Mathematisches Forschungsinstitut Oberwolfach

Report No. 16/2013

DOI: 10.4171/OWR/2013/16

Arbeitsgemeinschaft: Limits of Structures

Organised by
László Lovász, Budapest
Balázs Szegedy, Toronto

31 March – 5 April 2013

Abstract. The goal of the Arbeitsgemeinschaft is to review current progress in the study of very large structures. The main emphasis is on the analytic approach that considers large structures as approximations of infinite analytic objects. This approach enables one to study graphs, hypergraphs, permutations, subsets of groups and many other fundamental structures.

Mathematics Subject Classification (2010): 05Cxx.

Introduction by the Organisers

Built on decades of research in ergodic theory, Szemerédi's regularity theory and statistical physics, a new subject is emerging that considers very large finite structures as approximations of infinite analytic objects. More precisely, one can introduce various convergence notions and limit objects for growing sequences of graphs, hypergraphs, permutations, and for several kinds of other important structures. Many properties of these structures are easier to study in the limiting setting since powerful tools from analysis become available. This approach creates new connections between analysis, combinatorics and group theory. The goal of the Arbeitsgemeinschaft is to present a landscape of beautiful ideas developed by researchers from diverse fields. The subject is very rich and many of its aspects are covered in the recent book [1] by L. Lovász.

The presentations at the workshop discussed a number of applications in extremal combinatorics, Fourier analysis (also in a higher order version of it), group theory, ergodic theory, topology and probability. The workshop was well attended with over 40 participants. It brought together researchers with backgrounds in Probability, Combinatorics, Ergodic theory, group theory and logic. Besides talks there was a problem session and an informal discussion of recent progress in random regular graphs.

References

[1] L. Lovász, *Large networks and graph limits*, AMS (2012), ISBN-13: 978-0-8218-9085-1.

Arbeitsgemeinschaft: Limits of Structures

Table of Contents

Abstracts

Introduction to Szemerédi's regularity lemma

Heinrich-Gregor Zirnstein

The aim of this talk was to give a short introduction to the celebrated Szemerédi Regularity Lemma in the context of dense graphs limits [3]. To keep things simple, we will focus on a weaker version of the Regularity Lemma [1] which has a direct application to the problem of finding a maximum cut of a graph. The essence of its proof is a simple lemma about approximations in Hilbert space [3].

We make no attempt to explain traditional applications of the regularity lemma in graph theory; we refer to Ref. [2] for more on those.

1. A Weak Regularity Lemma and the Maximum Cut Problem

We now present a weak version of the Regularity Lemma due to Frieze and Kannan [1] and explain how it can help in approximating the maximum cut of a dense graph. But first, we must fix some notation.

To each graph, we can associate its *adjacency matrix* A, whose rows and columns are the vertices of the graph and whose entries $A(i,j)$ record whether the vertices i and j are directly connected to each other:

$$A(i,j) = \begin{cases} 1 & \text{if there is an edge between } i \text{ and } j; \\ 0 & \text{otherwise.} \end{cases}$$

Without loss of generality, we assume that the vertices of the graph are labelled by natural numbers $V = \{1, \ldots, n\}$.

Anticipating the limit of a large vertex count n, we can represent the adjacency matrix A just as well by a function $W : [0,1]^2 \to [0,1]$ on the unit square as follows

$$W(x,y) = A(\lceil nx \rceil, \lceil ny \rceil).$$

In other words, each graph corresponds to a symmetric, measurable function W on the unit square. Moreoever, this function is a $0,1$-valued step function supported on a grid of squares with side lengths $1/n$.

More generally, let $\mathcal{W}_0$ denote the set of all measurable and symmetric functions $W : [0,1]^2 \to [0,1]$. Elements of this set are also called *graphons*. They arise as limits of a sequence of dense graphs, but that's the subject of a subsequent talk; here we will focus on ordinary graphs represented as above.

Now, the Regularity Lemma is a statement about approximating any function in $\mathcal{W}_0$, which can be very complicated on a fine scale, by step functions which only vary on a coarse scale. A *step function* W is just a function on the unit square such that for some partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of the unit interval $[0,1]$, the values are constant $W(x,y) = W(x',y')$ whenever the two points (x,y) and (x',y') are contained in the same part $P_i \times P_j$.

The quality of the approximation is measured by the following norm.

Definition 1 (Cut norm). *Let $W : [0,1]^2 \to \mathbb{R}$ be a measurable and symmetric function, for instance a graphon $W \in \mathcal{W}_0$. Its* cut norm *is defined as*

$$\|W\|_\square = \sup_{S,T\subset[0,1]} \left| \int_{S\times T} W(x,y)\,dx\,dy \right|.$$

In other words, the cut norm measures whether two functions can be distinguished by averaging over all rectangles $S\times T$. These integrals naturally correspond to the number of edges between the vertex sets corresponding to S and T.

We can now state the weak form of the Regularity Lemma.

Lemma 1 (Weak Regularity Lemma). *Let $\varepsilon > 0$ be a real number. Then, for every function $W \in \mathcal{W}_0$, there exists a partition $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ of the unit interval $[0,1]$ into $k \le \lceil 2^{c/\varepsilon^2} \rceil$ many parts and a step function $\widetilde{W}$ constant on this partition such that*

$$\|W - \widetilde{W}\|_\square \le \varepsilon$$

for some constant $c \in \mathbb{R}$ independent of ε.

As the name may suggest, the cut norm is useful for controlling the maximum cut of a graph. Given a graph $G = (V, E)$, a *cut* is just a collection of edges obtained from a partition of the vertex set into two parts $V = S \uplus S^c$. If A is the adjacency matrix of the graph, then the quantity

$$A(S, S^c) = \sum_{i\in S, j\in S^c} A(i,j)$$

is called the *weight* of the cut. The *maximum cut problem* is the problem of finding a cut with maximum weight. In other words, the goal is to “cut away” the maximum number of edges by separating the graph into two parts. This problem is NP-hard and one of the most famous long-standing problems in computer science: currently, there is no known algorithm that can solve this problem in polynomial time, and if we had such an algorithm, then we would also be able to solve all other NP-hard problems in polynomial time.

Since the maximum cut problem is hard to solve exactly, one can try to find approximative solutions instead. Now, the cut norm controls the maximum cut since

$$|W(S,S^c) - \widetilde{W}(S,S^c)| \le \|W - \widetilde{W}\|_\square$$

by considering the rectangle $S \times T = S \times S^c$. In other words, if two graphs are close in the cut norm, then the weights of their cuts are also close. The weak regularity lemma says that the (scaled) adjacency matrix W of a graph can be approximated by a simpler step function $\widetilde{W}$ with a similar cut weights. Frieze and Kannan [1] have shown that this yields approximation algorithms for the maximum cut problem in a straightforward manner. Unfortunately, the Regularity Lemma, and hence these approximation algorithms only work well for dense graphs as the function W was obtained from the adjacency matrix A by scaling with a factor of $1/n$.

2. Hilbert-Space Proof of the Regularity Lemma

We can actually deduce the Regularity Lemma from a very general result about approximating vectors in Hilbert spaces [3].

Lemma 2 (Regularity Lemma in Hilbert Space). *Let H be a real Hilbert space and $A_1, A_2, \dots$ be arbitrary nonempty subsets. Then, for every $\varepsilon > 0$ and vector $f \in H$, we can find $m < \lceil 2/\varepsilon^2 \rceil$ many vectors $f_i \in A_i$ with coefficients $\lambda_i \in \mathbb{R}$ such that*

$$|\langle g, f - (\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m)\rangle| \leq \varepsilon \|f\| \|g\|$$

for every $g \in A_{m+1}$.

Essentially, the lemma says that we can approximate the vector f by a linear combination of elements from $A_1, \dots, A_m$ such that the remaining error is almost orthogonal to the set A_{m+1}. At first sight, this may seem too good to be true, because the set A_{m+1} can be arbitrary and has no relation to the previous sets $A_1, \dots, A_m$. However, keep in mind that the index m is not a parameter that we can choose; it is the lemma that will choose the linear combination to be as long as needed.

Proof. (Regularity Lemma in Hilbert Space)
Let

$$\eta_k := \inf_{\lambda_i \in \mathbb{R}, f_i \in A_i} \left\| f - \sum_{i=1}^{k} \lambda_i f_i \right\|^2$$

denote the squares of the norms of the errors. They form a decreasing sequence

$$\|f\|^2 \geq \eta_1 \geq \eta_2 \geq \cdots \geq 0.$$

Since the norms of the errors cannot decrease indefinitely, we can find an index $m < \lceil 2/\varepsilon^2 \rceil$ where the error fails to improve significantly

$$\eta_m - \eta_{m+1} < \frac{\varepsilon^2}{2} \|f\|^2.$$

Choosing a good approximation $\tilde{f} = \sum_{i=1}^{m} \lambda_i f_i$ with $\|f - \tilde{f}\|^2 \leq \eta_m + \frac{\varepsilon^2}{2}\|f\|^2$ yields, for any vector $g \in A_{m+1}$ and coefficient $\lambda \in \mathbb{R}$,

$$\|f - \tilde{f} - \lambda g\|^2 \geq \eta_{m+1} \geq \|f - \tilde{f}\|^2 - \varepsilon^2 \|f\|^2$$

$$\iff \lambda^2 \|g\|^2 - 2\lambda \langle g, f - \tilde{f}\rangle + \varepsilon^2 \|f\|^2 \geq 0.$$

The discriminant of the quadratic polynomial must be positive, which concludes the proof. □

We can now prove the Weak Regularity Lemma.

Proof. (Weak Regularity Lemma) Consider the Hilbert space $H = L^2([0,1]^2)$ and let thet set A_n be equal to the set of characteristic functions of squares

$$A_n = \{1_{S \times S} : S \subset [0,1] \text{ measurable}\} \subset H.$$

Applying the previous lemma to the function $W \in \mathcal{W}_0 \subset H$ yields a function

$$\widetilde{W} = \sum_{i=1}^{m} \lambda_i 1_{T_i \times T_i}, \quad m < \lceil c/\varepsilon^2 \rceil \text{ for a constant } c \in \mathbb{R}$$

with the property that

$$\left| \int_{S\times S} (W - \widetilde{W}) \right| \le \varepsilon$$

for every square $S \times S$. This is a step function with at most 2^m steps. Since a (symmetric) rectangle can be obtained by removing squares from a larger square, this implies that

$$\|W - \widetilde{W}\|_\Box \le \varepsilon,$$

though perhaps at the cost of increasing ε by some constant factor. □

References

[1] A. Frieze, R. Kannan, *Quick approximation to matrices and applications*, Combinatorica **19** (1999), 175–220.

[2] J. Komlós, M. Simonovits, *Szemerédi's Regularity Lemma and its applications in graph theory*, Technical Report DIMACS **96** (2007).

[3] L. Lovász, B. Szegedy, *Szemerédi's lemma for the analyst*, Geom. Funct. Anal. **17** (2007), 252–270.

Limits of dense graph sequences

Lluís Vena

The Regularity Lemma, both the strong [6] and the weak versions [1], has seen many applications in combinatorics and graph theory [2], as well as in number theory [5]. In [4], Lovász and Szegedy gave an analytical interpretation of the Regularity Lemma and deduced some consequences regarding the space of objects naturally arising as limits of homomorphisms densities of graph sequences. In this talk we present some of these analytic consequences.

Given finite graphs F and G, we define $t(F, G)$ as the probability that a function, picked uniformly at random from the vertex set of F to the vertex set of G, is a graph homomorphism; this is, that the function preserves the edge set of F as edges of G. A sequence of finite graphs $\{G_n\}$ is said to be convergent if $\lim_{n\to\infty} t(F, G_n)$ converges for every finite graph F (see [3]). However, it can be checked that there are some convergent sequences of graphs that converge to a limit that no finite graph attains. For instance, a sequence of random graphs with increasing number of vertices and with the probability of any edge to appear being $1/2$ is a convergent sequence, but there is no finite graph G that has $t(F, G) = (1/2)^{|E(F)|}$ for all finite graph F.

Let $\mathcal{W}_0$ be the set of symmetric measurable functions $W : [0, 1]^2 \to [0, 1]$, called *graphons*. A graph G over n vertices can be seen as an element of $\mathcal{W}_0$ by dividing the interval $[0, 1]$ into n equal-sized intervals $[i/n, (i+1)/n]$, $i \in \{0, \dots, n-1\}$, and

$W(x, y) = 1$ for any $(x, y) \in [(i-1)/n, i/n] \times [(j-1)/n, j/n]$ with $\{i, j\} \in E(G)$ and $W(x, y) = 0$ otherwise.

Let k be the size of the vertex set of F. Given $W \in \mathcal{W}_0$, we can extend the homomorphism density notion to the set of graphons with

$$t(F, W) = \int_{[0,1]^k} \prod_{(i,j) \in E(F)} W(x_i, x_j)\, dx_1 \cdots dx_k.$$

Thus, the graphons are candidates for the closure of the finite graphs with the homomorphism density as a limit notion.

In $\mathcal{W}$, the space of bounded symmetric measurable functions $W : [0,1]^2 \to \mathbb{R}$, we can introduce the cut norm

$$||W||_\square = \sup_{S,T \subset [0,1]} \left| \int_{S \times T} W(x, y) dx dy \right|, \; S, T \text{ measurable.}$$

In particular, in [4] it is shown that

$$|t(F, W) - t(F, U)| \leq |E(F)| \; ||W - U||_\square,$$

so the cut norm controls the differences in $t(F, \cdot)$. The space $\mathcal{W}$ can also be endowed with the L_1 norm. In particular, $||W||_\square \leq ||W||_1$.

Let $\phi : [0,1] \to [0,1]$ be a measure preserving transformation and denote $W^\phi = W(\phi(x), \phi(y))$. It can be noted that $t(F, W)$ is invariant under ϕ; $t(F, W^\phi) = t(F, W)$. Thus, it is natural to define

$$\delta_\square(U, W) = \inf_\phi ||U - W||_\square, \; \phi \text{ measure preserving bijection}$$

and

$$\delta_1(U, W) = \inf_\phi ||U - W||_1, \; \phi \text{ measure preserving bijection.}$$

After identifying points with zero distance in $(\mathcal{W}_0, \delta_\square)$, we obtain a metric space $\mathcal{X}_0$ that Lovász and Szegedy [4] showed to be compact. Moreover, the set of finite graphs is dense in this space.

The proof shows that any sequence of graphons has a subsequence that converges to a limit object. The analytic interpretation of the Regularity Lemma as an approximation result in the square norm, using step-functions with finitely-many steps, provides a finite-information approximation of the original graphons. For those approximations, a subsequence can be shown to converge in the L_1 norm. These approximations can be tweaked to provide a sequence of graphons that can be interpreted as a martingale. The existence of the limit object appears as a consequence of the Martingale Convergence Theorem.

References

[1] A. Frieze, R. Kannan, *Quick approximation to matrices and applications*, Combinatorica **19** (1999), no. 2, 175–220.

[2] J. Komlós, M. Simonovits, *Szemerédi's regularity lemma and its applications in graph theory*, Combinatorics, Paul Erdős is eighty (D. Miklós, et al., eds.) Bolyai Soc. Math. Stud. (1993), no. 2, 295–352.

[3] L. Lovász, B. Szegedy, *Limits of dense graph sequences*, J. Combin. Theory Ser. B **96** (2006), no. 6, 933–957.
[4] L. Lovász, B. Szegedy, *Szemerédi's lemma for the analyst*, Geom. Funct. Anal. **17** (2007), no. 1, 252–270.
[5] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Collection of articles in memory of Juriĭ Vladimirovič Linnik. Acta Arith. **27** (1975), 199–245.
[6] E. Szemerédi, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes, Colloq. Internat. CNRS, Univ. Orsay, Orsay, (1976), 399–401.

Benjamini–Schramm limits

Johannes Carmesin

Benjamini and Schramm define a metric on the connected bounded degree graphs with a specified vertex called the root. Two such rooted graphs are near if there is a large radius such that the balls of that radius around the two roots of the graphs are isomorphic.

This can be made a metric on the finite connected bounded degree graphs G without a root. For this, we associate with G the probability measure μ_G that for each vertex $o \in V(G)$ gives back the rooted graph (G, o) with probability $1/|V(G)|$. More generally, Benjamini and Schramm consider probability measures on the space of connected bounded degree rooted graphs. Such measures will be called random rooted graphs. Informally, a random rooted finite graph (G, o) is unbiased if given G, the root is distributed uniformly at random amongst the vertices. For example, μ_G is unbiased.

Benjamini and Schramm proved the following theorem. Let $M > 0$. Let (G, o) be the Benjamini–Schramm limit of a sequence (G_j, o_j) of random rooted finite planar unbiased graphs of degree bounded by M. Then G is recurrent with probability one.

The main tool in the proof of this theorem is that of circle packings. Very roughly, the circle packing for each (G_j, o_j) gives a random circle packing P_j with random root o_j. In a certain sense, (P_j, o_j) converges to a probability distribution P of circle packings for G with random root o.

Using a theorem of He and Schramm, it can be shown that it suffices to prove that P has at most one accumulation point with probability one. To prove this, Benjamini and Schramm introduce the key notion of a (δ, s)-supported point. Very informally, given a finite set of points in the plane, one of those points is (δ, s)-supported if the points nearby spread out a lot. This spreading out is measured by the real $\delta > 0$ and the natural number s, where it gets easier to be (δ, s)-supported the smaller δ is and the larger s is.

If P has at least 2 accumulation points with a positive probability, then it can be shown that from the set of midpoints of circles in P_j at least a constant fraction has to be (δ, s)-supported. Here δ is fixed and for every s there is some j such that the above is true for δ, s and j.

However, Benjamini and Schramm prove for any arbitrary arrangements of points in the plane that less than a constant fraction of these points are (δ, s)-supported (for sufficiently large s). Hence P has at most one accumulation point, which completes the very rough sketch.

References

[1] I. Benjamini, O. Schramm, *Recurrence of Distributional Limits of Finite Planar Graphs*, Electron. J. Probab. **6** (2001), 1–13.

[2] Z-X. He, O. Schramm, *Hyperbolic and parabolic packings*, Discrete Comput. Geom. **14** (1995), no. 2, 123–149.

Spectral aspects of the regularity lemma and dense graph limits

Oleg Pikhurko

This is a very brief and informal account of some of the ideas from the paper *"Limits of kernel operators and the spectral regularity lemma"* by Balázs Szegedy [1].

We view a graphon $M : [0,1]^2 \to [0,1]$ as a self-adjoint integral kernel operator on $L_2([0,1])$ that maps a function f to

$$(Mf)(x) := \int M(x,y) f(y)\, \mathrm{d}y.$$

Here $L_2([0,1])$ is the real Hilbert space with the scalar product

$$(f,g) := \int f(x) g(x)\, \mathrm{d}x.$$

In addition to the standard L_2-norm $\|f\|_2 = (f,f)^{1/2}$, we will use the following version of the cut-norm (for bounded measurable $M : [0,1]^2 \to \mathbb{R}$):

$$\|M\|_\square := \sup_{\|f\|_\infty \le 1, \|g\|_\infty \le 1} |(f, Mg)|.$$

The Cauchy-Schwartz inequality shows that the L_2-convergence implies the cut-norm and weak convergences while, for example, the adjacency functions of random graphs show that the converse is not true. Also, one can come up with examples that neither of the cut-norm and weak convergences implies the other. However, further implications can be shown under extra assumptions. For example,

(1) if $f_i \to f$ weakly, then $\limsup \|f_i\|_2 \ge \|f\|_2$ and the equality implies that $f_i \to f$ in L_2 (cf. [1, Lemma 1.1]);

(2) if $f_i \to f$ weakly, then $Mf_i \to Mf$ in L_2 [1, Lemma 1.2].

The graphon M (and more generally every symmetric $M \in L_2([0,1]^2)$) admits a *spectral decomposition*

$$M(x,y) = \sum_{i=1}^{\infty} \lambda_i f_i(x) f_i(y) \quad (\text{in } L_2),$$

where λ_i are reals tending to 0 and $\{f_i\}$ is an orthonormal basis made of eigenfunctions of M. This decomposition can be derived by considering the *spectral radius*

$$\mathrm{rad}(M) = \sup_{\|f\|_2 \le 1} |(f, Mf)|.$$

By using the above properties and the compactness of the unit ball in the weak topology, one can show that the supremum is in fact maximum. Some work shows that any optimal f is an eigenvector of M and $f^\perp$ is M-invariant, allowing to iterate the argument. In particular,

$$\mathrm{rad}(M) = \max_i |\lambda_i| = \max_{\|f\|_2 = 1} \|Mf\|_2.$$

The last formula implies, again by Cauchy-Schwartz, that $\|M\|_\square \le \mathrm{rad}(M)$ [1, Lemma 1.5]. This immediately leads to some version of a regularity lemma: the graphon M is approximated within ε (in the cut-norm) by a "simpler" function

$$[M]_\varepsilon := \sum_{i : |\lambda_i| > \varepsilon} \lambda_i f_i(x) f_i(y). \tag{1}$$

Note that $\|M\|_\infty \le 1$ implies that $\|f_i\|_\infty \le |\lambda_i|^{-1}$. Indeed,

$$|\lambda_i f_i(x)| = \left| \int M(x,y) f_i(y)\, \mathrm{d}y \right| \le \|f_i\|_2 = 1$$

for a.e. x. Also, by Parselval's identity, the number of terms in the right-hand side of (1) is at most $1/\varepsilon^2$.

This allows us to derive a version of the weak regularity lemma as follows. First, each f_i in (1) can be approximated within δ in L_∞-norm by a step function g_i with at most $\|f_i\|_\infty/\delta \le 1/(\varepsilon\delta)$ steps (just partition the essential range of f_i into at most $1/(\varepsilon\delta)$ intervals of length 2δ and take the pre-image of each). We can assume $\|g\|_\infty \le \|f\|_\infty$. Next,

$$\|g_i(x)g_i(y) - f_i(x)f_i(y)\|_\infty \le \|g_i - f_i\|_\infty \; \|f_i\|_\infty \le \frac{\delta}{\varepsilon}.$$

Combine the partitions of the step-functions g_i into one partition $\mathcal{P}$ of $[0,1]$ that has at most $(\varepsilon\delta)^{-1/\varepsilon^2}$ parts. We obtain a step-function

$$S(x,y) := \sum_{i : |\lambda_i| > \varepsilon} \lambda_i g_i(x) g_i(y),$$

which is constant on each element of $\mathcal{P} \times \mathcal{P}$ and (since each $|\lambda_i| \le 1$) satisfies that

$$\|M - S\|_\square \le \|M - [M]_\varepsilon\|_\square + \|[M]_\varepsilon - S\|_\infty \le \varepsilon + \frac{\delta}{\varepsilon} \times \frac{1}{\varepsilon^2}.$$

By taking, for example, $\delta = \varepsilon^4$, we derive that $\|M - S\|_\square \le 2\varepsilon$, where the number of steps of S can be bounded by a function of ε independent of M (recall that we assumed that $\|M\|_\infty \le 1$).

