upper half-space with in the growth restriction $|u(x, y)| \leq v(y)$. We define $h_v^\infty = \{u : \mathbf{R}_+^{n+1} \to \mathbf{R}, \Delta u = 0, |u(x, y)| \leq v(y)\}$. The corresponding spaces of harmonic functions in the unit disk were introduced by Shields and Williams in 1970s, [9]. Similar classes with $v(y) = \log 1/y$ and one-sided estimate were studied by Korenblum [5]. We consider these spaces as generalizations of the Besov spaces $B_\infty^{-s,\infty} = D_\infty(h_s)$. The wavelet characterization we obtain is now in terms of the projections E_j that give partial sums of the wavelet series.

Theorem 1 (Eikrem, Mozolyako,M,[4]). *Let T be a distribution of finite order s that admits convolutions with the Poisson kernel and let V_j be an r-regular MRA, $r > s$ and $2^r > D$. Then $T \in D_\infty(v)$ if and only if $\|E_j T\|_\infty \leq C(T)v(2^{-j})$.*

For slow growing weights like l_a there is no description in terms of wavelet coefficients or in terms of projections D_jT. The problem is $\|E_jT\|_\infty \leq C(T)v(2^{-j})$ implies $\|D_jT\|_\infty \leq C_1(T)v(2^{-j})$ but the opposite is not true. To obtain a more convenient characterization of $D_\infty(v)$ we project only on a subsequence of $V_j's$. We fix a sequence of positive integers n_j such that $v(2^{-n_{j+1}}) \geq 2v(2^{-n_j})$ and $v(2^{-n_{j+1}}) \leq 2Dv(2^{-n_j})$. Then we take partial sums of the wavelet series on levels n_j only. We obtain $T \in D_\infty(v)$ if and only if $(E_{n_{j+1}} - E_{n_j})T \leq C(T)v(2^{-n_j})$.

The main idea of the proof of Theorem 1 is to replace the Poisson kernel by the father wavelet.

Lemma. *Let $T \in D_\infty(v)$ and $g \in L^1(\mathbf{R}^n)$ such that* $\operatorname{supp}\widehat{g} \subset B_{1/y}$ *then*

$$\|T * g\|_\infty \leq C(T)\|g\|_1 v(y).$$

To prove the lemma we use Bourgain's construction from [2]. We take a function $\Sigma \in L^1(\mathbf{R}^n)$ such that $\widehat{\Sigma}\widehat{P} = 1$ on B_1. Then $\widehat{\Sigma_y}\widehat{P_y} = 1$ on $B_{y^{-1}}$ and we have for $g_0(\cdot) = g(x_0 - \cdot)$

$$|(T * g)(x_0)| = |(\widehat{T}, \widehat{g_0})| = |(\widehat{T}\widehat{P_y}, \widehat{\Sigma_y}\widehat{g_0})| = |(T * P_y, \Sigma_y * g_0)| \leq Cv(y)\|\Sigma\|_1\|g\|_1.$$

Our interest in these spaces of distributions originated in the study of boundary values of harmonic functions, see [1, 3, 6, 4]. Let us first consider the Bloch space $\mathcal{B}$. The wavelet description of $\mathcal{B}$ is well known, $T \in \mathcal{B}$ if and only if $\|D_jT\|_\infty \leq C$. This holds also for the Haar wavelet system when $T = E_0T + \sum_j D_jT$ is the martingale decomposition of T. For the case of Bloch functions the martingale differences are bounded. Then the growth is subject to the law of the iterated logarithm. The precise statement is the celebrated law of the iterated logarithm for Bloch functions due to Makarov, [7].

Theorem (Makarov)**.** *Let u be a harmonic function in $\mathbf{R}^{n+1}_+$. Suppose that u satisfies* $\sup_{(x,y)} y|\nabla u(x,y)| < \infty$. *Then*

$$\limsup_{y\to 0} \frac{|u(x,y)|}{\sqrt{\log y^{-1} \log\log\log y^{-1}}} \leq C$$

for almost every $x \in \mathbf{R}^n$.

Now we go back to the spaces h^∞_v. It is not difficult to construct examples of functions u in h^∞_v such that $\limsup_{y\to 0} v^{-1}(y)|u(x,y)| > 0$ almost everywhere. It can be also observed that for all such examples function $u(x,y)$ oscillates between $v(y)$ and $-v(y)$. We try to catch this oscillation by the following weighted average

$$I_u(x,s) = \int_s^1 u(x,y)d(v^{-1}(y)).$$

It is clear that $I_u(x,s) \leq C\log v(s)$, we observe the cancellation that leads to the following law of the iterated logarithm.

Theorem 2. *Let $u \in h^\infty_v$ then*

$$\limsup_{y\to 0} \frac{|I_u(x,s)|}{\sqrt{\log v(s) \log\log\log v(s)}} \leq C$$

for almost every $x \in \mathbf{R}^n$.

For the case of the unit disk in $\mathbf{R}^2$ and the weight $v(y) = \log y^{-1}$ the proof was given in [6]. We still use the Haar wavelet expansion of the boundary values of u (using results of Korenblum on premeasures) but we take superdyadic martingales, it corresponds to taking wavelet sums on the levels 2^{2^j} as it is explained after Theorem 1. For the case of the upper-half space and arbitrary v Theorem 2 is proved in [4], using smooth multiresolution approximation and appropriately chosen scale of wavelet generators, n_j. Theorem 2 gives an analog of the law of the iterated logarithm for spaces h_v^∞. Let us also point out three main differences between Theorem 2 and Makarov's law of the iterated logarithm. First, we measure the oscillation of the function and not its growth; second, to obtain the result we take an appropriate sub-scale of the dyadic scale and finally, we apply smooth multiresolution approximation instead of Haar wavelets.

Open problems We hope that other weighted spaces can be treated in a similar way, giving new results on description of dual spaces and multipliers. It would be also interesting to get results in the spaces of harmonic functions with non-homogeneous growth restrictions, $|u(x,y)| \leq v(x,y)$, where v now depends on x but changes slowly in x variable. Another problem is to give the description of the boundary values of functions in growth spaces in the unit ball. We would like to know if there is an appropriate wavelet decomposition (or a more soft tool) that can be applied to corresponding distributions on the unit sphere.

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A Construction of Locally Supported Tight Wavelet Frames on Sphere

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(joint work with T. Lyche)

Wavelets are functions in the Hilbert space $L_2(\mathbf{R}^d)$ whose integer translates and dilations form a basis for $L_2(\mathbf{R}^d)$, where $d \geq 1$. For example, if $\psi \in L_2(\mathbf{R})$ such that

$$\{2^{j/2}\psi(2^j x - k), j, k \in \mathbf{Z}\}$$

is an orthonormal basis for $L_2(\mathbf{R})$, ψ is called an orthonormal wavelet. A construction of such functions ψ with compact support was presented in [5] and see also [6]. However, when $d \geq 2$, compactly supported wavelet functions are difficult to construct. When the orthonormal conditions are relaxed, biorthogonal wavelets, prewavelets and even tight wavelet frames have been constructed in $L_2(\mathbf{R}^d), d \geq 1$ for any order of regularity. More precisely,

- Biorthogonal Box Spline Wavelets (cf. [9], [11], and [10])
- Prewavelets and pre-Riesz basis (cf. [20], [4], [12], [13]);
- Prewavelets on Sphere (cf. [19]);
- Tight Wavelet Frames on $\mathbf{R}^d$ (cf. [21], [1], [2], [7], [16], [15]);
- Tight Wavelet Frames over Bounded Intervals (cf. [3] and [14]).

It is well known that compactly supported wavelet functions are extremely useful for signal and image processing and many other applications. It is also known that tight frames perform just as well as the orthonormal wavelets. Indeed, if $\{f_j, j \in J\}$ is a tight frame for $L^2(\mathbf{R}^d)$, then for any $g \in L^2(\mathbf{R}^d)$,

$$g = \sum_{j \in J} \langle g, f_j \rangle f_j, \tag{1}$$

where $\langle g, f_j \rangle = \int_{\mathbf{R}^d} g f_j d\sigma$. Note that (1) is the same as the Parseval representation of functions in $L^2(\mathbf{R}^d)$ using orthonormal functions $f_j, j \in J$. A frame may not be a basis and hence it may contain some redundancy which can be useful for certain applications. Thus, it may have several representations for one function. However, any other representation $\{c_j, j \in J\}$ of a function g will have a larger norm than $\|g\|_2$, i.e. if $g = \sum_{j \in J} c_j f_j$ with $c_j \neq \langle g, f_j \rangle$ for some $j \in J$,

$$\|\{c_j, j \in J\}\|_2^2 = \sum_{j \in J} |c_j|^2 > \sum_{j \in J} |\langle g, f_j \rangle|^2 = \|g\|_2$$

(cf. [6]).

In this talk, we are interested in constructing smooth locally supported tight wavelet frames over the unit sphere $\mathbf{S}$ in $\mathbf{R}^3$. To the authors's knowledge, such tight wavelet frame functions have not been available in the literature so far. Our aim is to have a representation for $L^2(\mathbf{S})$ functions on the sphere like the standard orthonormal wavelet representation in $L^2(\mathbf{S})$ so that they will be useful to geoscience applications.

To construct smooth tight wavelet frames, we shall use the multi-resolution approximation (MRA) of the L^2 space on the sphere introduced in [19] which is built by using tensor product of polynomial spline space and trigonometric spline space. It is worthy pointing out that the trigonometric B-splines are indispensable for any tensor product functions on the sphere to be C^1, i,e, have continuous tangent planes at the north and south poles. See [19], [8] and [22] for the indispensability. As we have already known how to construct tight wavelet frames based on polynomial B-splines (cf. [3] and [14]), the key step is to construct tight wavelet frames based on trigonometric tight wavelet frames.

To do so, we need several properties of trigonometric B-splines (cf. [17]). In particular, we need Fourier transform of trigonometric B-splines since our constructive method is based on the refinement matrix between two nested spaces. To our knowledge, the Fourier transform of trigonometric B-splines is new and so is the refinable mask. We shall explain some sufficient conditions to construct tight wavelet frames over any bounded intervals. Several sufficient conditions will be discussed including one which is weaker than the existing one given in [3]. We verify that the refinement matrix associated with uniform trigonometric B-splines satisfies the weaker sufficient condition and hence these trigonometric B-splines can be used to construct tight wavelet frames. Together with a tight wavelet frame based on polynomial B-spline, a smooth locally supported tight wavelet frame on sphere is constructed.

