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Mini-Workshop: Spherical Varieties and Automorphic Representations

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Abstract. This workshop brought together, for the first time, experts on spherical varieties and experts on automorphic forms, in order to discuss subjects of common interest between the two fields. Spherical varieties have a very rich and deep structure, which leads one to attach certain root systems and, eventually, a "Langlands dual" group to them. This turns out to be important for automorphic forms, as it provides a (mostly conjectural) way to analyze periods of automorphic forms and related problems in local harmonic analysis.

Mathematics Subject Classification (2010): 22 (primary), 11, 14 (secondary).

Introduction by the Organisers

The goal of this workshop was to bring together, for the first time, experts on two different fields: spherical varieties, on one hand, and periods of automorphic forms, on the other.

If G is a reductive group over a global field k (e.g.: a number field), automorphic forms are (roughly speaking) elements of irreducible representations for the action of $G(\mathbb{A}_k)$ on $C^\infty([G])$, where $\mathbb{A}_k$ denotes the ring of adeles of k, and $[G] = G(k)\backslash G(\mathbb{A}_k)$. (The reader may think of $G(k)\backslash G(\mathbb{A}_k)$ as a homogeneous manifold such as $\mathrm{GL}_n(\mathbb{Z})\backslash \mathrm{GL}_n(\mathbb{R})$, whose functions are equipped with more symmetries than the action of $G(\mathbb{R})$, the so-called *Hecke operators.*)

Automorphic representations have important invariants which are very difficult to study, such as their L-functions. A common way to study automorphic L-functions is to take large enough – spherical – subgroups $H \subset G$ and to consider *period integrals* of automorphic forms, i.e. integrals of the form:

$$\int_{[H]} \phi(h)dh,$$

perhaps adding a continuously varying character of $[H]$. Such integrals are often equal to L-functions or special values of L-functions, and they also reveal other interesting properties of the automorphic representation π of ϕ, such as being a "functorial lift" in the sense of Langlands.

It turns out that these phenomena are related to the structure of spherical varieties discovered in works of Luna, Vust, Brion, Knop and others, and to the dual group $\check{G}_X$ that was associated by Gaitsgory and Nadler to any spherical variety X, based on this structure.

The goal of our workshop was to inform experts on spherical varieties about the general theory and relevant problems in automorphic forms, and vice versa. Automorphic representations appear in harmonic analysis of the homogeneous space $[G]$, and their local constituents appear in harmonic analysis on the symmetric space $G(k_v)$, where v is any completion of k; Wee Teck Gan gave a general introduction to non-abelian harmonic analysis, with an emphasis on the Plancherel decomposition of $L^2(G(k_v))$. This introduction was continued by Dipendra Prasad, who talked about automorphic representations, the local Langlands Conjecture, and automorphic L-functions. On the side of spherical varieties, Guido Pezzini gave an introduction to their structure and invariants, and Paolo Bravi described the "Luna systems", i.e. these combinatorial invariants which are used to classify wonderful and spherical varieties. The relation between harmonic analysis and the compactification theory started appearing in the talk of Yiannis Sakellaridis, who explained the role of "boundary degenerations", i.e. normal bundles to G-orbits in suitable compactifications, to the description of the continuous spectrum in the Plancherel formula for $L^2(X(k_v))$. There are several root systems that one can attach to a spherical variety, always with the same Weyl group and on the same vector space but with roots of different length, encoding different features of the variety; Bart Van Steirteghem compared the various root systems and explained their use. Proving that these invariants give rise to root systems, though, is quite involved, and Friedrich Knop recounted his analysis of the moment map (the graded version of his analysis of invariant differential operators), which provides a conceptual proof for the appearance of root systems. Bernhard Krötz described problems in real harmonic analysis related to spherical varieties and "real spherical varieties", a term he uses for homogeneous spaces for a real Lie group on which a minimal $\mathbb{R}$-rational parabolic (not necessarily a Borel) acts with an open orbit. One of the ways to understand the root system of a(n affine) spherical variety is as a measure of the failure of the coordinate ring – which is naturally filtered by dominant weights – to be graded; on the third day, Michel Brion described the invariant Hilbert scheme, which roughly describes the possible ring structures on a given G-module. On the automorphic side, Omer Offen introduced the relative trace formula of Jacquet, a basic tool for studying period integrals, but possibly also the natural setting in which conjectures about spherical varieties and their dual group should be formulated. On the fourth day, Yiannis Sakellaridis explained the Satake isomorphism, the work of Gaitsgory and Nadler, and the relevance of finer invariants, such as colors, in the study of unramified functions on spherical varieties. There is a very simple and telling way in which the Weyl group of a spherical variety appears, and this is by considering an action, defined by Friedrich Knop, of the full Weyl group on the set of Borel orbits; Jacopo Gandini described this action, as well as further results on the structure of Borel orbits. On the last day Farrell Brumley explained the number-theoretic importance of periods via some examples from analytic number theory, and Nicolas Templier described results on the asymptotic behavior of Whittaker functions. Finally, Stephanie Cupit-Foutou has used the invariant Hilbert scheme to classify spherical varieties, and she gave an alternative description of Luna data based on the invariant Hilbert scheme.

Mini-Workshop: Spherical Varieties and Automorphic Representations

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Non-abelian Harmonic Analysis, Automorphic Forms and the Langlands Program

Wee Teck Gan

In this expository talk, I gave a brief introduction to the Langlands program, using abelian harmonic analysis (i.e. the theory of Fourier series and Fourier transform) as a motivation.

0.1. **Abelian harmonic analysis.** More precisely, suppose that G is the additive group $\mathbb{G}_a$, then classical harmonic analysis concerns the study of the unitary representations $L^2(G(\mathbb{R}))$ and $L^2(G(\mathbb{Z})\backslash G(\mathbb{R}))$ under right translation. To understand how these representations decompose into irreducible ones, it is clearly useful to have explicit knowledge of the unitary dual $\widehat{G(\mathbb{R})}$ of $G(\mathbb{R})$. In this case, one knows that

$$\widehat{G(\mathbb{R})} = \{\chi_y : y \in \mathbb{R}\}$$

where

$$\chi_y(x) = e^{2\pi i x y}.$$

Then the theory of Fourier series gives a direct sum decomposition

$$L^2(G(\mathbb{Z})\backslash G(\mathbb{R})) \cong \bigoplus_{y \in \mathbb{Z}} \mathbb{C} \cdot \chi_y.$$

On the other hand, the theory of Fourier transform gives a direct integral decomposition

$$L^2(G(\mathbb{R})) \cong \int_{\widehat{G(\mathbb{R})}} \chi_y \otimes \chi_y^\vee \, dy$$

as a representation of $G(\mathbb{R}) \times G(\mathbb{R})$, with dy the Lebesgue measure on $\widehat{G(\mathbb{R})} = \mathbb{R}$.

0.2. **Non-abelian harmonic analysis.** Now one may replace the additive group with a more interesting group, such as a connected semisimple (or reductive) linear algebraic group over a local field F (such as $\mathbb{R}$ or $\mathbb{Q}_p$). In this case, one does not have a full knowledge of the unitary dual $\widehat{G(F)}$. Nonetheless, one can show that there is a canonical decomposition:

$$L^2(G(F)) \cong \int_{\widehat{G(F)}} \pi \otimes \pi^\vee \, d\mu_G(\pi)$$

for some measure $d\mu_G$ (the Plancherel measure of $G(F)$). When $F = \mathbb{R}$, Harish-Chandra gave a rather precise description of this Plancherel measure. The highlights are:

- the measure $d\mu_G$ has continuous as well as atomic parts; thus
$$L^2(G(F)) = L^2_{disc}(G(F)) \oplus L^2_{cont}(G(F))$$
is the sum of its discrete spectrum and its continuous spectrum. The representations π contributing to the discrete spectrum are called the discrete series representations. They have L^2 matrix coefficients and the embedding $\pi \otimes \pi^\vee \longrightarrow L^2(G(F))$ is canonically constructed by the formation of matrix coefficients.
- the support of the measure is not the full unitary dual; indeed, the support is on the set of tempered representations (those whose matrix coefficients are $L^{2+\epsilon}$ for every $\epsilon > 0$);
- there is a refined decomposition
$$L^2(G(F)) \cong \bigoplus_M L^2_M$$
as M runs over association classes of Levi subgroups of G, so that the discrete spectrum is the term corresponding to $M = G$. This decomposition reflects the behaviour of the matrix coefficients of tempered representations as one approaches infinity in different directions. Moreover, L^2_M can be described in terms of the discrete spectrum of $M(F)$ (with fixed central character), via the theory of Eisenstein integral.

0.3. **Local Langlands correspondence.** The point is that the study of $L^2(G(F))$ singles out certain natural classes of irreducible representations of $G(F)$. When $F = \mathbb{R}$, the work of Harish-Chandra et al gives a classification of the discrete series and tempered representations, and the work of Langlands then classifies all irreducible admissible representations (which are not necessarily unitary) in terms of the tempered ones. When F is p-adic, the analogous classification is known as the local Langlands conjecture for $G(F)$. We assume that G is F-split henceforth, for simplicity.

More precisely, let $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil-Deligne group of F, with W_F the Weil group. Then a local Langlands parameter is a conjugacy class of admissible homomorphisms
$$\phi : WD_F \longrightarrow G^\vee$$
where $G^\vee$ is the Langlands dual group of G. Thus, a local Langlands parameter is essentially a local Galois representation, valued in $G^\vee$.

The local Langlands conjecture then postulates that there is a finite-to-one map (with precisely determined fibers)
$$\mathcal{L} : \mathrm{Irr}(G(F)) \longrightarrow \Phi(G(F)).$$
This conjecture is now known for $G = \mathrm{GL}(n)$ by Harris-Taylor and Henniart, and for the classical groups by recent work of Arthur and Moeglin.

0.4. **Automorphic forms for real groups.** So far, we have focused on the representation $L^2(G(F))$. What about the representation $L^2(G(\mathbb{Z})\backslash G(\mathbb{R}))$? The study of this is the subject matter of the theory of automorphic forms.

As before, one has a decomposition

$$L^2(G(\mathbb{Z})\backslash G(\mathbb{R})) = L^2_{disc}(G(\mathbb{Z})\backslash G(\mathbb{R})) \oplus L^2_{cont}(G(\mathbb{Z})\backslash G(\mathbb{R})).$$

We call $L^2_{disc}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ the automorphic discrete spectrum; it decomposes as

$$L^2_{disc}(G(\mathbb{Z})\backslash G(\mathbb{R})) \cong \bigoplus_{\pi \in \widehat{G(\mathbb{R})}} m(\pi) \cdot \pi$$

for some multiplicity $m(\pi) < \infty$. Moreover, the continuous part $L^2_{cont}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ can be described in terms of the automorphic discrete spectrum of proper Levi subgroup of $G(\mathbb{R})$: this is the theory of Eisenstein series due to Langlands. Thus, the study of the automorphic discrete spectrum is the fundamental issue.

0.5. **Adelic automorphic forms.** The fact that the multiplicities $m(\pi)$ can be > 1 suggests that there are extra symmetries in the system to be exploited. As it turns out, there is a natural family of operators acting on $L^2(G(\mathbb{Z})\backslash G(\mathbb{R}))$ commuting with the action of $G(\mathbb{R})$. These are the Hecke operators. Thus, one should study $L^2(G(\mathbb{Z})\backslash G(\mathbb{R}))$ as a module over $G(\mathbb{R}) \times \mathcal{H}_{G(\mathbb{Z})}$, where the latter is the Hecke algebra associated to $G(\mathbb{Z})$. What does one know about this Hecke algebra? It turns out that the best framework to encompass both the action of $G(\mathbb{R})$ and that of the Hecke algebra is to use the ring of adeles $\mathbb{A}$. When $F = \mathbb{Q}$, for example,

$$\mathbb{A} = \mathbb{R} \times {\prod_p}' \mathbb{Q}_p$$

where the RHS is a restricted direct product.

0.6. **Global Langlands program.** Thus, one is finally led to consider the representation $L^2(G(F)\backslash G(\mathbb{A}))$ of $G(\mathbb{A})$ by right translation. The representations which intervene in the discrete spectrum of this are the so-called square-integrable automorphic representations. The theory of automorphic forms seeks to understand these automorphic representations. The global Langlands program connects them with Galois representations, objects of fundamental importance in number theory. To a first approximation, the global Langlands conjecture says that

Automorphic representations and Galois representations

are, in very precise ways, two sides of the same coin.

Basics of the structure of spherical varieties

Guido Pezzini

Let G be a reductive connected complex algebraic group, and X a normal irreducible complex algebraic variety equipped with an action of G.

In the paper [4], D. Luna and T. Vust introduced an invariant of such an action, with the goal of giving a sort of measure to which extent the properties of the G-action determine the geometry of the variety. This invariant, called the *complexity* of X and denoted by $c(X)$, is defined as the minimal codimension of a B-orbit on X, where B is a Borel subgroup of G. The idea is that the lower the complexity, the more influence the symmetries of X induced by G have on the geometry of X.

Under this point of view varieties of complexity zero are the simplest cases of G-varieties, yet they include many varieties that are classically studied in the theory of reductive groups. More precisely, *normal* varieties of complexity zero are called *spherical varieties*, and have been studied extensively in the last 30 years.

The assumption of normality is of technical nature, but so fundamental for the theory that only in the last few years the first results on some non-normal complexity zero varieties have appeared in the literature (see e.g. [2]). For the purposes of this report, we may underline that a useful consequence of normality is that X is covered by quasi-affine G-stable open subsets (see [6, Lemma 8]).

Examples of spherical varieties are complete homogeneous spaces $X = G/P$, where P is a parabolic subgroup of G; toric varieties, which is the case where $G = (\mathbb{G}_m)^n$ is an algebraic torus; symmetric homogeneous spaces $X = G/G^\theta$ where $\theta\colon G \to G$ is an involution. We report some other examples.

(1) $G = \mathrm{SL}(2) \times \mathrm{SL}(2) \times \mathrm{SL}(2)$ and $X = G/\mathrm{diag}(\mathrm{SL}(2))$ (notice that such a homogeneous space is not spherical if a semisimple group of rank higher than 1 is used instead of $\mathrm{SL}(2)$).
(2) $X = G/U$ where U is a maximal unipotent subgroup of G.
(3) $X = \mathrm{SL}(3)/H$ with $H = TU_{\alpha_1+\alpha_2}$, where $T \subset B$ is a maximal torus, α_1, α_2 are simple roots associated to T and B, and $U_{\alpha_1+\alpha_2}$ is the one dimensional unipotent subgroup of B associated to $\alpha_1 + \alpha_2$.
(4) The projective space of 2-by-2 matrices $\mathbb{P}(M_{2\times 2})$, with $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$ acting by left and right multiplication.

Sphericity is equivalent to various other properties, and has strong consequences for the G- and B-action on X. We summarize in the next two theorems some basic facts, and for other equivalent definition of sphericity we refer to Chapter 5 of [7].