One advantage of this approach is that if M has some symmetries, then these are inherited by $[M]_\varepsilon$ and almost inherited by S (within L_∞-error of at most $2\,\|S - [M]_\varepsilon\|_\infty$). So we obtain a "symmetry preserving" regularity lemma.

The paper [1] builds further upon this argument to derive a strong symmetry preserving regularity lemma [1, Theorem 3] and presents many other interesting results.

References

[1] B. Szegedy, *Limits of kernel operators and the spectral regularity lemma*, Europ. J. Combin. **32** (2011), 1156–1167.

Graph algebras and reflection positivity

Péter E. Frenkel

1. The moment problem in graph theory

As a basis for analogy, let us recall Hausdorff's solution [1] of the classical moment problem.

Theorem 1. *For a sequence $(a_i)_{i=0}^{\infty}$ of real numbers, the following are equivalent.*

(1) *There exists a random variable W such that $\mathbb{E}W^i = a_i$ for all i.*
(2) *We have $a_0 = 1$, and the matrix $(a_{i+j})_{i,j=0}^{\infty}$ is positive semidefinite.*

We shall be discussing the graph-theoretic analogue of this. Graphs will always be finite, undirected, and will have no loops, but may have multiple edges. In the graph F, the number of edges connecting node i to node j is denoted F_{ij}.

We write $[k] = \{1, 2, \dots, k\}$. A *partially k-labeled graph* is a graph whose vertex set contains $[k]$. We write $\mathcal{G}_k$ for the set of isomorphism types of partially k-labeled graphs, where isomorphisms are required to restrict to the identity on $[k]$. There is a natural map $\mathbb{E} : \mathcal{G}_k \to \mathcal{G}_0$ that forgets the labels. There is a natural semigroup structure on $\mathcal{G}_k$: if $F_1 \cap F_2 = [k]$, then we define $F_1F_2 = F_1 \cup F_2$. The *graph algebra* $\mathcal{Q}_k$ is the semigroup algebra $\mathbb{R}\mathcal{G}_k$.

A *graph parameter* $f : \mathcal{G}_0 \to \mathbb{R}$ is *normalized* if $f(K_1)$=1. It is *multiplicative* if $f(F_1F_2) = f(F_1)f(F_2)$ for all $F_1, F_2 \in \mathcal{G}_0$. The kth *connection matrix* of f is $M(k, f) = (f(\mathbb{E}(F_1F_2)))_{F_1,F_2\in\mathcal{G}_k}$. The parameter f is *reflection positive* if all its connection matrices are positive semidefinite.

Let Ω be a probability space. A *kernel* is a function $W \in \mathcal{L}^\infty(\Omega^2)$ such that $W(x, y) = W(y, x)$ for almost all $(x, y) \in \Omega^2$. When Ω has finitely many points, the kernel W is also called a *normalized weighted graph* on the node set Ω.

A kernel W has *moments,* also known as *homomorphism densities,*

$$t(F, W) = \mathbb{E} \prod_{\{i,j\}\subseteq V(F)} W^{F_{ij}}(X_i, X_j) \qquad (F \in \mathcal{G}_0),$$

where $X_i : \Omega^{V(F)} \to \Omega$ is the ith projection.

Theorem 2 (Freedman, Lovász and Schrijver [4, 5])**.** *For a graph parameter f : $\mathcal{G}_0 \to \mathbb{R}$ and a natural number q, the following are equivalent.*

(1) *There exists a normalized weighted graph W on at most q nodes such that $f = t(-, W)$.*
(2) *The graph parameter f is normalized, multiplicative, reflection positive and $M(k, f)$ has rank at most q^k for all k.*

One might naively hope that the theorem remains true if we allow arbitrary kernels on probability spaces in (1) and replace the rank condition in (2) by requiring that f grows at most exponentially with the number of edges. However, it turns out that for this, we have to allow *randomized kernels,* i.e., probability measures on $\Pi = \Omega^2 \times \Xi$ such that the first marginal is the product measure on Ω^2, and the reflection $(x, y, \xi) \mapsto (y, x, \xi)$ is measure-preserving. Here Ξ is a compact subset of $\mathbb{R}$. Let X, Y and W be the projections from Π to Ω, Ω and Ξ respectively, and let $W^{[n]} = \mathbb{E}(W^n|X, Y)$ be the nth moment considered as a function on Ω^2. We define the *moment*

$$t(F, W) = \mathbb{E} \prod_{\{i,j\} \subseteq V(F)} W^{[F_{ij}]}(X_i, X_j),$$

where $X_i : \Omega^{V(F)} \to \Omega$ is the ith projection.

Theorem 3 (Lovász and Szegedy [7]). *For a graph parameter $f : \mathcal{G}_0 \to \mathbb{R}$ and a number $d \in [0, \infty)$, the following are equivalent.*

(1) *There exist normalized weighted graphs, i.e. kernels W_n on suitable finite probability spaces Ω_n, with $|W_n| \leq d$ and $f(F) = \lim t(F, W_n)$ for all F.*
(2) *There exists a randomized kernel W with $\Omega = [0, 1]$ and $\Xi \subseteq [-d, d]$, such that $f = t(-, W)$.*
(3) *The parameter f is normalized, multiplicative, reflection positive and the inequality $|f(K_2^n)| \leq d^n$ holds for all n, where K_2^n is the graph with two nodes and n edges.*

Let us prove the easy implication (2) $\implies$ (3). (It will then clearly follow that (1) $\implies$ (3). We refer to the original paper for the proof of the difficult implications (3) $\implies$ (1), (2).) Only reflection positivity requires a proof. Endow $\Omega^k \times \Xi^{\binom{k}{2}}$ with the probability measure whose first marginal is the product measure on Ω^k and which satisfies that all projections W_{ij} $(1 \leq i < j \leq k)$ to Ξ are independent when conditioned on the projection $(X_1, \ldots, X_k)$ to Ω^k, with $(W_{ij}|X_1 = x_1, \ldots, X_k = x_k)$ having the same distribution as $(W|X = x_i, Y = x_j)$.

For $F \in \mathcal{G}_k$, define the random variable $F(W) \in \mathcal{L}^\infty\left(\Omega^k \times \Xi^{\binom{k}{2}}\right)$ to be

$$\mathbb{E}\left(\prod_{\{i,j\} \subseteq V(F), \{i,j\} \not\subseteq [k]} W^{[F_{ij}]}(X_i, X_j) \,\middle|\, X_1, \ldots, X_k \right) \cdot \prod_{\{i,j\} \subseteq [k]} W_{ij}^{F_{ij}}.$$

Then $f(\mathbb{E}(F_1 F_2)) = \mathbb{E}(F_1(W) F_2(W))$ and the claim follows.

2. Simple graphs

Let $\mathcal{G}_k^{\text{simp}} \subset \mathcal{G}_k$ be the set of partially k-labeled simple graphs. For $F \in \mathcal{G}_k$, let $F^{\text{simp}} \in \mathcal{G}_k^{\text{simp}}$ be the underlying simple graph.

A *simple graph parameter* $f : \mathcal{G}_0^{\text{simp}} \to \mathbb{R}$ is *reflection positive* if its *simple connection matrix* $\left(f(\mathbb{E}(F_1F_2)^{\text{simp}})\right)_{F_1,F_2\in\mathcal{G}_k^{\text{simp}}}$ is positive semidefinite for all k.

The limit objects for dense simple graphs are *graphons,* i.e., kernels W with $0 \le W \le 1$. Simple graph parameters representable as $t(-,W)$ with a suitable graphon W can be characterized by reflection positivity, or, alternatively, by a nonnegativity condition arising from an inclusion-exclusion formula. The latter description is analogous to the following classical

Theorem 4 (Hausdorff [1], new proof given by Diaconis and Freedman [2])**.** *For a sequence $(a_i)_{i=0}^{\infty}$ of real numbers, the following are equivalent.*

1. *There exists a random variable $0 \le W \le 1$ such that $\mathbb{E}W^i = a_i$ for all i.*
2. *We have $a_0 = 1$, and $\sum_{j=0}^{k}(-1)^j\binom{k}{j}a_{n+j} \ge 0$ for all n and k.*

The analogous signed sum for a simple graph parameter f is

$$f^\dagger(F) = \sum \left((-1)^{|E(F')-E(F)|} f(F') \middle| V(F') = V(F), E(F') \supseteq E(F) \right).$$

Theorem 5 (Lovász and Szegedy [6, 5])**.** *For a simple graph parameter $f : \mathcal{G}_0^{simp} \to \mathbb{R}$, the following are equivalent.*

1. *There exist simple graphs W_n such that $f(F) = \lim t(F, W_n)$ for all F.*
2. *There exists a graphon W with $\Omega = [0,1]$, such that $f = t(-,W)$.*
3. *The parameter f is normalized, multiplicative and reflection positive.*
4. *$f^\dagger(F) \ge 0$ for all F.*

Let us prove the easy implication (2) $\implies$ (3). Only reflection positivity requires a proof. Define a randomized graphon as follows. Let $\Xi = \{0,1\}$. Endow $\Pi = \Omega^2 \times \Xi$ with the probability measure such that the distribution of the projection $\tilde{W} : \Pi \to \Xi$ conditioned on the projection to Ω^2 is a Bernoulli distribution with parameter W. Then the kth simple connection matrix of f is the kth connection matrix of $t(-,\tilde{W})$, and the reflection positivity follows.

Even the easy implications (1), (2) $\implies$ (3), (4) have important applications in extremal graph theory. The simplest ones are the reproving [3] of the Goodman inequality $t(K_3) \ge t(K_2)(2t(K_2)-1)$ and the Katona–Kruskal inequality $t(K_3)^2 \le t(K_2)^3$, where $t = t(-,W)$ for a simple graph or a graphon W, and K_n is the complete graph on n nodes.

References

[1] F. Hausdorff, *Summationsmethoden und Momentfolgen I-II.*, Math. Z. **9** (1921), 74–109 and 280–299.

[2] P. Diaconis, D. Freedman, *The Markov moment problem and de Finetti's theorem I.*, Math. Z. **247** (2004), no. 1, 183–199.

[3] Ch. Borgs, J. Chayes, L. Lovász, V. T. Sós, K. Vesztergombi, *Counting graph homomorphisms*, Topics in discrete mathematics, Algorithms Combin. **26** (2006), Springer, Berlin, 315–371.

[4] M. Freedman, L. Lovász, A. Schrijver, *Reflection positivity, rank connectivity and homomorphism of graphs*, Journal of the AMS **20** (2007), no. 1, 37–51.
[5] L. Lovász, *Large networks and graph limits*, AMS, ISBN-13: 978-0-8218-9085-1.
[6] L. Lovász, B. Szegedy, *Limits of dense graph sequences*, Journal of Combinatorial Theory, Series B **96** (2006), 933–957.
[7] L. Lovász, B. Szegedy, *The graph theoretic moment problem*, preprint, arXiv:1010.5159.

Limits of permutations

Gergely Ambrus

Let S_n denote the set of permutations on $[n] = \{1, 2, \ldots, n\}$, and let $\mathcal{S}$ be the union of the S_ns for $n \in \mathbb{N}$. For $\sigma \in \mathcal{S}$, its *length* $|\sigma|$ is the cardinality of its base set. To any $\sigma \in S_n$, we associate a probability measure μ_σ on the unit square $[0,1]^2$, whose density is constant n on the squares $[(i-1)/n, i/n] \times [(\sigma(i)-1)/n, \sigma(i)/n]$, and 0 elsewhere.

A *sample* of length k from $\sigma \in S_n$ is the induced permutation on a subset with k elements of $[n]$. Accordingly, a sample of length k from a probability measure μ on $[0,1]^2$ is the permutation given by the y-coordinates of k points chosen independently according to μ, ordered by their x-coordinates – if μ is non-atomic, then the probability that two of the y-coordinates are the same is 0.

For given permutations $\sigma \in S_n$ and $\pi \in S_k$, where $k \leq n$, the *density of* π *in* σ, denoted by $t(\pi, \sigma)$, is the probability that a uniformly chosen random k-sample from σ agrees to π. Thus, the k-samples of σ determine a probability distribution on S_k. In case $k > n$, we set $t(\pi, \sigma) = 0$. Similarly, a probability measure μ on $[0,1]^2$ leads to the subpermutation densities $t(\pi, \mu)$ via k-samples from μ. It is not hard to show that the difference between $t(\pi, \sigma)$ and $t(\pi, \mu_\sigma)$ is of order $|\sigma|^{-1}$.

Let $\{\sigma_j\}_{j=1}^\infty$ be a sequence of permutations. We say that $\{\sigma_j\}$ is *convergent*, if the sequence $t(\pi, \sigma_j)$ converges for every fixed permutation π. This is equivalent to requiring that the measures μ_{σ_j} *weakly converge* to a probability measure μ on the unit square. If it exists, the limit measure satisfies an important property; namely, it belongs to the set $\mathcal{Z}$ of probability measures on $[0,1]^2$ which have *uniform marginals*, that is, $\mu(A \times [0,1]) = \mu([0,1] \times A) = \lambda(A)$ for any Borel set $A \subset [0,1]$. Also, it easily follows by a standard diagonal argument that any infinite sequence of permutations contains a convergent subsequence.

Hoppen et al. [3] proved that $\mathcal{Z}$ is exactly the set of *limit permutations*: If $\{\sigma_j\}$ is convergent, then $\mu_{\sigma_j} \Rightarrow \mu$ for some $\mu \in \mathcal{Z}$, and, on the other hand, every $\mu \in \mathcal{Z}$ arises as the limit in the above sense of a convergent permutation sequence. If $\mu_{\sigma_j} \Rightarrow \mu$, it also follows that $t(\pi, \sigma_j) \to t(\pi, \mu)$ for every $\pi \in \mathcal{S}$. This notion of convergence is also metrizable: Let $I[n]$ denote the set of *intervals* contained in $[n]$. The *rectangular distance* of $\sigma_1, \sigma_2 \in [n]$ is defined by

$$d_\square(\sigma_1, \sigma_2) = \frac{1}{n} \max_{S,T \in I[n]} \left| |\sigma_1(S) \cap T| - |\sigma_2(S) \cap T| \right|.$$

Accordingly, for $\mu_1, \mu_2 \in \mathcal{Z}$, their rectangular distance is expressed as

$$d_\square(\mu_1, \mu_2) = \sup_{S,T \in I[0,1]} |\mu_1(S \times T) - \mu_2(S \times T)|,$$

where $I[0,1]$ denotes the set of intervals on $[0,1]$. For given $\sigma_1, \sigma_2 \in S_n$, the difference $|d_\square(\sigma_1, \sigma_2) - d_\square(\mu_1, \mu_2)|$ is of order at most n^{-1}. Convergence of a permutation sequence is then equivalent to convergence of the associated measures with respect to $d_\square$.

Our final notion to be introduced is quasi-randomness, following Cooper [2]. The *discrepancy* of a permutation $\sigma \in S_n$ is

$$d(\sigma) = \frac{1}{n} \max_{S,T \in I[n]} \left| |\sigma(S) \cap T| - \frac{|S||T|}{n} \right|;$$

intuitively, it expresses how much σ jumbles the elements of $[n]$. For $\mu \in \mathcal{Z}$, its discrepancy is simply given by $d(\mu) = d_\square(\mu, \lambda)$, where λ is the Lebesgue measure on $[0,1]^2$. Once again, the difference between the discrepancy of σ and that of μ_σ is at most of order $|\sigma|^{-1}$.

A sequence of permutations $\{\sigma_j\}$ is *quasi-random*, if $|\sigma_j| \to \infty$ and $d(\sigma_j) \to 0$. By virtue of the above connections, this is equivalent to the fact that $\{\sigma_j\}$ converges to the Lebesgue measure λ on $[0,1]^2$.

A natural question of R. Graham asks if there exists a finite k, so that the density of permutations of length k in any sequence of permutations $\{\sigma_j\}$ determines the quasi-randomness of $\{\sigma_j\}$. To be precise, we say that $\{\sigma_j\}$ *satisfies the property* $P(k)$, if for every $\pi \in S_k$, $t(\pi, \sigma_j) = 1/k! + o(1)$. Using this notion, quasi-randomness is equivalent to the fact that $P(k)$ holds for every $k \geq 1$.

The analogue property for graphs was proven by Chung, Graham and Wilson [1], who showed that the density of subgraphs with 4 vertices already characterizes quasi-randomness. Král and Pikhurko proved that the same situation holds for permutations.

Theorem 1 (Král, Pikhurko [4])**.** If the sequence of permutations $\{\sigma_j\}$ with $|\sigma_j| \to \infty$ satisfies $P(4)$, then it is quasi-random.

On the other hand, examples [2, 4] show that neither $P(2)$ nor $P(3)$ imply $P(4)$, thus, the above theorem gives the optimal answer for Graham's question.

References

[1] F. R. K. Chung, R. L. Graham, R. M. Wilson, *Quasi-random graphs*, Combinatorica **9** (1989), 345–362.

[2] J. N. Cooper, *Quasirandom permutations*, Journal of Combinatorial Theory Series A **106** (2004), 123–143.

[3] C. Hoppen, Y. Kohayakawa, C. G. Moreira, B. Ráth, R. M. Sampaio, *Limits of permutation sequences*, Journal of Combinatorial Theory Series B **103** (2013), 93–113.

[4] D. Král, O. Pikhurko, *Quasirandom permutations are characterized by 4–point densities*, preprint, arXiv:1205.3074.

Sofic groups and invariant random networks

Andreas Thom

Let $\mathbb{G}_{d,\bullet}$ be the set of isomorphism classes of rooted, connected graphs with degree bounded by d. For every $r \in \mathbb{N}$, there are only finitely many possibilities for the r-neighborhood of the root (up to isomorphism). This easily implies that $\mathbb{G}_{d,\bullet}$ can be endowed with a natural topology which makes it into a compact, separable, and Hausdorff topological space. We denote by RN the space of probability measures on $\mathbb{G}_{d,\bullet}$. Every finite graph $G = (V, E)$ gives rise to an element $\mu_G \in \mathrm{RN}$ via the formula

$$\mu_G := \frac{1}{|V|} \sum_{v \in V} \delta_{(G,v)},$$

where $\delta_{(G,v)}$ denotes the rooted graph given by the connected component of $v \in G$.

A sequence of finite graphs $(G_n)_n$ is said to be Benjamini–Schramm convergent, if the probability measures $(\mu_{G_n})_n$ converge in the weak-$*$-topology on RN. An element $\mu \in \mathrm{RN}$ is called *sofic* if μ is a Benjamini–Schramm limit of a sequence of finite graphs.

It is a fundamental open question of characterize sofic measures in terms of intrinsic properties. For each $\mu \in \mathrm{RN}$, one can define two measures μ_1 and μ_2 on $\mathbb{G}_{d,\bullet,\bullet}$, the space of connected graphs equipped with two roots o_1 and o_2. This space is locally compact and comes endowed with two maps $\pi_i \colon \mathbb{G}_{d,\bullet} \to \mathbb{G}_{d,\bullet,\bullet}$ ($i \in \{1, 2\}$), which just forget one of the roots. For any Borel set $E \subset G_{d,\bullet,\bullet}$, we set

$$\mu_i(E) := \int_{G_{d,\bullet}} |E \cap \pi_i^{-1}(x)| \; d\mu(x).$$

We say that μ is *unimodular* if $\mu_1 = \mu_2$. For a finite graph $G = (V, E)$, a basic double counting argument shows that μ_G is unimodular. Hence, every sofic measure is unimodule. It is a famous open problem (first raised in [1]) whether all unimodular measures are sofic.

The notions above admit various generalizations. For example, one can put labels from a compact set on edges and vertices or direct edges; essentially using the same arguments. A finitely generated group Γ, generated by a finite symmetric set $S \subset \Gamma$ is called *sofic* if the Cayley graph $\mathrm{Cay}(\Gamma, S)$ is sofic as an S-labelled directed graph. This notion goes back to Gromov and was further clarified in work of Weiss [3]. As for now, there is no group known to be non-sofic.

Elek and Szabó [2] showed that sofic groups satisfy one of the longstanding conjectures of Kaplansky. Indeed, it can be shown that the group ring of a sofic group Γ with coefficients in an arbitrary skew field k is directly finite. This means, that for any $n \in \mathbb{N}$ and any $a, b \in M_n(k\Gamma)$, the equation $ab = 1$ in $M_n(k\Gamma)$ implies that $ba = 1$. This is known to hold for any group Γ if the characteristic of k is zero, and remains open in general.

References

[1] D. J. Aldous, R. Lyons, *Processes on unimodular random networks*, Electron. J. Probab. **12** (2007), no. 54, 1454–1508 (electronic).

[2] G. Elek, E. Szabó, *Sofic groups and direct finiteness*, J. Algebra **280** (2004), no. 2, 426–434.
[3] B. Weiss, *Sofic groups and dynamical systems. Ergodic theory and harmonic analysis (Mumbai, 1999)*, Sankhya Ser. A **62** (2000), no. 3, 350–359.

Graphings and local-global limits of bounded degree graphs

Endre Csóka

A graph sequence $(G_n)_{n=1}^{\infty}$ is Benjamini–Schramm convergent [1] if the distribution of the isomorphism types of neighborhoods of radius r (when a vertex is chosen uniformly at random in G_n) converges for every fixed r. Benjamini and Schramm described a limit object for locally convergent sequences in the form of an involution-invariant distribution on rooted countable graphs with bounded degree. One can also describe this limit object as a graphing [2], which is a bounded degree graph on a Borel probability space such that the edge set is Borel measurable and it satisfies a certain measure preservation property.