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Structure of the set of PFWs

Hrvoje Šikić

Consider a system $\{\psi_{jk}(x)\} := \{2^{j/2}\psi(2^j x - k) : j,k \in \mathbb{Z}\}$, where $\psi \in L^2(\mathbb{R})$. We shall say that $\psi \in P$, the set of *Parseval frame wavelets* (PFW), if the system $\{\psi_{jk}(x)\}$ satisfies the *reproducing property*, i.e. , for every $f \in L^2(\mathbb{R})$,

$$f = \sum_{j,k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}\,, \tag{1}$$

unconditionally in $L^2(\mathbb{R})$. Our approach to the analysis of P has been outlined already in [10], and utilizes *multipliers* in order to achieve *positivity*; which enables us to treat various objects in a simpler manner.

Following [9], we can illustrate the idea of multipliers on a simple example. We denote by $\langle\psi\rangle := \overline{\text{span}}\{\psi_{0k} : k \in \mathbb{Z}\}$ the *principal shift invariant space* generated by ψ (for more on shift invariant spaces in this contest see [2] and [6]). If $\varphi \in \text{span}\{\psi_{0k} : k \in \mathbb{Z}\}$, than $\varphi = \sum_{j=1}^{\ell} c_j \psi_{0k_j}$. Taking Fourier transforms on both sides leads to $\widehat{\varphi} = t\widehat{\psi}$, where t is a trigonometric polynomial (in particular, it is also 1-periodic). Hence, using the L^2-norm,

$$\|\widehat{\varphi}\|^2 = \int_{\mathbb{R}} |t|^2 \cdot |\widehat{\psi}|^2 = \int_{\mathbb{T}} |t|^2 \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi+k)|^2 \, d\xi\,,$$

where $\mathbb{T} := {}^{\mathbb{R}}/_{\mathbb{Z}}$ is a one-dimensional torus. We denote the periodization $\xi \longmapsto \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi+k)|^2$ by $p_\psi(\xi)$. Observe that p_ψ is a 1-periodic function and $p_\psi \in$

$L^1(\mathbb{T})$. If we extend above equality to the closure of trigonometric polynomials we obtain that

$$t \longmapsto (t\widehat{\psi})^{\vee} \tag{2}$$

is an isometric isomorphism between the L^2-space generated by measure $p_\psi(\xi)\,d\xi$, i.e., $L^2(\mathbb{T}; p_\psi)$ and $\langle\psi\rangle$. Hence, we can describe all elements in $\langle\psi\rangle$ via multipliers t; observe that multipliers arise through trigonometric polynomials.

In order to develop a successful theory of PFWs, one has to develop a corresponding notion of an MRA; for basic definitions see [4] and for a different approach to MRA PFWs, see [1]. Our approach, as developed in [5], relies on a notion of a *filter* (again, see [4] for definitions and basic results on filters with respect to wavelets). If we start with a *generalized filter*, i.e., a 1-periodic measurable function $m_\circ$ such that $|m_\circ(\xi)|^2 + |m_\circ(\xi+\frac{1}{2})|^2 = 1$ a.e., then the key problem in the construction of MRA wavelets is that the product

$$\prod_{j=1}^{\infty} m_\circ\Big(\frac{\xi}{2^j}\Big) \tag{3}$$

may not exist (and even if it does it may not converge to the desired function). We can avoid this by positivity (and multipliers). There is a filter multiplier μ such that $m_\circ(\xi) = \mu(\xi)|m_\circ(\xi)|$ a.e. We define a function $\varphi_{|m_\circ|}$ via its Fourier transform

$$\widehat{\varphi}_{|m_\circ|}(\xi) := \prod_{j=1}^{\infty} \Big|m_\circ\Big(\frac{\xi}{2^j}\Big)\Big|\,, \tag{4}$$

$\xi \in \mathbb{R}$; observe that the product in (4) always exists because $0 \leqslant |m_\circ(\xi)|$. Furthermore, it can be shown that $\lim_{n\to\infty} \widehat{\varphi}_{|m_\circ|}(2^{-n}\xi)$ is either 0 or 1. If it is always 0, then we say that $m_\circ$ is a *generalized low pass filter*. We continue by solving the functional equation

$$\nu(2\xi)\overline{\nu(\xi)} = \mu(\xi) \qquad a.e.\,, \tag{5}$$

where ν is measurable and unimodular (i.e., $|\nu(\xi)| = 1$ a.e.). We then define $\varphi \in L^2(\mathbb{R})$ so that $\widehat{\varphi}$ corresponds to the idea of the product in (3). More precisely,

$$\widehat{\varphi}(\xi) := \nu(\xi)\widehat{\varphi}_{|m_\circ|}(\xi)\,, \tag{6}$$

$\xi \in \mathbb{R}$. It follows that φ satisfies

$$\widehat{\varphi}(2\xi) = m_\circ(\xi)\widehat{\varphi}(\xi) \qquad a.e. \tag{7}$$

We then define a function $\psi \in L^2(\mathbb{R})$ by

$$\widehat{\psi}(2\xi) := e^{2\pi i\xi}\overline{m_\circ(\xi+\frac{1}{2})\widehat{\varphi}(\xi)}\,, \tag{8}$$

$\xi \in \mathbb{R}$. It can be shown that if we start with a generalized low pass filter $m_\circ$, then the construction above always leads to $\psi \in P$. Hence, we say that such ψ-s are MRA PFWs. For detailed analysis of P and MRA PFWs see [7]. For implications of these ideas with respect to the Zak transform see [3]. For the development of these ideas in the context of the sampling theory see [8].

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Support maximization with linear programming in the cone of nonnegative measures

Didier Henrion

(joint work with Milan Korda, Jean-Bernard Lasserre, Carlo Savorgnan)

We address the problem of maximizing (the volume) of the support of a linearly constrained nonnegative measure. We show that this decision problem admits an infinite-dimensional convex linear programming (LP) formulation, which implies that it can be solved numerically with a converging hierarchy of finite-dimensional semidefinite programming (SDP) problems if the problem data are semialgebraic. We describe applications of these techniques to the computation of the moments of a semialgebraic set, and to the estimation of the region of attraction of a polynomial dynamical system.

In [1] an infinite-dimensional LP approach was introduced to compute the moments of a given compact set $X_0 \subset \mathbb{R}^n$. The basic idea was to notice that the Lebesgue measure on X_0 is the measure μ_0 solving the LP

$$(1)\qquad \begin{array}{ll} \sup & \int \mu_0 \\ \text{s.t.} & \mu_0 + \hat{\mu}_0 = \lambda \\ & \mu_0 \geq 0, \ \text{spt}\, \mu_0 \subset X_0 \\ & \hat{\mu}_0 \geq 0, \ \text{spt}\, \hat{\mu}_0 \subset X \end{array}$$

where the supremum is over nonnegative measures $\mu_0(dx)$, $\hat{\mu}_0(dx)$ respectively supported on X_0 and a given compact set X (say, a ball) which contains X_0, and such that μ_0 and $\hat{\mu}_0$ sum up to λ, the Lebesgue measure on X. The dual to this LP is as follows

$$
\begin{array}{ll}
\inf & \int v_0 \lambda \\
\text{s.t.} & v_0(x) \geq 1 \text{ on } X_0 \\
& v_0(x) \geq 0 \text{ on } X
\end{array} \tag{2}
$$

where the infimum is over a continuous function $v_0(x)$ nonnegative on X, which can be interpreted as a dual Lagrange multiplier for the primal linear constraint $\mu_0 + \hat{\mu}_0 = \lambda$. Note that the supremum in problem (1) is attained by μ_0 equal to the Lebesgue measure on X_0. In contrast, the infimum in problem (2) is not attained, but it can be shown that the (continuous) function v_0 converges almost uniformly to the (discontinuous) indicator function of X_0 (equal to one on X_0 and zero elsewhere).

In [1] it is explained that if set X_0 is basic semialgebraic and described by finitely many given polynomial inequalities, infinite-dimensional primal-dual LP problems (1-2) can be solved by a converging hierarchy of finite-dimensional semidefinite programming (SDP) problems (in turn solved numerically with powerful primal-dual interior-point algorithms). At a given relaxation order d, the primal SDP is a moment relaxation of LP (1), whereas the dual SDP is a polynomial sum-of-squares restriction of LP (2). From the solution of the primal SDP of order d, we obtain a vector approximating the moments of degree up to $2d$ of the Lebesgue measure on X_0.

In [2] we extended this approach to compute the region of attraction of a constrained dynamical system, defined as the set

$$
X_0 := \{x_0 \in \mathbb{R}^n \, : \, \frac{dx(t)}{dt} = f(t, x(t)), \; x(0) = x_0, \; x(1) \in X_1, \; x(t) \in X, \; \forall t \in [0,1]\}
$$

where the smooth vector field f, the compact state constraint set X (say, a ball) and the target constraint set $X_1 \subset X$ are given. The basic idea was to notice that the Lebesgue measure on X_0 is the solution to the LP

$$
\begin{array}{ll}
\sup & \int \mu_0 \\
\text{s.t.} & \mu_0 + \hat{\mu}_0 = \lambda \\
& \frac{\partial \mu}{\partial t} + \operatorname{div}(f\mu) = \delta_0 \mu_0 - \delta_1 \mu_1 \\
& \mu_0 \geq 0, \; \operatorname{spt} \mu_0 \subset X \\
& \hat{\mu}_0 \geq 0, \; \operatorname{spt} \hat{\mu}_0 \subset X \\
& \mu_1 \geq 0, \; \operatorname{spt} \mu_1 \subset X_1 \\
& \mu \geq 0, \; \operatorname{spt} \mu \subset [0,1] \times X
\end{array} \tag{3}
$$

where the supremum is over nonnegative measures $\mu_0(dx)$, $\hat{\mu}_0(dx)$, $\mu_1(dx)$ and $\mu(dt, dx)$ respectively supported on X, X, X_1 and $[0,1] \times X$, such that μ_0 and $\hat{\mu}_0$ sum up to λ, the Lebesgue measure on X. In problem (3), the linear constraint $\frac{\partial \mu}{\partial t} + \operatorname{div}(f\mu) = \delta_0 \mu_0 - \delta_1 \mu_1$ is called Liouville's equation, or the advection equation, or the equation of conservation of mass in fluid dynamics, statistical physics or kinetic theory. It should be understood in the sense of distributions, i.e.

$$\begin{aligned} &\int_0^1 \int_X \left(\frac{\partial v(t,x)}{\partial t} + \operatorname{grad} v(t,x)\cdot f(t,x)\right)\mu(dt,dx) \\ = &\int_{X_1} v(1,x)d\mu_1(dx) - \int_X v(0,x)d\mu_0(dx) \end{aligned}$$

for all sufficiently smooth test functions $v(t,x)$ supported on $[0,1]\times X$. The dual to LP (3) is as follows

$$\begin{array}{ll} \inf & \int v_0\lambda \\ \text{s.t.} & v_0(x) \geq 1 \text{ on } X \\ & v_0(x) \geq 1 + v(0,x) \text{ on } X \\ & v(1,x) \geq 0 \text{ on } X_1 \\ & -\frac{\partial v(t,x)}{\partial t} - \operatorname{grad} v(t,x)\cdot f(t,x) \geq 0 \text{ on } [0,1]\times X \end{array} \tag{4}$$

where the infimum is over continuous function $v_0(x)$ supported on X, interpreted as a dual Lagrange multiplier for the primal linear constraint $\mu_0+\hat{\mu}_0 = \lambda$, and over continuous function $v(t,x)$ supported on $[0,1]\times X$, interpreted as a dual Lagrange multiplier for the primal Liouville equation. Note that the supremum in problem (3) is attained by μ_0 equal to the Lebesgue measure on X_0. In contrast, the infimum in problem (4) is not attained, but it can be shown that the (continuous) function v_0 converges almost uniformly to the (discontinuous) indicator function of X_0.

