Arbeitsgemeinschaft mit aktuellem Thema:
LIMITS OF DISCRETE STRUCTURES
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Introduction

A recently emerging and fast growing topic in mathematics considers very large finite structures as approximations of infinite analytic objects. This approach enables one to use a variety of tools from analysis to study graphs, hypergraphs, permutations, subsets of groups and many other fundamental structures. There are several applications in extremal combinatorics, Fourier analysis (also in a higher order version of it), property testing, group theory, topology and many other areas. The subject is very rich and many of its aspects are discussed in the recent book [29] by L. Lovász. The goal of the workshop is to present a landscape of beautiful ideas developed by researchers from various fields.

Talks

1. **Szemerédi's regularity lemma and generalizations**

The talk should explain Szemerédi's regularity lemma and its generalizations. Prove the Hilbertspace version of the Regularity lemma [32] and as an application prove the weak regularity lemma by Frieze and Kannan. (For this application the cut norm needs to be defined.) State Szemerédi's regularity lemma and present its proof ([32],[42]).

2. **Limits of dense graph sequences**

This talk should introduce the basic concepts and facts in (dense) graph limit theory [31],[29]. Define graph homomorphisms, homomorphism densities, restricted homomorphisms and the corresponding densities. Define local convergence, graphons and cut norm. Introduce the $\delta_\square$ and δ_1 distances on graphons. Show the compactness of the graph limit space. Prove that δ_1 is lower semi continuous w.r.t $\delta_\square$ (see [34]).

3. **Graph algebras and reflection positivity**

Definition of graph algebras, connection matrices, multiplicativity and reflection positivity. (discuss the relationship to the classical moment problem). Sketch the Lovász-Freedman-Schrijver result [16] and the Lovász-Szegedy results [31],[35]. Present a few applications in extremal combinatorics.

4. **Razborov's flag algebras with applications**

Razborov's flag algebras [38] are closely related to the so-called graph algebras. They provide a powerful tool to prove results in extremal combinatorics. Two main ingredients are: 1) A systematic way of formulating Cauchy-Schwartz inequalities 2) Variational principles on the graph limit space. The talk should give a detailed explanation of the calculus related to these algebras.

5. **Topological aspects of dense graph limits**

The equivalence class of every graphon can be represented on a unique topological space. The talk should present the concepts: topology, compactness and dimensionality of graphons [33]. Relationship between dimensionality and complexity of Szemerédi partitions shall be presented. It is natural to examine the underlying space for graphons that arise in extremal combinatorics. The topic creates a new link between combinatorics and topology [36]. The talk gives the definition of finitely forcible graphons and present several examples. Finitely forcible graphons represent extremal structures that arise as unique solutions of extremal problems in the dense setting. Rather surprisingly, in each case, the graphon is compact and finite dimensional, which motivates interesting open problems.

6. **Spectral aspects of the regularity lemma and dense graph limits**

A fruitful connection between (a weak version of) Szemerédi's regularity lemma and spectral decompositions was discovered by Frieze and Kannan [15]. This was later refined by Szegedy [41] who showed that arbitrarily strong versions of the regularity lemma can be proved using spectral approximation. It also turns out that there is a purely spectral interpretation of limits of dense graph sequences. The talk should present graph convergence in terms of the convergence of spectral information (eigenvalues and eigenfunctions) in kernel operators. The talk should describe Szemerédi's regularity lemma in terms of spectral decompositions.

7. **Large deviations and the exponential random graph model**

This talk should be about the results of Chatterjee, Diaconis and Varadhan on graph limits and random graph models [7],[8]. Chatterjee and Varadhan examined the question what random graphs look like when conditioned on rare events. Related to this, Chatterjee and Diaconis examined the exponential random graph model which can increase the probability of rare events. In both cases analysis on the graph limit space is used as a tool to study these problems.