Graphings are Borel graphs with an additional measure preserving property. They seem to be the right generalizations of finite bounded degree graphs to the infinite setting. It is a broad research direction to generalize facts from finite graph theory to this measurable setting. Graphings contain more information than local statistics of neighborhoods. There is a strengthening of the Benjamini–Schramm convergence called local-global convergence by Hatami, Lovász and Szegedy [3], and limit objects for this convergence notion can be represented by graphings.

In local-global convergence, instead of the neighborhood-statistics, we consider the set of all colored neighborhood-statistics of the graph with all possible vertex colorings. For example, an expander graph has the same neighborhood-statistics than the disjoint union of two instances. But the latter one can be 2-colored so that all colored neighborhoods are monochromatic and both colors have probability 1/2, while there is no coloring for the former case which provides approximately the same colored neighborhood distribution. For another example, a sequence of d-regular random graphs and d-regular random bipartite graphs have the same Benjamini–Schramm limits: the random d-regular infinite tree. In contrast, the local-global topology can distinguish them, because a d-regular random graph has no proper 2-coloring, not even almost proper 2-coloring with high probability, but a random bipartite graph does have a proper 2-coloring, by definition.

The formal definition is the following. For a finite graph G, let $K(k,G)$ denote the set of all vertex colorings with k colors. Fix integers k and r, and let $U^{r,k}$ be the set of all triples (H,o,c) where (H,o) is a rooted graph of radius at most r and c is an arbitrary k-coloring of $V(H)$. Consider a finite graph G together with a $c \in K(k,G)$. Pick a random vertex v from G. Then the restriction of the k-coloring to $N_{G,r}(v)$ is an element in $U^{r,k}$, and thus for the graph G, every $c \in K(k,G)$ induces a probability distribution on $U^{r,k}$ which we denote by $P_{G,r}[c]$.

$$Q_{G,r,k} = \{P_{G,r}[c] : c \in K(k,G)\} \subseteq M(U^{r,k}).$$

Definition 1. *A sequence of finite graphs* $(G_n)_{n=1}^{\infty}$ *with all degrees at most* d *is called locally-globally convergent if for every* $r, k \geq 1$*, the sequence* $(Q_{G_n,r,k})_{n=1}^{\infty}$ *converges in the Hausdorff distance in the compact metric space* $(M(U_{r,k}), d_{var})$*.*

Definition 2. *Let* X *be a Polish topological space and let* ν *be a probability measure on the Borel sets in* X*. A graphing is a graph* $\mathcal{G}$ *on* $V(\mathcal{G}) = X$ *with Borel measurable edge set* $E(\mathcal{G}) \subset X \times X$ *in which all degrees are at most* d *and*

$$\int_A e(x, B)\mathrm{d}\nu(x) = \int_B e(x, A)\mathrm{d}\nu(x)$$

for all measurable sets $A, B \subseteq X$*, where* $e(x, S)$ *is the number of edges from* $x \in X$ *to* $S \subseteq X$*.*

The following theorem says that for each sequence of graphs, there exists a limit graphing.

Theorem 1. *Let* $(G_i)_{i=1}^{\infty}$ *be a local-global convergent sequence of finite graphs with all degrees at most* d*. Then there exists a graphing* $\mathcal{G}$ *such that* $Q_{G_n,r,k} \to Q_{\mathcal{G},r,k}$ $(n \to \infty)$ *in Hausdorff distance for every* r *and* k*.*

Given a Benjamini–Schramm limit, we can define a hierarchy of the local-global limits, as follows.

Definition 3 (Local-global partial order)**.** *Assume that* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are two graphings of maximal degree at most* d*. We say that* $\mathcal{G}_1 \prec \mathcal{G}_2$ *if* $\mathrm{cl}(Q_{\mathcal{G}_1,r,k}) \subseteq \mathrm{cl}(Q_{\mathcal{G}_2,r,k})$ *for every* $r, k \geq 1$*. In particular,* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are locally-globally equivalent if and only if both* $\mathcal{G}_1 \prec \mathcal{G}_2$ *and* $\mathcal{G}_2 \prec \mathcal{G}_1$ *hold.*

Roughly, a graphing is "larger" if this contains more structure. It is known that this partial ordering forms a lattice with a smallest and a largest element, namely there exists a graphing with a richest, and a graphing with the smallest structure within the same Benjamini–Schramm equivalence class.

References

[1] I. Benjamini, O. Schramm, *Recurrence of distributional limits of finite planar graphs*, Electronic J. Probab. **6** (2001), no. 23, 1–13.

[2] G. Elek, *On limits of finite graphs*, Combinatorica **27** (2007), 503–507.

[3] H. Hatami, L. Lovász, B. Szegedy, *Limits of local-global convergent graph sequences*, preprint, arxiv:1205.4356.

A model theory approach to structural limits

Cameron Freer

While the theories of dense and of bounded-degree graph limits are often presented in parallel, even the key definitions of convergence are strikingly different. In this talk, we present an approach of Nešetřil and Ossona de Mendez [1, 3], which aims to unify these approaches using model theory and notions from first-order logic.

Fix a first-order language. Given a first-order formula ϕ and a finite structure G in this language, define $\langle \phi, G \rangle$ to be the probability that ϕ is satisfied in G when its free variables are instantiated by a tuple chosen uniformly independently at random from the elements of G. We say that a sequence of such structures $(G_n)_{n\in\mathbb{N}}$ is first-order convergent when the sequence $(\langle \phi, G_n \rangle)_{n\in\mathbb{N}}$ converges for every first-order formula ϕ. One may define a notion of a measurable structure, known as a *modeling*, to which the concept $\langle \phi, \cdot \rangle$ extends in a natural way, providing a notion of limit object.

By examining sub-boolean algebras of the boolean algebra of first-order formulas (over the relevant language), one can obtain corresponding weaker notions of convergence, which recover dense graph limits (for quantifier-free formulas), Benjamini–Schramm convergence (for local formulas), and elementary convergence (for sentences).

Besides providing a unifying framework for treating these various notions of convergence, the approach can also expand the class of sparse graphs that can be handled beyond the bounded-degree case. In particular, graphs of bounded tree-depth admit natural notions of convergence and limit structure via this theory. This may be viewed as one step towards treating the more general setting of *nowhere dense* graphs [2].

When constructing graph limits using ultralimits, one naturally obtains objects that retain first-order convergence information, and so this approach may also be viewed as the study of this additional structure, already present.

Finally, by describing the notions of convergence using model-theoretic methods, one obtains definitions that uniformly treat the case of multiple relations and other more complex settings.

References

[1] J. Nešetřil, P. Ossona de Mendez, *A Model Theory Approach to Structural Limits*, Comment. Math. Univ. Carolin. **53** (2012), no. 4, 581–603.

[2] J. Nešetřil, P. Ossona de Mendez, *Sparsity*. Springer, Heidelberg (2012).

[3] J. Nešetřil, P. Ossona de Mendez, *A unified approach to structural limits (with application to the study of limits of graphs with bounded tree-depth)*, preprint, arXiv:1303.6471.

The graph limit approach to Sidorenko's conjecture

Christian Reiher

A rather general problem from extremal graph theory asks the following: fix a (small) graph F and consider another (large) graph G on n vertices possessing at least $\varrho \cdot n^2/2$ edges. What is the minimum number of homomorphisms from F to G? What we mean here by a homomorphism from F to G is just a function from $V(F)$ to $V(G)$ sending edges of F to edges of G. Note that we did not demand our homomorphisms to be injective or to send non–edges to non–edges. It is plain that the number of such homomorphisms, which we denote by $t(F, G)$, can be bounded from above by n^{v_F}, where v_F refers to the number of vertices of F; traditionally one thus studies the normalized quotient $t(F, G)/n^{v_F}$ and seeks to give lower bounds for it that depend on ϱ alone, the ultimate goal being to find the pointwise minimal function $M_F(\varrho)$ arising in this way.

A naive guess as to what this function might be can be obtained by taking G to be a large quasirandom graph of edge density about ϱ. Due to the counting lemma, the quotient $t(F, G)/n^{v_F}$ is in this case known to roughly equal ϱ^{e_F}, where e_F stands for the number of edges of F. So optimistically one might hope that $M_F(\varrho) \geq \varrho^{e_F}$ holds in general. Another example, however, shows that this is most of the time false: taking G to be a balanced complete bipartite graph one realizes that the function M_F vanishes on the interval $\gamma \in [0, 1/2]$ provided that the chromatic number of F is at least 3.

Still this does not rule out that the following statement, which was conjectured by Sidorenko in [10], could be true: *If F denotes a bipartite graph, and G denotes an arbitrary graph on n vertices with at least $\varrho \cdot n^2/2$ edges, then $t(F, G) \geq \varrho^{e_F} n^{v_F}$.* This is known to hold by now for several large classes of bipartite graphs F recalled below. It is customary to remark at this point that a precise formula for the function M_F is known for just quite a few few non–bipartite graphs, the most notable examples being cliques (see [7] for $F = K_3$, [6] for $F \in \{K_3, K_4\}$ and [8] for $F = K_r$.)

It was already known to Sidorenko [10] that this problem is equivalent to some of its analytical reformulations, the most general of which appears to be the following: *Let $\Omega = (X, \mathfrak{B}, \mu)$ denote a measure space whose total measure equals 1, W a graphon on Ω, i.e. a symmetric measurable function from X^2 to the unit interval, and F a bipartite graph with set of vertices $\{v_1, \ldots, v_n\}$. Then $t(F, W) \geq \varrho^{e_F}$, where*

$$t(F, W) = \int \prod_{v_i v_j \in E(F)} W(x_i, x_j) d\mu^{\otimes n}(x_1, \ldots, x_n)$$

and $\varrho = \int W(x, y) d(\mu \otimes \mu)(x, y)$.

The cases of equality are easiest to discuss in this setting and conjecturally they depend in the following way on the structure of F:

- If F is a matching, then equality holds for all graphons W.
- If F is forest but not a matching, then equality holds if and only if W is regular in the sense of having essentially constant vertex degree, i.e. if the set of all x from X satisfying $\int W(x, y) d\mu(y) \neq \varrho$ has measure zero.
- Finally, if F contains any cycle, then equality holds precisely if W is essentially constant.

The first of these statements is, of course, trivial; as to the second one, it was proved in [1] that Sidorenko's conjecture holds for forests, and in the eight–author article [3] the case of equality was settled as well. The strengthening of Sidorenko's conjecture provided by the third statement is known as the *forcing conjecture* and as of today it is known to hold for all graphs F that are known to satisfy Sidorenko's conjecture.[1] This is the case for

[1]This has not always been the case and may temporarily change in the future again.

- All complete bipartite graphs (by Hölders inequality).
- All bipartite graphs one of whose vertex classes contains at most four vertices, [9].
- All forests, [1].
- Even cycles (applying the Cauchy–Schwarz inequality to the corresponding result for paths).
- Hypercubes, [4] (iterating the Cauchy–Schwarz inequality in some tricky way).
- Bipartite graphs having one vertex that is complete vertex class opposite to it (see [2] for a proof using the tensor power trick, and [5] for an entirely different proof yielding the forcing conjecture for such graphs as well).
- Reflection trees (a class of bipartite graphs defined and studied in [5]).

In the talk we mainly discuss the new approach to Sidorenko's conjecture developed in [5] dubbed the "logarithmic calculus" by Li and Szegedy. It is based on two well known cases of Jensen's inequality, namely

$$\textstyle\int f \log g d\mu \leq \log \int f g d\mu \quad \text{ and } \quad \int f g \log g d\mu \geq (\int f g d\mu) \log(\int f g d\mu),$$

that are valid for all measurable functions $f, g \colon X \longrightarrow \mathbb{R}^+$ with $\int f d\mu = 1$. Typically these estimates are applied to some power of Ω rather than to Ω itself. To illustrate how this approach works, we take T to denote any tree with set of vertices $\{v_1, \ldots, v_n\}$ and define f_T to be the function from X^n to $\mathbb{R}^+$ given by

$$f_T(x_1, \ldots, x_n) = \frac{\prod_{v_i v_j \in E(T)} W(x_i, x_j)}{\varrho \prod_{1 \leq i \leq n} d(x_i)^{1-r_i}},$$

where W refers to the graphon under discussion, ϱ is defined as above, $d(x) = \int W(x,y) d\mu(y)$ is the "degree of x", and r_i denotes the ordinary degree of v_i in T. Arguing by induction on T and removing leaves in the induction step it is straightforward to check that

$$\int f_T(x_1, \ldots, x_n) \mu^{\otimes(n-1)}(\widehat{x_i}) = \frac{d(x_i)}{\varrho} \quad \text{ for } i = 1, \ldots, n,$$

which in particular implies $\int f_T d\mu^{\otimes n} = 1$. More or less the same computation also shows

$$\int f_T(x_1, \ldots, x_n) \log(d(x_i)) \mu^{\otimes(n-1)}(\widehat{x_i}) = \frac{d(x_i)\log(d(x_i))}{\varrho} \quad \text{ for } i = 1, \ldots, n.$$

Thus we have

$$\begin{aligned}
\log t(T,W) &= \log \int \varrho f_T \prod_{1\le i\le n} d(x_i)^{r_i-1} d\mu^{\otimes n} \\
&\ge \log \varrho + \int f_T \log \prod_{1\le i\le n} d(x_i)^{r_i-1} \\
&= \log \varrho + \sum_{1\le i\le n} (r_i-1) \int f_T \log(d(x_i)) d\mu^{\otimes n} \\
&= \log \varrho + \frac{1}{\varrho} \sum_{1\le i\le n} (r_i-1) \int d(z) \log(d(z)) d\mu(z) \\
&\ge \log \varrho + \frac{n-2}{\varrho} \cdot \varrho \log \varrho = (n-1)\log \varrho,
\end{aligned}$$

proving that T is indeed Sidorenko[1]. The main advantage of this approach lies in the circumstance that it may be generalized to many other bipartite graphs that can be broken up into manageable pieces. The basic idea here is to prove inequalities that are stronger than Sidorenko's for certain graphs, that can be sticked together in such a manner that the resulting inequalities imply Sidorenko's conjecture for larger graphs that are in a corresponding appropriate way constructible from the pieces we started with. Employing this strategy, Li and Szegedy managed to prove, e.g., that if you take two bipartite graphs satisfying Sidorenko's conjecture and glue them together along an edge, then the graph you obtain is Sidorenko as well. An important concept appearing in such arguments, called "smoothness" in [5], is defined as follows: Let H be a bipartite graph and let T be an induced subtree of H. Enumerate the set of vertices of H in such a way as $\{v_1,\dots,v_n\}$ that $S=\{v_1,\dots,v_m\}$ is the set of vertices of T for some $m\le n$, set $H^*=H-E(T)$, and define the function $t_S(H^*,W)$ from X^m to $\mathbb{R}^+$ by

$$t_S(H^*,W)(x_1,\dots,x_m) = \int \prod_{v_iv_j\in E(H^*)} W(x_i,x_j) d\mu^{\otimes(n-m)}(x_{m+1},\dots,x_n).$$

Then T is said to be *smooth* in H provided that

$$\int f_T \log t_S(H^*,W) d\mu^{\otimes m}(x_1,\dots,x_m) \ge |E(H^*)|\cdot \log \varrho.$$

For example, the empty tree being smooth in H is equivalent to H being Sidorenko. It is not hard to see that if one glues two graphs together along isomorphic smooth subtrees that tree is going to be smooth in the resulting graph again. The above mentioned statement on gluing graphs together along edges follows from the non–trivial observation that every edge is smooth in any Sidorenko graph.

Now the natural question arises as to what pairs (T,H) have the property that T is smooth in H. A rather obvious necessary condition for this to happen

[1]Strictly speaking some care is required when dealing with "vertices of W of degree zero", but there are various standard ways to handle this minor hurdle.

is that H has to "retract to T", which is intended to mean that there has to exist some homomorphism from H to T whose restriction to T is the identity. A conjecture due to Li and Szegedy vastly generalizing Sidorenko's conjecture asserts that conversely if T is an induced subtree of another graph H retracting to T, then T is smooth in H. It seems conceivable that this more general hypothesis, which seems to be easier to approach inductively than the original conjecture, may play a decisive rôle, in the eventual proof of Sidorenko's conjecture (provided the latter will turn out to be true).

References

[1] N. Alon, I. Z. Ruzsa, *Non–averaging subsets and non-vanishing transversals*, Journal of Combinatorial Theory A **86** (1999), 1–13.

[2] D. Conlon, J. Fox, B. Sudakov, *An approximate version of Sidorenko's conjecture*, Geometric and Functional Analysis **20** (2010), 1354–1366.

[3] D. Dellamonica Jr., P. E Haxell, T. Łuczak, D. Mubayi, B. Nagle, Y. Person, V. Rödl, M. Schacht, *Tree–minimal graphs are almost regular*, Journal of Combinatorics **3** (2012), 49–62.

[4] H. Hatami, *Graph norms and Sidorenko's conjecture*, Israel Journal of Mathematics **175** (2010), 125–150.

[5] J. L. Xiang Li, B. Szegedy, *On the logarithmic calculus and Sidorenko's conjecture*, preprint, arXiv:1107.1153.

[6] V. Nikiforov, *The number of cliques in graphs of given order and size*, Transactions of the American Mathematical Society **363** (2011), 1599–1618.

[7] A. A. Razborov, *On the minimal density of triangles in graphs*, Combinatorics, Probability and Computing **17** (2008), 603–618.

[8] Ch. Reiher, *The Clique Density Theorem*, submitted.

[9] A. Sidorenko, *Inequalities for functionals generated by bipartite graphs*, Discrete Mathematics and Applications **2** (1992), 489–504.

[10] A. Sidorenko, *A correlation inequality for bipartite graphs*, Graphs and Combinatorics **9** (1993), 201–204.

Profinite actions and applications in cost and rank gradient

Nhan-Phu Chung

We discuss here one of the main results of Abért and Nikolov about relations of cost and rank gradient via profinite actions [1]. We also present two of its applications: the new calculation of cost for free groups [5] and an example of non-amenable groups containing no free subgroups in the class of residually finite groups [9].

First, we review the definition of rank gradient. It was first introduced by Lackenby with motivation from 3-dimensional geometry [6]. Let Γ be a finitely generated group. A *chain* in Γ is a sequence $\Gamma = \Gamma_0 \geq \Gamma_1 \geq \dots$ of subgroups of finite index in Γ.

Definition 1. *The rank gradient of* Γ *with respect to* (Γ_n) *is defined as*

$$RG(\Gamma, (\Gamma_n)) = \lim_{n\to\infty} \frac{d(\Gamma_n) - 1}{|\Gamma : \Gamma_n|},$$

where $d(G)$ *denotes the minimal number of generators (or rank) of* G.

Let $H, K \leq \Gamma$ be subgroups of finite index with $H \leq K$. From Neilsen–Schreier theorem on H and K, we get $d(H) - 1 \leq |K : H|(d(K) - 1)$ yielding

$$\frac{d(H) - 1}{|\Gamma : H|} \leq \frac{d(K) - 1}{|\Gamma : K|}.$$

Thus the definition of $RG(\Gamma, (\Gamma_n))$ makes sense.

Definition 2. *The absolute rank gradient of* Γ *is defined as*

$$RG(\Gamma) = \inf_{H \leq \Gamma, [\Gamma:H] < \infty} \frac{d(H) - 1}{[\Gamma : H]}.$$

Now we define briefly a definition of cost of a group. The cost was introduced by Levitt [7] and has been studied deeply by Gaboriau [4]. Let an infinite countable group Γ act on a standard Borel space $(X, \mathcal{B}, \mu)$ by measure preserving automorphisms. An action of Γ on $(X, \mathcal{B}, \mu)$ is called *essentially free* if for each $\gamma \in \Gamma \backslash \{e_\Gamma\}$, $\mu\{x \in X : \gamma x = x\} = 0$.

Example 1. *When* Γ *is infinite, the Bernoulli shift action of* Γ *on* $(\{0,1\}^\Gamma, \mu)$ *is essentially free, where* μ *is the product measure on* $\{0,1\}^\Gamma$ *having the* $(1/2, 1/2)$*-measure.*

A *relation* R on X is a set of ordered pairs from X, $R \subset X \times X$. Let us define the relation E on X by

$$xEy \ \Leftrightarrow \ \exists \gamma \in \Gamma, \ y = \gamma x.$$

Let $S \subset X \times X$ be an arbitrary relation on X. A path from x to y in S is a finite sequence $x_0 = x, x_1, \ldots, x_k = y \in X$ such that $(x_i, x_{i+1}) \in S$ or $(x_{i+1}, x_i) \in S$ for $0 \leq i \leq k - 1$. We say that a sub-relation S of E *spans* E, if for any $(x, y) \in E$ with $x \neq y$ there exists a path from x to y in S. The edge measure of a Borel sub-relation S of E is defined as $e(S) = \int_{x \in X} \deg_S(x) d\mu$, where $\deg_S(x) = |\{y \in X : (x, y) \in S\}|$. The cost of E is defined as $\mathrm{cost}(E) = \mathrm{cost}(\Gamma, X) = \inf e(S)$, where the infimum is taken over all Borel sub-relations S of E that span E.

If Γ is generated by $g_1, \ldots, g_d$ then the set $S = \bigcup_{i=1}^d \bigcup_{x \in X} \{(x, g_i x)\}$ is a spanning Borel sub-relation of E with $e(S) = d$. Thus $\mathrm{cost}(\Gamma, X) \leq d(\Gamma)$.

The cost of Γ is defined as $\mathrm{cost}(\Gamma) = \inf \mathrm{cost}(\Gamma, X)$, where the infimum is taken over all essentially free actions of Γ on a standard Borel space. We say that Γ has *fixed price* c if all essentially free actions of Γ on a standard Borel space X have cost c.

Example 2. (1) *Every infinite amenable group has fixed price 1.*
(2) *The free group* $\mathbb{F}_n, 1 \leq n < \infty$ *has fixed price* n [4].