Theorem 1 ([9]). *Let X be a normal irreducible G-variety. If X is affine, then X is spherical if and only if the ring of regular functions $\mathbb{C}[X]$ is a multiplicity-free G-module, i.e. any two distinct irreducible submodules are non-isomorphic. In general, X is spherical if and only if the space of global sections $\Gamma(X, \mathcal{L})$ is a multiplicity-free G-module, for every linearized line bundle $\mathcal{L}$ on X.*

Theorem 2 ([8] and [1]). *If X is a spherical G-variety, then G and B have a finite number of orbits on it, and the closure of any G-orbit is a spherical G-variety.*

We remark that in general the inequality $c(Z) \leq c(X)$ holds for every closed, irreducible, B-stable subset Z of X, without assuming X spherical. This implies the finiteness of the number of B- and G-orbits whenever $c(X) = 0$. On the other hand, if Y is also such a subset and $Z \subset Y$ holds, then the inequality $c(Z) \leq c(Y)$ is *not* true in general. A counterexample is found in the variety $\mathbb{P}(M_{2\times 2})$ under the action of $G = \mathrm{SL}(2)$ by left multiplication.

Several discrete invariants of a spherical G-variety X can be naturally defined. Due to the central role of Borel subgroups in the representation theory of G, most invariants involve the action of B:

(1) the set of B-eigenvalues of rational functions (on X) that are B-eigenvectors; it is a subgroup, denoted by $\Lambda(X)$, of the group of characters of B, and its rank is by definition the *rank* of X as a spherical variety;
(2) the vector space $\Lambda^*_{\mathbb{Q}}(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$;
(3) the (finite) set $\mathcal{D}(X)$ of all B-stable but not G-stable prime divisors of X, called *colors*.

A spherical variety X contains a dense G-orbit, which we denote by X_0, and X is also called an *embedding* of X_0. Then the above invariants actually depend only on X_0; this can be made precise also for colors, e.g. replacing a color by its generic point.

Notice that a B-eigenvector $f_\chi \in \mathbb{C}(X)$ is determined by its B-eigenvalue $\chi \in \Lambda(X)$ up to multiplication by a constant. Moreover, any $\mathbb{Q}$-valued discrete valuation $\nu\colon \mathbb{C}(X)^* \to \mathbb{Q}$ (over the constant functions) induces an element $\rho(\nu)$ of $\Lambda^*_{\mathbb{Q}}(X)$, by requiring that $\rho(\nu)$ take the value $\nu(f_\chi)$ on $\chi \in \Lambda(X)$.

One may apply this construction to the valuation associated with any prime divisor on X. The advantage of considering valuations is that they are defined on $\mathbb{C}(X_0) = \mathbb{C}(X)$ regardless of whether they come from some prime divisor or not.

Under this point of view G-invariant valuations are particularly useful in describing the difference set $X \setminus X_0$: since it is G-stable, the valuation associated to any prime divisor contained in $X \setminus X_0$ is G-invariant.

Invariant valuations are also strictly related to the *little Weyl group*, a crucial invariant of a spherical variety. The first result in this direction is the following.

Theorem 3 ([5]). *Let X be a spherical variety. Then*

$$V(X) = \{\rho(\nu) \mid \nu \textit{ is a } G\textit{-invariant valuation } \}$$

is a convex polyhedral cone in $\Lambda^*_{\mathbb{Q}}(X)$.

In analogy with the classification of toric varieties, embeddings of a fixed spherical G-homogeneous space X_0 can be classified by means of families of convex cones in the vector space $\Lambda^*_{\mathbb{Q}}(X_0)$. Some data has to be added taking into account the behaviour of colors. We outline this classification, referring to [3] and [4] for details.

The role of affine toric varieties is played here by *simple* spherical varieties, which are by definition spherical varieties with a unique closed G-orbit. Indeed, in general if $Y \subseteq X$ is any G-orbit, then

$$X_{Y,G} = \{x \in X \mid \overline{Gx} \supseteq Y\}$$

is open in X, quasi-projective and G-stable, spherical and simple with unique closed orbit Y. Therefore X is covered by simple spherical varieties.

Now to any simple spherical variety X with open orbit X_0 and closed orbit Y we associate two objects:

(1) the cone $\mathcal{C}_X$ generated in $\Lambda^*_{\mathbb{Q}}(X_0)$ by the image of the valuations associated to all B-stable prime divisors containing Y;
(2) the set $\mathcal{D}_X$ of colors containing Y.

The couple $(\mathcal{C}_X, \mathcal{D}_X)$ is called the *colored cone* of X. “Admissible” colored cones are defined combinatorially in [3, §3].

Theorem 4. *Let X_0 be a spherical G-homogeneous space. The map $X \mapsto (\mathcal{C}_X, \mathcal{D}_X)$ is a bijection between simple embeddings of X_0 (up to G-equivariant isomorphisms that are the identity on X_0) and colored cones in $\Lambda^*_{\mathbb{Q}}(X_0)$.*

If X is not simple, then we consider its G-orbits $Y_1, \ldots, Y_n$. The set

$$\mathcal{F}_X = \{(\mathcal{C}_{X_{Y_i,G}}, \mathcal{D}_{X_{Y_i,G}}) \mid i \in \{1, \ldots, n\}\}$$

is called the *colored fan* of X, and admissible colored fans are also defined combinatorially in [3, §3].

Theorem 5. *Let X_0 be a spherical G-homogeneous space. The map $X \mapsto \mathcal{F}_X$ is a bijection between embeddings of X_0 (up to G-equivariant isomorphisms that are the identity on X_0) and colored fans in $\Lambda^*_{\mathbb{Q}}(X_0)$.*

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Periods of Automorphic forms and distinguished representations

Dipendra Prasad

The Oberwolfach mini-workshop on 'Spherical varieties and Automorphic representations' brought together people in two rather different subjects: on the one hand people working on Spherical varieties, mostly from an algebraic geometric point of view, and mostly over algebraically closed field of characteristic 0, and then on the other hand, there were people pursuing Automorphic representations. That the two subjects have got a lot in common is through the recent work of Sakellaridis and Venkatesh, and it is clear that the detailed knowledge of Spherical varieties will be a very useful knowledge for people in Automorphic representations.

The program was therefore mainly instructional in nature, where people were expected not to talk to specialists in their own subject, but to people belonging to the other half of the group.

I gave an introductory lecture on Period of Automorphic forms and distinguished representations. My lecture was the 2nd lecture on Automorphic forms on the very first day, following the lecture of Wee Teck Gan. So I began by filling in a few details from his talk, and then continued with my topic.

I divided my talk into the following sections:

(1) L-group, and some basic examples.
(2) The Local Langlands Correspondence.
(3) The notion of Automorphic Representations.
(4) Local and Global L-functions, and their analytic properties.
(5) Principle of functoriality, in particular the symmetric powers.
(6) Distinguished representations: the relative theory.
(7) Period integrals, and its relation to local distinction.

In my lectures, I concentrated mostly on non-Archimedean local fields k, and defined a distinguished representation of a group G distinguished by a (spherical subgroup) H to be an irreducible representation for which there exists a nonzero H-invariant linear form. I stated the basic conjecture in the subject due to Sakellaridis and Venkatesh, and verified in several cases by Sakellaridis, and by Gan-Gomez that the representations of G distinguished by H should arise as functorial transfer from another group. I suggested in my lecture that if there exists a unique $H(k)$ orbit which is open in the (Hausdorff) topology coming from k on the flag variety $(G/P_0)(k)$ where P_0 is the minimal parabolic in G defined over k, then there is a unique H-invariant linear form. The audience clarified that this is too optimistic. First of all, if H is not reductive, there is no hope, such as for the unipotent radical of a Borel. But even if H is reductive it fails, for example, $G = SO(7)$ containing $H = GL(3)$.

My talk which was of 90 minutes fell short of the last topic on Period integrals, which were defined in several later talks such as that of Sakellaridis, and Omer Offen.

Luna's combinatorics

Paolo Bravi

We briefly present the combinatorial invariants introduced by D. Luna to classify the spherical homogeneous spaces, see [7] and the references therein.

Spherical closure. Let G be a reductive linear algebraic group over an algebraically closed field k of characteristic zero. Let T be a maximal torus and B a Borel subgroup of G containing T. Let S be the corresponding set of simple roots of (G, T).

Let H be a spherical subgroup of G. A color of G/H is by definition a prime B-stable divisor of G/H. The set of colors of G/H is denoted by $\mathcal{D}(G/H)$.

The normalizer $\mathrm{N}_G(H)$ of H acts on the set $\mathcal{D}(G/H)$. The kernel of this action is called the spherical closure of H, and denoted by $\overline{H}$. Clearly, $\mathcal{D}(G/\overline{H}) = \mathcal{D}(G/H)$ and $\overline{\overline{H}} = \overline{H}$. A subgroup H of G is called spherically closed if $\overline{H} = H$.

In general, the quotient $\overline{H}/H$ is diagonalizable and the spherical subgroups H of G, with a given spherical closure K, are classified by the lattices $\Lambda(G/H)$, of B-eigenvalues of rational B-eigenfunctions on G/H, which contain $\Lambda(G/K)$.

If H is spherically closed then it has finite index in its normalizer, therefore G/H has a canonical simple embedding associated with the cone of G-invariant valuations (which is strictly convex in this case) and no colors containing the closed G-orbit. By a theorem of F. Knop [5], every spherically closed subgroup H of G is wonderful, i.e. the canonical embedding G/H is wonderful.

Wonderful varieties. An algebraic G-variety X is called wonderful if: is smooth, complete, has an open G-orbit whose complementary is union of smooth prime G-stable divisors $X_1, \ldots, X_r$ with non-empty transversal intersections and, for all $x \in X$, $\overline{G.x} = \cap_{x \in X_i} X_i$.

A wonderful G-variety is spherical, it is a simple complete toroidal embedding of its open G-orbit.

To a wonderful variety X we attach essentially three invariants: the closed orbit, the spherical roots and the colors (viewed as functionals on $\Lambda(X)$).

There exists a (unique) point $z \in X$ stabilized by B^-. Clearly, z lies in the closed G-orbit and its stabilizer is a parabolic subgroup which is opposite to the stabilizer P_X of the open B-orbit. The subset of S of simple roots of the Levi of P_X is denoted by S^p_X.

Spherical roots. Let H be a spherically closed subgroup of G, in particular it contains the center of G. We can assume that G is semisimple and simply connected. Let X denote the wonderful embedding of G/H.

In [3] M. Brion has proved that the cone of G-invariant valuations is defined by a set of linear independent inequalities $\langle v, \sigma \rangle \leq 0$, for some primitive elements σ in $\Lambda(X)$. These elements σ are called spherical roots of X, their set is denoted by Σ_X.

Under our hypotheses, the spherical roots of X are non-negative integer combinations of simple roots, and generate $\Lambda(X)$ (which we will also denote by $\mathbb{Z}\Sigma_X$). They are equal to the T-weights occurring in $\mathrm{T}_z X/\mathrm{T}_z(G.z)$, the normal space at

z of the closed G-orbit in X. In particular, they are in correspondence with prime G-stable divisors, $\sigma \mapsto X_\sigma$, such that σ is the T-weight occurring in $\mathrm{T}_z X/\mathrm{T}_z X_\sigma$.

On the other hand the G-stable subvarieties of X are wonderful, and they are in bijective correspondence with subsets of Σ_X. In particular, the spherical roots of X are spherical roots of rank 1 wonderful G-varieties. Since the latter have been classified, the list of elements that can occur as spherical roots of wonderful G-varieties is finite and known for any G.

Colors. Recall we have assumed G to be simply connected. Let $\mathcal{D}_X$ be the set of prime B-stable not G-stable divisors of X; these are exactly the closures of the colors of G/H, so that as sets $\mathcal{D}_X = \mathcal{D}(G/H)$. Every B-stable divisor of G/H, i.e. an element $D \in \mathbb{N}\mathcal{D}(G/H)$, has an equation in G, i.e. a (unique up to a scalar) regular $B \times H$-eigenfunction (B acting on the left and H on the right) on G. We have an exact sequence

$$0 \to \mathbb{Z}\Sigma_X \to \mathbb{Z}\mathcal{D}_X \to \Lambda(H) \to 0.$$

Indeed, the Picard group of X is freely generated by the linear equivalence classes of the elements of $\mathcal{D}_X$, and the Cartan pairing is the $\mathbb{Z}$-bilinear pairing $c_X \colon \mathbb{Z}\mathcal{D}_X \times \mathbb{Z}\Sigma_X \to \mathbb{Z}$ such that for all $\sigma \in \Sigma_X$

$$[X_\sigma] = \sum_{D \in \mathcal{D}_X} c_X(D,\sigma)[D].$$

Spherical systems. The triple (S^p_X, Σ_X, c_X) is called the spherical system of X. It is a purely combinatorial datum. Notice that Luna's definition is slightly different, but equivalent.

There are constraints on c_X. For every simple root α, let P_α denote the minimal standard parabolic subgroup of G corresponding to α, and set $\mathcal{D}_X(\alpha) = \{D \in \mathcal{D}_X : P_\alpha.D \neq D\}$. The latter contains at most two elements. More precisely:

- p) $\alpha \in S^p_X$ if and only if $\mathcal{D}_X(\alpha) = \emptyset$;
- a) $\alpha \in S \cap \Sigma$ if and only if $\mathcal{D}_X(\alpha)$ contains two elements, say D^+_α, D^-_α, and $c_X(D^+_\alpha,\sigma) + c_X(D^-_\alpha,\sigma) = \langle \alpha^\vee, \sigma\rangle$ for all $\sigma \in \Sigma_X$;
- 2a) if $\alpha \in S \cap \frac{1}{2}\Sigma_X$, $\mathcal{D}_X(\alpha)$ contains one element, say $D_{2\alpha}$, and $c_X(D_{2\alpha},\sigma) = \frac{1}{2}\langle \alpha^\vee, \sigma\rangle$ for all $\sigma \in \Sigma_X$;
- b) otherwise $\mathcal{D}_X(\alpha)$ contains one element, say D_α, and $c_X(D_{2\alpha},\sigma) = \langle \alpha^\vee, \sigma\rangle$ for all $\sigma \in \Sigma_X$.

The sets $\mathcal{D}_X(\alpha)$ need not be disjoint, but there are strong constraints on their intersections.

This leads to certain combinatorial axioms which define an abstract spherical system. The Luna conjecture (which is now proved [7, 6, 4, 2]) states that the wonderful varieties are classified by the abstract spherical systems.

Morphisms. Let $\varphi \colon X \to Y$ be a G-equivariant surjective morphism with connected fibers between wonderful G-varieties. The subset of colors $\mathcal{D}_\varphi \subset \mathcal{D}_X$ that map dominantly is distinguished, that is, there exists $D \in \mathbb{N}_{>0}\mathcal{D}_\varphi$ such that $c_X(D,\sigma) \geq 0$ for all $\sigma \in \Sigma_X$. The spherical system of Y can then be combinatorially obtained from the spherical system of X and the set $\mathcal{D}_\varphi$. In particular, the set

Σ_Y is the basis of the (free) monoid $\{\sigma \in \mathbb{N}\Sigma_X : c_X(D,\sigma) = 0 \text{ for all } D \in \mathcal{D}_\varphi\}$. Moreover, quotient distinguished subsets of $\mathcal{D}_X$ classify such morphisms from X to other wonderful G-varieties.