8. **Ultrafilters and hypergraphs**

The so-called Hypergraph regularity theory by Rödl, Skokan, Nagle, Schacht, Gowers, Tao and others is a current breakthrough in modern combinatorics. In particular it gives a new proof of Szemerédi's famous theorem, even in the multi-dimensional setting. Give a survey on the topic. State the regularity and the removal lemmas and explain how the removal lemma implies Szemerédi's theorem on arithmetic progressions. To bring the topic together with hypergraph limits, Elek and Szegedy gave a new approach to the hypergraph regularity theory using ultraproducts of finite measure spaces. It also turns out that the measure and integral theory of ultraproduct spaces is a good framework for many other problems. The talk should give a survey on this topic. [14]

9. **The graph limit approach to Sidorenko's conjecture**

Roughly speaking, Sidorenko's conjecture says that the density of any given bipartite graph in a graph with fixed edge density is minimized by the random graph. The conjecture has various equivalent formulations and it is related to combinatorics, probability theory, analysis, and mathematical physics. A famous result of Conlon, Fox and Sudakov [10] says that if one point in a bipartite graph H is complete to the other side then H satisfies Sidorenko's conjecture. The talk should survey the problem and outline two different graph limit approaches. The first one is by László Lovász [28]. He proved a local version of the conjecture. Another approach given by Li and Szegedy [30] used Jensen's inequality for logarithmic functions to obtain the conjecture for various bipartite graphs. The talk should present the full proof of the Conlon-Fox-Sudakov result using the logarithmic calculus.

10. **Benjamini-Schramm limits**

Benjamini and Schramm introduced a convergence notion for bounded degree graphs. Give the definition of the convergence notion and the definition of an involution invariant probability distribution of rooted, connected (possibly infinite) graphs. Show examples for convergent sequences and their limits. Outline the proof that limits of bounded degree planar graphs are recurrent (with probability 1) [5].

11. **Gaphings and local-global limits of bounded degree graphs (basics)**

Graphings are Borel graphs with an additional measure preserving property. They seem to be the right generalizations of finite bounded degree graphs to the infinite setting. It is a broad research direction to generalize facts from finite graph theory to this measurable setting. Graphings contain more information than local statistics of neighborhoods. There is strengthening of the Benjamini-Schramm convergence called local-global convergence and limit objects for this convergence notion can be represented by graphons. [27]

12. **Local algorithms and factor of i.i.d processes**

Local algorithms are paralelized (randomized) algorithms that run in constant time. They can be used to produce independent sets, matchings, colorings and other crucial structures in graph theory. One can describe local algorithms by the so-called factor of i.i.d processes. This topic is fast growing and has many interesting open problems. The goal of the talk is to give an introduction to the subject. Introduce local algorithms on finite graphs. Introduce factor of i.i.d processes on infinite graphs. Show the simple construction which produces independent sets of density $1/(d+1)$. Discuss the connection between local algorithms, factor of i.i.d processes and Bernoulli graphings. [27].

13. **Statistical physics and graph limits (convergence from the right)**

This talk deals with the statistical physical aspects of graph limit theory. Define homomorphism entropy and present the Dobrushin uniqueness theorem [29]. Outline the proof of the theorem by C. Borgs, J. Chayes, J. Kahn, L. Lovász from [6] which states the equivalence of local convergence and homomorphism entropy convergence (with the Dobrushin condition).

14. **Limits of permutations**

Limits of permutations were introduced by Hoppen, Kohayakawa, Moreira, Rath, Menses Sampaio in [21] and [22]. Define limits and sampling from permutations. Define convergence notion and limit object. Introduce quasi randomness for permutations. Show the Kral-Pikhurko characterization for quasirandom permutations ([26]).