In [2] it is explained that if f is a given polynomial vector field and X, X_1 are basic semi-algebraic sets described by finitely many given polynomial inequalities, infinite-dimensional primal-dual LP problems (3-4) can be solved by a converging hierarchy of finite-dimensional semidefinite programming (SDP) problems. At a given relaxation order d, the primal SDP is a moment relaxation of LP (3), whereas the dual SDP is a polynomial sum-of-squares restriction of LP (4). From the solution of the dual SDP of order d, we obtain a polynomial $v_0^d(x)$ of degree $2d$ which is an approximation of $v_0(x)$ such that the semi-algebraic set $X_0^d := \{x \in X : v_0^d(x) \geq 1\}$ is a valid outer approximation of X_0, i.e. $X_0 \subset X_0^d$. Moreover, the approximation converges in Lebesgue measure, i.e. $\lim_{d\to\infty} \lambda(X_0^d) = \lambda(X_0)$.

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Real algebraic geometry and semidefinite programming

Tim Netzer

Many of the Positivstellensätze that have been proven in the context of real algebraic geometry have applications in other areas of mathematics. One example is Lasserre's optimization method, which uses Positivstellensätze to solve an arbitrary polynomial optimization problem by solving a sequence of semidefinite

optimization problems. Other such examples come from the classification theory of feasible sets of semidefinite programming, so-called spectrahedra and their projections. In my talk I will give an introduction to these topics, and an overview over some of the most important recent results.
During the last years, my research has mostly been concerned with the problem of classifying spectrahedra and their projections. The generalized Lax conjecture states that each hyperbolicity cone is spectrahedral. Another conjecture, going back to Helton and Nie, states that each convex semialgebraic set is the projection of a spectrahedron. The first conjecture translates into an algebraic problem of realizing polynomials as determinants of linear matrix polynomials. The second conjecture is related to finding weighted sums-of-squares representations of linear polynomials with additional degree bounds.

Degree bounds for sums of squares, and applications to convex sets

Claus Scheiderer

Let $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$, let $\Sigma \subseteq \mathbb{R}[x]$ be the cone of sums of squares. Fix polynomials $1 = g_0, g_1, \dots, g_r \in \mathbb{R}[x]$, let $M = \Sigma + \Sigma g_1 + \cdots + \Sigma g_r$ be the quadratic module generated by the g_i and $K = \{u \in \mathbb{R}^n \colon g_1(u) \geq 0, \dots, g_r(u) \geq 0\}$ the associated basic closed set. For $d \geq 0$ let $\mathbb{R}[x]_d = \{f \in \mathbb{R}[x] \colon \deg(f) \leq d\}$ and $M_d = M \cap \mathbb{R}[x]_d$. Moreover write

$$M(d) \;=\; \Big\{\sum_{i=0}^{r} s_i g_i \colon\; s_i \in \Sigma,\; \deg(s_i g_i) \leq d \;(i = 0, \dots, r)\Big\}.$$

Clearly $M(d) \subseteq M_d$, but in general these two are not equal. Under a weak technical assumption that we always assume to hold, $M(d)$ is closed in $\mathbb{R}[x]_d$ for all d. Write

$$\delta(f) \;:=\; \inf\Big\{d \geq 0 \colon\; f \in M(d)\Big\}.$$

Testing whether $f \in M(d)$ is (the feasibility question of) a semidefinite program, and hence is solved efficiently. Therefore, to have an effective membership test for M, one would like to know an a priori upper bound for $\delta(f)$, when $f \in M$.

If $\sup\{\delta(f) \colon f \in M_d\} < \infty$, we say that M is *stable in degree* $\leq d$. If this holds for all $d \geq 0$, M is called *stable*. Note that M_d is closed if M is stable in degree $\leq d$.

When K contains a non-empty open cone then it is easy to see that M is stable. More interesting are the cases where K is compact. For simplicity we always assume that M is archimedean (i.e., $c - \sum_i x_i^2 \in M$ for some $c > 0$). When $f > 0$ on K, there exist (large) upper bounds for $\delta(f)$ in terms of $\deg(f)$, $\min f(K)$ and the size of the coefficients of f [8].

Theorem 1. [4] *If M is archimedean and* $\dim(K) \geq 2$*, then M is not stable.*

When $\dim(K) \geq 3$, there is $f \in \mathbb{R}[x]$ with $f|_K \geq 0$ and $f \notin M$ [3]. For $\epsilon > 0$ we have $f + \epsilon \in M$ by the theorems of Schmüdgen [7] and Putinar [2], and it follows that $\delta(f + \epsilon) \to \infty$ as $\epsilon \to 0$. For $\dim(K) = 2$ the proof is harder, since there are cases when M contains every polynomial that is nonnegative on K (e.g. when K is a smooth surface or polygon or the like [5]). Here is a proof in a concrete sample case:

Let $n = 2$ and $M = \Sigma + \Sigma(1 - x_1^2 - x_2^2)$, so $K \subseteq \mathbb{R}^2$ is the closed unit disk. Choose $f \in \mathbb{R}[x]$ strictly positive on $\mathbb{R}^2$ with $f \notin \Sigma$, e.g. $f = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 2$. For $c > 0$ let $f_c(x) = f(cx)$, then $f_c \in M$ [7]. If $\delta(f_c)$ were bounded for $c \to \infty$, there would be identities

$$f(x) \;=\; s^{(c)}\Big(\frac{x}{c}\Big) + \Big(1 - \frac{x_1^2 + x_2^2}{c^2}\Big)\, t^{(c)}\Big(\frac{x}{c}\Big), \quad c > 0,$$

with sums of squares $s^{(c)}(x)$, $t^{(c)}(x)$ of uniformly bounded degrees. Passing to a suitable sequence $c_\nu \to \infty$, we could then assume

$$s^{(c_\nu)}\Big(\frac{x}{c_\nu}\Big) \;\to\; s(x) \;\in\; \Sigma, \quad t^{(c_\nu)}\Big(\frac{x}{c_\nu}\Big) \;\to\; t(x) \;\in\; \Sigma$$

coefficientwise (for $\nu \to \infty$). Then passage to the limit $\nu \to \infty$ would give $f = s + t \in \Sigma$, contradiction. Thus we have $\limsup_{c\to\infty} \delta(f_c) = \infty$.

On the other hand, stability holds for $\dim(K) = 1$ unless prevented by singularities of K. For simplicity we only state the most basic case:

Theorem 2. [6] *Let $C \subseteq \mathbb{R}^n$ be a compact and smooth real algebraic curve, let $I \subseteq \mathbb{R}[x]$ be its vanishing ideal. Then $M = \Sigma + I$ is stable.*

It is also known that M contains all polynomials nonnegative on C. Using Lasserre's method of obtaining semidefinite representations of convex hulls via moment relaxation [1], one deduces (see [6]):

Theorem 3. *For every compact one-dimensional semi-algebraic set $K \subseteq \mathbb{R}^n$, the convex hull of K is a linear projection of a spectrahedron.*

Theorem 4. *The Helton-Nie conjecture holds in dimension two: Every convex semi-algebraic subset of $\mathbb{R}^2$ is a linear projection of a spectrahedron.*

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Approximation Bounds for Sparse Principal Component Analysis

Alexandre d'Aspremont
(joint work with Francis Bach, Laurent El Ghaoui)

We study approximation bounds for a semidefinite relaxation of the sparse eigenvalue problem, written here in penalized form

$$\max_{\|x\|_2=1} x^T \Sigma x - \rho \, \mathbf{Card}(x)$$

in the variable $x \in \mathbb{R}^n$, where $\Sigma \in \mathbf{S}_n$ and $\rho \geq 0$. Sparse eigenvalues appear in many applications in statistics and machine learning. Sparse eigenvectors are often used, for example, to improve the interpretability of principal component analysis, while sparse eigenvalues control recovery thresholds in compressed sensing [4]. Several convex relaxations and greedy algorithms have been developed to find approximate solutions (see [5, 6, 10, 9] among others), but except in simple scenarios (where, e.g., ρ is small and the two leading eigenvalues of Σ are separated in [5]), very little is known about the tightness of these approximation methods. In particular, the results in [6] can test optimality *a posteriori* but do not produce generic approximation bounds.

Here, using randomization techniques based on [2], we derive simple approximation bounds for the semidefinite relaxation derived in []dAsp08b. We do not produce a constant approximation ratio and our bounds depend on the optimum value of the semidefinite relaxation: the higher this value, the better the approximation. A similar behavior was observed by [15] for the semidefinite relaxation to MAXCUT, who showed that the classical approximation ratio of [8] can be improved when the value of the cut is high enough.

We then show that, in some applications, it is possible to bound a priori the optimum value of the semidefinite relaxation, hence produce a lower bound on the approximation ratio. In particular, following recent works by [1, 3], we focus on the problem of detecting the presence of a (significant) sparse principal component in a Gaussian model, hence test the significance of eigenvalues isolated by sparse principal component analysis. More precisely, we apply our approximation results to the problem of discriminating between the two Gaussian models

$$\mathcal{N}(0, \mathbf{I}_n) \quad \text{and} \quad \mathcal{N}\left(0, \mathbf{I}_n + \theta v v^T\right),$$

where $v \in \mathbb{R}^n$ is a sparse vector with unit Euclidean norm and cardinality k. We use a convex relaxation for the sparse eigenvalue problem to produce a tractable statistic for this hypothesis testing problem and show that in a high-dimensional setting where the dimension n, the number of samples m and the cardinality k grow towards infinity proportionally, the detection threshold on θ remains finite.