The minimal morphisms of this kind are of three types, which can be easily described in terms of isotropy groups of the open orbits $H \subset K$ (let us denote by $L_H H^u$ and $L_K K^u$ the respective Levi decompositions with $L_H \subseteq L_K$):

- $\mathcal{P}$) $H^u \supset K^u$, then H is a (maximal) parabolic subgroup of K;
- $\mathcal{R}$) $H^u = K^u$, then H/H^u is contained in no proper parabolic subgroup of K/K^u;
- $\mathcal{L}$) $H^u \subset K^u$, then L_H and L_K are equal up to their connected centers, and K^u/H^u is a simple L_H-module.

These types can also be characterized combinatorially in terms of distinguished subsets [1]. A convenient way of describing the isotropy group H of the open orbit of the wonderful G-variety with a given spherical system is to find (since it is always possible) a full sequence of minimal distinguished subsets which correspond to minimal inclusions $H = H_0 \subset H_1 \subset \cdots \subset H_m = G$ such that first come all inclusions of type $\mathcal{L}$, then all those of type $\mathcal{R}$ and finally all of type $\mathcal{P}$.

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Boundary degenerations and their role in harmonic analysis.

Yiannis Sakellaridis

Given a spherical variety X for a reductive group G over a local field k, and assuming that $X(k)$ carries a $G(k)$-invariant measure (we will denote $X(k)$ simply by X etc, when there is no confusion), a basic goal of nonabelian harmonic analysis is to describe the decomposition of $L^2(X)$ into a direct integral of irreducible unitary representations of G, the so-called Plancherel decomposition. This is closely related, but not identical, to other "distinction" problems such as: Which irreducible admissible representations of G admit an embedding into $C^\infty(X)$?

In the group case, i.e. when $X = H$ under the action of $G = H \times H$ by left and right multiplication, a conjectural description of the Plancherel decomposition is known, due to Langlands. There should be a direct integral decomposition:

$$L^2(H) = \int_{\phi/\sim} \mathcal{H}_\phi \mu(\phi),$$

where ϕ varies over conjugacy classes of "tempered Langlands parameters". These are homomorphisms with bounded image from the Weil group $\mathcal{W}_k$ of k (in the archimedean case) or the Weil-Deligne group $\mathcal{W}_k \times \mathrm{SU}(2)$ (in the non-archimedean case – we will use the symbol $\mathcal{W}'_k$ to unify notation), into the L-group ${}^LH$, such that the image of any Frobenius element is semisimple, composition with the map ${}^LH \to \mathrm{Gal}(\bar{k}/k)$ factors through the natural map: $\mathcal{W}_k \to \mathrm{Gal}(\bar{k}/k)$, and conjugacy is taken with respect to the connected component $\check{H} \subset {}^LH$.

Each such class of tempered Langlands parameters is supposed, by the Local Langlands Conjecture, to correspond to a finite set of unitary irreducible admissible representations of H, and the Hilbert space $\mathcal{H}_\phi$ is supposed to be spanned by their matrix coefficients.

Harish-Chandra [4, 5, 6] developed a Plancherel formula for $L^2(H)$, both in the archimedean and in the non-archimedean case, which reduces the above conjecture to the arithmetic part of the Local Langlands Conjecture, namely the description of the discrete spectrum (i.e. representations π such that the image of matrix coefficients: $\pi \otimes \bar{\pi} \to C(H)$ lies in $L^2(H/Z, \omega_\pi)$, where Z denotes the center of H and ω_π is the central character of π). The continuous spectrum is reconstructed out of discrete spectra of Levi subgroups, but the description of the Plancherel measure $\mu(\phi)$ is quite involved. The theory of wonderful compactifications will give a cleaner answer to this problem.

A generalization of the above conjecture was proposed in [7] for arbitrary spherical varieties: First, one associates an L-group[1] ${}^LG_X$ to any spherical variety, which comes together with a distinguished homomorphism:

$${}^LG_X \times \mathrm{SL}_2 \to {}^LG.$$

Then the conjecture predicts a direct integral decomposition:

$$L^2(X) = \int_{\phi/\sim} \mathcal{H}_\phi \mu(\phi),$$

where ϕ now ranges over tempered Langlands parameters into ${}^LG_X$. The representations in $\mathcal{H}_\phi$ are supposed to belong to the *Arthur packet* corresponding to the parameter built out of ϕ and the above map from SL_2 to ${}^LG$; this is an extra complication that we will not discuss here.

The goal of this talk was to explain how the "continuous spectrum" can be constructed out of "discrete spectra" of simpler spherical G-varieties, the so-called "boundary degenerations" of X. First we explain this at the level of parameters, though: If one believes the above conjecture, one has parameters with various "degrees of continuity"; indeed, let us for simplicity talk about parameters into the connected component $\check{G}_X$ of the L-group of X. The maximal dimension of a

[1]To be precise, this has been written up only in the case that G is split, where the connected component of the L-group suffices.

torus centralizing the image of a parameter can be called the "degree of continuity", since one can then continuously perturb the parameter by parameters into such a torus. The least continuous parameters are those whose image does not belong to a Levi subgroup of $\check{G}_X$; these will be called "discrete". On the other hand, representations which embed into $L^2(X/Z, \omega)$, where Z denotes the connected torus of G-automorphisms of X and ω is a unitary character of Z, will be also be called "discrete" (although "discrete modulo center" is a more precise term). Of course, discrete parameters should correspond to discrete representations; but this correspondence is an arithmetic problem, and here we treat discrete spectra as black boxes.

On the other hand, if in the group case the "more continuous" parameters correspond to discrete (modulo center) representations of Levi subgroups, what do they correspond to in the general case? In the case of symmetric varieties, of course, the answer has been known to be certain symmetric varieties of certain Levi subgroups – for instance, in the Plancherel formula developed in the archimedean case by Delorme, Van den Ban and Schlichtkrull [3, 1, 2]. Still, the description of the Plancherel measure remains mysterious from this point of view.

In [7] we introduced a different description for the continuous spectrum, and proved a Plancherel formula up to discrete spectra (under some assumptions) when G is split and k is p-adic. The basic concept is that of a *boundary degeneration*. While this can be described without compactification theory, its role is more transparent using compactifications of spherical varieties. For simplicity, we will assume that X has a wonderful compactification, i.e. a smooth, proper embedding $\bar{X}$ such that the complement of the open G-orbit is a normal crossings divisor, with a unique closed G-orbit.

It is then known that G-orbits on $\bar{X}$ are in bijection with standard Levi subgroups in the dual group $\check{G}_X$, or subsets of the set of (simple) "spherical roots". For each such subset Θ, we denote by ∞_Θ the corresponding orbit; its normal bundle is spherical, and its open G-orbit is called a *boundary degeneration* of X, and is denoted by X_Θ. It has a G-action and the same dimension as X – hence, it is somehow a model for X close to ∞_Θ; however, it is a simpler space, in the sense that it has more symmetries: being a normal bundle to an intersection of divisors, there is a torus $A_{X,\Theta}$ acting by G-automorphisms, of dimension equal to the codimension of ∞_Θ.

For example, in the group case $X = H$ it can be seen that the isomorphism class of X_Θ is of the following form: there are two opposite parabolics P and P^- of H, with Levi $P \cap P^- = L$, and X_Θ is the $L \times L$-variety L "induced" from these parabolics:

$$X_\Theta = L \times^{(P^- \times P)} (H \times H).$$

The most degenerate X_Θ, corresponding to the closed orbit (which is a flag variety), is always horospherical, i.e. stabilizers contain maximal unipotent subgroups of G.

Now, it is easy to show:

Proposition 1. *The dual group of X_Θ is the standard Levi subgroup $\check{G}_{X,\Theta}$ of $\check{G}_X$ corresponding to Θ.*

Thus, in terms of our heuristics, the "discrete" spectrum of X_Θ, expected to correspond to parameters into $\check{G}_{X,\Theta}$, should be somehow related to the continuous spectrum of X with the same parameters. This is indeed the case:

Theorem 1. *There is a canonical morphism $\iota_\Theta : L^2(X_\Theta) \to L^2(X)$ such that the images of the discrete parts $\iota_\Theta(L^2(X_\Theta)_{\mathrm{disc}})$, for all Θ (including $X = X_\Theta$), span $L^2(X)$.*

The stated property does not justify the term "canonical"; we point the reader to [7] for that.

The "Bernstein map" ι_Θ is not injective, in general. Moreover, the spaces $\iota_\Theta(L^2(X_\Theta)_{\mathrm{disc}})$ and $\iota_{\Theta'}(L^2(X_{\Theta'})_{\mathrm{disc}})$, for $\Theta \neq \Theta'$, are not distinct, in general. Again, in terms of parameters, this is to be expected because parameters which are not conjugate inside of the Levi $\check{G}_{X,\Theta}$ (or inside of two different such Levi subgroups) can become conjugate in $\check{G}$. In terms of "scattering theory", we can think of the maps ι_Θ as "waves coming in from the direction of ∞_Θ"; we should also take into account that these waves will escape, either from this direction or from others.

This is accounted for by the following theorem, proved under some additional conditions:

Theorem 2. *For every element w of the little Weyl group W_X which takes a standard Levi $\check{G}_{X,\Theta} \subset \check{G}_X$ to another standard Levi $\check{G}_{X,\Omega}$ there is a distinguished unitary isomorphism $S_w : L^2(X_\Theta) \to L^2(X_\Omega)$ such that:*

$$\sum_\Theta \iota^*_{\Theta,\mathrm{disc}} : L^2(X) \to \oplus_\Theta L^2(X_\Theta)_{\mathrm{disc}}$$

is an isomorphism into invariants under all the scattering maps S_w.

This reduces the study of $L^2(X)$ to discrete spectra, which is an arithmetic problem. There are also explicit descriptions of the maps ι_Θ and S_w in terms of "normalized Eisenstein integrals" and their functional equations. However, the precise nature of the scattering maps is also a fine arithmetic issue, it seems: if discrete spectra are related to Langlands/Arthur parameters, then the scattering maps seem to be related to L-functions. This is a topic for further research.

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Various interpretations of the root system(s) of a spherical variety

Bart Van Steirteghem

The little Weyl group and the spherical roots. Let G be a complex connected reductive group and let B be a Borel subgroup of G. Recall that a normal irreducible complex algebraic variety X equipped with an action of G is called **spherical** if B has a dense orbit on it. We refer the reader to [13] or [14] for an introduction to spherical varieties. Throughout this paper, X will be a spherical G-variety and G/H will be its unique open G-orbit.

Two basic invariants of X are, using the notations of [14]:

- the subgroup $\Lambda(X)$ of the character group $X(B)$ of B consisting of the B-weights in the field $\mathbb{C}(X)$ of rational functions on X; and
- the so-called **valuation cone** $V(X)$, which is a convex polyhedral cone in $\Lambda^*_{\mathbb{Q}}(X) := \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$ ([12, Proposition 2.1] and [3, Corollaire 3.2]).

Note that these invariants only depend on the open G-orbit of X, that is, $\Lambda(X) = \Lambda(G/H)$ and $V(X) = V(G/H)$.

Another important birational invariant of X is its so-called little Weyl group W_X. It is defined by the following theorem, due to Brion [2, Theorem 3.5]. A completely different proof of (a generalization of) the theorem was given by Knop in [6, Theorem 7.4]. We combine the formulations of [5, Theorem 5.4] and [10, Theorem 1.1.4]. Let $T \subset B$ be a maximal torus, $W = N_G(T)/T$ the associated Weyl group and $N(\Lambda(X))$ the stabilizer in W of $\Lambda(X) \subset X(B) = X(T)$. We equip $\Lambda(X) \otimes \mathbb{Q}$ (and therefore its dual $\Lambda^*_{\mathbb{Q}}(X)$) with an inner product by restricting a W-invariant inner product on $X(T) \otimes \mathbb{Q}$ to $\Lambda(X) \otimes \mathbb{Q}$. As observed in [10], W_X does not depend on the choice of the W-invariant inner product on $X(T) \otimes \mathbb{Q}$.

Theorem 1. *(a) The valuation cone $V(X)$ is a simplicial cone: there exist linearly independent $\sigma_1, \sigma_2, \ldots, \sigma_s \in \Lambda(X)$ such that*

$$V(X) = \{v \in \Lambda^*_{\mathbb{Q}}(X) \colon \langle v, \sigma_i \rangle \leq 0 \text{ for all } i \in \{1, 2, \ldots, s\}\}.$$

*(b) The reflections over the codimension-one faces of $V(X)$ generate a finite subgroup W_X of $\mathrm{GL}(\Lambda^*_{\mathbb{Q}}(X))$. We call W_X the **little Weyl group** of X. In particular, $V(X)$ is a fundamental domain for the action of W_X on $\Lambda^*_{\mathbb{Q}}(X)$.*

(c) The lattice $\Lambda(X) \subset \Lambda(X) \otimes \mathbb{Q}$ is stable under the action of W_X on $\Lambda(X) \otimes \mathbb{Q}$. More precisely, W_X is a subgroup of the image of the map $N(\Lambda(X)) \to \mathrm{GL}(\Lambda^*_{\mathbb{Q}}(X))$ induced by the action of $N(\Lambda(X))$ on $\Lambda^*_{\mathbb{Q}}(X)$.

The theorem says that W_X is a crystallographic reflection group. Let $\Sigma(X)$ be the set of primitive elements $\sigma \in \Lambda(X)$ such that $\ker(\sigma) \subset \Lambda^*_{\mathbb{Q}}(X)$ is a wall of $V(X)$ and $\langle \sigma, V(X)\rangle \leq 0$. The elements of $\Sigma(X)$ are called the **spherical roots** of X. By construction, they are the simple roots of a root system in $\Lambda \otimes \mathbb{Q}$ with Weyl group W_X for which $V(X) \subset \Lambda^*_{\mathbb{Q}}(X)$ is the negative Weyl chamber. This definition is due to Luna [11, §1.2]. The set $\Sigma(X)$ of spherical roots of X is one of the three components of the 'spherical system' of X, a fundamental combinatorial invariant of X [11, §1.2 and §7.2]. For a given group G, the set $\{\sigma \in X(B) \colon \sigma \text{ is a spherical root of some spherical } G\text{-variety}\}$ is finite. If X is wonderful, then the set $\Sigma(X)$ has an elementary geometric description; see, e.g., [13, Definition 3.4.1].

Four other sets of simple roots for W_X. Other choices have been made with regards to the lengths of the simple roots associated to X. Let $\mathcal{L}$ be any $\mathbb{Z}$-submodule of $\Lambda(X) \otimes \mathbb{Q}$ generated by linearly independent vectors which satisfies the following two properties

(L1) $\mathcal{L}$ is W_X-stable; and

(L2) $\mathcal{L}^\perp := \{v \in \Lambda^*_{\mathbb{Q}}(X) \colon \langle v, \mathcal{L}\rangle = 0\}$ is contained in the linear part of $V(X)$.