15. **Limits of functions on Abelian groups and higher order Fourier analysis**

Gowers norms of functions on Abelian groups were introduced in the fundamental papers [17], [18] by W.T. Gowers. Since then a lot of progress has been made by Green, Gowers, Szegedy, Tao, Wolf, Ziegler (and many others) ([19],[23],[24],[25],[43],[20],[39],...) towards a good understanding of these norms. The topic is now called "Higher Order Fourier Analysis". The purpose of this talk is to present a limit approach to the subject developed by Szegedy which fits into the philosophy of the graph and hypergraph limit theories. Introduce measure and σ-topology on ultraproducts of Abelian groups. Point out that Gowers norms become semi-norms on ultra product groups. Define the σ-algebra corresponding to the U_k norm (In this part prove that there is a maximal σ-algebra on which U_k is a norm.) Prove the exsistence of the first order limit object of functions on Abelian groups. Show examples. Sketch the idea of higher order limit concepts and limit objects. [39]

16. **Vertex and edge coloring models**

In vertex (resp. edge) coloring models graphs can be viewed as particle systems in which a particle is assigned to each vertex (edge) and they interact along edges (resp. vertices). Each particle has a finite number of possible states and the interaction depends on the state of the particles. Partition functions in both models shall be introduced in the talk and their basic properties (reflection positivity, multiplicity, connection matrices and their rank growth) are discussed. The talk should present many examples (Matchings, colorings, flows, etc...). The main result of [16] (characterization theorem for finite vertex coloring models) will be discussed in the talk, and will be proved in the "non-degenerate case". The talk should also state the characterization theorem for edge coloring models [40].

17. **Combinatorial cost**

Introduce the notion of first Betti number and cost for graph sequences and prove their basic relations. Calculate the cost of a large girth sequence and hyperfinite families. Explain the relation with L_2-Betti numbers for residually finite groups. [13]

18. **Sofic groups**

A finitely generated group is called sofic if its Cayley diagram is the Benjamini Schramm limit of finite Schreier graphs. (Note that the notion does not depend on the chosen generating system.) Quite surprisingly, it is not known if every group is sofic. A paper by Elek and Szabó proves the Kaplansky conjecture and the Luck approximation theorem for sofic groups. The talk should present this paper. [12]

19. **Profinite actions as limits of finite actions**

Let Γ be a finitely presented group and let $\Gamma = \Gamma_0 > \Gamma_1 > \ldots$ be a normal chain of subgroups of finite index in Γ with $\cap\Gamma_n = 1$. The Lück Approximation Theorem () implies that

$$\lim_{n\to\infty} \frac{b_1(\Gamma_n)}{|\Gamma : \Gamma_n|} = \beta_1(\Gamma)$$

where $b_1(\Gamma)$ denotes the dimension of the first $\mathbb{Q}$-homology of Γ and $\beta_1(\Gamma)$ is the first L^2 Betti number of Γ. In particular, the growth of the first homology does not depend on the chain. When we substitute b_1 with the minimal number of generators (rank) d, it is easy to see that the limit

$$\mathrm{RG}(\Gamma, (\Gamma_n)) = \lim_{n\to\infty} \frac{d(\Gamma_n)}{|\Gamma : \Gamma_n|}$$

always exists: we call this the rank gradient of Γ with respect to the chain (Γ_n). It is an open problem whether the rank gradient is independent of the chain. Also, no one knows an example where $\mathrm{RG}(\Gamma,(\Gamma_n))$ and $\beta_1(\Gamma)$ are different. If they would be always equal, it would mean that the first homology asymptotically controls the rank, a very surprising phenomenon.