Before discussing the main theorem of [1] we review notions of profinite actions and Farber chains. Let (Γ_n) be a chain in Γ. Then we define the tree structure $T = T(\Gamma, (\Gamma_n))$ of Γ with respect to (Γ_n) as follows: the vertex set of T is $\{\Gamma_n \gamma : n \in \mathbb{N}, \gamma \in \Gamma\}$ and $(\Gamma_n g, \Gamma_m h)$ is an edge in T if $m = n + 1$ and $\Gamma_m h \subset \Gamma_n g$. The right actions of Γ on the coset space Γ / Γ_n respect to the tree structure and so Γ acts on T by automorphisms.

The boundary ∂T of T is defined as the set of infinite rays starting from the root. For any $t = \Gamma_n\gamma \in T$ we define $\mathrm{Sh}(t) \subset \partial T$, the shadow of t as $\mathrm{Sh}(t) := \{x \in \partial T : t \in x\}$. Set the base of topology on ∂T to be the set of shadows and define the measure of a shadow as $\mu(\mathrm{Sh}(t)) = \frac{1}{|\Gamma:\Gamma_n|}$. Then ∂T is a totally disconnected compact, metrizable space. The group Γ acts on ∂T by measure preserving homeomorphisms and this action is called the *profinite action* of Γ with respect to (Γ_n).

We say that the chain (Γ_n) is Farber if its profinite action is essentially free or equivalently $\forall g \in \Gamma, g \neq 1, \frac{|\{\overline{h}\in\Gamma/\Gamma_n : g\overline{h}=\overline{h}\}|}{[\Gamma:\Gamma_n]} \to 0$ as $n \to \infty$. Note that the Farber condition was introduced by Farber in another equivalent form [3] and the existence of Farber chain implies that Γ is residually finite: let $\Lambda_n = \bigcap_{h\in\Gamma} h\Gamma_n h^{-1}$ then $\bigcap_{n\in\mathbb{N}} \Lambda_n = \{1\}$. If (Γ_n) is a normal chain such that $\bigcap \Gamma_n = \{1\}$ then (Γ_n) is a Farber chain.

Theorem 1. [1] *Let (Γ_n) be a Farber chain. Then*

$$RG(\Gamma, (\Gamma_n)) = \mathrm{cost}(\Gamma \curvearrowright \partial T(\Gamma, (\Gamma_n)) - 1.$$

Let $\widehat{\Gamma}$ be the profinite completion of Γ which is defined as the inverse limit of the groups $\Gamma/N, N \rhd \Gamma, [\Gamma : N] < \infty$, i.e, $\widehat{\Gamma} = \{(g_i)_{i\in I} \in \prod_{i\in I} \Gamma/N_i :$ for all $N_j \leq N_i, g_i = g_j (\mathrm{mod}\ N_i)\}$. Then $\widehat{\Gamma}$ is a compact group. Let μ_Γ be its normalized Haar measure. Then the left translation action p_Γ of Γ on $(\widehat{\Gamma}, \mu_\Gamma)$ is a measure preserving action.

Corollary 2. [1] *Let Γ be a finitely generated infinite residually finite group. Then*

$$RG(\Gamma) = \mathrm{cost}(p_\Gamma) - 1.$$

Now we present a new calculation of cost for free groups [5]. Let $1 \leq n < \infty$ and let $\Gamma = \mathbb{F}_n$ be the free group with n generators. Let H be a subgroup of Γ of finite index, then $\frac{d(H)-1}{|\Gamma:H|} = n - 1$. And hence by Corollary 2, we have $\mathrm{cost}(p_{\mathbb{F}_n}) = n$. We also have $\mathrm{cost}(a) \geq \mathrm{cost}(p_{\mathbb{F}_n})$ [5] and $\mathrm{cost}(a) \leq n$ for any action a of $\mathbb{F}_n$. It follows that $\mathbb{F}_n$ has fixed price n.

Next we discuss Osin's examples for von Neumann–Day problem in the class of residually finite groups. Recall that the original problem asks if there exists a non-amenable group without non-abelian free subgroups. The affirmative answer was obtained by Olshanskii [8]. Recently, Ershov [2] gave the first example for the problem in the class of residually finite groups. Another example of the problem in the class of residually finite groups has been constructed by Osin [9] when he combined the two following theorems.

Theorem 3. [1] *If Γ is a finitely generated infinite residually finite and amenable group then $RG(\Gamma) = 0$.*

Theorem 4. [9] *There exists a finitely generated infinite residually finite torsion group with positive rank gradient.*

References

[1] M. Abért, N. Nikolov, *Rank gradient, cost of groups and the rank versus Heegaard genus problem*, J. Eur. Math. Soc. **14** (2012), no. 5, 1657–1677.
[2] M. Ershov, *Golod-Shafarevich groups with property (T) and Kac-Moody groups*, Duke Math. J. **145** (2008), no. 2, 309–339.
[3] M. Farber, *Geometry of growth: approximation theorems for L^2 invariants*, Math. Ann. **311** (1998), no. 2, 335–375.
[4] D. Gaboriau, *Coût des relations d'équivalence et des groupes*, Invent. Math. **139** (2000), no. 1, 41–98.
[5] A. S. Kechris, *Weak containment in the space of actions of a free group*, Israel J. Math. **189** (2012), 461–507.
[6] M. Lackenby, *Expanders, rank and graphs of groups*, Israel J. Math. **146** (2005), 357–370.
[7] G. Levitt, *On the cost of generating an equivalence relation*, Ergodic Theory Dynam. Systems **15** (1995), no. 6, 1173–1181.
[8] A. Y. Ol'šanskiĭ , *On the question of the existence of an invariant mean on a group*, Uspekhi Mat. Nauk **35** (1980), no. 4 (214), 199–200.
[9] D. Osin, *Rank gradient and torsion groups*, Bull. Lond. Math. Soc. **43** (2011), no. 1, 10–16.

Vertex and edge coloring models

Peter Gmeiner

1. Vertex Coloring Models

A *vertex coloring model* is a weighted graph H equipped with weights $\alpha_H(u) \in \mathbb{R}_{\geq 0}$ for each vertex $u \in V(H)$ and $\beta_H(u,v) \in \mathbb{R}$ for each edge $uv \in E(H)$. Without loss of generality we can completely describe H by $d := |V(H)|$, $a := (\alpha_H(1), \ldots, \alpha_H(d)) \in \mathbb{R}^d_{\geq 0}$ and $B := (\beta_H(i,j))_{i,j=1}^d \in \mathbb{R}^{d \times d}$. The vertices $V(H)$ are often interpreted as colors or states of the vertices of another unweighted graph G (which can have multiple edges but no loops). A *coloring* of the vertices $V(G)$ is then given by a mapping $\phi : V(G) \to V(H)$. To ϕ we assign a weight

$$\alpha_\phi := \prod_{u \in V(G)} \alpha_H(\phi(u))$$

and a homomorphism function

$$\hom_\phi(G,H) := \prod_{uv \in E(G)} \beta_H(\phi(u), \phi(v)).$$

The *partition function* or homomorphism function of a vertex coloring model is defined as

$$f_H(G) := \hom(G,H) := \sum_{\substack{\phi: V(G) \to V(H) \\ \text{homomorphism}}} \alpha_\phi \hom_\phi(G,H).$$

If $\alpha_H(u) = 1$ for all $u \in V(H)$ and $\beta_H(u,v) = 1$ for all $uv \in E(H)$ then $\hom(G,H)$ is the number of homomorphisms from G to H. A *graph parameter* is a real-valued function $f : \mathcal{G} \to \mathbb{R}$ on the set of finite graphs $\mathcal{G}$ (which are invariant under graph isomorphism) and is called *multiplicative* if $f(G_1 \cup G_2) = f(G_1)f(G_2)$ for graphs

G_1, G_2, where $G_1 \cup G_2$ is the disjoint union of the graphs. The set of isomorphism classes of k-labeled graphs (a graph is k-labeled if k vertices are labeled by $1, \dots, k$) is denoted by $\mathcal{G}_k$. By K_k we denote a k-labeled complete graph on k vertices.

We define a natural gluing operation for two k-labeled graphs G_1, G_2 with the *gluing-operator* $g_k : \mathcal{G}_k \times \mathcal{G}_k \to \mathcal{G}_k$ with $g_k(G_1, G_2) := G_1 \cup G_2$ where we identify vertices with the same labels. A k-labeled graph G is *vertex reflection symmetric*, if $G = g_k(H, H)$ for some k-labeled graph H and $k \geq 0$, in particular $f_H(G)$ is vertex reflection positive. The (infinite) matrix $f \circ g_k : \mathcal{G}_k \times \mathcal{G}_k \to \mathbb{R}$ is called *vertex connection matrix* for a graph parameter f and an entry is given by $M(f,k)_{G_1,G_2} := f(g_k(G_1, G_2))$, where the rows and columns are indexed by base elements of $\mathcal{G}_k$. A graph parameter f is *vertex reflection positive*, if $M(f,k)$ is positive semidefinite for all $k \geq 0$. The function $k \mapsto r(f,k) := \mathrm{rk}(\mathrm{M(f,k)})$ is the *rank connectivity function* of f.

We list some properties of the vertex connection matrix [1]:

- Let $f \neq 0$ be a graph parameter. Then f is multiplicative if and only if $M(f,0)$ is positive semidefinite, $f(K_0) = 1$ and $r(f,0) = 1$.
- Let f be a multiplicative graph parameter, $k, l \in \mathbb{N}_0$, then $r(f, k+l) \geq r(f,k)r(f,l)$.

For a k-labeled graph G and a partial coloring $\phi : [k] \to V(H)$ (with $[k] := \{1, \dots, k\}$) let $\hom_\phi(G,H) := \sum_{\substack{\psi: V(G) \to V(H) \\ \psi \text{ extends } \phi}} \frac{\alpha_\psi}{\alpha_\phi} \hom_\psi(G,H)$. The homomorphism function then reads $\hom(G,H) := \sum_{\phi:[k]\to V(H)} \alpha_\phi \hom_\phi(G,H)$.

Let G be any k-labeled graph then $f_H(G) := \hom(G,H)$ is vertex reflection positive and $r(f_H, k) \leq |V(H)|^k$. This condition is also sufficient.

Theorem 1 ([1, 2], Characterization for finite vertex coloring models)**.** *Let f be a vertex reflection positive graph parameter for which there exists a number $q \in \mathbb{N}$ such that $r(f,k) \leq q^k$ for all $k \geq 0$. Then there exists a weighted graph H with $|V(H)| \leq q$ such that $f(G) = f_H(G)$.*

Examples.

- Matching. Let $\Phi(G)$ be the number of perfect matchings in a graph G. By definition it follows that $\Phi(G)$ is multiplicative. We can show that $r(\Phi, k) \leq 2^k$. On the other hand for 1-labeled graphs K_1, K_2 we have $M(\Phi, 1)_{K_1,K_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is not positive semidefinite and hence $M(\Phi, k)$ is not positive semidefinite for all $k \geq 0$. In particular Φ is not vertex reflection positive and cannot be represented as a homomorphism function.
- Chromatic polynomial. Let $p(G,x)$ be the chromatic polynomial for a graph G. For x fixed, $p(G,x)$ is multiplicative. For $k, q \in \mathbb{Z}_+$ set $B_{k,q}$ as the number of partitions of a k-element set into at most q parts ($B_k := B_{k,k}$ is the kth Bell number), then one can show that

$$r(p,k) = \begin{cases} B_{kx} & \text{if } x \in \mathbb{Z}_+, \\ B_k & \text{else.} \end{cases}$$

In particular $M(p,k)$ is vertex reflection positive if and only if $x \in \mathbb{Z}_+$ and so $p(G,x) = \hom(G, K_x)$ if and only if $x \in \mathbb{Z}_+$.

- Flows. Let Γ be a finite abelian group, $S \subseteq \Gamma$ which is closed under inversion ($S = -S$). For a directed graph G we fix the orientation of the edges. An *S-flow* is a function $f : E(G) \to S$ such that $f(uv) = -f(vu)$ for each edge $uv \in E(G)$ and $\sum_{u \in N(v)} f(uv) = 0$ for all $v \in V(G)$, where $N(v)$ is the set of all neighbour vertices of v.

  flo(G) is the number of S-flows in G and independent from the orientation of G. Let Γ^* be the character group of Γ and H a complete, looped directed graph on Γ^*. Define $\alpha_\chi := \frac{1}{|\Gamma|}$ for all $\chi \in \Gamma^*$ and $\beta_{\chi,\chi'} := \sum_{s \in S} \chi(-s)\chi'(s)$ for all $\chi, \chi' \in \Gamma^*$. From $S = -S$ it follows that β is symmetric and $\beta_{\chi,\chi'} \in \mathbb{R}$. We can show that $\mathrm{flo(G)} = \mathrm{hom(G,H)}$.

2. Edge coloring models

A *circle edge* is an edge which is not incident to any vertex. It is an edge with no endpoints. Let $\mathcal{G}^E$ be the set of isomorphism classes of graphs with circles, loops and multiple edges.

We call an edge an *outgoing edge* or *open end* if it goes out from a graph but is not finished (it is possible that both ends are open). We define $\mathcal{G}_k^E := \{G \in \mathcal{G}^E \mid G \text{ has } k \text{ open ends labeled by } 1, \ldots, k\}$. A *gluing operator* along edges is defined by $g_k^E : \mathcal{G}_k^E \times \mathcal{G}_k^E \to \mathcal{G}^E$, where $g_k^E(G_1, G_2)$ is the disjoint union of G_1, G_2 and then we glue together the labeled open ends of both graphs (we identify open ends with same labels and replace them by an edge). Note that $g_k^E(G_1, G_2)$ has no open ends, but can have circles (gluing together two edges with both ends open gives a circle).

With $Q^E := \{G \mid G = \sum_{i=1}^n \mu_i \tilde{G}_i, \quad \mu_i \in \mathbb{R}, \ \tilde{G}_i \in \mathcal{G}^E \ \forall i \in \{1, \ldots, n\}\}$ we define the space of quantum graphs. Analogously we define the space Q_k^E of quantum graphs with k-labeled open ends. The gluing operator g_k^E extends linearly to $g_k^E : Q_k^E \times Q_k^E \to Q^E$ and is symmetric and bilinear. $G \in Q^E$ is *edge reflection symmetric*, if $G = g_k^E(H, H)$ for some $H \in Q_k^E$ and $k \geq 0$. A graph parameter f is *edge reflection positive*, if its linear extension $f : Q^E \to \mathbb{R}$ takes nonnegative values on all edge reflection symmetric quantum graphs. The matrix $f \circ g_k^E : Q_k^E \times Q_k^E \to \mathbb{R}$ is called *edge connection matrix* and is given by $M^E(f,k)_{G_1,G_2} := f(g_k^E(G_1, G_2))$ where the rows and columns are indexed by base elements of $\mathcal{G}_k^E$. An $\mathbb{R}$-valued *edge coloring model* is given by a function $t : \mathbb{N}^d \to \mathbb{R}$ (with colorset $\mathcal{C} = \{c_1, \ldots, c_d\}$). For a coloring of edges $\psi : E(G) \to \mathcal{C}$ in G and a vertex $v \in V(G)$ we define $\deg_c(\psi, v)$ as the number of edges e incident to v with color $\psi(e) = c$. We get a vector $\deg(\psi, v) := (\deg_{c_1}(\psi, v), \ldots, \deg_{c_d}(\psi, v)) \in \mathbb{N}^d$. Furthermore we define $t_\psi(G) := \prod_{v \in V(G)} t(\deg(\psi, v))$ and the *partition function* for edge coloring models is $t(G) := \sum_{\psi : E(G) \to \mathcal{C}} t_\psi(G)$ and is a multiplicative graph parameter. The following theorems give a characterization of graph parameters in terms of edge coloring models.

Proposition 2 ([3]). *Let $t : \mathbb{N}^d \to \mathbb{R}$ be an edge coloring model then the partition function $t : \mathcal{G}^E \to \mathbb{R}$ is edge reflection positive.*

Theorem 3 ([3]). *Let $f : \mathcal{G}^E \to \mathbb{R}$ be an edge reflection positive and multiplicative graph parameter. Then there exists an edge coloring model $t : \mathbb{N}^d \to \mathbb{R}$ such that $t : \mathcal{G}^E \to \mathbb{R}$ equals f.*

References

[1] M. Freedman, L. Lovász, A. Schrijver, *Reflection positivity, rank connectivity, and homomorphism of graphs*, J. Amer. Math. Soc. **20** (2007), 37–51.
[2] L. Lovász, *Large networks and graph limits*, American Mathematical Society (2012).
[3] B. Szegedy, *Edge coloring models and reflection positivity*, J. Amer. Math. Soc. **20** (2007), 969–988.

Factor of i.i.d. processes

Ágnes Backhausz

1. Local algorithms on finite graphs

Local algorithms on bounded degree graphs are strongly connected to parallelized algorithms in constant running time. In this case the vertices of a large graph can send messages to their neighbours, and this is repeated constantly many times. Then every vertex produces a label based on the information it has received.

In a local algorithm each vertex produces a new label based on the isomorphism class of its neighbourhood of radius r. Each vertex applies the same rule.

For symmetric graphs (e.g. for a large circle) every vertex gets the same label. Therefore we use randomness to break symmetry. More precisely, we start from an independent identically distributed random labelling of the graph, and then we apply a local rule: each vertex produces a new label based on the isomorphism class of its labelled neighbourhood of radius r. For example, we can get independent sets, dominating sets, matchings, colorings this way; see e.g. [1, 2, 6].

We need some notation for the precise definition. Let S, T be arbitrary sets. An S-labelled graph G is given by $h : V(G) \to S$. A rooted graph is (G, o), where o is a distinguished vertex of G. For $r, d \in \mathbb{N}$ the set $\mathcal{N}(r, d, s)$ consists of the isomorphism types of rooted S-labelled graphs of maximum degree d such that each vertex is of distance less than or equal to r from the root.

A function $f : \mathcal{N}(r, d, s) \to T$ is the rule. For a labelling h we define $h^f : V(G) \to T$ such that $h^f(v)$ is the value of f on the S-labelled rooted neighbourhood of radius r with the root placed on v.

Definition 1. *A randomized local algorithm of radius r and degree d is as follows. A measurable function $f : \mathcal{N}(r, d, \Omega) \to L$ is given, where Ω is a probability space and L is a measure space. The input is a graph G of degrees bounded by d; the output is h^f, where h is a labelling of G with independent random elements of Ω.*

This works on arbitrary finite graphs, since f is defined for all possible configurations. Moreover, it works on infinite graphs with bounded degree. However, for bounded degree infinite graphs one can do more: using measurable functions as rules, the finiteness of the radius can be omitted. We will discuss the case of the infinite d-regular tree for sake of simplicity.

2. Factor of i.i.d. processes on the d-regular tree

Let T be the infinite d-regular tree with a distinguished root o, and $\Omega = [0,1]^{V(T)}$. This is a compact topological space.

We choose a measurable function $f : \Omega \to \{1, \dots, k\}$, which depends only on the isomorphism class of the labelled rooted tree. To put it in another way, it is invariant under the action of the root preserving automorphisms of T.

We put random independent uniformly distributed elements from $[0, 1]$ on the vertices of T. This is a random element of Ω. Then for every $v \in T$ we define a color (new label) $c(v) \in \{1, \dots, k\}$ as the value of f on the labelled rooted tree obtained from T by assigning labels ω and placing the root on v. f is the rule of the coloring. It has radius r if it depends only on the labels of vertices of T that are of distance less than or equal to r from the root. We get a random labelling of T; this is called a factor of i.i.d. process.

We use the weak topology for random processes on the tree. Two processes are close to each other if the neighbourhood statistics are close to each other. Factor of i.i.d. processes can be approximated in the weak topology with factor of i.i.d. processes given by a rule of finite radius. These rules are defined on trees, therefore they work only on locally tree-like graphs. That is, if the rule f has radius r, and the shortest circle in the graph has length larger than $2r$, then every neighbourhood of radius r is a tree, and we can apply f. Thus these algorithms may be applied on finite large girth graphs.

This will help us to check that not every process on the tree is factor of i.i.d. Put a random 0 or 1 on the root, and put labels periodically on the infinite tree such that the endpoints of edges have always different labels. Note that the correlation does not decay, as it does in factor of i.i.d. processes. On the other hand, if it were a factor of i.i.d. process, we would be able to approximate it with an f of finite radius, and then apply f for finite random large girth graphs. We would obtain that these finite graphs are close to a bipartite graph, which contradicts a result of Béla Bollobás. This states that the size of the maximal independent set that can be produced by local algorithms in a random d-regular graph is at most C times the number of vertices, where C is a constant strictly less than $1/2$ which does not depend on the algorithm.

3. Independent sets and other structures

It is not hard to construct an independent set of density $\frac{1}{d+1}$ in d-regular graphs. Let $f : \Omega \to [0, 1]$ such that $f(\omega) = 1$ holds if and only if the label of the root is smaller than the labels on all the neighbouring vertices. This is a rule of radius 1. $f^{-1}(1)$ is an independent set: at most one of the endpoints of an edge can belong to it. Each vertex is in the set with probability $\frac{1}{d+1}$: if it gets the smallest label in the set consisting of its neighbours and the vertex itself.

One can do much better. Csóka, Gerencsér, Harangi and Virág have a result for finding an independent set containing $43,52\%$ of the vertices with a local algorithm in d-regular large girth graphs. It is an open question if it is the maximum that can be obtained by a local algorithm. See also the recent paper of Gamarnik and Sudan [4].

4. Graphings

We now show the connection between factor of i.i.d. processes and graphings. We use the definition of [5].

Definition 2. *X is a Polish topological space, ν is a probability measure on the Borel sets in X. A graphing is a graph on $V(\mathcal{G}) = X$ with Borel measurable edge set $E(\mathcal{G}) \subset X \times X$ in which all degrees are at most d and*

$$\int_A e(x,B)d\nu(x) = \int_B e(x,A)d\nu(x)$$

for all measurable sets $A, B \subseteq X$, where $e(x,S)$ is the number of edges from $x \in X$ to $S \subseteq X$.

Every finite graph is a graphing with the uniform distribution on the sets of vertices. For example, take the unit circle and connect points that are at distance α.

The Bernoulli graphing of the infinite d-regular tree T is defined as follows. Let X be the set of isomorphism classes of T with vertices labelled with elements of $[0,1]$; two labelled trees are isomorphic if there is a root-preserving automorphism from one to the other. We get measure ν by putting uniform labels from $[0,1]$ independently on the vertices of T. Finally, two points of X are connected if they can be obtained from each other by moving the root to a neighbouring vertex.