More broadly speaking, in the spirit of smoothed analysis [13], this shows that analyzing the performance of semidefinite relaxations on random problem instances is sometimes easier and provides a somewhat more realistic description of typical approximation ratios. Another classical example of this phenomenon is a MAXCUT-like problem arising in statistical physics, for which explicit (asymptotic) formulas can be derived for certain random instances, e.g., the Parisi formula [12, 11, 14] for computing the ground state of spin glasses in the Sherrington-Kirkpatrick model. It thus seems that comparing the performance of convex relaxations on random problem instances (e.g. in detection problems) often yields a more nuanced understanding of their performance in cases where uniform approximation ratios are either impossible to derive, or excessively conservative.

1. Sparse eigenvalues

We first formally define sparse eigenvalues. Let $\Sigma \in \mathbf{S}_n$ be a symmetric matrix. We define the sparse maximum eigenvalues of the matrix Σ as

$$\lambda^k_{\max}(\Sigma) \triangleq \begin{array}{ll} \max & x^T\Sigma x \\ \text{s.t.} & \mathbf{Card}(x) \leq k \\ & \|x\|_2 = 1, \end{array} \tag{1}$$

in the variable $x \in \mathbb{R}^n$ where the parameter $k > 0$ controls the sparsity of the solution. Starting from a penalized version of problem (1), written

$$\phi(\rho) \triangleq \max_{\|x\|_2=1} \; x^T\Sigma x - \rho\,\mathbf{Card}(x), \tag{2}$$

it was shown in [6] that

$$\phi(\rho) = \max_{\substack{\mathbf{Rank}(X)=1\\ X\succeq 0,\mathbf{Tr}(X)=1}} \sum_{i=1}^n \mathbf{Tr}\left(X^{1/2}(a_ia_i^T - \rho\mathbf{I})X^{1/2}\right)_+$$

and we write $\psi(\rho)$ the semidefinite relaxation of this last problem

$$\psi(\rho) \triangleq \begin{array}{ll} \max & \sum_{i=1}^n \mathbf{Tr}(X^{1/2}a_ia_i^TX^{1/2} - \rho X)_+ \\ \text{s.t.} & \mathbf{Tr}(X) = 1,\; X \succeq 0, \end{array} \tag{3}$$

which is equivalent to a semidefinite program [6] In the next section, we use this quantity as a test statistic for detecting significant sparse eigenvectors.

2. Approximation Bounds

Using the randomization argument detailed in [2, 7], we can derive an explicit bound on the quality of the semidefinite relaxation (3).

Proposition 1. *Let us call X the optimal solution to problem* (3) *and let $r = \mathbf{Rank}(X)$, we have*

$$n\rho\;\vartheta_r\left(\frac{\psi(\rho)}{n\rho}\right) \leq \phi(\rho) \leq \psi(\rho), \tag{4}$$

where

$$\vartheta_r(x) \triangleq \mathbf{E}\left[\left(x\xi_1^2 - \tfrac{1}{r-1}\sum_{j=2}^r \xi_j^2\right)_+\right] \tag{5}$$

controls the approximation ratio, with $\xi_1, \ldots, \xi_r$ i.i.d. standard normal.

When r is large, we can approximate $\vartheta_r(\cdot)$ by the function

$$\vartheta(x) \triangleq \mathbf{E}\left[\left(x\xi^2-1\right)_+\right] = \frac{2e^{-1/2x}}{\sqrt{2\pi x}} + 2(x-1)\mathcal{N}\left(-x^{-\frac{1}{2}}\right), \tag{6}$$

where $\xi \sim \mathcal{N}(0,1)$.

3. Detection problems

In this section, we focus on the problem of detecting the presence of a sparse leading component in a Gaussian model. It was shown in [3] that the sparse eigenvalue statistic is minimax optimal in this setting. Computing sparse maximum eigenvalues is NP-Hard, but we show here that the relaxation detailed in the previous section achieve detection rates that are a multiple of the minimax optimum, in a high-dimensional setting where the ambient dimension n, the number of samples m and the sparsity level k all grow towards infinity proportionally. More specifically, we focus on the following hypothesis testing problem, where

$$\begin{cases} \mathcal{H}_0: & x \sim \mathcal{N}\left(0, \mathbf{I}_n\right) \\ \mathcal{H}_1: & x \sim \mathcal{N}\left(0, \mathbf{I}_n + \theta v v^T\right), \end{cases} \tag{7}$$

where $\theta > 0$ and $v \in \mathbb{R}^n$ is a sparse vector satisfying $\mathbf{Card}(v) \leq k^*$ and $\|v\|_2 = 1$. Given m sample variables $x_i \in \mathbb{R}^n$, we let $\hat{\Sigma} \in \mathbf{S}_n$ be the sample covariance matrix, with

$$\hat{\Sigma} = \frac{1}{m}\sum_{i=1}^{m} x_i x_i^T.$$

We will now seek to bound the value of the statistics $\phi(\rho)$ and $\psi(\rho)$ defined in (2) and (3) respectively, under the two hypotheses above. The following proposition shows that if θ is high enough, then this test discriminates between $\mathcal{H}_0$ and $\mathcal{H}_1$ with probability $1-3\delta$.

Proposition 2. *Suppose we set*

$$\Delta = 4\log(9en/k^*) + 4\log(1/\delta) \quad \textit{and} \quad \rho = \frac{\Delta}{m} + \frac{\Delta}{\sqrt{k^* m(\Delta + 4/e)}} \tag{8}$$

and define θ_ϕ such that

$$\theta_\phi = \left(2\sqrt{\frac{k^*(\Delta+4/e)}{m}} + \frac{k^*(\Delta+4/e)}{m} + 2\sqrt{\frac{\log(1/\delta)}{m}}\right)\left(1 - 2\sqrt{\frac{\log(1/\delta)}{m}}\right)^{-1} \tag{9}$$

then if $\theta > \theta_\phi$ in the Gaussian model (7)*, the test statistic based on $\phi(\rho)$ discriminates between $\mathcal{H}_0$ and $\mathcal{H}_1$ with probability* $1-3\delta$*.*

This detection level was shown to be minimax optimal in [3]. This is not surprising, since the statistic $\phi(\rho)$ is simply a penalized formulation of $\lambda^k_{\max}(\cdot)$ which was shown to reach a similar detection level in [3]. Both $\phi(\rho)$ and $\lambda^k_{\max}(\cdot)$ are intractable however, and we will now focus on an efficiently computable statistic based on $\psi(\rho)$. We will control the quality of the approximation of $\phi(\rho)$ by $\psi(\rho)$

for the value of ρ used in computing θ_ϕ. We suppose $n = \mu m$ and $k^* = \kappa n$, where $\mu > 0$ and $\kappa \in (0,1)$. Setting ρ as in (8), we get

$$n\rho = \mu\Delta + \frac{\sqrt{\mu}\Delta}{\sqrt{\kappa(\Delta + 4/e)}} \quad \text{and} \quad \psi(\rho) \geq 1 - \mu\Delta\kappa - \frac{\sqrt{\mu\kappa}}{\sqrt{(\Delta + 4/e)}} - 2\sqrt{\frac{\log(1/\delta)}{m}}.$$

This means that the approximation ratio is bounded below by $\beta(\mu, \kappa)$, with

$$\beta(\mu,\kappa) = \frac{\vartheta(c)}{c} \quad \text{with} \quad c = \frac{1 - \mu\Delta\kappa - \frac{\sqrt{\mu\kappa}}{\sqrt{(\Delta+4/e)}} - 2\sqrt{\frac{\log(1/\delta)}{m}}}{\mu\Delta + \frac{\mu\Delta}{\sqrt{\kappa(\Delta+4/e)}}}, \tag{10}$$

The following proposition shows that if θ is high enough, then this test discriminates between $\mathcal{H}_0$ and $\mathcal{H}_1$ with probability $1 - 3\delta$.

Theorem 3. *Suppose $n = \mu m$ and $k^* = \kappa n$, where $\mu > 0$ and $\kappa \in (0,1)$ are fixed and n is large. Define the detection threshold θ_ψ such that*

$$\theta_\psi \geq \beta(\mu,\kappa)^{-1}\theta_\phi \tag{11}$$

where $\beta(\mu,\kappa)$ is defined in (10) and θ_ϕ is defined in (9), then if $\theta > \theta_\psi$ in the Gaussian model (7) the test statistic based on $\psi(\rho)$ discriminates between $\mathcal{H}_0$ and $\mathcal{H}_1$ with probability $1 - 3\delta$.

Observe that whenever μ is small enough, $\beta(\mu,\kappa) > 0$ for all values of $\kappa \in (0,1)$ and the approximation ratio converges to one as μ goes to zero. This means that the detection threshold θ of the statistic $\psi(\rho)$ remains finite when n goes to infinity in the proportional regime. By contrast, the detection threshold of the MDP statistic in [3] blows up to infinity when k goes to infinity in this scenario.

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The moment problem of closed semi-algebraic sets

Konrad Schmüdgen

Let K be a closed subset of R^d and let $M(K)$ denote the set of positive Borel measures μ on R^d such that *supp* $\mu \subseteq K$ and all polynomials $p \in R[x] := R[x_1, \cdots, x_d]$ are in $L^1(R^d, \mu)$. For $\mu \in M(K)$, the number $s_\alpha = \int x^\alpha d\mu(x)$, where $\alpha \in N_0^d$, is the α-th moment and the multisequence $s = (s_\alpha)_{\alpha \in N_0^d}$ is called the *moment sequence* of μ. The *K-moment problem* asks whether or not a given multisequence s is the moment sequence of some measure $\mu \in M(K)$. An equivalent formulation is that the linear functional L_s defined by $L_s(x^\alpha) := s_\alpha$, $\alpha \in N_0^d$, is of the form $L_s(p) = \int_K p\, d\mu$ for all $p \in R[x]$.

Such a K-moment sequence s and the corresponding measure $\mu \in M(K)$ are called *determinate* if there is no other measure $\nu \in M(K)$ which has the same moment sequence s as μ. If the set K is compact, then the polynomials are dense in $C(K)$ and hence each K-moment sequence is determinate.

This talk is concerned with the existence and the uniqueness of the K-moment problem with a particular emphasize on non-compact basic closed semi-algebraic sets. The main aim is to discuss the two fibre theorems obtained in [1] and [2].

Let $f = \{f_1, \ldots, f_k\}$ be a k-tuple of polynomials $f_k \in R[x]$. Then

$$K_f := \{x \in R^d : f_1(x) \geq 0, \ldots, f_k(x) \geq 0\}$$

is the associated *basic closed semi-algebraic set* and

$$T_f = \{\text{sums of } f_1^{\varepsilon_1} \cdots f_k^{\varepsilon_k} g^2; g \in R[x], \varepsilon_j = 0, 1\}$$

is the corresponding *preorder*. We consider the following properties of f:

(MP): Moment Property

For each linear functional L on $R[x]$ such that $L(T_f) \geq 0$ there is a measure $\mu \in M(R^d)$ such that $L(p) = \int p(x) d\mu(x)$ for all $p \in R[x]$.