Then the set $\Sigma(\mathcal{L})$ of primitive elements of $\mathcal{L}$ such that $\ker(\sigma) \subset \Lambda^*_{\mathbb{Q}}(X)$ is a wall of $V(X)$ and $\langle \sigma, V(X)\rangle \leq 0$ is also set of simple roots of a root system with Weyl group W_X.

Besides the standard choice $\mathcal{L} = \Lambda(X)$ mentioned above, four other natural choices are given below. We indicate afterwards why each $\mathcal{L}$ satisfies (L1) and (L2) and briefly discuss the role of each $\Sigma(\mathcal{L})$.

1. $\mathcal{L} = \Lambda(G/N_G(H))$; then $\Sigma(\mathcal{L})$ is denoted $\Sigma^N(X)$;
2. $\mathcal{L} = \Lambda(G/\overline{H})$, where $\overline{H}$ is the spherical closure (see below) of H; then $\Sigma(\mathcal{L})$ is denoted $\Sigma^{sc}(X)$;
3. $\mathcal{L} = \Lambda(X) \cap \Lambda_R = \Lambda(G/(ZH))$, where Λ_R is the root lattice of (G,T) and Z is the center of G; then $\Sigma(\mathcal{L})$ is denoted $\Sigma^K(X)$;
4. $\mathcal{L} = (\Lambda(X) \otimes \mathbb{Q}) \cap \Lambda_R$; then $\Sigma(\mathcal{L})$ is denoted $\Sigma^{SV}(X)$.

Recall that $N_G(H)$ acts on G/H by $n \cdot (gH) = gHn^{-1} = gn^{-1}H$. In fact, the induced map from $N_G(H)$ to the group of G-equivariant automorphisms of G/H is surjective and has kernel H, whence $\mathrm{Aut}^G(G/H) \cong N_G(H)/H$. It follows that $N_G(H)$ acts on the set $\mathcal{D}(G/H)$ of B-stable prime divisors (or colors, see [14]) of G/H. The kernel of this action, which contains H and Z, is called the **spherical closure** $\overline{H}$ of H. Luna introduced this notion and used it to reduce the classification of spherical varieties to that of wonderful varieties [11]. Knop proved that $G/\overline{H}$ has a wonderful compactification in [7, Corollary 7.6].

We now indicate why the four choices for $\mathcal{L}$ above satisfy (L1) and (L2). If K is a subgroup of G containing H then the surjection $G/H \to G/K$ implies that we have an inclusion $\Lambda(G/K) \subset \Lambda(G/H)$ and a surjective linear map $\pi\colon \Lambda^*_{\mathbb{Q}}(G/H) \to \Lambda^*_{\mathbb{Q}}(G/K)$. Moreover $\pi(V(G/H)) = V(G/K)$, see [5, §4]. One can show (using [5, Theorem 4.4] for example) that $\Lambda(G/N_G(H))^\perp \subset \Lambda^*_{\mathbb{Q}}(X)$ is the linear part of $V(X)$, which is also the invariant subspace of $\Lambda^*_{\mathbb{Q}}(X)$ for the action of W_X. It is now straightforward to show that if K is a subgroup of $N_G(H)$ containing H, then $\Lambda(G/K)$ satisfies (L1) and (L2). This takes care of the first three choices for $\mathcal{L}$. For the fourth choice, $\mathcal{L} = (\Lambda(X) \otimes \mathbb{Q}) \cap \Lambda_R$, condition (L1) follows from the second assertion in part (c) of Theorem 1. Condition (L2) follows from the fact that $\Lambda(X) \cap \Lambda_R$ satisfies it.

We briefly discuss the role of the four alternative sets of simple roots, in the same order as above.

1. $\Sigma^N(X)$: The subgroup $\Lambda(G/N_G(H)) \subset \Lambda(X)$ is the 'root lattice' of X, defined in [7, §6], and $\Sigma^N(X)$ is a basis of $\Lambda(G/N_G(H))$ and of the root system Δ_X Knop associates to X. If X is homogeneous or quasi-affine (see [7, Remark 6.6]), then the natural map $\mathrm{Aut}^G(X) \to \mathrm{Hom}(\Lambda(X), \mathbb{C}^\times)$ of [7, Theorem 5.4] induces an isomorphism $\mathrm{Aut}^G(X) \to \mathrm{Hom}(\Lambda(X)/\Lambda(G/N_G(H)), \mathbb{C}^\times)$. If X is quasi-affine, then there is a very simple construction of $\Sigma^N(X)$, see [7, Theorem 1.3]. This set also plays an important role in the geometry of Alexeev and Brion's moduli scheme of affine spherical varieties with a given weight monoid, see [1, Prop 2.13 and Cor 2.14].
2. $\Sigma^{sc}(X)$: We already mentioned the importance of the notion of spherical closure in Luna's classification program of spherical varieties. To be a bit more specific, his theory of augmentations allows one to combinatorially classify all spherical subgroups H of G with a given spherical closure [11, §6.4].
3. $\Sigma^K(X)$: This choice of normalization of the simple roots of W_X is the one in [8, §1]. In this paper, Knop defines the set of spherical roots of a spherical variety over a field of arbitrary characteristic and $\Sigma^K(X)$ is that set when the characteristic is zero.
4. $\Sigma^{SV}(X)$: This is the set of 'normalized simple spherical roots' of [15, §3.1], where the authors also conjecture that it is the set of simple roots of the 'dual group' of X defined by Gaitsgory and Nadler in [4].

From $\mathbf{\Sigma(X)}$ to $\mathbf{\Sigma^N(X)}$, $\mathbf{\Sigma^{sc}(X)}$ and $\mathbf{\Sigma^K(X)}$. The precise relationship between $\Sigma(X)$ and $\Sigma^N(X)$ was described by Losev in [9, Theorem 2]. Given $\sigma \in \Sigma(X)$, either $\sigma \in \Sigma^N(X)$ or $2\sigma \in \Sigma^N(X)$, and Losev's theorem says that $\sigma \in \Sigma(X)$ is doubled in $\Sigma^N(X)$ if and only if $\sigma \notin \Lambda_R$ or σ satisfies one of the conditions (1), (2) or (3) of [9, Definition 4.1.1]. The sets $\Sigma^{sc}(X)$ and $\Sigma^K(X)$ are obtained in a similar fashion from $\Sigma(X)$: for the latter one only doubles those $\sigma \in \Sigma(X)$ that do not belong to the root lattice Λ_R; for $\Sigma^{sc}(X)$ one doubles those $\sigma \in \Sigma(X)$ that do not belong to Λ_R or that satisfy condition (2) or (3) of [9, Definition 4.1.1].

Examples. The following examples were taken from [16]. For $X = \mathrm{SL}(2)/T$ one has $\Sigma(X) = \Sigma^{sc}(X) = \Sigma^K(X) = \Sigma^{SV}(X) = \{\alpha\}$ and $\Sigma^N(X) = \{2\alpha\}$, where α is the simple root of SL(2). For $X = (\mathrm{SL}(2) \times \mathrm{SL}(2))/\mathrm{SL}(2)$, we have $\Sigma(X) = \{\frac{\alpha+\alpha'}{2}\}$, whereas $\Sigma^N(X) = \Sigma^{sc}(X) = \Sigma^K(X) = \Sigma^{SV}(X) = \{\alpha + \alpha'\}$. When $X = \mathrm{SL}(3)/\mathrm{SO}(3)$ we have that $\Sigma^{SV}(X)$ is the set of simple roots of SL(3), whereas $\Sigma(X) = \Sigma^N(X) = \Sigma^{sc}(X) = \Sigma^K(X)$ consists of the doubles of the simple roots. Finally, when $X = \mathsf{G}_2/\mathrm{SL}(3)$, then $\Sigma(X) = \Sigma^K(X) = \Sigma^{SV}(X) = \{\alpha_1 + 2\alpha_2\}$, while $\Sigma^{sc}(X) = \Sigma^N(X) = \{2\alpha_1 + 4\alpha_2\}$, where α_1 and α_2 are the simple roots of G_2.

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Real spherical varieties

Bernhard Krötz
(joint work with Henrik Schlichtkrull)

Our concern is with homogeneous spaces $Z = G/H$ attached to a real reductive group. Here $H < G$ is a closed subgroup with finitely many connected components.

In the sequel we let $P = MAN < G$ be a minimal parabolic subgroup of G. We call Z *real spherical* provided there exists an open P-orbit on Z.

In joint work with H. Schlichtkrull [1] we gave a proof of the Matsuki conjecture which asserts that Z is real spherical if and only if the number of P-orbits on Z is finite.

Real spherical spaces can be characterized by means of representation theory: For that let V be a Harish-Chandra module, $V/\mathfrak{n}V$ the associated finite dimensional Casselman-Jacquet module and V^∞ be the smooth completion of V. Then

$$\dim \operatorname{Hom}_H(V^\infty, \mathbb{C}) \leq \dim(V/\mathfrak{n}V)^{M\cap H}$$

by [1].

Let us say that Z is roughly polar, provided that for all P with PH open there exists a compact subset $\Omega = \Omega(P) \subset G$ and a finite subset $F = F(P) \subset G$ such that

$$G = \Omega A F N_G(H)$$

where $N_G(H)$ is the normalizer of H. If G is a split reductive group, then we show that Z real spherical implies that it is roughly polar [3].

Finally a consequence of the local structure theorem for real spherical homogeneous spaces implies that the stabilizer of an open P-orbit PH is a parabolic subgroup $Q \supset P$ such that $Q \cap H$ is reductive in G – this intersection can be made very precise and falls into two basic classes: $[L, L] < Q \cap H < L$ for a Levi $L < Q$, or $Q \cap H$ is compact mod its center, see [2].

The geometric results assembled above allow analytic constructions such as a Harish-Chandra Schwartz space on Z and a fairly precise description of the Z-tempered spectrum. Future goals are to give a geometric characterization of the discrete spectrum of Z and to establish a Plancherel theorem.

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Moment maps and invariant differential operators

Friedrich Knop

We explained how the little Weyl group of a spherical variety influences the geometry of the moment map on the cotangent bundle and the algebra of invariant differential operators. A good reference is Timashev's textbook [9].

The key to understand the geometry of the moment map is the following construction. Let G be a connected reductive group (everything defined over $\mathbb{C}$) with Lie algebra $\mathfrak{g}$ and $B \subseteq G$ a Borel subgroup. Let X be a G-variety and $f \in \mathbb{C}[X]$ a B-semiinvariant regular function of X. Then $P := \{g \in G \mid gf \in \mathbb{C}f\}$ is a parabolic subgroup of G. On the open subset $X_0 := \{x \in X \mid f(x) \neq 0\}$ one defines the morphism

$$m_f : X_0 \to \mathfrak{g}^* : x \mapsto \left[\xi \mapsto \frac{\xi f(x)}{f(x)}\right].$$

It has the following properties:

Lemma 1. *Let $x_0 \in X_0$ generic, $a_0 = m_f(x_0)$ is image in $\mathfrak{g}^*$ and $\Sigma := m_f^{-1}(a_0)$. Then the centralizer of a_0 in G is a Levi subgroup L of P, the "slice Σ is L-invariant and*

$$P \times^L \Sigma \to X_0 : [p, x] \mapsto px$$

is an isomorphism.

This one of many variants of the Local Structure Theorem. The first one was proved by Brion-Luna-Vust in [2]. The version here was first stated in [5]. It is important that the function f is regular and not just rational. But only quasiaffine varieties have enough functions. Workarounds are: pass to an affine cone or state Lemma 1 for semiinvariant sections of equivariant line bundles. A version which is valid in full generality is stated in [8].

A typical application of Lemma 1 is to choose f as general as possible. Then L becomes small and acts on Σ only via a torus quotient $A = L/L_0$. We state the result for spherical varieties:

Corollary 1. *Every spherical variety X contains an open subset X_0 which is isomorphic to $\overline{A} \times P_u$ where $\overline{A}$ is an affine embedding of A (hence toroidal) and P_u is the unipotent radical of a parabolic P.*

An important point is that one has a lot control over the open subset X_0. For example it can be chosen to meet certain specified G-invariant subvarieties of X.

Assume X to be smooth and spherical, and let $\pi : T_X^* \to X$ be the cotangent bundle. The moment map is defined as

$$m : T_X^* \to \mathfrak{g}^* : z \mapsto \left[\xi \mapsto \langle z, \xi_{\pi(z)}\rangle\right]$$

The function f defines a 1-form $\frac{df}{f}$ hence a section $\sigma_f : X_0 \to T_X^*$ on X_0. This way, m_f factorizes through the moment map: $m_f = m \circ \sigma_f$. Since the union of the images of the various σ_f is Zariski dense, we get

Theorem 1 ([5])**.** *For generic $z \in T_X^*$ let $a := m(z)$ and $S_z := m^{-1}(a)$. Then the centralizer L of a in G is a Levi subgroup which acts on F_z only via a torus quotient $A_z := L/L_0$ which is (non-canonically) isomorphic to A above. Moreover, S_z is a toroidal embedding of A_z.*

The variety S_z projects isomorphically to a subvariety $\Sigma_z \subseteq X$ and generalizes the construction in Corollary 1. The main difference is that the construction of Σ_z does not depend on a choice of a Borel subgroup of G.

This can be used as follows. Every toroidal embedding is determined by the limit behavior of the orbits of 1-parameter subgroups. For Σ_z, Corollary 1 allows to control this behavior for certain 1-PSGs, namely those lying in the valuation cone of X (see e.g. [4] for the valuation cone). On the other hand, the family of tori A_z on T_X^* defines a local system whose monodromy group W_X is finite. Since it acts on Σ_z (or, more precisely, the fan attached to it), this allows to determine the limit behavior of all 1-PSGs which are W_X-conjugate to one in the valuation cone. It turns out that these are, in fact, all. This way, one deduces:

Theorem 2 ([5]). *The "little Weyl group" W_X of X is generated by reflections and the valuation cone of X is one of its Weyl chambers.*

The observation, that the valuation cone is the fundamental domain of a finite reflection group is due to Brion [1]. His proof is entirely different and less conceptual but has the virtue that, at least "in spirit", it carries over to fields of arbitrary characteristic $\neq 2$. See [7]. Vinberg proposed in [10] a construction of W_X which is essentially the same but has a more geometric flavor.

The description of W_X as a monodromy group allows also to describe the algebra of G-invariant functions on T_X^*. For this observe that W_X acts on the vector space $\mathfrak{a} := \mathrm{Lie}A$. Then we have:

Theorem 3 ([3]). *There is a canonical isomorphism $\mathbb{C}[T_X^*]^G \cong \mathbb{C}[\mathfrak{a}^*]^{W_X}$.*

Observe, that if $X = G/H$ is homogeneous then $T_X^* = G \times^H \mathfrak{h}^\perp$ where $\mathfrak{h}^\perp = (\mathfrak{g}/\mathfrak{h})^*$ with $\mathfrak{h} = \mathrm{Lie}H$. Hence

$$\mathbb{C}[\mathfrak{h}^\perp]^H \cong \mathbb{C}[T_X^*]^G \cong \mathbb{C}[\mathfrak{a}^*]^{W_X}$$

This generalizes the classical Chevalley isomorphism which is recovered by choosing $G = G_0 \times G_0$ and $H = \Delta(G_0)$.