5. Connection of graphings and factor of i.i.d. processes

We have seen that the vertices of the Bernoulli graphing are the isomorphism classes of the infinite d-regular tree. Hence a random vertex of the Bernoulli graphing is a random labelling of the tree; moreover, in its connected component we see the labellings that can be obtained by moving the root along the edges. We consider the measurable colourings of the vertices of the Bernoulli graphing. This gives a measurable function, which may be used as the rule of a factor of i.i.d. process, based on the previous argument.

On the other hand, we can take graphings as operators [5]. Let $f : X \to \mathbb{C}$ be a measurable function, and

$$\mathcal{G}f(x) = \sum_{(x,v)\in E(\mathcal{G})} f(v).$$

The operator $\mathcal{G}$ acts on the Hilbert space $L^2(X,\nu)$ as a bounded self-adjoint operator, and its norm is smaller or equal to the maximal degree. The self-adjointness of $\mathcal{G}$ follows from the graphing axiom in Definiton 2.

Spectral properties of the operator corresponding to the Bernoulli graphing of T may be used to prove certain properties of factor of i.i.d. processes, which have consequences for randomized local algorithms on finite graphs as well.

References

[1] E. Csóka, G. Lippner, *Invariant random matchings in Cayley graphs*, preprint, arXiv:1211.2374.

[2] G. Elek, G. Lippner, *Borel oracles. An analytic approach to constant time algorithms*, Proc. AMS **138** (2010), no. 8, 2939–2947.

[3] D. Gaboriau, R. Lyons, *A Measurable-Group-Theoretic Solution to von Neumann's Problem*, Inventiones Mathematicae **177** (2009), 533–540.

[4] D. Gamarnik, M. Sudan, *Limits of local algorithms over sparse random graphs*, preprint, arXiv:1304.1831.

[5] H. Hatami, L. Lovász, B. Szegedy, *Limits of local-global convergent graph sequences*, preprint, arXiv:1205.4356.

[6] R. Lyons, F. Nazarov, *Perfect matchings as IID factors on non-amenable groups*, Europ. J. Combin. **32** (2011), 1115–1125.

Ramanujan graphs

Péter Csikvári

1. Introduction

In this talk, we surveyed the classical theory of Ramanujan graphs together with certain recent developments.

In this abstract, G is a d-regular graph. Let A be its adjacency matrix: A_{ij} is the number of edges between the vertices i and j. Let

$$d = \lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$$

be the eigenvalues of the adjacency matrix of G, where the number of vertices is $v(G) = n$.

The d-regular graph G is said to be Ramanujan if all of the nontrivial eigenvalues of its adjacency matrix lie in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. If the graph G is nonbipartite then its only trivial eigenvalue is $\lambda_1(G) = d$, whereas if the graph G is bipartite then its trivial eigenvalues are $\lambda_1(G) = d$ and $\lambda_n(G) = -d$.

The classical theory of Ramanujan graphs started with a theorem of Alon and Boppana claiming that if (G_n) is a sequence of d-regular graphs such that $v(G_n) \to \infty$, then

$$\limsup_{n\to\infty} \lambda_2(G_n) \geq 2\sqrt{d-1}.$$

Subsequently, Noga Alon published refinements of the above theorem under the name Alon Nilli [9, 10]. From these theorems one can deduce the following theorem of Serre which is also a strengthening of the original Alon–Boppana theorem.

Theorem 1 (Serre [11]). *For each $\varepsilon > 0$, there exists a positive constant $c = c(\varepsilon, d)$ such that for any d-regular graph G, the number of eigenvalues λ of G with $\lambda \geq (2-\varepsilon)\sqrt{d-1}$ is at least $c \cdot v(G)$.*

A particularly simple proof of this theorem was given by Sebastian Cioaba [3] together with a nice counterpart of the Alon–Boppana theorem which relates the smallest eigenvalue and the so-called odd girth of the graph. In the talk, we outline Cioaba's proof to Serre's theorem.

2. Expanders, random and pseudorandom graphs

Ramanujan graphs are strongly related to expanders and pseudorandom graphs. In fact, it turns out that Ramanujan graphs are the best possible pseudorandom graphs. A graph G is called (n, d, λ)-pseudorandom if it is a d-regular graph on n vertices such that all nontrivial eigenvalues are at most λ. Since the concept of pseudorandom graphs was introduced to give a better insight to random graphs, it is natural to ask whether random graphs are Ramanujan graphs. It turns out that they are indeed close to being Ramanujan. J. Friedman [4] proved that for every positive ε, a random d-regular graph on n vertices satifies that

$$|\lambda_i(G)| \leq 2\sqrt{d-1} + \varepsilon \quad (i \geq 2)$$

with very high probability. In fact, it is conjectured that positive proportion of the d-regular graphs are Ramanujan.

After all, it can be surprising that there are not many known constructions for Ramanujan graphs. In fact, the first such constructions were given by Lubotzky, Phillips and Sarnak [5] and independently by Margulis [7]. The Lubotzky–Phillips–Sarnak construction was subsequently extended to all d of the form $q+1$, where q is a prime power. It is still an open problem to construct infinite family of d-regular nonbipartite Ramanujan graphs for all d.

3. Girth of the Ramanujan graphs

In a recent paper of Abért, Glasner and Virág [1], the authors found several connections between the number of cycles and the second largest eigenvalue. Their results imply that Ramanujan graphs has large essential girth. The precise statement is the following.

Theorem 2 (Abért–Glasner–Virág [1]). *Let $d \geq 3$ and $\beta = (30\log(d-1))^{-1}$. Then for any d-regular finite Ramanujan graph G, the proportion of vertices in G whose $\beta \log\log|G|$-neighborhood is a d-regular tree is at least $1 - c(\log|G|)^{-\beta}$.*

The authors also introduced the concept of weakly Ramanujan graph sequences. We say that a sequence (G_n) of finite d-regular graphs is weakly Ramanujan if

$$\lim_{n\to\infty} \mu_{G_n}([-2\sqrt{d-1}, 2\sqrt{d-1}]) = 1.$$

They showed that if (G_n) is a weakly Ramanujan sequence of finite d-regular graphs, then (G_n) has essentially large girth. In fact, it follows from their another theorem claiming that if G is a d-regular unimodular random graph that is infinite and Ramanujan almost surely, then G is the infinite tree almost surely.

4. Addendum to the talk and to the abstract

At the end of the talk, I had a few minutes left and László Lovász suggested me to speak about Nathan Linial's signing conjecture [2]. The signing conjecture says that for every d-regular graph G, it is always possible to put ± 1 values on the edges such that all eigenvalues of the obtained signed matrix are in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. One can deduce from the signing conjecture that from a Ramanujan graph G, one can construct another Ramanujan graph on two times as many vertices. Hence the conjecture implies that there are infinitely many d-regular Ramanujan graphs. Only two weeks after the Arbeitsgemeinschaft, the signing conjecture was proved for bipartite graphs by Adam Marcus, Daniel A. Spielman and Nikhil Srivastava [6]. Hence there are infinitely many d-regular bipartite Ramanujan graphs for all d.

References

[1] M. Abért, Y. Glasner, B. Virág, *The measurable Kesten theorem*, preprint, arXiv:1111.2080.
[2] Y. Bilu, N. Linial, *Lifts, discrepancy and nearly optimal spectral gap*, Combinatorica **26** (2006), no. 5, 495–519.
[3] S. Cioaba, *On the extreme eigenvalues of regular graphs*, J. Combin. Theory Ser. B **96** (2006), no. 3, 367–373.
[4] J. Friedman, *A proof of Alon's second eigenvalue conjecture and related problems*, Amer. Mathematical Society (2008).
[5] A. Lubotzky, R. Phillips, P. Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), no. 3, 261–277.
[6] A. Marcus, D. Spielman, N. Srivastava, *Interlacing families I: bipartite Ramanujan graphs of all degrees*, preprint, arXiv:1304.4132.
[7] G. A. Margulis, *Explicit group theoretical constructions of combinatorial schemes and their application to the design of expanders and concentrators*, Problems of Information Transmission **24** (1988), no. 1, 39–46.
[8] M. Morgenstern, *Existence and explicit constructions of $q+1$ regular Ramanujan graphs for every prime power q*, Journal of Combinatorial Theory Series B, **62** (1994), 44–62.
[9] A. Nilli, *On the second eigenvalue of a graph*, Discrete Math. **91** (1991), 207–210.
[10] A. Nilli, *Tight estimates for eigenvalues of regular graphs*, Electron. J. Combin. **11** (2004), Note 9, pp. 4.
[11] J.-P. Serre, Private letters to W. Li dated October 8, 1990 and November 5, 1990.

Statistical physics and graph limits convergence from the right

Fiona Skerman

The convergence of a sequence of bounded degree graphs $(G_n)_n$ can be defined by counting the number of maps $F \to G_n$ from a fixed graph F into G_n or by a weighted count of maps $G_n \to H$ into a fixed weighted graph H. These two notions are referred to as left and right convergence respectively. In their paper [3] Borgs, Chayes, Kahn and Lovász show that left convergence implies right convergence under some conditions on the weights of the graph H.

A sequence is *left convergent* if for all connected graphs F, the sequence

$$\frac{\hom(F, G_n)}{v(G_n)}$$

converges. It will be easier to work with the equivalent notion of convergence defined by Benjamini and Schramm [1]. Let $B_G(v,r)$ denote the subgraph of G on vertices at distance $\leq r$ from root vertex v.

Definition 1 (local/Benjamini–Schramm convergence)**.** *A sequence of bounded degree graphs* $(G_n)_n$ *is* locally convergent *if* $\forall r$, $\forall$ *rooted graphs* U, $\exists \mu(U)$ *such that*

$$\mu(G_n, U) := |\{v \in G_n \ : B_{G_n}(v,r) = U\}|/v(G_n) \to \mu(U).$$

We define right convergence via homomorphism entropy, $\operatorname{ent}(G,H)$. Let $\phi : G \to H$ be a graph homomorphism from graph G to weighted graph H, where a non-edge is indicated by weight zero. Write $\hom(G,H)$ to denote the weighted counts of maps ϕ based on the weightings of image vertices (α_i) and edges (β_{ij}) in H, using $W(\phi)$ as shorthand.

$$\operatorname{ent}(G,H) = \frac{\ln \hom(G,H)}{v(G)}, \qquad \hom(G,H) = \sum_{\phi : V(G) \to V(H)} W(\phi),$$

$$W(\phi) := \prod_{u \in V(G)} \alpha_{\phi(u)} \prod_{uv \in E(G)} \beta_{\phi(u),\phi(v)}.$$

Definition 2 (right convergent)**.** *A sequence of bounded degree graphs* $(G_n)_n$ *is* right convergent *if the homomorphism entropy,* $\operatorname{ent}(G_n, H)$*, converges for weighted graphs* H*.*

Example 1 (cycles)**.** Take the sequence of cycles $\{C_n\}$ where C_n is the cycle on n vertices. Then $\{C_n\}$ is left convergent, as seen by showing local convergence.

$$\mu(C_n, U) \to \begin{cases} 1 & \text{if } U \text{ is a path rooted at central vertex;} \\ 0 & \text{otherwise.} \end{cases}$$

However, the number of graph homomorphisms from a cycle C_n into the graph K_2 depends on the parity of n (K_2 is the graph with two vertices u, v and an edge between them, set $\alpha_u = \alpha_v = \beta_{uv} = 1$).

$$\hom(C_n, K_2) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \Rightarrow \operatorname{ent}(C_n, K_2) = \begin{cases} \frac{1}{n} \ln 2 & n \text{ even} \\ \infty & n \text{ odd} \end{cases}$$

Note $\operatorname{ent}(C_n, K_2)$ does not converge. Thus left-convergence need not imply right-convergence. The result by Borgs et.al. [3] shows this implication holds after imposing conditions on the weights of the image graph H.

Theorem 1 (Left convergence implies right convergence [3])**.** *Let* $(G_n)_n$ *be a left convergent sequence with max degree* $\leq D$*. Let* H *be a graph with positive vertex weights* $\sum_j \alpha_j = 1$*, edge weights* $0 \leq \beta_{ij} \leq 1$ *and for each* i*,* $\sum_{j \in V(H)} \alpha_j (1 - \beta_{ij}) < 1/2D$*. Then* $\operatorname{ent}(G_n, H)$ *converges as* $n \to \infty$ *i.e.* $(G_n)_n$ *is right convergent with respect to* H*.*

Two proofs are given in [3], one using the cavity method which relies on the Dobrushin uniqueness Theorem. To state Dobrushin we need some new terms.

Definition 3. Given a fixed partial graph homomorphism $\alpha : \Lambda \to V(H)$ for some $\Lambda \in V(G)$ we define a random extended map $\phi_\alpha : V(G) \to V(H)$. This is done by conditioning on an underlying distribution of maps $\phi : V(G) \to V(H)$ defined by the weights of H.

$$\mathcal{P}(\tilde{\phi_\alpha} = \tau) := \mathcal{P}_G(\phi = \tau \mid \phi = \alpha \text{ off } \Lambda), \quad \text{where} \quad \mathcal{P}_G(\phi) := W(\phi)/t(G).$$

Also, for each subset $Z \subseteq V(H)$, we define a distribution ν_Z on $V(H)$ which is proportional to the product of the weight of the vertex and the weights of edges incident to it in the subgraph induced by Z.

$$\mathcal{P}(\nu_Z = i) = \frac{\alpha_i \prod_{z \in Z} \beta_{i,z}}{\sum_h \alpha_h \prod_{z \in Z} \beta_{h,z}}, \quad \forall i \in V(H).$$

We are now in a position to state the Dobrushin uniqueness theorem, note $d(v, \Omega')$ is the length of the shortest path from vertex v to any vertex in Ω'. The following formulation of the theorem is from [3].

Theorem 2 (Dobrushin Uniqueness Theorem)**.** Let graph G have maximum degree $\triangle(G) \leq D$ and graph H have non-negative vertex weights α_i, and real edge weights β_{ij}. Suppose there exists $0 < \kappa < 1$ so that for any $s \leq D$ and s-subsets $Z, Z' \subseteq V(H)$ with set difference one $|Z \triangle Z'| = 1$,

$$\sum_{u \in V(H)} |\nu_Z(u) - \nu_{Z'}(u)| \leq 2\kappa/D. \tag{1}$$

Then for any bipartition of the vertex set of G, i.e. $\Lambda \dot{\cup} \Lambda' = V(G)$, and partial maps from G to H, $\alpha, \beta : \Lambda' \to V(H)$, there exists a coupling $\tilde{\phi_\alpha}, \tilde{\phi_\beta}$ of the random extended maps $\phi_\alpha, \phi_\beta : V(G) \to V(H)$ such that for $\Omega \subseteq \Lambda \subseteq V(G)$;

$$\sum_{v \in \Omega} |\tilde{\phi_\alpha}(v) - \tilde{\phi_\beta}(v)| \leq 2 \sum_{v \in \Omega} \kappa^{d(v,\Omega')}.$$

In Proposition 3.4 in [3], the following statement is proved and so the Dobrushin Uniqueness Theorem applies.

Proposition 3. *Conditions on H in Theorem 1 imply* (1) *is satisfied with* $\kappa = 2D \max_{u \in V(H)} \sum_{w \in V(H)} \alpha_u (1 - \beta_{uw})$.

Intuitively, a consequence of the Dobrushin Uniqueness Theorem can be described as follows. If our large graph G is a large cubic lattice, say, then the values (image in $V(H)$) of the vertices on the outer shell (Λ') have little affect on the images of the values of vertices around a small neighbourhood about the centre of the cube (Ω) and that this dependence drops off exponentially. So we are bounding the affect of boundary values on the values inside the *cavity.* The Dobrushin proof of Theorem 1 in [3] is made by application of Dobrushin Uniqueness Theorem as well as analysis that shows certain properties of G are predominantly local, i.e. only dependent on their r-neighbourhoods within small error bounds.

References

[1] I. Benjamini, O. Schramm, *Recurrence of distributional limits of finite planar graphs*, Electronic J. Probab. **6** (2001), no. 23, 1–13.

[2] L. Lovász, *Large networks and graph limits*, AMS, (2012), ISBN-13: 978-0-8218-9085-1.

[3] C. Borgs, J. Chayes, J. Kahn, L. Lovász, *Left and right convergence of graphs with bounded degree*, Random Structures Algorithms, **42** (2013), no. 1, 1–28.

On the growth of L^2-invariants for sequences of lattices in Lie groups

Miklós Abért

In the talk we discussed Benjamini–Schramm (BS) convergence for sequences of finite graphs of bounded degree and finite volume Riemannian manifolds (orbifolds) of bounded geometry, with a special emphasis on locally symmetric spaces.

In both cases, we first turn the object to a random rooted or pointed object (by choosing the root uniformly randomly against the volume). Then BS convergence can be defined as weak convergence of the random rooted objects. So the BS limit of a sequence is a random rooted object. The absolute bound on the degree (or the geometry) ensures that every sequence has a BS convergent subsequence (see [2] and [3]).

A general question is to understand which invariants are continuous in the BS topology. For graphs, examples are the normalized size of a maximal matching, or the normalized log of the number of proper colorings with a large enough number of colors (compared to the degree, see [4]). A negative example is the normalized maximal size of independent subsets.

The general picture is that BS convergence implies spectral convergence. For graphs, by the work of Lyons [6], this implies that the normalized log of the number of spanning trees is BS-continuous. For locally symmetric spaces, we get the following.

Let G be a connected center-free semisimple Lie group without compact factors, $K \leq G$ a maximal compact subgroup and $X = G/K$ the associated Riemannian symmetric space. Let (Γ_n) be a sequence of lattices in G. We say that the X-orbifolds $M_n = \Gamma_n \backslash X$ *BS-converge* to X if for every $R > 0$, the probability that the R-ball centered around a random point in M_n is isometric to the R-ball in X tends to 1 when $n \to \infty$. In other words, if for every $R > 0$, we have

$$\lim_{n \to +\infty} \frac{\mathrm{vol}((M_n)_{<R})}{\mathrm{vol}(M_n)} = 0,$$

where $M_{<R} = \{x \in M \ : \ \mathrm{InjRad}_M(x) < R\}$ is the R-thin part of M.

A straightforward, and well studied, particular case is when $\Gamma \leq G$ is a lattice and $\Gamma_n \leq \Gamma$ is a chain of normal subgroups with trivial intersection; in this case, the R-thin part of $\Gamma_n \backslash X$ is empty for large n.

A family of lattices (resp. the associated X-orbifolds) is *uniformly discrete* if there is an identity neighborhood in G that intersects trivially all of their conjugates. For torsion-free lattices Γ_n, this is equivalent to saying that there is a uniform lower bound for the injectivity radius of the manifolds $M_n = \Gamma_n \backslash X$. In particular, any family (M_n) of covers of a fixed compact orbifold is uniformly discrete. Margulis has conjectured that the family of all cocompact torsion-free arithmetic lattices in G is uniformly discrete.

For an irreducible unitary representation $\pi \in \widehat{G}$ and a uniform lattice Γ in G let $\mathrm{m}(\pi, \Gamma)$ be the multiplicity of π in the right regular representation $L^2(\Gamma\backslash G)$. Define the relative Plancherel measure of $\Gamma\backslash G$ as the measure

$$\nu_\Gamma = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \sum_{\pi \in \widehat{G}} \mathrm{m}(\pi, \Gamma)\delta_\pi$$

on $\widehat{G}$. Finally denote by ν^G the Plancherel measure of the right regular representation $L^2(G)$.

In a joint work with Bergeron, Biringer, Gelander, Nikolov, Raimbault and Samet [1] we show the following.

Theorem 1. *Let (Γ_n) be a uniformly discrete sequence of lattices in G such that the spaces $\Gamma_n\backslash X$ BS-converge to X. Then for every relatively quasi-compact ν^G-regular subset $S \subset \widehat{G}$, we have:*

$$\nu_{\Gamma_n}(S) \to \nu^G(S).$$

The next theorem shows that there is also pointwise convergence. Let $d(\pi)$ be the multiplicity of π in the regular representation $L^2(G)$ with respect to the Plancherel measure of G. Thus, $d(\pi) = 0$ unless π is a discrete series representation.

Theorem 2. *Let (Γ_n) be a uniformly discrete sequence of lattices in G such that the spaces $\Gamma_n\backslash X$ BS-converge to X. Then for all $\pi \in \widehat{G}$, we have*

$$\frac{m(\pi, \Gamma_n)}{\mathrm{vol}(\Gamma_n\backslash G)} \to d(\pi).$$

In the special situation when (Γ_n) is a chain of normal subgroups with trivial intersection in some fixed cocompact lattice $\Gamma \leq G$, this is the classical theorem of DeGeorge and Wallach. In that very same situation Theorem 1 is due to Delorme [5].

References

[1] M. Abért, N. Bergeron, I. Biringer, T. Gelander, N. Nikolov, J. Raimbault, I. Samet, *On the growth of Betti numbers of locally symmetric spaces*, Comptes Rendus MATH **349** (2011), 831–835.

[2] D. Aldous, R. Lyons, *Processes on unimodular random networks*, Electr. J. Probab. **12** (2007), no. 54, 1454–1508.

[3] I. Benjamini, O. Schramm, *Recurrence of distributional limits of finite planar graphs*, Electron. J. Probab. **6** (2001), no. 23, 1–13.

[4] C. Borgs, J. Chayes, J. Kahn, L. Lovász, *Left and right convergence of graphs with bounded degree*, preprint, arXiv:1002.0115.

[5] P. Delorme, *Formules limites et formules asymptotiques pour les multiplicit es dans $L^2(G)$*, Duke Math. J. **53** (1986), no. 3, 691–731.

[6] R. Lyons, *Asymptotic enumeration of spanning trees*, Combinatorics, Probability and Computing **14** (2005), 491–522.

Root measures of graph polynomials and Benjamini–Schramm convergence

László Miklós Lovász

In this talk, we examine what happens to the uniform measure on the roots of a graph polynomial in a Benjamini–Schramm convergent graph sequence.