(SMP): Strong Moment Property

For each linear functional L on $R[x]$ such that $L(T_f) \geq 0$ there is a measure $\mu \in M(K_f)$ such that $L(p) = \int p(x) d\mu(x)$ for all $p \in R[x]$.

Suppose that $h_1, \cdots, h_n \in R[x]$ are **bounded** polynomials on the set K_f. Put $h = (h_1, \ldots, h_n)$ and $H := h(K_f)$. For $\lambda \in H$ set

$$f(\lambda) = (f_1, \cdots, f_k, -(h_1 - \lambda_1)^2, \cdots, -(h_n - \lambda_n)^2)$$

The following result is the *first fibre theorem* [1].

Theorem 1: *If the fibre sequence $f(\lambda)$ has property (MP) (resp. (SMP)) for all $\lambda \in H$, then f has property (MP) (resp. (SMP)) as well.*

The original proof in [1] uses Stengle's Positivstellensatz and unbounded reduction theory. Elementary proofs are given by T. Netzer (2007) and M. Marshall (2009).

The second fibre theorem is based on the following *disintegration theorem*: *Suppose that X and T are closed subsets of Euclidean spaces. Let ν be a finite positive Borel measure on X, $p : X \to T$ a ν-measurable mapping, $\mu := p(\nu)$ the push-forward of ν by the map p. Then there exist a mapping $t \to \lambda_t$ of T into the set of positive Borel measures on X such that:*

(i) $\operatorname{supp} \lambda_t \subseteq p^{-1}(t)$,
(ii) $\lambda_t(p^{-1}(t)) = 1 \ \mu - \text{a.e.}$,
(iii) $\int_X f(x)\, d\nu(x) = \int_T d\mu(t) \int_X f(x)\, d\lambda_t(x)$.

Now we specialize the preceding. Let X and T be closed subsets of R^d and R^m, respectively, and let $p : X \to T$ be a polynomial mapping $p(x) = (p_1(x), \ldots, p_m(x))$. In the case $X = R^m$ the fibres are just the real algebraic varieties

$$p^{-1}(t) = \{x \in R^n : p_1(x) = t_1, \ldots, p_k(x) = t_k\},\ t \in T.$$

Then the *second fibre theorem* [2] is the following result.

Theorem 2: *Suppose that $\nu \in M(X)$ and let μ and λ_t be the corresponding measures. If $\mu \in M(K)$ is determinate, the polynomials $C[x_1, \cdots, x_m]$ are dense in $L^2(T, \mu)$, and $\lambda_t \in M(p^{-1}(t))$ is determinate for μ-almost all $t \in T$, then the measure $\nu \in M(X)$ is determinate on R^d.*

We mention two important applications of this theorem.

1. Suppose that $p(x) = x_1^2 + \cdots + x_n^2$, $X = R^n$ and $T := p(X) = [0, \infty)$. Then *ν is determinate if $\mu = p(\nu) \in M([0, +\infty))$ is determinate.*

2. Suppose $p_1, \ldots, p_m$ are *bounded* polynomials on X. Put $T = p(X)$. Then *$\nu \in M(X)$ is determinate if the fibre measures $\lambda_t \in M(p^{-1}(t))$ are determinate μ-a.e. on T.*

Finally we restate another result from [2] which is based on some growth condition for the sequence s. Recall that a positive semi-definite 1-sequence $s = (s_n)_{n \in N_0}$ is said to satisfy *Carleman's condition* if

$$\sum_{n=1}^{\infty} s_{2n}^{-1/2n} = +\infty.$$

This condition holds if there exists a $M > 0$ such that $|s_{2n}| \le M^n (n!)^2$ for $n \in N_0$.

Theorem 3: *Suppose that $s = \{s_n; n \in N_0^d\}$ is a positive semi-definite multi-sequence such that the first $d-1$ marginal 1-sequences*

$$\{s_{(n,0,\ldots,0))}; n \in N_0\}, \{s_{(0,n,\ldots,0)}; n \in N_0\}, \ldots, \{s_{(0,\ldots,0,n,0)}; n \in N_0\}$$

satisfy Carleman's condition. Then s is a R^d-moment sequence.
If in addition the last marginal 1-sequence $\{s_{(0,\dots,0,n)}; n \in N_0\}$ satisfies also Carleman's condition, then s is determinate.

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Rational moment generating functions and nonconvex compact polyhedra

Dmitrii Pasechnik
(joint work with N.Gravin, J.B.Lasserre, S.Robins, B.Shapiro, M.Shapiro)

We report on an approach to inverse moment problems for polynomial density measures supported on compact polyhedra, i.e. finite unions of convex polytopes, via Fantappiè transformations. The corresponding moment generating function turns out to be rational, with denominator encoding vertices of the polyhedron. This leads to exact procedures for reconstructing these measures, as well as applications in geometry of non-convex polyhedra.

A similar technique applied to harmonic moments of plane measures supported on polygons allows to quantify the well-known phenomenon of absense of determinancy for such measures.

The material presented here is based on [2, 3, 4].

Let $P \subset \mathbb{R}^d$ be compact, and μ a measure supported on P. It is natural to consider the transform $F(u) = \int_P \frac{dx}{1-\langle u,x\rangle} = \sum_m \binom{|m|}{m}\mu_m u^m$, relating P and the moments $\mu_m := \int x^m d\mu(x)$ of μ. It is well-known that if μ is atomic with finite support then it can be recovered from sufficiently many moments, as $F(u)$ has a particularly simple form in this case. This is related to the moment matrix of μ having finite rank. However, this idea does not work for more general measures. We can nevertheless remedy this for a large class of measures, as follows.

Definition. The Fantappiè transformation ("rationalized" moment generating function) of $P \subset \mathbb{R}^d$ w.r.t. the measure μ supported on P is

$$\mathcal{F}_\mu^k \quad := \quad (d + k)! \int_P \frac{d\mu(x)}{(1 - \langle u, x\rangle)^{d+1+k}} \quad = \quad \sum_m \frac{(|m| + d + k)!}{\prod_j m_j!}\mu_m u^m.$$

In fact, the following example is well-known, and can be found in e.g. [1].

Example. Fantappiè transformation of the unit density measure supported on a simplex $\Delta = \operatorname{conv}(v_1, \ldots, v_{d+1}) \subset \mathbb{R}^d$ is

$$\mathcal{F}^0_\Delta(u) = \int_\Delta \frac{d!dx}{(1 - \langle u, x\rangle)^{d+1}} = \frac{d!\mathrm{Vol}(\Delta)}{\prod_k (1 - \langle v_k, u\rangle)}.$$

Thus we have the following.

Theorem 1. Let $P \subset \mathbb{R}^d$ be a compact polyhedron, i.e. a finite union of convex polytopes. Then $\mathcal{F}^0_P(u)$, the Fantappiè transformation of the unit density measure supported on P, is a rational function with denominator $\Omega(u)$ dividing $\prod_{v\in V}(1 - \langle v, u\rangle)$, where V is the set of vertices of a triangulation of P.
Equally, $V := V(P)$ is the intersection of the sets of vertices of triangulations of P.

Usually, but not always, as the following example shows, $\Omega(u) = \prod_{v\in V}(1 - \langle v, u\rangle)$.

Example. Let $A = \{0, a_1, a_2, a_3\} \subset \mathbb{R}^3$ be a spanning set, and $v \in \mathbb{R}^3$. Let $P_\pm := \operatorname{conv}(v \pm A)$ and $P := P_+ \cup P_-$. Then $1 - \langle u, v\rangle$ does not appear in $\Omega(u)$, as

$$\mathcal{F}^0_P(u) = \mathcal{F}^0_{P_+}(u) + \mathcal{F}^0_{P_-}(u) = K \frac{\sum\limits_{1\le i<j\le 3} \langle u, a_i\rangle\langle u, a_j\rangle + (1 - \langle u, v\rangle)^2}{\prod\limits_{1\le i\le 3} ((1 - \langle u, v\rangle)^2 - \langle u, a_i\rangle^2)},$$

where $K \neq 0$ is a real constant.

Question 1. Give a geometric characterization for the case $\Omega(u) = \prod_{v\in V}(1 - \langle v, u\rangle)$.

The set $V(P)$ has geometric significance, i.e. it is basically a generalization of the set of vertices of a *polytope*. Recall that a polytope can be triangulated without adding any extra vertices—this ceases to be true in the non-convex case as soon as $d \ge 3$, (classical example is *Schönhardt polyhedron*). Thus the uniform probability measure μ_P supported on a polytope $P = \operatorname{conv}(V)$ can be written as a sum of uniform probability measures μ_T supported on simplices T with vertices in V.

Theorem 2. Let $P \subset \mathbb{R}^d$ be a compact polyhedron, with $V = V(P)$ as in Theorem 1, such that any $d + 2$-subset of T (affinely) spans $\mathbb{R}^d$, and $|V| \ge d + 2$. Then $\mu_P = \sum_{T\subseteq V} \alpha_T \mu_T$, where the sum is taken over $d + 1$-subsets of V spanning $\mathbb{R}^d$, and $\alpha_T \in \mathbb{R}$.

Question 2. Theorem 2 says that μ_P can be represented as a signed measure, with coefficients $\alpha_T \in \mathbb{R}$. In all examples we know nonzero α_T are ± 1. Is this always the case? If not, is it possible to say more about α_T?

Question 3. Can the spanning condition on $d+2$-subsets of V be removed or weakened?

Theorem 1 is not limited to uniform measures. Indeed, the following holds, implying that in the case of polynomial density measure supported on a polyhedron one can still choose an appropriate transform with the rational moment generating function.

Theorem 3. Let $P \subset \mathbb{R}^d$ be compact, μ a measure supported on P, and ρ a homogeneous polynomial of degree δ. Then

$$\mathcal{F}^{\delta}_{\rho\mu}(u) = (d+\delta)! \int_P \frac{\rho(x)d\mu(x)}{(1-\langle x,u\rangle)^{d+\delta+1}} = \rho\left(\frac{\partial}{\partial u}\right) \circ \mathcal{F}^0_P(u).$$

Thus, if one knowns, *exactly*, sufficiently many moments of a polynomial density measure μ supported on a polyhedron P, then $\mathcal{F}_\mu(u)$ by can be recovered by multivariate Padé approximation, and μ and P by factoring the denominator $\Omega(u)$ of $\mathcal{F}_\mu(u)$. However, in practice moments are always known with an error, and thus approximately recovered $\Omega(u)$ would not factor into linear factors. An approach to overcome this for convex P, based on a sequence of projections of P onto lines, is developed in [2].

Problem. Develop efficient algorithms for direct recovery of μ and P, in particular in the case of non-convex P.