There is a non-commutative version, as well. For this, let $\mathcal{D}(X)$ be the algebra of differential operators on X. Let, as usual, ρ be the half-sum of positive roots. Then:

Theorem 4 ([6]). *There is a canonical isomorphism $\mathcal{D}(X)^G \cong \mathbb{C}[\rho + \mathfrak{a}^*]^{W_X}$.*

This is a generalization of the Harish Chandra isomorphism. To see this, use again $X = G_0 = (G_0 \times G_0)/\Delta(G_0)$, as above. Then the enveloping algebra $U(\mathfrak{g}_0)$ coincides with the left invariant differential operators on X. Thus, the set of G-invariant differential operators equals its center $Z(U(\mathfrak{g}_0))$.

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Invariant Hilbert schemes

Michel Brion

The Hilbert scheme is a fundamental object of projective algebraic geometry; it parameterizes the closed subschemes of projective space $\mathbb{P}^n$, having a prescribed Hilbert polynomial. Many moduli spaces (for example, the space of curves of a fixed genus) can be constructed from the Hilbert scheme by taking locally closed subschemes and geometric invariant theory quotients.

The invariant Hilbert scheme, introduced in [1], is an analogue of the Hilbert scheme in the setting of reductive group actions on affine varieties. This talk presented the definition of the invariant Hilbert scheme as well as some basic properties, examples, and applications; we refer to [7] for a recent survey.

For simplicity, we follow an approach via commutative algebra. Let G be a linear algebraic group over the base field k. A *G-algebra* is a finitely generated k-algebra A equipped with an action of G by automorphisms; we further assume that the G-module A is *rational*, i.e., a union of finite-dimensional submodules on which G acts algebraically. If A is reduced, then it is the coordinate ring of an affine G-variety X; in general, A corresponds to an affine *G-scheme*, $X = \mathrm{Spec}(A)$.

We now assume that k has characteristic 0 and G is reductive; then every rational G-module is semi-simple. Given a G-algebra A, we thus have a canonical isomorphism of G-modules

$$A \cong \bigoplus_{M \in \mathrm{Irr}(G)} \mathrm{Hom}^G(M, A) \otimes M,$$

where $\mathrm{Irr}(G)$ denotes the set of isomorphism classes of simple G-modules, and $\mathrm{Hom}^G(M, A)$ is a k-vector space with dimension being the multiplicity of M in A. For the trivial module M_0, we obtain $\mathrm{Hom}^G(M_0, A) = A^G$, the subalgebra of G-invariants in A; moreover, each $\mathrm{Hom}^G(M, A)$ is an A^G-module. By the Hilbert-Nagata theorem, the algebra A^G is finitely generated, and each module of covariants $\mathrm{Hom}^G(M, A)$ is finitely generated as well.

We say that the G-algebra A is *multiplicity-finite*, if every module of covariants is finite-dimensional as a k-vector space; equivalently, A^G is a finite-dimensional k-vector space. The *Hilbert function* of A is then the assignment $h : \mathrm{Irr}(G) \to \mathbb{N}$, $M \mapsto \dim_k \mathrm{Hom}^G(M, A)$. We say that A is *multiplicity-free* if $h(M) \leq 1$ for all $M \in \mathrm{Irr}(G)$.

For example, the coordinate ring $\mathcal{O}(G)$ is multiplicity-free relative to the action of $G \times G$ by left and right multiplication, since $\mathcal{O}(G) \cong \bigoplus_{M \in \mathrm{Irr}(G)} M^* \otimes M$. Given a closed subgroup $H \subset G$, the G-algebra $\mathcal{O}(G/H) = \mathcal{O}(G)^H$ is multiplicity-finite with Hilbert function $M \mapsto \dim_k(M^*)^H$. Also, note that the normal multiplicity-free G-algebras are exactly the coordinate rings of affine spherical G-varieties.

The classification of G-algebras having a prescribed Hilbert function is a generally hopeless problem: examples show that such algebras may have an arbitrary large number of generators. But that classification becomes tractable when one considers G-algebras generated by a prescribed G-module.

More specifically, *given a Hilbert function h and a G-module V, the G-algebras A having Hilbert function h and equipped with a surjective homomorphism of G-algebras $\mathcal{O}(V) \to A$ are parameterized by a quasi-projective scheme,* $\mathrm{Hilb}_h^G(V)$. In other words, the *invariant Hilbert scheme* $\mathrm{Hilb}_h^G(V)$ parameterizes the closed G-subschemes $X \subset V$ with Hilbert function h.

When G is the multiplicative group $\mathbb{G}_m$, we may identify $\mathrm{Irr}(G)$ with $\mathbb{Z}$, and G-modules with graded vector spaces. Let $V := k^{n+1}$ on which G acts with weight -1, and $h : \mathbb{Z} \to \mathbb{N}$ a Hilbert function; then $\mathrm{Hilb}_h^G(V)$ parametrizes the homogeneous ideals $I \subset k[x_0, \ldots, x_n]$ such that $\dim_k(k[x_0, \ldots, x_n]_m / I_m) = h(m)$ for all m. If such ideals exist, then there exists a polynomial P such that $h(m) = P(m)$ for all $m \gg 0$. Conversely, to any polynomial P taking integral values at all large integers, one can assign a function h as above such that $\mathrm{Hilb}_h^G(V)$ is just the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$. Thus, the invariant Hilbert scheme generalizes the classical one; it also generalizes the *multigraded Hilbert scheme* constructed by Haiman and Sturmfels in [9] (but the construction of the invariant Hilbert scheme relies on that of the multigraded one).

We now sketch how to derive from the invariant Hilbert scheme, a moduli space for affine spherical varieties. We assume that k is algebraically closed and G is connected; we choose a Borel subgroup B of G, and identify $\mathrm{Irr}(G)$ with the monoid Λ^+ of dominant weights. Consider an affine spherical G-variety X, and its coordinate ring A; then $A \cong \bigoplus_{\lambda \in \Gamma} V(\lambda)$ as a G-module, where $V(\lambda)$ denotes the simple G-module with highest weight λ, and Γ is a finitely generated submonoid of Λ^+. Moreover, one can choose highest weight vectors $v_\lambda \in V(\lambda)$, where $\lambda \in \Gamma$, such that $v_\lambda v_\mu = v_{\lambda+\mu}$ for all λ, μ. We may now consider the G-algebra structures on the G-module $V(\Gamma) := \bigoplus_{\lambda \in \Gamma} V(\lambda)$ which satisfy the above compatibility relation for highest weight vectors. Any such G-algebra has Hilbert function the characteristic function of Γ, and is generated by $V(\lambda_1) \oplus \cdots \oplus V(\lambda_n)$, where $\lambda_1, \cdots, \lambda_n$ generate the monoid Γ. In fact, one can show that *these G-algebra structures are parameterized by an affine scheme of finite type,* M_Γ, a locally closed subscheme of the corresponding invariant Hilbert scheme.

We may view M_Γ as the scheme of linear maps

$$m : V(\Gamma) \otimes V(\Gamma) \longrightarrow V(\Gamma)$$

which are associative, commutative, G-equivariant and satisfy $m(1, v) = v$ for all v, and $m(v_\lambda, v_\mu) = v_{\lambda+\mu}$ for all λ, μ. Any such map is the sum of isotypical components

$$m^{\nu}_{\lambda,\mu} : V(\lambda) \otimes V(\mu) \longrightarrow V(\nu).$$

An example is the "Cartan multiplication", where $m^{\nu}_{\lambda,\mu} = 0$ whenever $\nu \neq \lambda + \mu$, and $m^{\lambda+\mu}_{\lambda,\mu}$ is the unique G-equivariant map $V(\lambda) \otimes V(\mu) \to V(\lambda + \mu)$ compatible with the choice of highest weight vectors. The corresponding affine spherical variety X_0 is *horospherical*, i.e., the stabilizer of any point contains a maximal unipotent subgroup of G.

Also, M_Γ is equipped with an action of a maximal torus T of B, defined by

$$t \cdot (m^{\nu}_{\lambda,\mu}) = (t^{\lambda+\mu-\nu} m^{\nu}_{\lambda,\mu})$$

with an obvious notation. If $m^{\nu}_{\lambda,\mu} \neq 0$, then the G-module $V(\lambda) \otimes V(\mu)$ contains $V(\nu)$, and hence $\lambda+\mu-\nu$ is a linear combination of simple roots with nonnegative integer coefficients. It follows that the T-orbit closure of any closed point $X \in M_\Gamma$ contains X_0 as its unique T-fixed point. In other words, X_0 is the *horospherical degeneration* of X. Moreover, the T-orbit closure of X is an affine toric variety (for a quotient of T), possibly non-normal. The corresponding monoid is generated by the $\lambda+\mu-\nu$ such that the product $V(\lambda)V(\mu) \subset \mathcal{O}(X)$ contains $V(\nu)$. By a result of Knop (see [11, Thm. 1.3]), it follows that the normalization of this orbit closure is an affine space on which T acts linearly with weights the *spherical roots*.

One can show that M_Γ has only finitely many T-orbits; it is conjectured that the closure of any such orbit is an affine space. This conjecture has been confirmed in several cases of interest: when Γ is the weight monoid of a spherical G-module for G of type A (see [13]), or when Γ is "saturated" (see [10, 2]). In the latter case, the invariant Hilbert scheme yields an approach to the classification of "strict" wonderful varieties; see [3], and [8] for the general case. This geometric approach complements that of Luna (see [12]), Bravi and Pezzini (see [4, 5, 6]), based on Lie theory and combinatorics.

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Introduction to the Relative Trace Formula

Omer Offen

The Relative Trace Formula (RTF) is a tool introduced by Jacquet to study *period integrals* of automorphic forms. (See e.g. [Jac05].) In this talk we explain the notion of a period integral and the RTF associated to the study of periods.

Let F be a number field with ring of addles $\mathbb{A} = \mathbb{A}_F$. Let G be an algebraic group defined over F and let $[G] = G(F)\backslash G(\mathbb{A})$ denote the *automorphic* quotient space.

Assume from now on that G is a connected reductive group and let H be a closed subgroup of G defined over F. For a continuous automorphic function $\varphi : [G] \to \mathbb{C}$ and a character $\chi : [H] \to \mathbb{C}^*$, whenever convergent, we define the period integral

$$\mathcal{P}_{H,\chi}(\varphi) = \int_{[H]} \varphi(h)\chi(h)\ dh.$$

If π is (an automorphic realisation of) an automorphic representation of $G(\mathbb{A})$ then $\mathcal{P}_{H,\chi}|_\pi \in \mathrm{Hom}_{H(\mathbb{A})}(\pi, \chi)$.

Definition 1. An automorphic representation π of $G(\mathbb{A})$ is called (H,χ)-*distinguished* if $\mathcal{P}_{H,\chi}|_\pi \not\equiv 0$.

If χ is the trivial character we simply set $\mathcal{P}_{H,\chi} = \mathcal{P}_H$ and call (H,χ)-distinction also H-distinction.

Distinction often characterizes the image of a functorial transfer. For example, Jacquet showed that if E/F is a quadratic extension, $G = R_{E/F}(GL_n)$ and $H = U_n(E/F)$ is the associated quasi-split unitary group then an irreducible cuspidal representation π of $G(\mathbb{A})$ is H-distinguished if and only if π is a quadratic base-change of an irreducible cuspidal representation of $GL_n(\mathbb{A})$ [Jac10].

Furthermore, for (H,χ)-distinguished representations the value of the period is often related to special values of L-functions. This is expected, in particular, (but not only, as Jacquet's example above shows (cf. [Jac01]) when the data $(G,(H,\chi))$ satisfies *local multiplicity one*. That is, for any place v of F and for every irreducible representation π_v of $G(F_v)$ we have $\dim \mathrm{Hom}_{H(F_v)}(\pi_v, \chi_v) \leq 1$. Ichino and Ikeda formulated a refinement of the Gross-Prasad conjectures, describing the relation between periods and special values of L-functions in this context. Sakellaridis and Venkatesh observed the relation with the local harmonic analysis on the associated G-variety G/H and generalised the conjectures to the context of G-spherical varieties satisfying local multiplicity one.

Now that the study of period integrals is motivated, we turn to the RTF. Let X be a G-variety defined over F and let $\mathcal{S}(X(\mathbb{A}))$ be the space of Schwartz functions on $X(\mathbb{A})$. Form the *automorphic kernel:*

$$\mathcal{K}_\Phi(g) = \sum_{x \in X(F)} \Phi(g^{-1} \cdot x), \quad \Phi \in \mathcal{S}(X(\mathbb{A})).$$

That is, consider a linear map $\Phi \mapsto \mathcal{K}_\Phi : \mathcal{S}(X(\mathbb{A})) \to \mathcal{A}([G])$ to the space of automorphic functions.

Example 1. Consider the action of $G \times G$ on G by $(g_1, g_2) \cdot x = g_1^{-1} x g_2$. Then the automorphic kernel associated with $f \in \mathcal{S}(G(\mathbb{A}))$ is Arthur's kernel function (e.g. [Art78]):

$$\mathcal{K}_f(g_1, g_2) = \sum_{\gamma \in G(F)} f(g_1^{-1} \gamma g_2).$$

For any triple (G, X_1, X_2) where X_i is a homogeneous G-variety $i = 1, 2$ we associate a distribution (the RTF) as follows. Consider $X = X_1 \times X_2$ as a G-space with the diagonal action $g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2)$ and set

$$\mathrm{RTF}(\Phi) = \mathrm{RTF}^G_{X_1, X_2}(\Phi) = \int_{[G]} \mathcal{K}_\Phi(g)\; dg, \quad \Phi \in \mathcal{S}(X(\mathbb{A})).$$

In general, this integral need not converge and requires a regularisation. For the sake of introduction we avoid this complication by assuming that G is anisotropic.