A *graph polynomial* is a function $f(G,z)$, where G is a graph, z is a (complex) variable, and for a fixed G, the function $z \to f(G,z)$ is a polynomial, we will denote this as f_G. A classic example is the chromatic polynomial ch_G. For a positive integer q, define $\mathrm{ch}_G(q)$ to be the number of proper colorings of the vertices of G with colors from the set $[q]$. It can be shown that this is a graph polynomial. Another example is the (modified) matching polynomial: let $m_k(G)$ be the number of matchings of size k, and let

$$M(G,x) = x^n - m_1(G)x^{n-1} + m_2(G)x^{n-2} - \dots$$

There are a few properties we can define for these polynomials. First, we want it to only depend on the isomorphism class of G. We call a graph polynomial f *monic* if f_G has degree $|V(G)|$, and the leading coefficient is 1. A graph polynomial is said to be of *exponential type* if the following polynomial equation holds:

$$f(G,x+y) = \sum_{S\subset V(G)} f(G[S],x)f(G[V(G)-S],y).$$

It is easy to see that this holds for the chromatic polynomial, and it also holds for the matching polynomial. A graph polynomial is said to be multiplicative if for any G_1, G_2:

$$f(G_1 \uplus G_2, x) = f(G_1,x)f(G_2,x).$$

Again, it is not hard to see that both the chromatic polynomial and the matching polynomial are of exponential type.

Given a (complex) graph polynomial, we can look at the uniform distribution on its roots (with multiplicities), call this $\mu_{(f,G)}$, and what happens with a Benjamini–Schramm convergent graph sequence.

The best equivalent definition of Benjamini–Schramm convergence to use here is the following: a sequence $G_1, G_2, \dots, G_n, \dots$ is convergent if and only if for any connected graph F, the sequence $\frac{\mathrm{inj}(F,G_n)}{|V(G_n)|}$ converges. We can divide by the number of automorphisms of F to define the function $F(G)$, the number of subgraphs of G isomorphic to F.

The question arises whether given a Benjamini–Schramm convergent graph sequence $\{G_n\}$, and a graph polynomial f, does $\mu_{(f,G_n)}$ converge? This is not the case, for example, in the case of the chromatic polynomial: as shown in [1]. If we look at the uniform distribution on the chromatic polynomial of P_n, and the chromatic polynomial of C_n (the path and the cycle on n vertices), they converge to a different distribution, but the sequence $P_1, C_1, P_2, C_2, P_3, C_3, \ldots$ is Benjamini–Schramm convergent.

However, in [1], Abért and Hubai prove a weaker result: for the chromatic polynomial, the root moments converge, that is, $\int z^k d\mu_{\mathrm{ch},G_n}$ is convergent for each positive integer k. This also implies that if there is a compact set K containing all the roots of all f_{G_n}, then for any holomorphic function g, the sequence $\int g d\mu_{\mathrm{ch},G_n}$ is convergent. The way they prove this is by showing with some calculation that given a positive integer k, sum of the k-th powers of the roots of ch_G can be expressed as a finite sum of the form $\sum_{F \in H_k} F(G)$, where H_k is a finite set of graphs depending on k, and contains only connected graphs. Since the k-th moment is the sum of the k-th powers divided by $|V(G)|$, this shows that Benjamini–Schramm convergence implies that the k-th moments converge.

In [2], Csikvári and Frenkel generalize this to all monic, multiplicative graph polynomials of exponential type. First, they classify all monic graph polynomials of exponential type, and use this to show that for a fix l, the coefficient of x^{n-l} can be expressed as a linear combination of functions $F(G)$, for a finite set of graphs F. Using the fact that the product of two such functions can also be expressed as a linear combination of such functions, this implies that for any k, the sum of the k-th powers of the roots can be expressed as a linear combination of functions $F(G)$. Now, if the polynomial is multiplicative, then the sum of the k-th powers is clearly *additive*: given a disjoint union $G = G_1 \uplus G_2$, the sum of the k-th powers of the roots of f_G is the sum of the two sums of k-th powers on G_1, G_2. The last step in the proof is to show that if we have a linear combination of functions of the form $F(G)$, then it is additive if and only if each F that has a nonzero coefficient is connected. All these together imply that for a monic graph polynomial of exponential type, if it is multiplicative, then the root moments converge.

References

[1] M. Abért, T. Hubai, *Benjamini–Schramm convergence and the distribution of chromatic roots for sparse graphs*, preprint, arxiv:1201.3861.

[2] P. Csikvári, P. Frenkel, *Benjamini–Schramm continuity of root moments of graph polynomials*, preprint, arXiv:1204.0463.

Ultrafilters and hypergraphs

Nathan Bowler

There are a number of different versions of the hypergraph regularity lemma (Rödl–Skokan [4], Rödl–Schacht [3], Gowers [2], Tao [6]). The idea being captured by these lemmas is that large k-uniform hypergraphs can be approximated to any resolution ϵ by boundedly many random-looking parts. In each case, there is a corresponding counting lemma, saying roughly what the homomorphism density from a given hypergraph H to a hypergraph built in this way from random-looking parts will be. (Recall that a function from the vertex set of K to the vertex set of another k-uniform hypergraph H is a *homomorphism* if and only if the image of any edge of H is an edge of K, and that the *homomorphism density* $t(K, H)$ from K to H is the proportion of functions $V(K) \to V(H)$ which are homomorphisms.)

As you might expect, the applications of hypergraph regularity lemmas are similar to the applications of graph regularity lemmas, but for hypergraphs. Thus, for example, the framework sketched below has applications in property-testing for hypergraphs. We will state a version of the hypergraph regularity lemma at the end after giving some motivation, but since the statement is technical we will limit ourselves for now to stating a consequence of this Lemma which is sufficient for many of the applications. The hypergraph removal lemma states that for any $\epsilon > 0$ and any k-regular hypergraph K there is some δ such that for any k-regular hypergraph H with $t(K, H) < \delta$ there is a subhypergraph I of H such that the number of edges in $H \setminus I$ is at most $\epsilon\binom{|V(H)|}{k}$ but $t(K, I) = 0$. Using an argument of Solymosi [5], we derive the multidimensional Szemerédi theorem from the hypergraph removal lemma.

We can get a helpful perspective on the hypergraph regularity lemma by considering measures on ultralimits, as explained by Elek and Szegedy [1]. Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For any formula ϕ we define $(\forall_{\mathcal{U}} n)\phi$ to mean $\{n \in \mathbb{N} | \phi\} \in \mathcal{U}$. For any sequence $(\delta_i \in [0,1] | i \in \mathbb{N})$, we define the ultralimit $\lim_{\mathcal{U}} \delta_n$ to be $\sup\{\delta \in [0,1] | (\forall_{\mathcal{U}} n)\delta \leq \delta_n\}$. For any sequence $(X_n | n \in \mathbb{N})$ of nonempty sets, we define the relation $\sim$ on $\prod_{n\in\mathbb{N}} X_n$ by $x \sim y$ if and only if $(\forall_{\mathcal{U}} n) x_n = y_n$, and we define the *ultraproduct* $\mathbf{X}$ to be the quotient of $\prod_{n\in\mathbb{N}} X_n$ by $\sim$.

If each X_i is finite, and we have subsets $Y_i \subseteq X_i$, then we can identify $\mathbf{Y}$ with a subset of $\mathbf{X}$. Although the subsets arising in this way do not form a σ-algebra, we can extend them to a σ-algebra $\sigma_{\mathbf{X}}$ with a measure $\mu_{\mathbf{X}}$ on it such that for any such $\mathbf{Y}$ we have $\mu_{\mathbf{X}}(\mathbf{Y}) = \lim_{\mathcal{U}} \frac{|Y_n|}{|X_n|}$ and such that for any set N of measure 0 there are such $\mathbf{Y}$ of arbitrarily small measure with $N \subseteq \mathbf{Y}$. We use $\mu_{\mathbf{X}^k}$ to denote the measure obtained by considering $\mathbf{X}^k$ as the ultraproduct of the X_i^k rather than as the kth power of $\mathbf{X}$.

The key fact we need is that separable parts of these measures can be simulated using the standard measure on powers of $[0,1]$. More precisely, let $\mathcal{A}$ be any separable sub-σ-algebra of $\sigma_{\mathbf{X}^k}$. Then there is a measure-preserving function $f : \mathbf{X}^k \to [0,1]^{\mathcal{P}^*[k]}$, where $\mathcal{P}^*[k]$ is the set of nonempty subsets of $[k]$, such that

- for any $H \in \mathcal{A}$ there is a Borel subset W of $[0,1]$ with $H \triangle f^{-1}W$ of measure 0;
- for A a nonempty subset of k, the A^{th} component f_A of f only depends on the restriction of the input to A;
- for any measurable subset W of $[0,1]$, the set $f_A^{-1}(W)$ is independent of the σ-algebra σ_A^* generated by the σ-algebras $\sigma_{\mathbf{X}^B}$ with B a nonempty proper subset of A;
- f is equivariant with respect to the natural actions of the symmetric group S_k on $\mathbf{X}^k$ and $[0,1]^{\mathcal{P}^*[k]}$.

The reason for introducing additional coordinates for subsets of $[k]$ of size at least two is that the σ-algebra σ_A^* is always a proper subalgebra of $\sigma_{\mathbf{X}^A}$. Thus, for example, if $k = 2$, (X_n) is a sequence of sets whose sizes tend to infinity, and a sequence $(Y_i \subseteq X_i^2)$ is obtained by picking elements at random with probability $\frac{1}{2}$, then $\mathbf{Y}$ almost surely has measure $\frac{1}{2}$ and is independent of $\sigma^*_{[2]}$.

The fact mentioned above gives a method for proving lemmas like the regularity lemma. Take a sequence (H_n) of counterexamples, where we consider H_n as a subset of the kth power of its vertex set X_n, and derive a contradiction by applying standard measure-theoretic results to a set $W \subseteq [0,1]^{\mathcal{P}^*[k]}$ simulating $\mathbf{H} \subseteq \mathbf{X}^k$.

For example, the removal lemma can be proved in this way, using the Lebesgue density theorem, which entails that for any measurable subset W of $[0,1]^p$ the set $\{w \in W | \lim_{\epsilon\to 0} \lambda(B_\epsilon(w) \cap W)/\lambda(B_\epsilon(w)) = 1\}$ of density points of W has the same measure as W itself.

To obtain a regularity lemma with such an argument, we use instead the fact that any measurable subset of $[0,1]^p$ can be approximated arbitrarily closely by unions of l-blocks (an l-block is a product of intervals of the form $[\frac{i}{l}, \frac{i+1}{l}]$). Consideration of what the inverses of l-blocks under a function f of the kind introduced above would look like suggests the following regularity lemma:

Definition 1. *If $B_1, \ldots, B_r$ are $(r-1)$-uniform hypergraphs on X, their* r-cylinder intersection *is the r-uniform hypergraph consisting of those sets $e = \{x_1, \ldots, x_r\}$ such that for each i we have $e - x_i \in B_i$.*

Lemma 1 (Hypergraph regularity)**.** *Let $k > 0$, $\epsilon > 0$, and $F\colon \mathbb{N} \to (0,1)$. Then there are c and N_0 such that for any k-uniform hypergraph H on a set X of size at least N_0 there are $l \le c$, partitions $K_X^{(r)} = \dot{\bigcup} P_r^i$ for each $0 < r \le k$ and a k-uniform hypergraph T on X such that*

- $|H \triangle T| < \epsilon |K_X^{(k)}|$;
- *T is a union of equivalence classes for the relation $\sim$, where $\{x_1 \ldots x_k\} \sim \{y_1 \ldots y_k\}$ if $\{x_i | i \in A\}$ is in the same class of the partition of $X^{|A|}$ as $\{y_i | i \in A\}$ for every nonempty subset A of $[k]$;*
- *for any i and r and any r-cylinder intersection L with $|L| \ge F(l)|K_X^{(r)}|$,*

$$\left| \frac{|P_r^i \cap L|}{|L|} - \frac{1}{l} \right| < F(l).$$

References

[1] G. Elek, B. Szegedy, *A measure-theoretic approach to the theory of dense hypergraphs*, Advances in Mathematics **231** (2012), no. 3-4, 1731–1772

[2] T. Gowers, *Quasirandomness, counting and regularity for 3-uniform hypergraphs*, Combin. Probab. Comput. **15** (2006), no. 1-2, 143–184.

[3] V. Rödl, M. Schacht, *Regular partitions of hypergraphs: regularity lemmas*, Combin. Probab. Comput. **16** (2007), no. 6, 833–885.

[4] V. Rödl, J. Skokan, *Regularity lemma for k-uniform hypergraphs*, Random Structures Algorithms **25** (2004), no. 1, 1–42.

[5] J. Solymosi, *A note on a question of Erdős and Graham*, Combin. Probab. Comput. **13** (2004), no. 2, 263–267.

[6] T. Tao, *A variant of the hypergraph removal lemma*, J. Combin. Theory Ser. A **113** (2006), no. 7, 1257–1280.

Topological aspects of dense graph limits

Cameron Freer

Although graphons are defined as measurable functions, it turns out that they have natural topological aspects as well. In this talk we present the basic concepts and results of the topological aspects of graphons, as developed by Lovász and Szegedy [1].

Let $J = (\Omega, \mathcal{A}, \pi)$ be a probability space. Given a graphon $W : J \times J \to [0, 1]$, we endow the kernel (J, W) with the distance function

$$r_W(x, y) = ||W(x, \cdot) - W(y, \cdot)||_1.$$

We say that (J, W) is *pure* when (J, r_W) is a complete separable metric space and the probability measure has full support. In this case, we may consider the topology of this metric space, and examine its properties.

This notion of the topology of a graphon leads to many interesting and useful developments. It simplifies some proofs, as the purified kernel is sometimes better behaved. In many cases, naturally defined graphons have topologies other than $[0, 1]$, and are more intuitively considered as kernels on their corresponding topological space. Furthermore there is a notion of dimension, which corresponds closely with the complexity of Szemerédi partitions.

Finally, the topology of graphons allows us to describe a surprising situation with *finitely forcible* graphons [2], which so far always seem to be compact and finite-dimensional.

References

[1] L. Lovász, B. Szegedy, *Regularity partitions and the topology of graphons*, in: An irregular mind, Bolyai Soc. Math. Stud. **21** János Bolyai Math. Soc. Budapest (2010), 415–446.

[2] L. Lovász, B. Szegedy, *Finitely forcible graphons*, J. Combin. Theory Ser. B **101** (2011), no. 5, 269–301.

The combinatorial cost

Gábor Kun

We study a combinatorial analogue of the cost. We calculate the cost of hyperfinite and large girth graph sequences. We introduce the finite L^2 Betti number and the rank gradient of residually finite groups (with a given Farber chain of normal subgroups). We show that Betti number $\leq$ combinatorial cost $\leq$ rank gradient. Based on the paper of Gábor Elek [2].

We will work with the graph sequences $\mathcal{G} = \{G_n\}_{n=1}^{\infty}, \mathcal{H} = \{H_n\}_{n=1}^{\infty}$, where $V(G_n) = V(H_n)$ for every n. We will assume that these graphs are connected and have uniformly bounded degree. We will define an equivalence relation on graph sequences that is the analogue of bi-Lipschitz equivalence.

Definition 1. *We say that the graph sequences* $\{G_n\}_{n=1}^{\infty}$ *and* $\{H_n\}_{n=1}^{\infty}$ *are equivalent if* $V(G_n) = V(H_n)$ *and there is an* L *such that for every* n *and* $(x, y) \in E(G_n)$ *the inequality* $\mathrm{dist}_{H_n}(x, y) \leq L$ *holds, and for every* n *and* $(x, y) \in E(H_n)$ *the inequality* $\mathrm{dist}_{G_n}(x, y) \leq L$ *holds.*

Definition 2. *Given a graph sequence* $\mathcal{G}$ *set* $e(\mathcal{G}) = \liminf_{n\to\infty} \frac{|E(G_n)|}{|V(G_n)|}$. *The cost of* $\mathcal{G}$ *is* $c(\mathcal{G}) = \inf\{e(\mathcal{H}) : \mathcal{H}$ *is equivalent to* $\mathcal{G}\}$. *We say that* $\mathcal{G}$ *has large girth if the length of the shortest cycle in* G_n *goes to infinity (as* $n \to \infty$*).*

The cost is not less mysterious in the finite case than for measurable group actions. First we show that a large girth graph sequence realizes its cost. This is the finite analogue of the theorem about the cost of a free group [3].

Theorem 1. *If* $\mathcal{G}$ *has large girth then* $c(\mathcal{G}) = e(\mathcal{G})$.

We introduce finite β-invariants: this will turn out to be handier and gives a lower bound on the cost. Let K be an arbitrary field and G a connected graph. Let $C_K(G)$ denote the cycle space of G: a subspace of $K^{E(G)}$ generated by cycles. (We use the convention $(x, y) = -(y, x)$ for every edge $(x, y) \in E(G)$.) Let $C_K^q(G)$ denote the subspace generated by cycles shorter than q. Set $s_K^q(\mathcal{G}) = \liminf_{n\to\infty} \frac{|E(G_n)| - \dim_K C_K^q(G_n) - |V(G_n)|}{|V(G_n)|}$. Finally, we can define the beta-invariant:

Definition 3. $\beta_K(\mathcal{G}) = \inf_q s_K^q(\mathcal{G})$.

Note that $\beta_K(\mathcal{G}) = \beta_L(\mathcal{G})$ if $\mathrm{char}(K) = \mathrm{char}(L)$.

Remark 2. $\beta_{\mathbf{Q}}(\mathcal{G}) \geq \beta_{\mathbf{F}_p}(\mathcal{G})$.

Question 1: Is this inequality strict for any graph sequence $\beta_K(\mathcal{G})$?

Theorem 3. *If* $\mathcal{G}$ *and* $\mathcal{H}$ *are equivalent then* $\beta_K(\mathcal{G}) = \beta_K(\mathcal{G})$

Corollary 4. $\beta_K(\mathcal{G}) + 1 \leq c(\mathcal{G})$.

Question 2: Is this inequality strict for any graph sequence $\beta_K(\mathcal{G})$?

Theorem 1 follows from Theorem 3, since $C_K^q(G) = \{0\}$ if the girth is larger than q.

Given a Farber chain $\{\Gamma_n\}_{n=1}^{\infty}$ of normal subgroups of Γ of finite index consider the sequence $\mathcal{G}$ of Cayley graphs: $G_n = \Gamma/\Gamma_n$.

Theorem 5. [1, 2] $\mathrm{rkgrad}\{\Gamma, \{\Gamma_n\}_{n=1}^{\infty}\} = c(\mathcal{G}) - 1$.

Question 3: Does the rank gradient depend on the Farber chain or on Γ only?

Theorem 6. *The finite beta invariant of the graph sequence equals to the* L^2 *Betti number of* Γ*:* $\beta_{\mathcal{Q}}(\mathcal{G}) = \beta^1_{(2)}(\Gamma)$.

We know that the cost of amenable groups is 1, see e.g. [3]. Now we prove this for the finite graph theoretical analogue, hyperfinite sequences.

Definition 4. *We say that* $\mathcal{G}$ *is hyperfinite if for every* $\varepsilon > 0$ *there is a* K *such that every* G_n *has components of size* $< K$ *after the removal of* $< \varepsilon|V(G_n)|$ *edges.*

Planar graphs are hyperfinite. Hyperfinite graphs play an important role in the theory of property testing.

Theorem 7. *Hyperfinite graph sequences have cost* 1*. A residually finite group is amenable if and only if the sequence* $\mathcal{G}$ *is hyperfinite.*

References

[1] M. Abért, N. Nikolov, *Rank gradient, cost of groups and the rank vs Heegaard genus conjecture*, preprint, arXiv:math/0701361 [math.GR].

[2] G. Elek, *The combinatorial cost*, l'Enseignement Math. **53** (2007), 225–236.

[3] D. Gaboriau, *Cout des relations d'equivalence et des groupes* (in French), [Cost of equivalence relations and of groups], Inventiones Mathematicae **139** (2000), no. 1, 41–98.

Large deviations and the exponential graph model

Júlia Komjáthy

In the paper [2], Chatterjee and Varadhan develop the large deviation principle (LDP) for the Erdős–Rényi random graph on the space of graphons with the cut metric. Here we summarize their results. Let $G(n,p)$ be the random graph on n vertices where each edge is present with probability p independently. Let us introduce the notation $\mathcal{W}$ for the space of measurable symmetric functions from $[0,1]^2$ to $[0,1]$. A graph $G=(V,E)$ on n vertices can be embedded in this space with the natural map, i.e.

$$f^G(x,y) := \begin{cases} 1 & \text{if } (\lceil nx\rceil, \lceil ny\rceil) \in E(G); \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We define the cut distance on the space of $\mathcal{W}$ by

$$d_\square(f,g) := \sup_{S,T\subset\mathcal{W}} \left\{ \int_{S\times T} |f(x,y)-g(x,y)|\mathrm{d}x\mathrm{d}y \right\}.$$

However, this distance is not stable under the relabeling of vertices, so let us introduce the equivalence relation by $f(x,y)\sim g(x,y)$ if there exists a measure preserving bijection $\sigma:[0,1]\to[0,1]$ such that $f(x,y)=g_\sigma(x,y):=g(\sigma x,\sigma y)$. We denote the space of measure preserving bijections from $[0,1]\to[0,1]$ by Σ. We write $\widetilde{f}$ for the closure of the orbit $\{f_\sigma\}$ in $(\mathcal{W},d_\square)$. We denote the quotient space by $\widetilde{\mathcal{W}} := \mathcal{W}_{/\sim}$ and since $d_\square$ is invariant under σ, we can define on $\widetilde{\mathcal{W}}$ the metric

$$\delta_\square(f,g) := \inf_\sigma d_\square(f,g_\sigma) = \inf_\sigma d_\square(f_\sigma,g) = \inf_{\sigma_1,\sigma_2} d_\square(f_{\sigma_1},g_{\sigma_2}).$$

A very important theorem is the following:

Theorem 1 ([1, Theorem 5.1])**.** *The space* $(\widetilde{\mathcal{W}},\delta_\square)$ *is compact.*

The main result of the paper is as follows. The random graph $G(n,p)$ induces the probability distributions $\mathbb{P}_{n,p}$ on the space $(\mathcal{W},d_\square)$ and $\widetilde{\mathbb{P}}_{n,p}$ on $(\widetilde{\mathcal{W}},\delta_\square)$ by the maps $G\to f^G$ and $G\to f^G\to\widetilde{f}^G$. The space $\mathcal{W}$ is compact in the weak topology and the large deviation principle for $\mathbb{P}_{n,p}$ on $\mathcal{W}$ is as follows:

Theorem 2 ([2, Theorem 2.2]). *The sequence $\mathbb{P}_{n,p}$ on $\mathcal{W}$ satisfies a large deviation principle in the weak topology. That is, for every weakly closed set $F \subset \mathcal{W}$ and weakly open set $U \in \mathcal{W}$*

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_{n,p}(F) \leq - \inf_{f\in F} I_p(f),$$
$$\liminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_{n,p}(U) \geq - \inf_{f\in U} I_p(f),$$

where the rate function $I_p(f)$ is well defined and lower semicontinuous on $\mathcal{W}$ and given by

$$I_p(f) = \frac{1}{2}\int_0^1 \int_0^1 \left| f(x,y)\log\frac{f(x,y)}{p} + (1-f(x,y))\log\frac{1-f(x,y)}{1-p}\right| \mathrm{d}x\mathrm{d}y. \tag{2}$$

Proof. The proof follows from applying the abstract Gärtner–Ellis theorem on the Bernoulli random variables arising through the map in (1) and seeing that the log-moment generating function properly scaled converges. □

However, the LDP on $\mathcal{W}$ is not so strong since it is not stable under re-labeling the vertices. The stronger result is

Theorem 3 ([2, Theorem 2.3]). *The sequence $\widetilde{\mathbb{P}}_{n,p}$ on $(\widetilde{\mathcal{W}}, \delta_\Box)$ satisfies a large deviation principle with rate function defined in (2). That is, for every weakly closed set $\widetilde{F} \subset \widetilde{\mathcal{W}}$ and weakly open set $\widetilde{U} \in \widetilde{\mathcal{W}}$*

$$\limsup_{n\to\infty} \frac{1}{n^2} \log \widetilde{\mathbb{P}}_{n,p}(\widetilde{F}) \leq - \inf_{\widetilde{f}\in \widetilde{F}} I_p(\widetilde{f}),$$
$$\liminf_{n\to\infty} \frac{1}{n^2} \log \mathbb{P}_{n,p}(\widetilde{U}) \geq - \inf_{\widetilde{f}\in \widetilde{U}} I_p(\widetilde{f}),$$

where the rate function $I_p(f)$ is well defined and lower semicontinuous on $(\widetilde{\mathcal{W}}, \delta_\Box)$.