Similar technique can be used to reconstruct polynomial density measures μ supported on plane polygons from their *harmonic* moments $\mu_k := \int z^k dxdy$, where $z := z(x,y) = x+iy$. An interesting feature of this class of problems is *non-determinancy*, i.e. two unequal measures can have the same sequence of harmonic moments. In [4] we were able to quantify this non-uniqueness, cf. Theorem 4 below.

A measure μ can be described by its transform

$$\Psi_\mu(u) = \sum_{j=0}^{\infty} \binom{j+2}{2} \mu_j u^j = \int\int \frac{d\mu(z)}{(1-uz)^3},$$

which is our main tool. Let $V \subset \mathbb{R}^2$ be finite. If $\mu := \mu_P$ is a uniform measure supported on a polygon P with vertices from V then $\Psi_\mu(u)$ is a rational function with denominator dividing $\prod_{v\in V}(1-vu)$.

Similarly to the general case above, if V is non-degenerate, then μ_P can be expressed as a weighted linear combination

$$(1) \qquad \mu_P = \sum_{T\subseteq V} \alpha_T \mu_T, \qquad \alpha_T \in \mathbb{R}$$

of uniform measures μ_T supported on triangles T with vertices in V. Let μ and μ' be two uniform measures supported on polygons with vertices from V, so that they have the same harmonic moments. Then $\mu - \mu'$ has all its harmonic moments equal to 0. This prompts the question of investigating such kinds of signed measures, which, as we just mentioned, can be written as in (1).

Theorem 4. Let $|V| = n$ and no 3 points of V are collinear. Then the dimension of the space of signed measures as in (1) equals $\binom{n-1}{2}$.

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Moment problems and entropy functionals

Tryphon T. Georgiou

Moments are linear constraints on a nonnegative measure and the moment problem is to determine whether such a measure exists, and if so, to provide a parametrization of admissible solutions. Entropy functionals on the other hand represent barrier functions on the positive cone of measures, and therefore, explicit solutions to moment problems are often presented as extrema of such functionals; these extrema are typically absolutely continuous.

Our interest in entropy functionals stems from the fact that a converse is often true, namely, absolutely continuous solutions to moment problems can be expressed as extrema of suitably chosen functionals. This provides an approach to the parametrization of rational solutions to moment problems –a problem motivated by R.E. Kalman in the context of stochastic partial realization theory [1]. The parametrization of rational solutions in the context of the truncated trigonometric moment problem is as follows ([2, 3, 4]): *Let* $\{r_k \mid r_k = \bar{r}_{-k} \in \mathbb{C},\ 0 \le k \le n\}$ *be such that the Toeplitz matrix* $[r_{k-\ell}]_{k,\ell=0}^{n}$ *is positive definite. Then, for any set* $\{\sigma_k \mid \sigma_k = \bar{\sigma}_{-k} \in \mathbb{C},\ 0 \le k \le n\}$ *such that*

$$\sigma(x) := \sum_{k=-n}^{n} \sigma_k e^{ikx} > 0$$

for $x \in [-\pi, \pi]$, *there exists a unique set of values* $\{\lambda_k \mid \lambda_k = \bar{\lambda}_{-k} \in \mathbb{C},\ 0 \le k \le n\}$ *such that*

$$\rho(x) := \frac{\sum_{k=-n}^{n} \sigma_k e^{ikx}}{\sum_{k=-n}^{n} \lambda_k e^{ikx}} > 0$$

for $x \in [-\pi, \pi]$ *and satisfies* $r_k = \int_{-\pi}^{\pi} e^{ikx} \rho(x) dx$ *for* $k \in \{0, \pm 1, \ldots, \pm n\}$. Thus, modulo possible cancellations between the numerator and denominator polynomials of $\rho(x)$, this result characterizes rational solutions of degree $\le n$ of the truncated trigonometric moment problem.

The original approach in [2, 3] was based on topological degree theory whereas the subsequent one in [4] identified $\rho(x)$ as the minimizer of

$$-\int_{-\pi}^{\pi}\left(\sum_{k=-n}^{n}\sigma_k e^{ikx}\right)\log\left(\rho(x)\right)dx$$

subject to the moment constraints and established uniqueness by appealing to the convexity of the functional; the λ's correspond to Lagrange multipliers. The implications of this result and its extension to Nevanlinna-Pick and Sarason-Nagy-Foias analytic interpolation have been carried out in [5, 6, 7].

This program has also been followed up in [8] for matrix-valued moment problems

$$R=\int_{\mathcal{S}} g_{\text{left}}(x)\rho(x)g_{\text{right}}(x)dx,$$

i.e., for problems where ρ is a matrix-valued distribution and g_{left}, g_{right} are likewise well-behaved matrix-valued functions. In this, solutions were sought that correspond to minimizers of the (quantum) relative entropy

$$\int_{\mathcal{S}}\text{trace}\left(\rho_0(x)\log(\rho_0(x))-\rho_0(x)\log(\rho_1(x))\right)dx$$

between (matrix-valued) density functions ρ_0, ρ_1 defined on more general, yet compact, domains $\mathcal{S}$. Two alternative functional representations for solutions were identified, an exponential and a rational one. The exponential form of solutions corresponds to minimizers of the relative entropy with respect to $\rho=\rho_0$ for a choice of a (matrix-valued) $\rho_1=\sigma$. The rational form on the other hand, which equally well parametrizes solutions by a choice of a (matrix-valued) σ, is of the form $\sigma^{1/2}L^*(\lambda)^{-1}\sigma^{1/2*}$ where L represents the linear map $\rho\mapsto R$ and L^* its adjoint (see [8, Section IV]). However, this form of solution corresponds to minimizers of the relative entropy with respect to $\rho=\rho_1$ for $\rho_0=\sigma$ *only when* σ *is a scalar multiple of the identity. It is an open question as to whether this form of rational solutions for matrix-valued moment problems corresponds to minimizers of a new type of "non-commutative" analog of the relative entropy functional.* Finally, inverse barrier functions of the form $\int_{\mathcal{S}}\rho(x)^{-1}dx$ are also relevant to moment problems and have been considered in a one-dimensional context in [9].

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Gabor Frames and Complex Analysis II

Yurii Lyubarskii

This is a survey lecture closely related to the previous lecture of Karlheinz Gröchenig with the same title. The lecture includes the following topics:

- Gabor frames generated by time frequency shifts of the Gaussian functions
- Fock space and Bargman transform
- Reduction to problems of sampling and interpolation in the Fock space
- Beurling densities
- Various approaches to proofs of sampling theorems
- Frame constants

Semidefinite Relaxations for the Grassmann Orbitope

Philipp Rostalski
(joint work with Raman Sanyal, Bernd Sturmfels)

The Grassmann orbitope is the convex hull over the Grassmann variety of decomposable skew- symmetric tensors of unit length. This variety parametrizes k-dimensional linear subspaces of $\mathbb{R}^n$, and it is the highest weight orbit under the k-th exterior power representation of the group $\mathrm{SO}(n)$. In this talk we discuss semidefinite relaxations of the Grassmann orbitope. That convex body can be approximated and represented surprisingly well by projections of spectrahedra (using Lasserre's moment matrices). We show that the first relaxation is exact for $k = 2$, we present numerical evidence that this result extends to higher k, and we discuss relations to a longstanding conjecture on calibrations by Harvey and Lawson, cf. [1, conj. 6.5, p.68].

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From Gabor frames to wavelet frames and vice versa

Ole Christensen
(joint work with Say Song)

We will discuss a procedure that allows us to construct dual pairs of wavelet frames based on certain dual pairs of Gabor frames. The talk is based on the paper [1].

A *Gabor system* in $L^2(\mathbb{R})$ has the form $\{e^{2\pi imbx}g(x-na)\}_{m,n\in\mathbb{Z}}$ for some parameters $a,b>0$ and a given function $g\in L^2(\mathbb{R})$. Using the *translation operators* $T_af(x):=f(x-a), a\in\mathbb{R}$, and the *modulation operators* $E_bf(x):=e^{2\pi ibx}f(x), b\in\mathbb{R}$, we will denote a Gabor system by $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$. A *wavelet system* in $L^2(\mathbb{R})$ has the form $\{a^{j/2}\psi(a^jx-kb)\}_{j,k\in\mathbb{Z}}$ for some parameters $a>1, b>0$ and a given function $\psi\in L^2(\mathbb{R})$. Introducing the *scaling operators* $(D_af)(x):=a^{1/2}f(ax)$, $a>0$, the wavelet system can be written as $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$.

Let $\theta>1$ be given. Associated with a function $g\in L^2(\mathbb{R})$ for which $g(\log_\theta|\cdot|)\in L^2(\mathbb{R})$, we define a function $\psi\in L^2(\mathbb{R})$ by

$$\widehat{\psi}(\gamma)=\begin{cases}g(\log_\theta(|\gamma|)), & \text{if } \gamma\neq 0,\\ 0, & \text{if } \gamma=0.\end{cases} \tag{1}$$

For fixed parameters $b,\alpha>0$, consider two bounded compactly supported functions $g,\tilde{g}\in L^2(\mathbb{R})$ and the associated Gabor systems $\{E_{mb}T_{n\alpha}g\}_{m,n\in L^2(\mathbb{R})}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$. For a fixed $\theta>1$, define the functions $\psi,\widetilde{\psi}\in L^2(\mathbb{R})$ by (1) from $g,\tilde{g}$ respectively.

Theorem 1 *Let $b>0$, $\alpha>0$, and $\theta>1$ be given. Assume that $g,\tilde{g}\in L^2(\mathbb{R})$ are bounded functions with support in the interval $[M,N]$ for some $M,N\in\mathbb{R}$ and that $\{E_{mb}T_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{n\alpha}\tilde{g}\}_{m,n\in\mathbb{Z}}$ form dual frames for $L^2(\mathbb{R}$. With $a:=\theta^\alpha$, if $b\leq\frac{1}{2\theta^N}$, then $\{D_{a^j}T_{kb}\psi\}_{j,k\in\mathbb{Z}}$ and $\{D_{a^j}T_{kb}\widetilde{\psi}\}_{j,k\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$.*

Based on Theorem 1, the rich theory for construction of dual pairs of Gabor frames enables us to provide explicit constructions of wavelet frame pairs. Consider, for example, exponential splines of the form

$$\mathcal{E}_N(\cdot) \quad := \quad e^{\beta_1(\cdot)}\chi_{[0,1]}(\cdot)*\cdots*e^{\beta_N(\cdot)}\chi_{[0,1]}(\cdot),$$

where $\beta_k=(k-1)\beta$, $k=1,\dots,N$, for some $\beta>0$. It is well known that for any $N\geq 2$ and $b\leq\frac{1}{2N-1}$, the Gabor system $\{E_{mb}T_n\mathcal{E}_N\}_{m,n\in\mathbb{Z}}$ is a Gabor frame, having a dual $\{E_{mb}T_n\widetilde{\mathcal{E}_N}\}_{m,n\in\mathbb{Z}}$ for a function $\widetilde{\mathcal{E}_N}$ of the form

$$\widetilde{\mathcal{E}_N} \quad = \quad \sum_{k=-N+1}^{N-1}a_nT_n\mathcal{E}_N.$$

Using Theorem 1 we obtain a pair of dual wavelet frames, generated by functions $\psi,\widetilde{\psi}$ for which $\widehat{\psi}$ and $\widehat{\widetilde{\psi}}$ are explicitly given compactly supported splines with geometrically distributed knot sequences.