Note that for $\Phi = \Phi_1 \otimes \overline{\Phi}_2$ with $\Phi_i \in \mathcal{S}(X_i(\mathbb{A}))$ we have

$$\mathcal{K}_\Phi = \mathcal{K}_{\Phi_1} \overline{\mathcal{K}_{\Phi_2}}$$

and therefore

$$\mathrm{RTF}_{X_1, X_2}(\Phi) = \langle \mathcal{K}_{\Phi_1}, \mathcal{K}_{\Phi_2} \rangle_{[G]}\,.$$

Assume for simplicity that $X_i(F) = G(F) \cdot \xi_i$, $i = 1, 2$ and set $H_i = G_{\xi_i}$, then there are natural projections $\mathrm{pr}_i : \mathcal{S}(G(\mathbb{A})) \to \mathcal{S}(X_i(\mathbb{A})) \to 0$ defined by

$$\mathrm{pr}_i(f)(g \cdot \xi_i) = \int_{H_i(\mathbb{A})} f(h_i g)\; dg, \quad i = 1, 2.$$

For $f_i \in \mathcal{S}(G(\mathbb{A}))$, $i = 1, 2$ let $f = f_1 * f_2^\vee$ where $f^\vee(g) = f(g^{-1})$ and let $\Phi_i = \mathrm{pr}_i(f_i)$. A formal computation shows that

$$\mathrm{RTF}^G_{X_1, X_2}(\Phi_1 \otimes \Phi_2) = \int_{[H_1 \times H_2]} \mathcal{K}_f(h_1, h_2)\; dh_1\; dh_2.$$

The *geometric expansion* of the relative trace formula is based on decomposing $X(F)$ in terms of $G(F)$-orbits. We note that there is a natural bijection

$$G(F) \cdot x \leftrightarrow H_1(F) g H_2(F) : G(F) \backslash X(F) \simeq H_1(F) \backslash G(F) / H_2(F)$$

with isomorphic stabilizers

$$G_x = H_1 \cap g H_2 g^{-1} \simeq (H_1 \times H_2)_g.$$

where $x = (g_1 \cdot \xi_1, g_2 \cdot \xi_2)$ and $g = g_1^{-1} g_2$. The geometric expansion is

$$\mathrm{RTF}_{X_1,X_2}(\Phi) = \sum_{\gamma \in [G(F)\backslash X(F)]} \mathrm{vol}([G_\gamma])\mathcal{O}(\Phi;\gamma)$$

where the orbital integrals are given by

$$\mathcal{O}(\Phi;\gamma) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} \Phi(g^{-1}\cdot\gamma)\; dg.$$

If $\Phi_i = \mathrm{pr}_i(f_i)$ and $f = f_1 * f_2^\vee$ as before then

$$\mathcal{O}(\Phi_1 \otimes \Phi_2; (\xi_1, g\xi_2)) = \int_{(G_{\xi_1}\times G_{\xi_2})_g(\mathbb{A})\backslash(G_{\xi_1}\times G_{\xi_2})(\mathbb{A})} f(h_1^{-1} g h_2)\; d(h_1,h_2).$$

The spectral decomposition of the relative trace formula is based on the decomposition of $L^2([G])$ into automorphic representations. It is of the form

$$\mathrm{RTF}_{X_1,X_2}(\Phi) = \sum_\pi \int_{[G]} B_\pi(\Phi)$$

where the sum (in general an integral) is over the automorphic spectrum and

$$B_\pi(\Phi_1\otimes\Phi_2) = \sum_{\varphi\in\mathrm{ob}(\pi)} \langle \mathcal{K}_{\Phi_1}, \varphi\rangle \langle \varphi, \mathcal{K}_{\Phi_2}\rangle = \sum_{\varphi\in\mathrm{ob}(\pi)} \mathcal{P}_{H_1}(\pi(f_1)\varphi)\overline{\mathcal{P}_{H_2}(\pi(f_2)\varphi)}$$

where the sum is over an orthonormal basis of π. It follows that π contributes to the RTF (i.e. $B_\pi \not\equiv 0$) if and only if π is both H_1 and H_2-distinguished.

We further discuss the example [Jac86], where a comparison between RTFs was used to study distinguished representations and periods in the context of the Jacquet-Langlands correspondence.

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Satake transform and the L-function of a spherical variety.

Yiannis Sakellaridis

In this talk, we discussed some "generalized Satake transforms", understood as follows: Let G be a reductive group over the ring of integers $\mathfrak{o}$ of a non-archimedean local field k, and let X be a homogeneous spherical G-scheme over $\mathfrak{o}$. The group $K = G(\mathfrak{o})$ is called hyperspecial, and it is a maximal compact subgroup of $G(k)$. For simplicity, we will assume that G is split.

There are some standard invariants of X, such as its character group $\mathcal{X}^*(X)$ (the group of Borel-eigencharacters on rational functions on X) and its dual, the cocharacter group $\Lambda_X = \mathcal{X}_*(X)$. Under some assumptions on X [8], the K-orbits on $X(k)$ are parametrized by the set Λ_X^+ of *antidominant* elements of Λ_X – antidominant in the sense that they belong to the "cone of invariant valuations" of X. Here are some examples:

Example 1. $X = U\backslash G$, where U is a maximal unipotent subgroup of G. Then $\Lambda_X^+ = \Lambda_X = \Lambda_A$, the group of coweights of $A = B/U$, where B is the Borel containing U, and we get the *Iwasawa decomposition*: $G(k) = U(k)\Lambda_B K$.

Example 2. $X = H$, a reductive group under the $G = H \times H$-action. Then Λ_X^+ corresponds to the set Λ_H^+ of antidominant coweights into a universal Cartan of H, and we get the *Cartan decomposition*: $H(k) = K_H \Lambda_H^+ K_H$.

From now on, we denote $X(k)$ by X, $G(k)$ by G etc.

The *spherical/unramified Hecke algebra* $\mathcal{H}(G,K)$ is the algebra of compactly supported, K-biinvariant measures on G. It acts on the space of K-invariant vectors of a G-representation π as:

$$\pi(\mu)(v) = \int_G \pi(g)(v)\mu(g);$$

this integral is really a finite sum.

The following can be considered as the problem at hand:

Problem 1. Consider the space $C_c^\infty(X)^K$ (or another "nice" space of K-invariant functions); it has a basis indexed by Λ_X^+ (the characteristic functions of these cosets). Describe its structure as a module for the spherical Hecke algebra $\mathcal{H}(G,K)$.

The problem includes the description of $\mathcal{H}(G,K)$ itself, which is the classical Satake isomorphism:

Example 3. The space $C_c^\infty(U\backslash G)^K$ has an additional action of the spherical Hecke algebra $\mathcal{H}(A, A_0)$ of the "universal Cartan" $A = B/U$ "on the left", which commutes with the action of $\mathcal{H}(G,K)$. We normalize the action of A so that it is L^2-unitary, and we denote it by $\cdot$ for elements of the group and by $\star$ for measures:

$$a \cdot f(Ug) = \delta^{\frac{1}{2}}(a) f(Uag),$$

where δ is the modular character of the Borel (the quotient of left and right Haar measure).

It is immediate that $\mathcal{H}(A, A_0)$ is canonically the group ring of Λ_A (notation as above). Pick a *basic vector*, Φ_0 = the characteristic function of $X(\mathfrak{o})$ =the characteristic function of UK. Then the Iwasawa decomposition immediately implies:

Lemma 2. *As an $\mathcal{H}(A, A_0) = \mathbb{C}[\Lambda_A]$-module, $C_c^\infty(U\backslash G)^K \simeq \mathbb{C}[\Lambda_A]$, where the map is given by the action on the basic vector:*

$$\mathcal{H}(A, A_0) = \mathbb{C}[\Lambda_A] \ni h \mapsto h \star \Phi_0.$$

The theorem of the Satake isomorphism is, then:

Theorem 1. *Consider the action map: $\mathcal{H}(G, K) \ni h \mapsto h \star \Phi_0 \in C_c^\infty(U\backslash G)^K \simeq \mathbb{C}[\Lambda_A]$. It is injective, with image in $\mathbb{C}[\Lambda_A]^W$ (invariants of the Weyl group).*

It is easy to see that this is actually a ring homomorphism, hence we get an isomorphism of rings:

$$\mathcal{H}(G, K) \xrightarrow{\sim} \mathbb{C}[\Lambda_A]^W.$$

Langlands interpreted the latter as invariant polynomials on the dual group $\check{G}$ of G. Indeed, the group ring $\mathbb{C}[\Lambda_A]$ is, essentially by definition, the coordinate ring of the maximal torus $\check{A} \subset \check{G}$, hence by the Chevalley isomorphism:

$$\mathbb{C}[\Lambda_A]^W = \mathbb{C}[\check{A}]^W = \mathbb{C}[\check{G}]^{\check{G}}.$$

Finally, a point $\chi \in \check{A}$ defines a character of Λ_A, i.e. an unramified character of A, to be denoted by the same letter. On the other hand, its W-orbit defines a maximal ideal of $\mathcal{H}(G, K)$, and hence a character:

$$\mathcal{H}(G, K) \to \mathbb{C}.$$

Tracing back the construction, it is easy to see how to realize this explicitly: it is the character by which $\mathcal{H}(G, K)$ acts on the (one-dimensional space of) K-invariant functions on $U\backslash G$ which satisfy:

$$f(bg) = \chi\delta^{\frac{1}{2}}(b)f(g)$$

for $b \in B$. This induced representation is the *principal series* obtained by (normalized) induction from χ.

This doesn't quite complete the description of the Hecke algebra $\mathcal{H}(G, K)$; one needs to compare this abstract isomorphism with the Cartan decomposition, which is accomplished via MacDonald's formula for zonal spherical functions (see [2]). Another famous formula for spherical functions on the space of an induced representation is the Shintani-Casselman-Shalika formula for Whittaker functions [3].

The Satake isomorphism can be thought of as the p-adic version of the Harish-Chandra isomorphism, which equates the center of the universal enveloping algebra of G with invariant polynomials on the dual Lie algebra:

$$\mathfrak{z}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\check{g}]^{\check{G}}.$$

Unfortunately, unlike the real case where differential operators often suffice to study all representations, the spherical Hecke algebra is a very crude algebra which kills most of the representations of the p-adic group. The unramified representations captured by the Satake isomorphism are very useful for global purposes (an adelic representation such as an automorphic representation is unramified at all but finitely many places), but they miss a lot of the local representation theory.

The Satake isomorphism also has a deeper version, where the algebra $\mathbb{C}[\check{G}]^{\check{G}}$ is considered as the complexification of the Grothendieck ring of the category of finite-dimensional representations of $\check{G}$ – to every (virtual) representation we simply associate its character. This might seem a bit artificial at first, but this category can actually be *reconstructed* in the Geometric Langlands program, where functions on $U\backslash G$ are replaced by perverse sheaves on schemes and stacks modeling the points of $U\backslash G$.

I have neither the expertise nor the space to explain this throughly, however I can explain in a similar manner the work of Gaitsgory and Nadler [4]: the issue at hand is to describe unramified (K-invariant) "functions" on a spherical variety X as a module for the Hecke algebra. Of course, "functions" become "sheaves" in the geometric setting; moreover, it turns out that one can intrinsically define a "convolution" of these sheaves, which is not possible at the level of functions. It turns out that suitable categories of sheaves have the structure of a semisimple Tannakian category, and hence by the Tannakian formalism the category of representations of a complex reductive group $\check{G}_X$ – this is the "dual group" of X. Taking into account the action of the Hecke algebra, as well, this dual group turns out, in the work of Gaitsgory and Nadler, to be a subgroup of $\check{G}$.

A "function theoretic" analog of Gaitsgory and Nadler would suggest a commutative diagram:

$$\begin{tikzcd} \mathcal{H}(G,K) \arrow[r] \arrow[d, "\sim"] & C_c^\infty(X)^K \arrow[d, "\sim"] \\ \mathbb{C}[\check{G}]^{\check{G}} \arrow[r] & \mathbb{C}[\check{G}_X]^{\check{G}_X} \end{tikzcd} \tag{1}$$

where the upper horizontal arrow is given by the action map on a distinguished element, such as the characteristic function of $X(\mathfrak{o})$.

Compare with the Harish-Chandra isomorphism of Knop [5], which describes the ring of invariant differential operators on a complex spherical variety X:

$$\begin{tikzcd} \mathfrak{z}(\mathfrak{g}) \arrow[r] \arrow[d, "\sim"] & \mathcal{D}(X)^G \arrow[d, "\sim"] \\ \mathbb{C}[\check{g}]^{\check{G}} \arrow[r] & \mathbb{C}[\check{g}_X]^{\check{G}_X} \end{tikzcd} \tag{2}$$

However, (1) is not always true. Instead of formulating a full theorem, which is not yet available, we describe several of the issues arising, some of which lead to open problems:

(1) In Knop's work, the restriction map in the lower horizontal arrow of (2) is not the natural one coming from the embedding $\check{\mathfrak{g}}_X \subset \check{\mathfrak{g}}$, but contains

a "shift"; the same shift will appear in (1). If X is "non-degenerate", e.g.: quasi-affine, this shift is trivial if and only if $P(X)$, the parabolic stabilizing the open Borel orbit, is equal to the Borel. In the general case, except for the embedding $\check{G}_X \hookrightarrow \check{G}$, there is also a commuting map: $\mathrm{SL}_2 \to \check{G}$, which explains this shift in terms of "Arthur parameters".

(2) The "correct" dual group $\check{G}_X$ is not always a subgroup of $\check{G}$, but sometimes the map $\check{G}_X \to \check{G}$ has finite kernel, in order to account for multiplicities of representations in the harmonic analysis on X, and sometimes cannot even be defined in such a way that multiplicities are taken into account! Thus, the algebra $\mathbb{C}[\check{G}_X]^{\check{G}_X}$ of Gaitsgory and Nadler only takes care of a subspace of $C_c^\infty(X)^K$, in general. For a discussion of the root datum of $\check{G}_X$, cf. [9].

(3) As the Geometric Langlands program shows, the characteristic function of $X(\mathfrak{o})$ is not always the correct "basic function", and the space $C_c^\infty(X)$ is not always the correct space of test functions. One has to consider *affine embeddings* of X, and if those have singularities, the correct "basic function" will mirror some kind of intersection cohomology, rather than being the characteristic function of a set. This opens up exciting, but uncharted, territory. For instance, it is known from [1] that one gets "better" Eisenstein series by using such functions when X is the quotient of G by the unipotent radical of a parabolic (or, rather, its affine closure); it was suggested in [8] to use these spaces of functions in order to extend the Rankin-Selberg method of integral representations of L-functions; and it was conjectured in [6] that, when we consider affine embeddings of a group, the space of functions obtained in this manner gives rise to certain unramified L-functions.

(4) Nevertheless, it turns out that in some cases (1) is true, cf. [7]; typically those cases are homogeneous affine varieties whose k-points have a unique open $B(k)$-orbit.

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B-orbits on spherical homogeneous spaces

Jacopo Gandini

Let G be a reductive algebraic group over an algebraically closed field k and fix a Borel subgroup $B \subset G$. A subgroup $H \subset G$ is called *spherical* if B acts with finitely many orbits on G/H, or equivalently if H acts with finitely many orbits on the falg variety G/B. We denote by $\mathcal{B}(G/H)$ the set of the B-orbits on G/H, the talk surveyed some of the main results concerning this set.

The set $\mathcal{B}(G/H)$ comes naturally endowed with the *Bruhat order*, namely the partial order $\leqslant$ induced by the inclusion of orbit closures. For instance, if $H = B$ and if T is a maximal torus contained in it, then there is a bijection between $\mathcal{B}(G/H)$ and the Weyl group $W = N/T$ (where N denotes the normalizer of T in G), and the partial order $\leqslant$ coincides with the classical Bruhat order. When H is a symmetric subgroup of G (namely the set of points fixed by an algebraic involution of G), the partially ordered set $\mathcal{B}(G/H)$ was studied by R. W. Richardson and T. A. Springer in [6].