The proof of this theorem relies on two things: the compactness of the space and Szemerédi's regularity lemma. For the lower bound it is enough to show that for any $\widetilde{f} \in \widetilde{\mathcal{W}}$ and for any $\varepsilon > 0$

$$\liminf_{n\to\infty} \frac{1}{n^2} \log \widetilde{\mathbb{P}}_{n,p}(B_{\delta_\Box}(\widetilde{f},\varepsilon)) \geq -I_p(\widetilde{f}),$$

with $B_{\delta_\Box}(\widetilde{f},\varepsilon) = \{\widetilde{g} : \delta_\Box(\widetilde{g},\widetilde{f}) < \varepsilon\}$. The sketch of the proof goes by a tilting argument: one generates a sequence of stepfunctions f_n approximating $\widetilde{f}$ on an $\frac{1}{n}\times\frac{1}{n}$ grid of $[0,1]$ and shows that the corresponding inhomogeneous graph $G(n,f_n)$ sequence converges to the graphon $\widetilde{f}$ in the $d_\Box$ distance. Then, the rest is an entropy-cost tilting argument by taking the Radon–Nikodym derivative of the measures $\mathbb{P}_{n,p}$ and $\mathbb{P}_{n,\widetilde{f}_n}$. For the upper bound one needs to show that for any $\widetilde{f} \in \widetilde{\mathcal{W}}$

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n^2} \log \widetilde{\mathbb{P}}_{n,p}(B_{\delta_\Box}(\widetilde{f},\varepsilon)) \geq -I_p(\widetilde{f}).$$

To do so one can make use of the Regularity Lemma: every ball $B_{\delta_\square}(\widetilde{f},\varepsilon)$ meets a Szemerédi-regular $g \in \mathcal{W}$, i.e. a step function with at most $M \times M$ different possible values, on a grid of $\frac{1}{M} \times \frac{1}{M}$. Then, the LDP for $\widetilde{f}$ follows from the LDP of g on $\mathcal{W}$ established by Theorem 2 and the fact that we only loose a factor of $n!$ by the re-labelings of a given graph, which clearly vanishes in the LDP since $\frac{1}{n^2}\log(n!) \to 0$.

The other main result is about conditional distributions. Let us introduce the notation

$$\widetilde{F}^* := \{\widetilde{f} \in \widetilde{F} : I_p(\widetilde{f}) = \inf_{\widetilde{g}\in\widetilde{F}} I_p(\widetilde{g})\}.$$

Since $\widetilde{\mathcal{W}}$ is compact and I_p is lower semicontinuous, the set of minimizers $\widetilde{F}^* \neq \emptyset$. The following theorem says that conditioned on that some rare event happens, the graph with high probability looks like on of the minimizers in the set defined by the rare event in $\widetilde{\mathcal{W}}$.

Theorem 4. *Let $p \in (0,1)$ fixed, $\widetilde{F} \subset \widetilde{\mathcal{W}}$ satisfying*

$$\inf_{\widetilde{f}\in\int(\widetilde{F})} I_p(\widetilde{f}) = \inf_{\widetilde{f}\in\widetilde{F}} I_p(\widetilde{f}) > 0.$$

Then for each $n, \varepsilon > 0$ there exists a $C(\varepsilon,\widetilde{F})$ such that

$$\mathbb{P}\left(\delta_\square(\widetilde{f}^{G(n,p)},\widetilde{F}^*) > \varepsilon \;\middle|\; \widetilde{f}^{G(n,p)} \in \widetilde{F}\right) \leq \exp\{-C(\varepsilon,\widetilde{F})n^2\}.$$

The proof follows from the lower-semicontinuity of I_p: the minimizer set $\widetilde{F}^*$ is compact, and the infinum value of I_p on $\widetilde{F} \setminus B_{\delta_\square}(\widetilde{F}^*,\varepsilon)$ must be larger than the value of I_p on F^*.

The rest of the paper is devoted to determine the rate function for upper tail for $T_{n,p}$, the number of triangles in $G(n,p)$: An immediate consequence of Theorem 3 is that the rate function defined as

$$\phi(p,t) := -\lim_{n\to\infty}\frac{1}{n^2}\log\mathbb{P}\left(T_{n,p} \geq tn^3\right)$$

satisfies

$$\phi(p,t) = \inf\{I_p(f) : f \in \mathcal{W}, \frac{1}{6}\int f(x,y)f(y,z)f(z,x)\mathrm{d}x\mathrm{d}y\mathrm{d}z \geq t\}. \tag{3}$$

The authors further show that ϕ is continuous, increasing in $(p^3/6, 1/6)$ and determine a sufficient criterion when the constant function $c_t := (6t)^{1/3}$ gives the minimizer for (3). Further, there is a p_0 such that for all $p < p_0$ there is a region (t,t') with $p^3/6 < t < t' < 1/6$ where the constant function c_t is not the minimizer of (3).

References

[1] L. Lovász, B. Szegedy, *Szemerédi's Lemma for the analyst*, Geom. Funct. Anal **17** (2007), 252-270.

[2] S. Sourav, S.R.S. Varadhan, *The large deviation principle for the Erdős–Rényi random graph*, European Journal of Combinatorics **32** (2011), 1000–1017.

Higher order Fourier analysis

Gábor Elek

The talk is based on the first (easier!) part of a paper of Balázs Szegedy [1]. Our objects of study are pairs (A, f), where A is a compact Abelian group and f is a complex Borel function on A with $\|f\|_\infty \leq 1$. First, we define a notion of convergence on these objects. For a positive natural n, we sample out a symmetric matrix from A the following way. Choose n elements $x_1, x_2, \ldots, x_n$ of A uniformly randomly (with respect to the Haar probability measure) and consider the $n \times n$ matrix

$$M_{i,j} = f(x_i, x_j)\,.$$

The sampling procedure gives rise to a random matrix of coefficents bounded by one in absolute value. That is, we obtain a probability measure $\mu_{A,f,n}$ on the compact space of symmetric complex $n \times n$ matrices of entries bounded by one. We say that a sequence of objects $\{(A_k, f_k)\}$ converges, if for any fixed n, the sequence $\mu_{A_k,f_k,n}$ converges weakly. Note that any element (A_k, f_k) represents a graphon [3]. Hence, we know that the limit object can be represented by a graphon as well. However, the point is, that the limit object can be represented on an Abelian group. Note that even if all the groups A_k are isomorphic to S^1, it is possible that the limit group is a higher dimensional torus.

The construction of the limit group uses the ultraproduct technique developed in [2]. Let us consider the ultraproduct of our Abelian groups $\{A_k\} = \mathbf{A}$. This is, in general, an enormous Abelian group with a σ-algebra $\mathcal{A}$ and a probability measure μ_A. Szegedy observes the following fact.

Fact 1: The ultralimits of the characters of A_k are measurable functions with respect to $\mathcal{A}$.

Then he considers the smallest σ-algebra $\mathcal{B}_1$ such that all the limit characters are measurable.

Fact 2: If $\mathbf{B} \in \mathcal{B}_1$ then the translates of $\mathbf{B}$ generate a separable σ-algebra. Vice versa, this property characterizes the elements of $\mathcal{B}$.

Now let us consider the ultralimit $\mathbf{f}$ of the functions f_k. This function is, in general, not $\mathcal{B}_1$-measurable. For an example, it is possible that $\mathbf{f}$ is orthogonal to all the limit characters. This is the case, if the Fourier coefficients of f_k tend uniformly to zero. Nevertheless, there is always at most countably many limit characters $\mathbf{f}$ is not orthogonal to. These characters generate a countable Abelian group T. Its dual group $\hat{T}$ will be the limit of the groups A_k. The limit function $\hat{f}$ is the measure valued Radon–Nikodym derivative of $\mathbf{f}$ with respect to the separable σ-algebra $\mathcal{C}$, where $\mathcal{C}$ is the algebra generated by the countably many characters above. In the case when all the f_k's are indicator functions, the Radon–Nikodym derivative is a scalar function and the random matrices $\mu_{\hat{T},\hat{f},\mu_A}$ associated to the pair $(\hat{T}, \hat{f})$ are indeed the weak limits of the measures $\mu_{A_k,f_k,n}$.

References

[1] B. Szegedy, *On higher order Fourier analysis*, preprint, arXiv:1203.2260.

[2] G. Elek, B. Szegedy, *A measure-theoretic approach to the theory of dense hypergraphs*, Adv. Math. **231** (2012), no. 3–4, 1731–1772.
[3] L. Lovász, B. Szegedy, *Limits of dense graph sequences*, J. Combin. Theory Ser. B **96** (2006), no. 6, 933–957.

Flag algebras

Benjamin Matschke

Asymptotic extremal combinatorics studies densities of small combinatorial structures in large ones of the same type. This talk surveys Razborov's so called flag algebras [2], which formalize common proof and calculation methods in that area.

For simplicity we restrict to the category of simple undirected graphs (and some subcategories). Razborov treats more generally any finite model theory.

1. Definitions

Definition 1. *A* **type** *σ is a graph with labeled vertices $1,\dots,|\sigma|$. A* **flag** *F over σ is a pair of graphs (G,σ), σ being an induced subgraph of G (G is unlabeled). We write $|F| := |V(G)|$. A* **morphism** *between two flags $F=(G,\sigma)$ and $F'=(G',\sigma)$ is an injective graph homomorphism $m : G \to G'$ that is the identity on σ. This also clarifies what we mean by isomorphisms. We define the sum $F \cup_\sigma F'$ as $(G \cup_\sigma G', \sigma)$. A* **sunflower** *over σ is a sum of σ-flags $F_1 \cup_\sigma \dots \cup_\sigma F_n$; here $F_1,\dots,F_n$ are called* **petals**. *Let $\mathcal{F}^\sigma_\ell$ be the set of flags with $|F|=\ell$, and $\mathcal{F}^\sigma := \bigcup_\ell \mathcal{F}^\sigma_\ell$. For $F_1,\dots,F_n,F \in \mathcal{F}^\sigma$, define*

$$p(F_1,\dots,F_n;F)$$

as the probability that a uniformly randomly chosen injective map $V(F_1 \cup_\sigma \dots \cup_\sigma F_n) \to V(F)$ extending id_σ yields an induced subgraph of F whose restriction to $V(F_i)$ is isomorphic to F_i, for all i.

Lemma 1 (Chain rule). *If $|F_1 \cup_\sigma \dots \cup_\sigma F_n| \le \ell \le |F|$,*

$$p(F_1,\dots,F_n;F) = \sum_{\widetilde{F}\in\mathcal{F}^\sigma_\ell} p(F_1,\dots,F_n;\widetilde{F})\,p(\widetilde{F};F).$$

Definition 2. *For a type σ, we define the* **flag algebra**

$$\mathcal{A}^\sigma := (\mathbb{R}\mathcal{F}^\sigma)/K^\sigma,$$

where $K^\sigma := \langle F - \sum_{\widetilde{F}\in\mathcal{F}^\sigma_\ell} p(F;\widetilde{F})\widetilde{F} \mid F \in \mathcal{F}^\sigma \text{ and } \ell \ge |F|\rangle$.

It is instructive to think of the basis elements $F \in \mathbb{R}\mathcal{F}^\sigma$ as densities of F in some large fixed σ-flag X. Modding out K^σ then implements the chain rule.

Lemma 2 (Product). *There is a product $\mathcal{A}^\sigma \otimes \mathcal{A}^\sigma \to \mathcal{A}^\sigma$ defined by*

$$F_1 \cdot F_2 := \sum_{F\in\mathcal{F}^\sigma_\ell} p(F_1,F_2;F)$$

for any $\ell \geq |F_1 \cup_\sigma F_2|$. *This makes* $\mathcal{A}^\sigma$ *into a commutative* $\mathbb{R}$*-algebra with unit* $1_\sigma := (\sigma, \sigma)$.

Lemma 3. *A flag* $F = (G, \sigma)$ *is called connected if* $G \backslash \sigma$ *is a connected graph. Fix a connected flag* $F_0 \in \mathcal{F}^\sigma_{|\sigma|+1}$. *Then* $\mathcal{A}^\sigma$ *is a polynomial algebra over* $\mathbb{R}$, *freely generated by all connected flags except for* 1_σ *and* F_0.

2. Motivation

Let $\mathrm{Hom}(\mathcal{A}^\sigma; \mathbb{R})$ denote all algebra homomorphisms from $\mathcal{A}^\sigma$ to $\mathbb{R}$. Define

$$\mathrm{Hom}^+(\mathcal{A}^\sigma; \mathbb{R}) := \{\varphi \in \mathrm{Hom}(\mathcal{A}^\sigma; \mathbb{R}) \mid \varphi(F) \geq 0 \text{ for all } F \in \mathcal{F}^\sigma\}.$$

We define the **semantic cone** as

$$C_{\mathrm{sem}}(\mathcal{A}^\sigma) := \{f \in \mathcal{A}^\sigma \mid \varphi(f) \geq 0 \text{ for all } \varphi \in \mathrm{Hom}^+(\mathcal{A}^\sigma; \mathbb{R})\}.$$

Thus, $C_{\mathrm{sem}}(\mathcal{A}^\sigma)$ is obtained by polarizing twice the cone in $\mathcal{A}^\sigma$ spanned by all σ-flags. We write $f \succeq_\sigma g$ if $f - g \in C_{\mathrm{sem}}(\mathcal{A}^\sigma)$.

The following theorem is Razborov's version of a theorem of Lovász and Szegedy [1]. It follows from the fact that $\mathrm{Hom}^+(\mathcal{A}^\sigma; \mathbb{R}) \subseteq [0,1]^{\mathcal{F}^\sigma}$ is the set of all limit point (with respect to the product topology in $[0,1]^{\mathcal{F}^\sigma}$) of sequences $(p(\_; F_i))_{i \in \mathbb{N}}$.

Theorem 4. *Let* $f \in \mathbb{R}[x_1, \ldots, x_n]$. *Then* $f(F_1, \ldots, F_n) \in C_{\mathrm{sem}}(\mathcal{F}^\sigma)$ *if and only if*

$$\liminf_{F \in \mathcal{F}^\sigma} f(p(F_1, F), \ldots, p(F_n, F)) \geq 0. \tag{1}$$

Several interesting statements in asymptotic extremal combinatorics can be written in the form (1). Theorem 4 then gives us a reformulation of that in terms of $C_{\mathrm{sem}}(\mathcal{F}^\sigma)$. Below we review some criteria for when an element of $\mathcal{A}^\sigma$ lies in $C_{\mathrm{sem}}(\mathcal{F}^\sigma)$.

3. Cauchy–Schwarz inequality

Definition 3 (Restriction operator)**.** *Let* $\sigma_0 \subseteq \sigma$ *be a sub-type. We define a linear map (in general not an algebra homomorphism)* $[\_]_{\sigma,\sigma_0} : \mathcal{A}^\sigma \to \mathcal{A}^{\sigma_0}$ *via* $[F]_{\sigma,\sigma_0} := q_{\sigma,\sigma_0}(F) F|_{\sigma_0}$, *where* $F = (G, \sigma)$, $F|_{\sigma_0} := (G, \sigma_0)$, *and* $q_{\sigma,\sigma_0}(F) \in [0,1]$ *is the probability that a (uniformly) random extension* $V(\sigma) \hookrightarrow G$ *of the embedding* $V(\sigma_0) \hookrightarrow G$ *induces a flag that is isomorphic to* F.

For $\sigma_0 \subseteq \sigma_1 \subseteq \sigma_2$, we have $[F]_{\sigma_2,\sigma_0} = [[F]_{\sigma_2,\sigma_1}]_{\sigma_1,\sigma_0}$.

Theorem 5 (Cauchy–Schwarz inequality for $\mathcal{A}^\sigma$)**.** *For any* $f, g \in \mathcal{A}^\sigma$ *and* $\sigma_0 \subseteq \sigma$,

$$[f^2]_{\sigma,\sigma_0} \cdot [g^2]_{\sigma,\sigma_0} \succeq_{\sigma_0} [fg]^2_{\sigma,\sigma_0}.$$

As an application, one obtains Goodman's bound relating the asymptotic edge– and triangle densities, which states (as flags over $\sigma = \emptyset$) that $K_3 \succeq K_2(2K_2 - K_1)$. For the proof one applies the Cauchy–Schwarz inequality for the flags (K_2, K_1) and (K_1, K_1) with $\sigma = K_1$ and $\sigma_0 = K_0$.

4. Differential method

We write the types K_0, K_1, K_2 and $\bar{K}_2$ as 0, 1, E and $\bar{E}$, respectively.

We define a linear map (in general not multiplicative) $\partial_1 : \mathcal{A}^0 \to \mathcal{A}^1$ by

$$\partial_1 G := \ell\Big(\sum_{\substack{(H,1)\in\mathcal{F}^1_{\ell+1} \\ G\cong H\setminus 1}} (H,1) - \sum_{\substack{(H,1)\in\mathcal{F}^1_{\ell} \\ H\cong G}} (H,1)\Big),$$

where $\ell := |G|$.

Further, define a linear map $\partial_E : \mathcal{A}^{\bar{E}} \to \mathcal{A}^E$ by

$$\partial_E(G,\bar{E}) := \binom{\ell}{2}\Bigg(\sum_{\substack{(H,\bar{E})\in\mathcal{F}^{\bar{E}}_{\ell} \\ H\cong G}} (H \cup_{\bar{E}} E, E) - \sum_{\substack{(H,E)\in\mathcal{F}^{E}_{\ell} \\ H\cong G}} (H,E)\Bigg).$$

Theorem 6. *Let $G_1,\dots,G_n$ be finite graphs. Consider $\varphi_0 \in \operatorname{Hom}^+(\mathcal{A}^0;\mathbb{R})$ and $f \in C^1(U)$ for some open subset $U \subseteq \mathbb{R}^n$, such that $\Phi : \operatorname{Hom}^+(\mathcal{A}^0;\mathbb{R}) \to \mathbb{R}$ given by $\Phi(\varphi) := f(\varphi(G_1),\dots,\varphi(G_n))$ is maximal at φ_0 among all φ such that $(\varphi(G_1),\dots,\varphi(G_n)) \in U$. Then, for any $g \in \mathcal{A}^1$,*

$$\varphi_0([g\cdot\partial_1\langle\nabla f,(G_1,\dots,G_n)\rangle]_{1,0}) = 0.$$

Furthermore, for any $g \in C_{\mathrm{sem}}(\mathcal{A}^E)$,

$$\varphi_0([g\cdot\partial_E\langle\nabla f,(G_1,\dots,G_n)\rangle]_{E,0}) \geq 0.$$

As an application, Razborov [2, 3] calculated the asymptotically minimal possible triangle density in a graph for any given edge density. Based on a similar ideas, Reiher [4] calculated more generally the minimal possible K_k-density in a graph with a given edge density, and this not only asymptotically.

References

[1] L. Lovász, B. Szegedy, *Limits of dense graph sequences*, Technical Report TR-2004-79, Microsoft Research, August 2004, 27 pages.

[2] A. Razborov, *Flag algebras*, J. Symbolic Logic **72** (2007), no. 4, 1239–1282.

[3] A. Razborov. *On the minimal density of triangles in graphs*, Combin. Probab. Comput. **17** (2008), no. 4, 603–618.

[4] C. Reiher, *The Clique Density Theorem*, preprint, arXiv:1212.2454 (2012), 25 pages.

Finite graphs and amenability

Ostap Chervak

Notion of convergence of bounded degree graphs was defined by Benjamini and Schramm. For the rest of our talk we will suppose that degree of mentioned graphs is bounded by some constant d. This talk is based on [2].

Let G be a finite graph and $H_\odot$ be a finite rooted graph of radius r. By a *local statistics* $t(G,H_\odot)$ we denote a number $\frac{|\{v\in V(G)\colon B_r(v)\ H_\odot\}|}{|V(G)|}$ which is a probabitity that an r-ball around a random vertex is isomorphic to a fixed graph $H_\odot$.