It is possible to reverse the above process and construct Gabor frames based on certain wavelet frames. This can, e.g., be applied to the Mayer wavelet, which yields a compactly supported smooth function that generates a Gabor frame with redundancy two. We refer to [1] for details.

Note that the idea of the log-transform appears already in the paper [2], in a less elaborated form and in the setting of tight frames. The exact relationship to [2] is explained in [1].

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Directional Tight Framelets and Image Denoising

Bin Han

(joint work with Qun Mo, Zhenpeng Zhao)

High-dimensional wavelets and framelets are often obtained from one-dimensional wavelets and framelets through the simple tensor product. The main advantages of tensor product wavelets and framelets lie in that they have a straightforward fast numerical algorithm and the construction of one-dimensional wavelets and framelets is often relatively easy. To our best knowledge, almost all successful wavelet-based methods in applications have used tensor product wavelets and framelets, mainly due to their simplicity and fast implementation. Despite the fact that real-valued tensor product wavelets and framelets have been widely used in many applications, they have some shortcomings, in particular, they lack directionality: they cannot capture directionality very well except the horizontal and vertical directions. There are many different approaches to try to improve the performance of real-valued tensor product wavelets and framelets, for example, bandlets, contourlets, curvelets, dual-tree complex wavelets, shearlets, steerable filter banks, etc.

In this talk, we propose a simple approach by using tensor product complex tight framelets. For any sequence $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$, we define $\widehat{u}(\xi) := \sum_{k\in\mathbb{Z}} u(k)e^{-ik\xi}$, which is a 2π-periodic measurable function if $u \in l_2(\mathbb{Z})$. We say that $\{a; b_1, \dots, b_s\}$ (or simply $\{\widehat{a}; \widehat{b_1}, \dots, \widehat{b_s}\}$) is a tight 2-framelet filter bank if

$$|\widehat{a}(\xi)|^2 + |\widehat{b_1}(\xi)|^2 + \cdots + |\widehat{b_s}(\xi)|^2 = 1,$$
$$\widehat{a}(\xi)\overline{\widehat{a}(\xi+\pi)} + \widehat{b_1}(\xi)\overline{\widehat{b_1}(\xi+\pi)} + \cdots + \widehat{b_s}(\xi)\overline{\widehat{b_s}(\xi+\pi)} = 0$$

for almost every $\xi \in \mathbb{R}$. The tensor product complex tight framelet filter banks in 2D are given by

$$\{a \otimes a; a \otimes b_1, \dots, a \otimes b_s, \dots, b_s \otimes b_1, \dots, b_s \otimes b_s\}.$$

Let $\theta \in C^\infty(\mathbb{R})$ such that $(\theta(x))^2 + (\theta(-x))^2 = 1$ and $\theta(x) = 0, x < -1$. Define

$$\chi_{[c_L,c_R];\epsilon_L,\epsilon_R}(\xi) := \begin{cases} \theta(\frac{\xi - c_L}{\epsilon_L}), & \xi < c_L + \epsilon_L, \\ 1, & c_L + \epsilon_L \le \xi \le c_R - \epsilon_R, \\ \theta(\frac{c_R - \xi}{\epsilon_R}), & \xi > c_R - \epsilon_R. \end{cases}$$

A (complex) tight framelet filter bank $\{a; b_1^p, \dots, b_s^p, b_1^n, \dots, b_s^n\}$ is constructed in the frequency domain: for $\ell = 1, \dots, s$,

$$\widehat{a} := \chi_{[-c,c];\epsilon,\epsilon}, \quad \widehat{b_\ell^p} := \chi_{[c+\frac{\pi-c}{s}(\ell-1), c+\frac{\pi-c}{s}\ell];\epsilon,\epsilon}, \quad \widehat{b_\ell^n} := \widehat{b_\ell^p}(-\cdot).$$

and have $2(s^2 - s + 2)$ directions.

Another family of tight framelet filter banks $\{a^p, a^n; b_1^p, \dots, b_s^p, b_1^n, \dots, b_s^n\}$ with two low-pass filters is constructed in the frequency domain: for $\ell = 1, \dots, s$,

$$\widehat{a^p} := \chi_{[0,c];\epsilon,\epsilon}, \quad \widehat{a^n} := \widehat{a^p}(-\cdot), \quad \widehat{b_\ell^p} := \chi_{[c+\frac{\pi-c}{s}(\ell-1), c+\frac{\pi-c}{s}\ell];\epsilon,\epsilon}.$$

The tensor product tight framelet in 2D has $2(s^2 + s + 1)$ directions. In particular, when $s = 1$, the tight framelet has 6 directions and its performance for image denoising with $s = 1$ (6 directions) is almost the same as dual-tree complex wavelet transform (DT-CWT) in [6].

	UD-DWT	DT-CWT	6d Frame	4d Frame	TP-CWT
Lena					
$\sigma = 10$	34.83	35.21	35.17	34.83	35.49
$\sigma = 20$	31.87	32.26	32.37	31.98	32.57
$\sigma = 30$	30.14	30.49	30.65	30.36	30.83
$\sigma = 50$	27.89	28.22	28.46	28.27	28.58
Barbara					
$\sigma = 10$	32.67	33.53	33.66	33.22	34.17
$\sigma = 20$	28.72	29.91	29.96	29.34	30.52
$\sigma = 30$	26.60	27.82	27.80	27.11	28.35
$\sigma = 50$	24.25	25.32	25.26	24.62	25.74

TABLE 1. PSNR values for several directional transforms using the same bivariate shrinkage thresholding as in DT-CWT. UD-DWT means undecimated discrete wavelet transform using Haar wavelet filter bank, DT-CWT means dual tree complex wavelet transform. 6d Frame means using the complex tight framelets in the second family with $s = 1$ having 6 directions. 4d Frame means using the complex tight framelets in the first family with $s = 1$ having 4 directions. TP-CWT means using the complex tight framelets in the second family with $s = 2$ having 14 directions.

To measure directionality of any tight framelet filter bank, we have the following result:

Theorem: For any tight 2-framelet filter bank $\{a; b^p, b^n\}$,

$$|\widehat{b^p}(\xi+\pi)|^2 + |\widehat{b^n}(\xi)|^2 \geq B(\xi), \qquad \forall\, \xi \in [0,\pi]$$

with the lower bound being sharp, where

$$2B(\xi) := 2 - |\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2 - \sqrt{4(1-|\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2) + (|\widehat{a}(\xi)|^2 - |\widehat{a}(\xi+\pi)|^2)^2}.$$

That is, there exists a tight framelet filter bank $\{\widehat{a}; \widehat{\mathring{b}^p}, \widehat{\mathring{b}^n}\}$ such that

$$|\widehat{\mathring{b}^p}(\xi+\pi)|^2 + |\widehat{\mathring{b}^n}(\xi)|^2 = B(\xi), \qquad a.e. \xi \in [0,\pi]. \tag{1}$$

For many known filters, it turns out that $B(\xi)$ is very small. Using optimization techniques, we can construct directional finitely supported complex tight framelets such that (1) holds. For more detail on directional tensor product complex tight framelets, see [4].

In this talk, we also propose another approach by directly using 2D tight framelets having filter banks.

Let M denote a $d \times d$ real-valued invertible matrix. We shall use the following notation:

$$f_{M;k}(x) := |\det M|^{1/2} f(Mx-k), \; x, k \in \mathbb{R}^d.$$

Theorem: For any sequence $\{s_j\}_{j=0}^{\infty}$ in $\mathbb{N}$ (that is, at the scale level j, we require s_j directions), $AS_0(\{\phi\}; \{\eta^{s_j,\ell} : \ell = 1, \ldots, s_j\}_{j=J}^{\infty})$ is a tight $2I_2$-framelet for $L_2(\mathbb{R}^2)$, that is,

$$\sum_{k\in\mathbb{Z}^2} |\langle f, \phi_{2^J I_2;k}\rangle|^2 + \sum_{j=J}^{\infty} \sum_{\ell=1}^{s_j} \sum_{k\in\mathbb{Z}^2} |\langle f, \eta^{s_j,\ell}_{2^j I_2;k}\rangle|^2 = \|f\|^2_{L_2(\mathbb{R}^2)}, \; \forall\, f \in L_2, \; J \in \mathbb{N} \cup \{0\},$$

If $s_j \approx 2^{j/2}$, then the width and length of the support of $\eta^{s_j,0}_{2^j I_2;k}$ are approximately 2^{-j} and $2^{-j/2}$, obeying the hyperbolic rule width$^{1/2}$ = length. Moreover, ϕ is a refinable function and the affine system has an underlying filter bank and fast algorithm.

See [5] for more details on directional tight framelets.

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Problem Session

1. An extension problem for wavelet frames

O. Christensen

A sequence $\{f_k\}_{k\in I}$ in a separable Hilbert space $\mathcal{H}$ is called a frame if there exist constants $A, B > 0$ such that

$$A\,||f||^2 \le \sum_{k\in I} |\langle f, f_k\rangle|^2 \le B\,||f||^2,\ \forall f \in \mathcal{H}. \tag{1}$$

The sequence $\{f_k\}_{k\in I}$ is a Bessel sequence if at least the upper inequality is satisfied. For any frame $\{f_k\}_{k\in I}$, there exists at least one dual frame, i.e., a frame $\{g_k\}_{k\in I}$ such that

$$f = \sum_{k\in I} \langle f, g_k\rangle f_k \ \forall f \in \mathcal{H}. \tag{2}$$

We consider systems of functions in $L^2(\mathbf{R})$ having wavelet structure, and ask the following

Question: Assume that $\psi_1, \widetilde{\psi_1} \in L^2(\mathbf{R})$ and that the wavelet systems

$$\{2^{j/2}\psi_1(2^jx-k)\}_{j,k\in\mathbf{Z}} \text{ and } \{2^{j/2}\widetilde{\psi_1}(2^jx-k)\}_{j,k\in\mathbf{Z}}$$

are Bessel sequences in $L^2(\mathbf{R})$. Is it always possible to find functions $\psi_2, \widetilde{\psi_2} \in L^2(\mathbf{R})$ such that the multi-generated wavelet systems

$$\{2^{j/2}\psi_1(2^jx-k)\}_{j,k\in\mathbf{Z}} \cup \{2^{j/2}\psi_2(2^jx-k)\}_{j,k\in\mathbf{Z}}$$

and

$$\{2^{j/2}\widetilde{\psi_1}(2^jx-k)\}_{j,k\in\mathbf{Z}} \cup \{2^{j/2}\widetilde{\psi_2}(2^jx-k)\}_{j,k\in\mathbf{Z}}$$

form dual frames for $L^2(\mathbf{R})$?