Let $H \subset G$ be a spherical subgroup. Fix a maximal torus $T \subset B$, let W be the Weyl group of T and let $R \supset S$ resp. be the attached sets of roots and of simple roots. The *Richardson-Springer monoid* is the monoid W^* generated by the simple reflection s_α with the relations $s_\alpha^2 = s_\alpha$ for all $\alpha \in S$ and the braid relations. As a set, W^* is the Weyl group W of G but with a different multiplication. An action of W^* on $\mathcal{B}(G/H)$ was defined by Richardson and Springer in [6] as follows: if $w \in W^*$ and $\mathcal{O} \in \mathcal{B}(G/H)$, then $w * \mathcal{O}$ is the unique open B-orbit contained in the B-stable subset $Bw\mathcal{O}$. The *weak order* is the partial order $\preceq$ on $\mathcal{B}(G/H)$ induced by the action of W^*: if $\mathcal{O}, \mathcal{O}' \in \mathcal{B}(G/H)$, then $\mathcal{O} \preceq \mathcal{O}'$ if and only if $\mathcal{O}' = w * \mathcal{O}$ for some $w \in W^*$. The Bruhat order is compatible with the W^*-action and with the dimension function, namely the following properties hold for all $\alpha \in S$ and for all $\mathcal{O}, \mathcal{O}' \in \mathcal{B}(G/H)$:

i) $\mathcal{O} \leqslant s_\alpha * \mathcal{O}$,
ii) If $\mathcal{O} \leqslant \mathcal{O}'$, then $s_\alpha * \mathcal{O} \leqslant s_\alpha * \mathcal{O}'$,
iii) If $\mathcal{O} \leqslant \mathcal{O}'$ and if $\dim(\mathcal{O}) = \dim(\mathcal{O}')$, then $\mathcal{O} = \mathcal{O}'$.

Theorem 1 ([6], [7]). *Suppose that H is a symmetric subgroup. Then the Bruhat order is the weakest partial order on $\mathcal{B}(G/H)$ which is compatible with the W^*-action and with the dimension function.*

Let $\mathrm{rk}(H)$ the dimension of a maximal torus of H. Given a B-variety Z, denote by $\mathcal{X}(Z) = \{\text{weights of } B\text{-eigenfunctions } f \in k(Z)\}$ the *weight lattice* of Z and define the *rank* of Z as the rank $\mathcal{X}(Z)$. These are invariants of Z under birational B-morphisms. If $\mathcal{O} \in \mathcal{B}(G/H)$, then we have the inequalities

$$\mathrm{rk}(G) - \mathrm{rk}(H) \leqslant \mathrm{rk}(\mathcal{O}) \leqslant \mathrm{rk}(G/H) :$$

while the latter is the rank of the open B-orbit, the first one coincide with the rank of any closed orbit. If $\alpha \in S$ and $\mathcal{O} \in \mathcal{B}(G/H)$, then we have $\mathrm{rk}(\mathcal{O}) \leqslant \mathrm{rk}(s_\alpha * \mathcal{O}) \leqslant \mathrm{rk}(\mathcal{O}) + 1$. More precisely, if $P_\alpha \supset B$ is the minimal parabolic subgroup associated to α and if $x \in G/H$, then we have the following possibilities:

Type G: $P_\alpha x = Bx$. Then we also set $s_\alpha \cdot Bx = Bx$.

Type U: $P_\alpha x = Bx_0 \sqcup Bx_\infty$, with $\dim(Bx_0) = \dim(Bx_\infty) + 1$ and $\mathrm{rk}(Bx_0) = \mathrm{rk}(Bx_\infty)$. Then we also set $s_\alpha \cdot Bx_1 = Bx_0$ and $s_\alpha \cdot Bx_0 = Bx_1$.

Type T: $P_\alpha x = Bx_1 \sqcup Bx_0 \sqcup Bx_\infty$, with $\dim(Bx_1) = \dim(Bx_0) + 1 = \dim(Bx_\infty) + 1$ and $\mathrm{rk}(Bx_1) = \mathrm{rk}(Bx_0) + 1 = \mathrm{rk}(Bx_\infty) + 1$. Then we also set $s_\alpha \cdot Bx_1 = Bx_1$, $s_\alpha \cdot Bx_0 = Bx_\infty$ and $s_\alpha \cdot Bx_\infty = Bx_0$.

Type N: $P_\alpha x = Bx_1 \sqcup Bx_0$, with $\dim(Bx_1) = \dim(Bx_0) + 1$ and $\mathrm{rk}(Bx_1) = \mathrm{rk}(Bx_0) + 1$. Then we also set $s_\alpha \cdot Bx_1 = Bx_1$ and $s_\alpha \cdot Bx_0 = Bx_0$.

This cases follow by analysing the action of the subgroups of $\mathrm{PGL}(2) \simeq \mathrm{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^1 \simeq P_\alpha/B$ acting with finitely many orbits (see [2], [6]). Given $\alpha \in S$ and $x \in G/H$, the map $(s_\alpha, \mathcal{O}) \longmapsto s_\alpha \cdot \mathcal{O}$ defines an action of s_α on the set of B-orbits contained in $P_\alpha x$, hence we get an action of s_α on the whole set $\mathcal{B}(G/H)$. F. Knop showed that these actions of the simple reflections glue together to an action of the Weyl group.

Theorem 2 ([2])**.** *The actions of the simple reflections defined above induce an action of the Weyl group W on $\mathcal{B}(G/H)$.*

One can define an action of the Hecke algebra attached to W on a module which is tightly related to the set $\mathcal{B}(G/H)$. This module, which was constructed by G. Lusztig and D. A. Vogan in the case of a symmetric homogeneous space in [3], is a main tool in the proof of previous theorem (see also [4], [8]).

If $\mathcal{O} \in \mathcal{B}(G/H)$ denote $\mathcal{O}_\leqslant = \{\mathcal{O}' \in \mathcal{B}(G/H) \,:\, \mathcal{O}' \leqslant \mathcal{O}\}$. A property which links the actions of W and of W^* with the Bruhat order is the *one-step property*:

$$\text{if } s_\alpha * \mathcal{O} \neq \mathcal{O}, \text{ then } (s_\alpha * \mathcal{O})_\leqslant = \textstyle\bigcup_{\mathcal{O}' \leqslant \mathcal{O}} \{\mathcal{O}, s_\alpha \cdot \mathcal{O}, s_\alpha * \mathcal{O}\}.$$

This reduces the description of the Bruhat order on $\mathcal{B}(G/H)$ to the description of the sets $\mathcal{O}_\leqslant$ when the orbit $\mathcal{O}$ is minimal w.r.t. the weak order. In the case of a symmetric subgroup these orbits are always closed, this is false however in the general case.

As it follows by the definition of the action of the simple reflections, we have that the action of W on $\mathcal{B}(G/H)$ preserves the rank of the orbits. In two special cases, namely in the maximal and in the minimal rank case, the rank determines uniquely the W-orbit in $\mathcal{B}(G/H)$.

Denote $(G/H)^\circ \subset G/H$ the open B-orbit and denote by $P(G/H)$ the stabilizer of $(G/H)^\circ$ in G, namely $P(G/H) = \{g \in G \,:\, g(G/H)^\circ = (G/H)^\circ\}$. Let $W_{P(G/H)}$ the Weyl group of the Levi of $P(G/H)$. If the characteristic of k is zero, then Knop showed that one can recover the little Weyl group $W_{G/H}$ of G/H from the stabilizer of $(G/H)^\circ$ w.r.t the action of W.

Theorem 3 ([2])**.** *The set of elements in $\mathcal{B}(G/H)$ of maximal rank is an orbit under W. If moreover* $\mathrm{char}(k) = 0$*, then the stabilizer of $(G/H)^\circ$ respect to the* *action of W is described as follows:*

$$W_{(G/H)^\circ} = W_{G/H} \ltimes W_{P(G/H)}.$$

Denote W_H the Weyl group of H, namely the quotient $N_H(T_H)/C_H(T_H)$ where T_H is a maximal torus in H and where $N_H(T_H)$ and $C_H(T_H)$ are resp. its normalizer and its centralizer in H. N. Ressayre showed that W_H is recovered from the stabilizer in W of an orbit of minimal rank.

Theorem 4 ([5]). *The set of elements in $\mathcal{B}(G/H)$ of minimal rank is an orbit under W. The stabilizer of any such an element is isomorphic to W_H.*

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Examples of periods in analytic number theory

Farrell Brumley

We look at two historical applications of period formulae to problems in analytic number theory, the class number problem and the equidistribution of integer points on spheres. The former appeals to the Hecke formula, the latter to the Waldspurger formula.

Gauss class number problem. Let $h(d)$ denote the class number of the imaginary quadratic field F of discriminant d. Gauss [1] conjectured that $h(d) \to \infty$ as $|d| \to \infty$. In 1933, Deuring [2] proved that if the classical Riemann hypothesis is false, then there are only finitely many imaginary quadratic fields with class number 1. As it is easy to describe, we present here Deuring's proof. It is based on a period formula, one of the first one encounters in analytic number theory.

Let $E(s,z)$ be the real analytic Eisenstein series for the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. This is defined for $\mathrm{Re}(s) > 1$ by the expression

$$E(s,z) = \frac{1}{2}\sum_{(m,n)=1} \frac{y^s}{|mz+n|^{2s}} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y(\gamma z)^s,$$

where $\Gamma_\infty = \left(\begin{smallmatrix}1 & * \\ & 1\end{smallmatrix}\right)$, and one can meromorphically extend $E(z,s)$ to all of $\mathbb{C}$ (with simple poles at 0 and 1). It a $\mathrm{SL}_2(\mathbb{Z})$-invariant eigenfunction for the hyperbolic Laplacian $\Delta = -y^2(\partial_x^2 + \partial_y^2)$ of eigenvalue $s(1-s)$; indeed, it is an averaging of the the Γ_∞-invariant eigenfunction y^s on $\mathbb{H}$.) It is also a Hecke eigenfunction $T_pE(z,s) = (p^{s-1/2} + p^{-s+1/2})E(s,z)$ and of moderate growth. It defines a real analytic automorphic form on the modular curve $Y_0(1) = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. It is not a cusp form (we will use the constant term later in the argument). One calls $E(s,z)$ a Maass form.

Let $\xi_{\mathbb{Q}(i)}(s) = \Gamma_{\mathbb{C}}(s)\zeta_{\mathbb{Q}(i)}(s)$ be the completed Dedekind zeta function for $\mathbb{Q}(i)$ and $\xi(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$ be the completed Riemann zeta function. Here we have used the notation $\Gamma_{\mathbb{C}}(s) = (2\pi)^{1-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. By explicit calculation, one can check that

$$E(i,s) = \frac{\xi_{\mathbb{Q}(i)}(s)}{\xi(2s)}. \tag{1}$$

This readily verifiable formula admits a generalization, which we now describe.

Fix a fundamental discriminant d (i.e., the discriminant of a quadratic field F). Then $\mathrm{SL}_2(\mathbb{Z})$ acts on the set of integral binary quadratic forms $ax^2 + bxy + cy^2$ of discriminant $d = b^2 - 4ac$ by unimodular substitutions. Let Λ_d be a complete set of $\mathrm{SL}_2(\mathbb{Z})$-inequivalent forms. If $d < 0$ we consider the finite subset of $Y_0(1)$ be the set of associated CM points on the modular curve. These are give by the roots (with positive imaginary part) of $q(z,1) = az^2 + bz + c = 0$: $z = (-b+\sqrt{d})/2a$. These points form a principal homogeneous space for the ideal class group C_F of F. We have a bijective correspondence between ideal classes and CM points given by $\mathfrak{a} = [a, \frac{b+\sqrt{d}}{2}] \leftrightarrow z_{\mathfrak{a}} = (-b+\sqrt{d})/2a$.

For any character of ψ of C_F, Hecke [3] showed that

$$\sum_{\mathfrak{a}\in C_F} E(z_{\mathfrak{a}}, s)\psi(\mathfrak{a}) = \frac{\Lambda(s,\psi)}{\xi(2s)}, \tag{2}$$

where $\Lambda(s,\psi) = \Gamma_{\mathbb{C}}(s)L(s,\psi)$ and $L(s,\psi) = \prod_{\mathfrak{p}}(1-\psi(\mathfrak{p})\mathrm{Norm}_{F/\mathbb{Q}}(\mathfrak{p})^{-s})^{-1}$. If F has class number 1, the singleton CM point $z_d \in Y_0(1)$ is determined uniquely by the property that the ring of integers of F is given by $\mathbb{Z}[z_d]$. Explicitly, $z_d = (-\delta+\sqrt{d})/2$ where $\delta = 0, 1$, $\delta \equiv d \bmod 4$. Inserting this into (2) we find

$$E(z_d, s) = \frac{\xi_F(s)}{\xi(s)}, \tag{3}$$

which recovers (1) when $F = \mathbb{Q}(i)$.

Now recall that we want to show that if the Riemann hypothesis is false, then there are only finitely many F of class number 1. So let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$

be such that $\zeta(s) = 0$. Then the right-hand side of (3) vanishes for any F. In particular, for any d such that $h(d) = 1$ we have $E(z_d, s) = 0$. We will see that for $|d|$ large enough, this vanishing is impossible. The idea is to look at the constant term of the Eisenstein series: $E(z, s) = y^s + c(s)y^{1-s} + O(e^{-2\pi y})$, where $c(s) = \frac{\xi(2s-1)}{\xi(2s)}$. When $\mathrm{Re}(s) > 1/2$ we have $E(z, s) \sim y^s$ as $y \to \infty$. Since $y(z_d) = \sqrt{|d|}/2 \to \infty$ as $|d| \to \infty$ one sees that for $|d|$ large enough and $\mathrm{Re}(s) > 1/2$ the point evaluation $E(z_d, s)$ cannot be zero.

Problems of Linnik type. Let S^2 be the unit 2-sphere $x^2+y^2+z^2 = 1$ in $\mathbb{R}^3$. For any integer $d \neq 0$ we project the set of integer solutions (a, b, c) of $a^2+b^2+c^2 = |d|$ to the sphere S^2 by rescaling by $|d|^{-1/2}$. Recall that Gauss showed that an integer $|d| \geq 1$ is representable as a sum of three squares if and only if $|d| \neq 4^a(8b + 7)$. Let $\mathcal{G}_d$ be the resulting finite subset of S^2. A theorem of Duke [4] states that for any sufficiently nice subset Ω of S^2, $\lim |\mathcal{G}_d \cap \Omega|/|\mathcal{G}_d| = m(\Omega)$, for $d \to -\infty$ along $d \not\equiv 0, 1, 4 \bmod 8$, where m is Lebesgue measure on S^2 normalized to give S^2 volume 1.