A sequence (G_n) of finite graphs is called Benjamini–Schramm convergent (BS-convergent) if for all rooted graphs $H_\odot$ local statistics $t(G_n, H_\odot)$ is convergent.

In fact, one can introduce a metric on the set of finite graphs, such that sequence is BS-convergent iff it is a Cauchy sequence. To study the complection of this space we will introduce a notion of graphing [6].

Let X be a standard Borel set endowed with probability measure μ, we will say that triple $\mathcal{G} = (X, E, \mu)$ is a graphing iff

- $E \subset X \times X$ is a measurable symmetric subset which does not intersect the diagonal $\{(x,x) : \ x \in X\}$.
- Degrees of all vertices are bounded by d.
- Whenever $A, B \subset X$ the following equality holds (double counting principle)

$$\int\limits_{x\in A} |\{y : \ (x,y) \in E\}| d\mu_A = \int\limits_{y \in B} |\{x : \ (x,y) \in E\}| d\mu_B.$$

By a local statistics of graphing $t(\mathcal{G}, H_\odot)$ one may define a probability $p(\mathcal{G}, H_\odot)$ that an r-ball around random (corresponding to measure μ) vertex is isomorphic to $H_\odot$. That way, a notion of BS-convergence extends to set of graphings.

Two graphings will be called *weakly equivalent* if they have same local statistics.

Theorem 1. *For every convergent sequence of finite graphs there exist a limit graphing* [3].

One of examples of graphings are "convex combinations" of finite graphs. Let $G_1, G_2, \dots, G_k$ be a finite graphs and let $\alpha_1, \dots \alpha_n$ be a sequence of positive reals such that $\sum \alpha_i = 1$. Let X_i be a union of products $G_i \times [0, \alpha_i]$ (endowed with a product measure, such that $\mu_i(X_i) = \alpha_i$), and let X be a disjoint union of X_i endowed with measure $\mu(S) = \sum \mu_i(S \cap X_i)$. Let E be a set

$$E = \{((v_1, t_1), (v_2, t_2)) : \ \exists i : \ v_1, v_2 \in G_i, \ t_1 = t_2, \ (v_1, v_2) \in E(G_i)\}.$$

It is easy to see that $(G) = (X, E, \mu)$ is indeed a graphing, and local statistics of $\mathcal{G}$ is a convex combination of local statistics of G_n. If all α_is are rational then $\mathcal{G}$ is weakly equivalent to a finite graph.

Conjecture 2 (Aldous-Lyons)**.** *Assume that $\mathcal{G}$ is a graphing. Does there exist a sequence of finite graphs converging to $\mathcal{G}$?*

Though Aldous–Lyons conjecture [1] is still open, it is true for many families, for example for "convex combinations" of graphs defined above. Much wider class of graphings with positive answer to Aldous–Lyons conjecture is the main object of this talk.

Definition 1. *Graphing $\mathcal{G} = (X, E, \mu)$ will be called* hyperfinite *if for every positive real ε there exist a natural number K and a set $T_\varepsilon \subset X$ of measure $\mu(T_\varepsilon) < \varepsilon$ such that after removing all edges incident to points in T_ϵ, all components in resulting graphing $\mathcal{G}_\varepsilon$ will have diameter smaller than K.*

Definition 2. *Sequence of finite graphs G_n will be called* hyperfinite *if for every positive real ε there exist a natural number K and a set $T_{\varepsilon,n} \subset G_n$ with number of elements $|T_{\varepsilon,n}| < \varepsilon |V(G_n)|$ such that after removing all endes encident to points in T_ϵ, all components in resulting graphs will have diameter smaller than K.*

Important examples of hyperfinite families are Folner sequences of amenable group, planar graphs.

To see that Aldous–Lyons conjecture is true for hyperfinite graphings note that $\mathcal{G}$ is a limit of graphings $\mathcal{G}_\varepsilon$ and the later are weakly equivalent to convex combination of finite graphs (there are only finite number of graphs with diameter smaller than K, so this combination will be finite).

Theorem 3 (Kaimanovich)**.** *A graphing is hyperfinite iff for every subgraphing of positive measure almost all components have isoperimetric constant zero* [5].

As a consequence of previous theorem one can prove that if there exist a measure-preserving map $\mathcal{G}_1 \to \mathcal{G}_2$ graphing $\mathcal{G}_1$ will be hyperfinite iff $\mathcal{G}_2$ is hyperfinite. In fact, hyperfiniteness is preserved by the weak equivalence of graphings.

Theorem 4. *Assume that $\mathcal{G}_1$ and $\mathcal{G}_2$ are weakly equivalent, then there exists a graphing $\mathcal{G}_3$ and two measure preserving maps $f_1 : \mathcal{G}_3 \to \mathcal{G}_1$ and $f_3 : \mathcal{G}_3 \to \mathcal{G}_2$.*

For bounded degree graphs there exists an other, stronger notion of convergence, the so-called *local-global convergence.* To define it, we need to introduce some technical notions.

Let G be finite k-colored graph, and $H_\odot$ be rooted k-colored graphs. Define local statistic for colored graphs $t_k(G, H\odot)$ similarly to it for ordinary graphs, namely put

$$t_k(G, H_\odot) = \frac{|\{v \in V(G) :\ B_r(v)\ H_\odot\}|}{|V(G)|}.$$

Now let us define a metric on space of all k-colored graphs. Enumerate all rooted graphs $H_\odot$ to form a sequence $H_n^\odot$ and put

$$d_{\text{colored}}(G_1, G_2) = \sum \frac{|t_k(G_1, H_n^\odot) - t_k(G_2, H_n^\odot)|}{2^n}.$$

Now, if G_1 and G_2 are two graphs, define their k-color local-global distance to be smallest such number $d_k(G_1, G_2)$ such that for every k-coloring of G_1 there exists a coloring of G_2 (and vice versa) such that for those two colorings $d_{\text{colored}}(G_1, G_2) \leq d_k(G_1, G_2)$.

Now, local-global distance of two graphs $d_{lg}(G, H)$ is a $\sum \frac{d_k(G,H)}{2^k}$. Notion of local-global convergence (convergence with respect to local-global metric) is rather strong, for example it is easy to see (k=2 is sufficient) that a local-global convergent hyperfinite family converges to a hyperfinite graphing. Though much more is true.

Theorem 5. *If a sequence of finite graphs G_n converges to $\mathcal{G}$ then limit graphing is hyperfinite iff sequence G_n is hyperfinite and in this case G_n converges locally-globally.*

References

[1] D. Aldous, R. Lyons, *Processes on unimodular random networks*, Electron. J. Probab. **12** (2007), no. 54, 1454–1508.
[2] G. Elek, *Finite graphs and amenability*, preprint, arXiv:1204.0449.
[3] G. Elek, *On limits of finite graphs*, Combinatorica **27** (2007), no.4, 503–507.
[4] H. Hatami, L. Lovász, B. Szegedy, *Limits of local-global convergent graph sequences*, preprint, arXiv:1205.4356.
[5] V. Kaimanovich, *Amenability, hyperfiniteness, and isoperimetric inequalities*, C. R. Acad. Sci. Paris Sr. I Math. **325** (1997), no. 9, 999–1004.
[6] A. S. Kechris, B. D. Miller, *Topics in orbit equivalence*, Lecture Notes in Mathematics, Springer-Verlag, **852** (1990).

Nilmanifolds and nilspaces

Yonatan Gutman

Recently Camarena and Szegedy have developed a beautiful theory of *nilspaces* - a certain generalization of nilmanifolds. The goal of this report is to present very succinctly, the elements of a new proof of the main case of a fundamental theorem appearing in [1], about the relation between nilmanifolds and nilspaces. I would like to thank Freddie Manners and Péter Varjú who worked with me on the new proof. I am grateful to Ben Green, Bernard Host and Balázs Szegedy for helpful discussions.

1. The prenilspace and k-step nilspace axioms

Define the functions $\rho_i : \{0,1\} \to \{0,1\}$, $i = 0,1,2,3$, by $\rho_0(x) \equiv 0\,, \rho_1(x) \equiv 1\,, \rho_2(x) = x\,, \rho_3(x) = 1-x$. Let $m,n \in \mathbb{N}$. A map $f : \{0,1\}^m \to \{0,1\}^n$ between *discrete cubes* is called a **discrete cube morphism** if for every $1 \le i \le n$ there exist $1 \le j \le m$ and $k \in \{0,1,2,3\}$ (depending on i) so that for any $(x_1,\ldots,x_m) \in \{0,1\}^m$, $f(x_1,\ldots,x_m)_{|i} = \rho_k(x_j)$. Observe that discrete cube morphisms are closed under composition. Let (X,d) be a compact metric space. Let $C^n(X) \subset X^{\{0,1\}^n}$, $n \in \mathbb{Z}_+$ be closed sets. The elements of $C^n(X)$ are referred to as the **(n)-cubes**. X is referred to as the **base space**. We define the following *axioms* $(n,k \in \mathbb{Z}_+)$: n**-Cube invariance** $(I)_n$: For any $m \in \mathbb{Z}_+$, $f \in C^m(\{0,1\}^n)$ and $c \in C^n(X)$ $c \circ f \in C^m(X)$. k**-Ergodicity** $(E)_k$: $C^k(X) = X^{\{0,1\}^k}$. n**-Completion** $(C)_n$: If $\tilde{f} : \{0,1\}^n \backslash \{\vec{1}\} \to X$ has the property that for every $1 \le i \le n$, $\tilde{f}_{|F_i} \in C^{n-1}(X)$ where $F_i = \{\vec{x} \in \{0,1\}^n |\, x_i = 0\}$, then there exists $c \in C^n(X)$ with $c^* := c_{|\{0,1\}^n \backslash \{\vec{1}\}} = \tilde{f}$. c is referred to as a *completion* of $\tilde{f}$. n**-Uniqueness** $(U)_n$: If $h,f \in C^n(X)$ and $h^* = f^*$ then $h = f$. Let $\mathcal{X} = (X, \{C^n(X)\}_{n=0}^{\infty})$. Define the following objects: **Prenilspace**: $[(I)_n$ for all $n \in \mathbb{Z}_+]$, $(E)_1$, $[(C)_n$ for all $n \in \mathbb{N}]$. k**-step Nilspace**: $[(I)_n$ for all $n \in \mathbb{Z}_+]$, $(E)_1$, $[(C)_n$ for all $n \in \mathbb{N}]$, $(U)_{k+1}$. A morphism between two prenilspaces $f : \mathcal{X} \to \mathcal{Y}$ consists of a continuous mapping $f : X \to Y$ such that $f(C^n(X)) \subset C^n(Y)$ for all $n \in \mathbb{N}$.

2. The structure of k-step nilspaces

Let $\mathcal{X}$ be a 1-step nilspace. Fix an arbitrary element $e \in X$ and let $a, b \in X$ arbitrary. Taking advantage of the fact that $\tilde{a} : \{0,1\}^2 \setminus \{\vec{1}\} \to X$ given by $\tilde{a}(0,0) = e$, $\tilde{a}(1,0) = a$, $\tilde{a}(0,1) = b$ has a unique completion, one obtains a continuous binary operation on X. It is not hard to show, using the axioms, this binary operation turn X into a compact Abelian group. For k-step nilspaces with $k > 1$ the situation is more complicated. Define a **principal bundle** to be a quadruple $\mathcal{E} = (E, B, \pi, G)$, where E, B are topological spaces, G is a topological group acting continuously on E and $\pi : E \to B$ a is continuous surjection such that, G preserves the fibers $\pi^{-1}(b)$, $b \in B$ and acts freely and transitively on each one of them. A (G)-**bundle map** $\phi : E \to E$ is a continuous G-equivariant map.

The Camarena–Szegedy Structure Theorem. Given a k-step nilspace $\mathcal{X}_k$, there is a finite series of finite-step nilspaces $\mathcal{X}_{k-1}, \dots \mathcal{X}_0 = \{\bullet\}$ and compact Abelian groups $A_k, \dots A_1$ as well as continuous prenilspace epimorphisms $\mathcal{X}_k \overset{\pi_k}{\to} \mathcal{X}_{k-1} \overset{\pi_{k-1}}{\to} \mathcal{X}_{k-2} \to \cdots \overset{\pi_1}{\to} \mathcal{X}_0$ such that $(X_j, X_{j-1}, \pi_j, A_j)$ is a A_j-principal bundle for $j = 1, \dots, k$. □

$\mathcal{X}_k$ is said to be **toral** if all structure groups $A_1, \dots A_k$ are tori (of various dimensions). Recall that a **nilmanifold** X is a quotient $X = G/\Gamma$ where G is a finite-step nilpotent Lie group and Γ a cocompact discrete subgroup. Our goal is to prove the following theorem:

Theorem 1. *The base space of a toral k-step nilspace is a nilmanifold.*

Proof. (Sketch.) The result is proven by induction. The base case $k = 1$: From the Camarena–Szegedy Structure Theorem it follows $X_1 = A_1$ is a torus. Assume the theorem has been established for $k - 1$. Let $\mathcal{X}_k = (X_k, \{C^n(X_k)\}_{n=0}^{\infty})$ be a k-step compact nilspace. We call a homeomorphism $\alpha : X_k \to X_k$ a **translation** if for any $c \in C^k(X_k)$, $[c, \alpha(c)] \in C^{k+1}(X_k)$ and $[c, \alpha^{-1}(c)] \in C^{k+1}(X_k)$ where the *concatenation* $[c_0, c_1] : \{0,1\}^{k+1} \to X_k$ is given by $[c_0, c_1](v, 0) = c_0(v)$ and $[c_0, c_1](v, 1) = c_1(v)$ for all $v \in \{0,1\}^k$. Note that translations are A_k-bundle maps. Let $\tilde{G}_k$ be the group of translations of $\mathcal{X}_k$ equipped with the supremum metric d_∞. Let G_k be the identity component of $\tilde{G}_k$. G_k will turn out to be the desired nilpotent Lie group for which $X_k = G_k/\Gamma_k$ (for suitable cocompact discrete Γ_k). Going through the proof of [1, Theorem 7] it is clear that the difficulty lies in establishing that the natural projection $\pi_k : G_k \to G_{k-1}$ is onto. Let $\alpha_{k-1} \in G_{k-1}$. By the inductive assumption $G_{\mathcal{X}_{k-1}}$ is a connected Lie group and therefore path connected. As a consequence one can find a (continuous) homotopy between Id and α_{k-1}, $H : X_{k-1} \times I \to X_{k-1}$. By *Gleason's Theorem* ([2, Theorem 3.3]), $(X_k, X_{k-1}, \pi_k, A_k)$ is a *fiber bundle.* Thus according to the *First Covering Homotopy Theorem* ([3, §11.3]), as X_{k-1} is compact, one can *lift* the homotopy H to a homotopy which is a bundle map. In particular there is a bundle map lift $h_k : X_k \to X_k$ of α_{k-1} ($\pi_k \circ h_k(x) = \alpha_{k-1} \circ \pi_k(x)$). However h_k may not be a translation. We associate to h_k the "cocycle" $\rho_k : C^k(X_k) \to A_k$, measuring its deviation from being a translation, defined by $\rho_k(c) = a$ iff $[c, (h_k(c^*), h_k(c(\vec{1})) +$

$a)] \in C^{k+1}(X_k)$, where $(h_k(c^*), h_k(c(\vec{1})) + a)$ is the configuration achieved from $h_k(c)$ by adding the element a to $h_k(c(\vec{1}))$. Using the $(C)_{k+1}$ and $(U)_{k+1}$ axioms, one can easily show that such an element a exists and that it is unique. This implies $\rho_k(c)$ is *continuous*. As ρ_k is constant on cubes with identical projection on $C^k(X_{k-1})$, one obtains a map $\rho_k : C^k(X_{k-1}) \to A_k$. It turns out that if $d_\infty(Id, h_k)$ is small enough (which can be assumed w.l.o.g) then there exists a continuous $g : X_{k-1} \to A_k$ such that the α_{k-1}-lift $\alpha_k := h_k + g : X_k \to X_k$ is a translation iff one can solve the equation $\rho_k(c) = \partial^k(g)(c) := \sum_{v\in\{0,1\}^k} g(c(v))(-1)^{\sum_i v_i}$ for all c. This equation is indeed solvable following the procedure in [1, Lemma 3.19] as one can explicitly write g as a certain average of ρ_k. Without getting into the details let us point out that the continuity of g is a consequence of the continuity of ρ_k.

References

[1] O. A. Camarena, B. Szegedy, *Nilspaces, nilmanifolds and their morphisms*, preprint, arXiv: 1009.3825.

[2] A. M. Gleason, *Spaces with a compact Lie group of transformations*, Proc. Amer. Math. Soc. **1** (1950), 35–43.

[3] N. Steenrod, *The Topology of Fibre Bundles*, Princeton Mathematical Series **14**, Princeton University Press, Princeton, N. J., (1951), viii+224 pp.

Reporter: Ágnes Backhausz

Participants

Prof. Dr. Miklos Abert
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
1364 Budapest
HUNGARY

Andrey Alpeev
St. Petersburg State University
Chebyshev Laboratory
14th Line 29B, Vasilievsky Island
199 178 St. Petersburg
RUSSIAN FEDERATION

Dr. Gergely Ambrus
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
13-15 Realtanoda u.
1053 Budapest
HUNGARY

Agnes Backhausz
Department of Probability Theory
and Statistics
Eötvös Lorand University
Pázmány Péter sétány 1/c., D 3-415
1117 Budapest
HUNGARY

Nathan Bowler
Mathematisches Seminar
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Pablo Candela
Département de Mathématiques
École Normale Superieure
45, rue d'Ulm
75230 Paris Cedex 05
FRANCE

Johannes Carmesin
Mathematisches Seminar
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Ostap Chervak
Department of Geometry and Topology
Faculty of Mechanics and Mathematics
Ivan Franko National University
ul. Universytetska 1
79000 Lviv
UKRAINE

Dr. Nhan-Phu Chung
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig
GERMANY

Peter Csikvari
Department of Computer Science
ELTE TTK
Pazmany Peter setany 1/c
1117 Budapest
HUNGARY

Dr. Endre Csoka
Mathematics Institute
Zeeman Building
University of Warwick
Coventry CV4 7AL
UNITED KINGDOM

Evan DeCorte
Delft Institute of Applied Mathematics
Delft University of Technology
Mekelweg 4
2628 CD Delft
NETHERLANDS

Prof. Dr. Christopher Deninger
Mathematisches Institut
Universität Münster
Einsteinstr. 62
48149 Münster
GERMANY

Prof. Dr. Gabor Elek
Mathematical Institute of the
Hungarian Academy of Sciences
P.O. Box 127
1364 Budapest
HUNGARY

Dr. Cameron Freer
MIT CSAIL
The Stata Center
32 Vassar Street G480
Cambridge MA 02139
UNITED STATES

Dr. Peter Frenkel
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
13-15 Realtanoda u.
1053 Budapest
HUNGARY

Prof. Dr. Damien Gaboriau
Dept. de Mathématiques, U.M.P.A.
École Normale Superieure de Lyon
46, Allee d'Italie
69364 Lyon Cedex 07
FRANCE

Dr. Roman Glebov
Mathematics Institute
University of Warwick
Coventry CV4 7AL
UNITED KINGDOM

Peter Gmeiner
Department Mathematik
Universität Erlangen-Nürnberg
Cauerstr. 11
91058 Erlangen
GERMANY

Andrzej Grzesik
Jagiellonian University
Theoretical Computer Science
Department
Lojasiewicza 6
30-348 Krakow
POLAND

Yonatan Gutman
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Jan Hladky
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Lucas Hosseini
10 rue Lavoisier
75008 Paris
FRANCE

Dr. Hao Huang
Princeton University
School of Mathematics
Institute for Advanced Study
Einstein Drive
Princeton NJ 08540
UNITED STATES

Julia Komjathy
Dept. of Mathematics & Computer Science
Eindhoven University of Technology
5600 MB Eindhoven
NETHERLANDS

Prof. Dr. Gabor Kun
Institute of Mathematics
Eötvös Loránd University
Pázmány sétány 1/C
1117 Budapest
HUNGARY

Prof. Dr. Laszlo Lovasz
Institute of Mathematics
Eötvös Lorand University
ELTE TTK
Pazmany Peter setany 1/C
1117 Budapest
HUNGARY

Laszlo Miklos Lovasz
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139-4307
UNITED STATES

Dr. Benjamin Matschke
I.H.E.S.
Le Bois Marie
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Viola Mészáros
Bolyai Institute
University of Szeged
Aradi Vertanuk Tere 1
6720 Szeged
HUNGARY

Prof. Dr. Jaroslav Nesetril
Department of Applied Mathematics
Charles University
Malostranske nam. 25
118 00 Praha 1
CZECH REPUBLIC

Prof. Dr. Patrice Ossona de Mendez
CAMS
EHESS
90 avenue de France
75013 Paris
FRANCE

Prof. Dr. Gabor Pete
Mathematical Institute
Technical University of Budapest
Egry József utca 1
1111 Budapest
HUNGARY

Prof. Dr. Oleg Pikhurko
Mathematics Institute
University of Warwick
Coventry CV4 7AL
UNITED KINGDOM

Dr. Christian Reiher
Department Mathematik
Universität Hamburg
Bundesstr. 55
20146 Hamburg
GERMANY

Rik Sengupta
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge MA 02139
UNITED STATES

Fiona Skerman
Department of Statistics
University of Oxford
1 South Parks Road
Oxford OX1 3TG
UNITED KINGDOM

Lluis Vena
Department of Mathematics
University of Toronto
100 St. George Street
Toronto, Ontario M5S 1A1
CANADA

Prof. Dr. Balazs Szegedy
Department of Mathematics
University of Toronto
40 St George Street
Toronto, Ont. M5S 2E4
CANADA

Prof. Dr. Balint Virag
Department of Mathematics
University of Toronto
40 St George Street
Toronto, Ont. M5S 2E4
CANADA

Prof. Dr. Andreas B. Thom
Mathematisches Institut
Universität Leipzig
Postfach 100920
04009 Leipzig
GERMANY

Jan Volec
Mathematics Institute
University of Warwick
Coventry CV4 7AL
UNITED KINGDOM

Heinrich-Gregor Zirnstein
Lutherstr. 18
04315 Leipzig
GERMANY