The open question is clearly strongly connected with the following conjecture by Deguang Han:

Conjecture by Deguang Han: Let $\{2^{j/2}\psi_1(2^jx-k)\}_{j,k\in\mathbf{Z}}$ be a wavelet frame with upper frame bound B. Then there exists $D > B$ such that for each $K \ge D$, there exists $\psi_2 \in L^2(\mathbf{R})$ such that

$$\{2^{j/2}\psi_1(2^jx-k)\}_{j,k\in\mathbf{Z}} \cup \{2^{j/2}\psi_2(2^jx-k)\}_{j,k\in\mathbf{Z}}$$

is a tight frame for $L^2(\mathbf{R})$ with bound K.

Questions of the above type are well studied in the literature in the context of multiresolution analysis, typically starting with the unitary extension principle by Ron & Shen or one of its variants. However, the problems above are stated for general wavelet systems without the assumption of an underlying refinable function. Much less is known about this case.

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2. Comparing supports of measures from their respective moments

J. B. Lasserre

To motivate the above research issue, consider two finite Borel measures μ_1, μ_2 on a box $\mathbf{B} \subset \mathbb{R}^2$, with $\mu_1(O) > 0$ for some open Borel set $O \in \mathcal{B}(\mathbf{B})$, and where μ_2 absolutely continuous with respect to the Lebesgue measure on $\mathbf{B}$. Hence it is fair to say that μ_1 and μ_2 are *qualitatively* different.

If we let $\mathbf{y}^i = (y^i_\alpha)$, $\alpha \in \mathbb{N}^2$ and $i = 1, 2$, with

$$y^i_\alpha = \int \mathbf{x}^\alpha \, d\mu_i, \quad \forall \alpha \in \mathbb{N}^2; \quad i = 1, 2,$$

be the respective moment sequences associated with μ_1 and μ_2,

is this qualitative difference between μ_1 and μ_2
reflected in the sequences $\mathbf{y}^1$ and $\mathbf{y}^2$? And if yes, how?

In other words:

- What kind and what amount of information on the support $\sup(\mu)$ of a measure μ on $\mathbb{R}^n$, can we extract from the only knowledge of:
  - the whole sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, of its moments
  - a truncated sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}^n_d$, of its moments (i.e., when $|\alpha| \leq d$).
- If extraction of some information is possible, is there any efficient computational procedure to extract this information?
- How to detect whether μ_1 is singular with respect to another measure μ_2?

Of course some results are already available.

• In particular, if the vector $\mathbf{p}$ of coefficients of some polynomial $p \in \mathbb{R}[\mathbf{x}]$ belongs to the kernel of some moment matrix $\mathbf{M}_d(\mathbf{y})$ associated with the moment sequence $\mathbf{y}$, then $\sup(\mu) \subset \{\mathbf{x} : p(\mathbf{x}) = 0\}$ because

$$0 \leq \int p^2(\mathbf{x}) d\mu(\mathbf{x}) \;=\; \langle \mathbf{p}, \mathbf{M}_d \, \mathbf{p} \rangle \;=\; 0,$$

and of course $\mathbf{p} \in \mathrm{Ker}(\mathbf{M}_\ell(\mathbf{y}))$ for all $\ell \geq d$.

• Also, if $\sup(\mu)$ is compact, then for any polynomial $p \in \mathbb{R}[\mathbf{x}]$, the smallest (resp. the largest) generalized eigenvalue associated with the moment matrix $\mathbf{M}_d(\mathbf{y})$ and the localizing matrix $\mathbf{M}_d(\mathbf{y}\, p)$, provides a lower bound $\underline{p}_d$ (resp. an upper bound $\overline{p}_d$) on $\min\{p(\mathbf{x}) : \mathbf{x} \in \sup(\mu)\}$ (resp. $\max\{p(\mathbf{x}) : \mathbf{x} \in \sup(\mu)\}$). Hence by playing with several polynomials p, one may distinguish between two different supports.

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3. Factorization of nonnegative matrix polynomials on the 2D torus

C. Scheiderer

Let $M = M(\omega_1, \omega_2)$ be a symmetric trigonometric matrix polynomial in two variables, i.e., a symmetric $m \times m$-matrix whose coefficients are trigonometric polynomials in $\omega = (\omega_1, \omega_2)$. Assume that $M(\omega)$ is positive semidefinite for all ω. Does there exist a trigonometric matrix polynomial $S = S(\omega)$ of size $N \times m$ (for some N) such that $M = S^t S$?

More generally, one may consider the analogous question for trigonometric matrix polynomials in d variables, for any integer $d \geq 1$. It has an affirmative answer for $d = 1$ (by the celebrated Fejér-Riesz theorem), and a negative answer for $d \geq 3$, even in the scalar case $m = 1$ (see [1]). For $d = 2$, the answer is known to be positive in the scalar case $m = 1$ (see [2]). But the question is open for $d = 2$ and $m \geq 2$.

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4. K. Schmüdgen

Suppose that A is a (complex or real) unital $*$-algebra with involution denoted by $a \to a^+$. Let $\sum A^2$ denote the set of all finite sums of hermitian squares a^+a, where $a \in A$. Let R be a fixed separating family of $*$-representation π of A. (A $*$-representation π of A on a unitary space $(D(\pi), \langle \cdot, \cdot \rangle)$ is an algebra homomorphism of A into the algebra $L(D(\pi))$ of linear operators on $D(\pi)$ such that $\pi(1)\varphi = \varphi$ and $\langle \pi(a)\varphi, \psi \rangle = \langle \varphi, \pi(a^+)\psi \rangle$ for all $a \in A$ and $\varphi, \pi \in D(\pi)$. That R is separating means that $\pi(a) = 0$ for all $\pi \in R$ implies that $a = 0$.) Let

$$A_+(R) = \{a = a^+ \in A : \langle \pi(a)\varphi, \varphi \rangle \geq 0 \text{ for } \varphi \in D(\pi),\ \pi \in R\}.$$

The $*$-algebra of $n \times n$ matrices over A is denoted by $M_n(A)$. Any $*$-representation π gives a unique $*$-representation of $M_n(A)$, denoted also by π, on $D(\pi) \oplus \cdots \oplus D(\pi)$ (n times) by $\pi((a_{ij})) := (\pi(a_{ij}))$, where $(a_{ij}) \in M_n(A)$.

The following problem is stated as Problem 4 on p. 786 in [1]:

Problem: *Does* $A_+(R) \subseteq \sum A^2$ *imply that* $M_n(A)_+(R) \subseteq \sum M_n(A)^2$ *?*

In particular, this problem is of interest when A is real $*$-algebra of polynomials of the 2-sphere in R^3 or the $*$-algebra of trigonometric polynomials in two variables. In these cases R consists of all point evaluations of A and the assumption $A_+(R) \subseteq \sum A^2$ holds a result of C. Scheiderer [2]. Various versions of Positivstellensätze for algebras of matrices has been investigated in [1].

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5. Schauder bases of translates and A_p-weights

H. Šikić

This problem developed from the collaboration with Morten Nielsen from Aalborg (Denmark). Consider $\psi \in L^2(\mathbb{R})$ and $B_\psi := \{\psi_k(x) := \psi(x-k) \,|\, k \in \mathbb{Z}\}$. Define $\langle\psi\rangle := \overline{\text{span}}\, B_\psi$ and $p_\psi(\xi) := \sum_{k\in\mathbb{Z}} |\widehat{\psi}(\xi+k)|^2$, $\xi \in \mathbb{R}$; where $\widehat{\psi}$ denotes the Fourier transform of ψ. Suppose that B_ψ is a Schauder basis for $\langle\psi\rangle$. This is equivalent to p_ψ being an A_2-weight; see [2] for details. For the definition of A_p-weight see [4]. For extensions of these notations to higher dimensions, vector – and matrix-valued functions, as well as for the extension of the present problem consult [5], [1] and [3].

Define $q_\psi := \inf\{p \geqslant 1 \,|\, p_\psi \text{ is an } A_p\text{-weight}\}$. It can be shown that $1 \leqslant q_\psi < 2$. Given $q \in [1,2\rangle$, what are the properties of B_ψ such that $q_\psi = q$? Can one characterize, through properties of Schauder bases, those B_ψ such that $q_\psi = q$?

Observe that it is known that if p_ψ and $1/p_\psi$ are both A_1-weight, then B_ψ is a Riesz basis for $\langle\psi\rangle$. Observe also that $q_\psi = 1$ does not necessarily imply that p_ψ is an A_1-weight.

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6. Rank minimization for Gram matrices

J. Stöckler

Let $p \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$ be a Laurent polynomial which is nonnegative on the torus, that is

$$p(a_1, \ldots, a_d) \geq 0 \qquad \text{for all} \qquad |a_1| = \cdots = |a_d| = 1.$$

By definition, p is a *sum of hermitian squares*, if there exists $r \in \mathbb{N}$ and $q_k \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$, $k = 1, \ldots, r$, such that

$$p = \sum_{k=1}^{r} q_k^* q_k,$$

where involution is defined by $(c_\alpha z^\alpha)^* = \overline{c_\alpha} z^{-\alpha}$ (in usual multi-index notation). Equivalently, in terms of *Gram matrices*, there exists a finite set $K \subset \mathbb{Z}^d$ and a positive semi-definite matrix $M_K \in \mathbb{C}^{|K| \times |K|}$ of rank $\leq r$, such that $p = \mathbf{z}^* M_K \mathbf{z}$, where $\mathbf{z} = (z^\alpha;\, \alpha \in K)$ is a column vector of monomials and $\mathbf{z}^* = (z^{-\alpha};\, \alpha \in K)^T$.

Problem: Among all Gram matrices that are associated with index set K, find one with minimal rank r.

For $d = 1$ and $K = \{0, 1, \ldots, n\}$, the minimal rank is $r = 1$ according to the Riesz-Fejer lemma. A constructive proof can be obtained by the "method of Schur complements" following Dritschel and Woerdemann, Trans. AMS 357 (2005) pp. 4661-4679.

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