A corresponding problem of Gaboriau is whether the cost of a probability measure preserving free action of a countable group Γ always equals the first L^2 Betti number of Γ. The connection between these problems is established in a paper of Abert and Nikolov, where they show that the rank gradient of a chain can be expressed from the cost of the corresponding profinite action. Using this connection, and assuming that Gaboriau's problem has a positive solution, one can show that the ratio of the Heegaard genus and the rank gets arbitrarily large for closed hyperbolic 3-manifolds. [2]

20. **Ramanujan graphs**

The talk starts with a short introduction to spectral theory of graphs: Alon-Boppana theorem, Ramanujan graphs, eigenvalue distribution and the expected spectral measure under Benjamini-Schramm convergence. Then it presents the main ideas of the result by Abért, Glasner and Virág that says that Ramanujan graphs have essentially large girth. In other terms, the eigenvalue distribution of a d-regular Ramanujan graph is close to the spectral measure of the d-regular tree. [3]

21. **On the growth of L^2-invariants for sequences of lattices in Lie groups**

This talk is about a recent paper by Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault and Samet. Let G be a semisimple Lie group, $K \leq G$ a maximal compact subgroup and $X = G/K$ the associated Riemannian symmetric space. One can study the asymptotic behaviour of invariants (like Betti numbers, twisted torsion, or multiplicities of unitary representations) of the spaces $\Gamma\backslash X$, where Γ varies over the space of lattices of G. For regular covering towers of such spaces, there are classical results due to DeGeorge–Wallach, Delorme, Sarnak-Xue and others.

Adapting the notion of Benjamini-Schramm convergence to the realm of Riemannian manifolds with bounded geometry, one can extend these results to arbitrary sequences of lattices. One can show that for compact symmetric spaces, BS-convergence implies a convergence of the relative Plancherel measures, which leads to convergence of volume normalized multiplicities of unitary representations and Betti numbers.

On the other hand, it turns out that when the volume tends to infinity, every sequence of higher rank locally symmetric G-spaces BS-converges to the universal cover X. An important role here is played by the notion of an Invariant Random Subgroup (IRS). An IRS of G is a random subgroup of G whose distribution is invariant under conjugation. When G is higher rank and simple, the Nevo-Stuck-Zimmer theorem and Kazhdan‘s property (T) leads to a complete description of the space of IRSs of G.

22. **Root measures of graph polynomials and Benjamini Schramm convergence**

There are some natural polynomials one can associate to a finite graph that reflect important properties of the graph. Examples are the chromatic polynomial, more generally the Tutte polynomial, the matching polynomial etc. Roots of these polynomials are also important. Birkhoff initiated the study of roots of the chromatic polynomial (he planned to solve the 4 coloring theorem with this tool). Chromatic roots also turn out to control the behaviour of the antiferromagnetic Potts model at zero temperature. A new direction here is to study the root measure, that is, the uniform measure on the roots of the graph polynomial versus Benjamini-Schramm convergence. This is natural from the statistical physics point of view, as BS convergence on lattices corresponds to taking a thermodynamical limit.

The eigenvalue distribution naturally arises from this point of view, as the root measure of the characteristic polynomial. Here, BS convergence implies weak convergence of the eigenvalue distribution. This is not true for the chromatic measure, but as Abert and Hubai [4] showed, it is true in holomorphic moments. By the recent work of Csikvari and Frenkel [9], this turns out to be the general phenomenon for a large class of graph polynomials, including the Tutte polynomial and the matching polynomial. Since the matching measure is concentrated on the real line, convergence in holomorphic moments implies weak convergence. This leads to a quick analytic proof that the matching ratio is testable.

23. **Finite graphs and amenability**

Hyperfiniteness is a version for amenability for bounded degree graph families. Explain the basic definition and give examples. Introduce hyperfinite graphings. Prove the Kaimanovich’s characterization for hyperfiniteness. Prove that hyperfinite graphings are the limits of hyperfinite graph sequences. Prove the equipartition theorem and conclude that all reasonable parameters on hyperfinite graph families are testable. [11]

24. **A model theory approach to structural limits**

This talks explains the results from the paper [37]. The paper deals with a unification of various graph limit notions using methods from model theory. The talk shoud explain the general concept and its relation to specific limit concepts. In particular the extension of Benjamini-Schramm limit should be presented and illustrated with concrete examples.

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