One way of expressing the equidistribution condition by harmonic analysis. We want to show that a certain sequence of probability measures on the sphere tends to the uniform measure. We define a measure μ_d by

$$\int_{S^2} \varphi\mu_d = \sum_{(a,b,c)\in\mathbb{Z}^3, a^2+b^2+c^2=|d|} \varphi\left(\frac{a}{\sqrt{|d|}}, \frac{b}{\sqrt{|d|}}, \frac{c}{\sqrt{|d|}}\right).$$

By Weyl's criterion, if we fix an orthonormal basis $\{\varphi_n\}$ of $L^2_0(S^2, m)$ (zonal spherical harmonics, say), then the equidistribution of $\mathcal{G}_d$ towards Lebesgue measure m is equivalent to the statement that for every n

$$\int_{S^2} \varphi_n\mu_d = o(1), \qquad \text{for } |d| \to \infty. \tag{4}$$

The above integral can be expressed as an automorphic period. Recall that for the definite quadratic form $Q(x, y, z) = x^2 + y^2 + z^2$ we can identify the quadratic space $(\mathbb{Q}^3, Q)$ with the trace-zero subspace of the quaternion algebra $\mathrm{B}_{2,\infty}$, ramified at 2 and ∞, endowed with the reduced norm form $z\bar{z}$. (At infinity, $\mathrm{B}_{2,\infty}(\mathbb{R})$ is Hamilton's quaternions.) This allows us to identify SO_Q with $\mathbf{G} = \mathrm{PG}(\mathrm{B}_{\mathbf{2},\infty}) = \mathrm{B}^\times_{\mathbf{2},\infty}/\mathbf{Z}(\mathrm{B}^\times_{\mathbf{2},\infty})$. Given a solution of $Q(a, b, c) = d$, one embeds K_d into $B_{2,\infty}$ by sending $s + \sqrt{d}t$ to $s + (a.i + b.j + c.j)t$. This yields an embedding of $\mathbb{Q}$ algebraic groups $\mathbf{H}_d = \mathrm{Res}_{K_d/\mathbb{Q}}\mathbb{G}_m/\mathbb{G}_m \hookrightarrow \mathbf{G}$. If K_f denotes a maximal compact subgroup of $\mathbf{G}(\mathbb{A}_f)$ and $K_\infty = \mathbf{H}_d(\mathbb{R}) = \mathrm{SO}_2(\mathbb{R})$, we may identify the sphere with $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K$, where $K = K_f K_\infty$.

With this notation set up, we can then describe $\mathcal{G}_d$ as $\mathbf{H}_d(\mathbb{Q})\backslash z_d\mathbf{H}_d(\mathbb{A})/K_{\mathbf{H}_d}$ inside the space $\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K$. The integral (4) can then be described as the automorphic period integral

$$W(\varphi, d) = \int_{\mathbf{H}_d(\mathbb{Q})\backslash\mathbf{H}_d(\mathbb{A})/K_{\mathbf{H}_d}} \varphi(z_d.t)dt,$$

where dt is the probability Haar measure on the quotient. Our job is therefore to show that for any choice of basis $\{\varphi_n\}_{n\geq 1}$ of $L_0^2(\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A}_{\mathbb{Q}})/K)$, we have $W(\varphi_n, d) = o(1)$ as $|d| \to \infty$.

One is free to take a basis of Hecke eigenfunctions, and doing so allows one to exploit the arithmetic symmetries of the underlying manifolds. One needs a formula linking the period integral to an L-function. This is given by Waldspurger's theorem [5], which states that for a non-constant L^2-normalized form φ generating an automorphic representation π of $\mathbf{G}(\mathbb{A}_{\mathbb{Q}})$ we have

$$|W(\varphi, d)|^2 = \frac{1}{4}\xi_{K_d}(2)C_0\frac{\Lambda(1/2, \mathrm{bc}(\pi'))}{\Lambda(s, \chi_d)\Lambda(1, Ad, \pi')},$$

where $C_0 = \Lambda(1, \chi_d)^{-1}$ is a normalizing constant for the measure on the torus, π' is the Jacquet-Langlands lift of π to GL_2. From Siegel's (ineffective) lower bound $L(1, \chi_d) \gg_\varepsilon |d|^{-\varepsilon}$ we find that

$$|W(\varphi, d)|^2 = O_\varepsilon(|d|^{-1/2+\varepsilon}|L(1/2, \pi' \otimes \chi_d)|).$$

Basic principles in complex analysis show that $L(1/2, \pi' \otimes \chi_d) = O(|d|^{1/2})$. One wants to prove the subconvex bound $L(1/2, \pi' \otimes \chi_d) = O(|d|^{1/2-\delta})$ for some $\delta > 0$. Such a bound was proved in the early 90's by Duke-Friedlander-Iwaniec [6].

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Tangent spaces of invariant Hilbert schemes and Spherical systems

Stéphanie Cupit-Foutou

In this talk, we describe some features of invariant Hilbert schemes with emphasis on the underlying combinatorics and its connection to spherical systems.

Spherical systems were discussed in Bravi's talk whereas the definition and some properties of invariant Hilbert schemes were recalled in Brion's talk. We refer to the corresponding abstracts for these concepts and primary references.

The ground field is the field of complex numbers. Let G denote a connected reductive algebraic group, B a Borel subgroup of G and $T \subset B$ a maximal torus of G. We denote the relative set of simple roots (resp. dominant weights) by S (resp. Λ^+).

Let Γ be a submonoid of Λ^+ finitely generated by $\lambda_1, \ldots, \lambda_d$. Let $\mathbf{M}_\Gamma^G$ denote the so-called moduli scheme of multiplicity-free varieties with weight monoid Γ. As seen in Brion's talk, $\mathbf{M}_\Gamma^G$ is a subscheme of some peculiar invariant Hilbert scheme; it is acted on by the adjoint torus T_{ad} of G.

Consider the following G-module

$$V = V(\lambda_1^*) \oplus \ldots \oplus V(\lambda_d^*)$$

where $V(\lambda_i^*)$ stands for the dual of the simple G-module with highest weight λ_i. Let $v_{\lambda_i^*}$ be a highest weight vector of $V(\lambda_i^*)$ and set

$$v_{\underline{\lambda}^*} = v_{\lambda_1^*} + \ldots + v_{\lambda_d^*}.$$

The horospherical G-variety

$$X_0 = \overline{G.v_{\underline{\lambda}^*}} \subset V$$

is a closed point of $\mathbf{M}_\Gamma^G$. Further, X_0 is fixed by the torus T_{ad}.

Following [1], we consider the action of T_{ad} on V given by

$$t.v_\mu = (\lambda_i^* - \mu)(t) v_\mu \quad (\forall t \in T)$$

where $v_\mu \in V(\lambda_i^*)$ is a T-weight vector with weight μ. This action on V descends to an action of T_{ad} on $V/\mathfrak{g}.v_{\underline{\lambda}^*}$ with $\mathfrak{g}$ denoting the Lie algebra of G.

Theorem 1 ([1]). *Let $G_{v_{\underline{\lambda}^*}}$ be the stabilizer of $v_{\underline{\lambda}^*}$ in G.*

The Zariski-tangent space $T_{X_0}\mathbf{M}_\Gamma^G$ of $\mathbf{M}_\Gamma^G$ at X_0 is a T_{ad}-submodule of the $G_{v_{\underline{\lambda}^*}}$-invariants of $V/\mathfrak{g}.v_{\underline{\lambda}^*}$, where $G_{v_{\underline{\lambda}^*}}$ acts on $V/\mathfrak{g}.v_{\underline{\lambda}^*}$ via its linear action on V.

Moreover, these two modules are equal whenever the boundary $Z \setminus G.v_{\underline{\lambda}^*}$ has codimension 2 in X_0.

In the remainder, we focus on peculiar monoids: $\Gamma = \Gamma(\mathcal{S})$ is a monoid canonically attached to a given spherically closed spherical system $\mathcal{S}$ of G. To avoid too many technicalities, the precise definition of the monoid $\Gamma(\mathcal{S})$ is omitted; we refer to [4] for details. Let us point out that the monoid $\Gamma(S)$ is a free submonoid of the set of dominant weights of some extension $\tilde{G}$ of G by an algebraic torus $\mathbb{G}_m^{d'}$. In particular, the generators of $\Gamma(S)$ consist of couples (λ_i, χ_i) with $\lambda_i \in \Lambda^+$; $i = 1, \ldots, d$ and $d \geq d'$. Note also that the λ_i's may not be linearly independent as the next example shows.

In the following, ϖ_α denotes the fundamental weight of G associated to some simple root $\alpha \in S$. Let $\langle \cdot, \cdot \rangle$ be the Killing form and $\alpha^\vee$ be the coroot corresponding to $\alpha \in S$.

Example 1. Let $G = SL_2$ and $\mathcal{S}$ be the spherical system of SL_2/T. Then $\tilde{G} = G \times \mathbb{G}_m$ and $\Gamma(\mathcal{S})$ is generated by (ϖ_α, χ) and $(\varpi_\alpha, -\chi)$.

Let us now highlight the main properties of the generators (λ_i, χ_i) of Γ.

Lemma 3. (i) *$\langle \lambda_i, \alpha^\vee \rangle \leq 2$ for every i and every $\alpha \in S$.*

(ii) *If $\langle \lambda_i, \alpha^\vee \rangle = 2$ for some generator and some $\alpha \in S$ then $\lambda_i = 2\varpi_\alpha$ and $\langle \lambda_j, \alpha \rangle = 0$ for all $j \neq i$.*

(iii) *Suppose* $\langle \lambda_i, \alpha \rangle \langle \lambda_j, \alpha' \rangle \neq 0$ *holds for some* i *and distinct* α *and* α' *in* S. *Then* $\langle \lambda_j, \alpha \rangle \langle \lambda_j, \alpha' \rangle = 0$ *for every* $j \neq i$.

Note that the adjoint torus of $\tilde{G}$ equals that of G.

Theorem 2 ([4]). *Let* $\mathcal{S}$ *be a spherically closed spherical system of* G *and* $\Gamma = \Gamma(\mathcal{S})$.

(i) *The* $\tilde{G}_{v_{\underline{\lambda}^*}}$*-invariants of* $V/\tilde{\mathfrak{g}}.v_{\underline{\lambda}^*}$ *is a multiplicity-free* T_{ad}*-module. Furthermore, its* T_{ad}*-weights are spherical roots of* G.

(ii) *The* T_{ad}*-weights of* $T_{X_0}\mathbf{M}_\Gamma^{\tilde{G}}$ *are the spherical roots of the spherical system* $\mathcal{S}$.

To conclude, let us illustrate the importance of the properties stated in Lemma 3 through a few examples of weight monoids not sharing these properties.

(1) Let $G = SL_2$ and Γ be generated by a single dominant weight λ. The scheme $\mathbf{M}_\Gamma^G$ is the affine line if λ equals $2\varpi_\alpha$ or $4\varpi_\alpha$; otherwise it is a reduced point. See [5].

(2) Let $\Gamma \subset \Lambda^+$ be free and such that $\mathbb{Z}\Gamma \cap \Lambda^+ = \Gamma$. Let $S(\Gamma)$ be the set of simple roots orthogonal to every element of Γ and $\Sigma(\Gamma)$ be the T_{ad}-weights of $T_{X_0}\mathbf{M}_\Gamma^G$. Then $(S(\Gamma), \Sigma(\Gamma), \emptyset)$ is a spherical system of G; see [3].

Such monoids may not satisfy the properties of Lemma 3 (e.g. Γ as in (1) generated by $4\varpi_1$). In (1), the two aforementioned monoids give rise to the same spherical system.

(3) Let $G = SL_2 \times SL_2$ and Γ be generated by $2\varpi_{\alpha_1}$ and $2\varpi_{\alpha_1} + 2\varpi_{\alpha_2}$. Obviously, Lemma 3-(ii) does not hold here. The moduli scheme $\mathbf{M}_\Gamma^G$ is not smooth. Specifically, $\mathbf{M}_\Gamma^G$ is the subvariety of the affine plane defined by $xy = 0$. See [2].

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Singularities and large values of Whittaker functions

Nicolas Templier

We report on some recent works on large values of Whittaker functions which is motivated by questions in geometric analysis on the size of eigenfunctions. Consider a locally symmetric space $M = \Gamma\backslash G/K$ of finite volume. Let f is a cuspidal automorphic form on M. Thus f is an eigenfunction of the algebra of invariant differential operators and it is automatically smooth and real analytic [11]. It is also rapidly decreasing at infinity [5, Chap. 1]. In particular f is bounded.

In what follows we focus on two situations:

1. The group $G = \mathrm{SL}_2(\mathbb{R})$ and the Hecke congruence lattices for each integer $N \geq 1$,
$$\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ c \equiv 0(N) \right\}.$$
2. The group $G = \mathrm{SL}_n(\mathbb{R})$ and the lattice $\Gamma = \mathrm{SL}_n(\mathbb{Z})$.

We normalize $\mathrm{vol}(M) = 1$ and $\|f\|_2 = 1$. In each case we shall be interested in estimating the sup-norm $\|f\|_\infty$.

1. Let us consider the first situation with $G = \mathrm{SL}_2(\mathbb{R})$. Let Δ be the hyperbolic Laplacian. Then $\Delta f = \lambda f$ with $\lambda > 0$. We have
$$\|f\|_\infty \ll_\epsilon \lambda^{\frac{5}{24}+\epsilon} N^{\frac{1}{3}+\epsilon} \ll \lambda^{\frac{1}{4}} N^{\frac{1}{2}}.$$
The upper bound on the far right follows from local considerations in harmonic analysis. For example Hörmander [7] has established a much more general estimate for eigenfunctions of elliptic pseudo-differential operators. The modest improvement in the middle is achieved using a delicate combination of arithmetic and analytic arguments, notably the amplification method. For the λ-aspect, see the celebrated paper of Iwaniec–Sarnak [8] and for the N-aspect see Harcos–Templier [4, 13].

Concerning lower bounds we have
$$1 \ll \lambda^{\frac{1}{12}-\epsilon} N^{\frac{1}{4}-\epsilon} \ll_\epsilon \|f\|_\infty \tag{1}$$
for some special forms f and N a square. This is a recent result in [14] the details of which we shall report below.

2. In the second situation of $G = \mathrm{SL}_n(\mathbb{R})$ we let only the eigenvalue λ of the Laplace operator vary. If Ω is any bounded subset of M, then the local upper bound for the supremum of f on Ω reads
$$\|f\|_{\infty,\Omega} := \sup_{g \in \Omega} |f(g)| \ll_{\Omega,\epsilon} \lambda^{\frac{n(n-1)}{8}+\epsilon}. \tag{2}$$

Example 1. For $n = 3, 4, 5, 6$ the power of λ in the upper-bound reads respectively $\lambda^{\frac{3}{4}}, \lambda^{\frac{3}{2}}, \lambda^{\frac{5}{2}}, \lambda^{\frac{15}{4}}$.

In a forthcoming article by Brumley-Templier [1], we prove that for $n = 3$ and the spectral parameter of f away from the walls,
$$1 \ll \lambda^{\frac{1}{2}-\epsilon} \ll_\epsilon \|f\|_\infty. \tag{3}$$
For general $n \geq 3$ we establish by a different method that
$$\lambda^{\frac{n(n-1)(n-2)}{24}-\epsilon} \ll_\epsilon \|f\|_\infty. \tag{4}$$

Example 2. For $n = 3$ the power of λ in this lower-bound is $\lambda^{\frac{1}{4}}$ which is weaker than (3). For $n = 4, 5, 6$ the lower-bound is respectively $\lambda, \lambda^{\frac{5}{2}}, \lambda^5$. Thus for $n = 6$ and higher the lower-bound in (4) is greater than the upper-bound in (2)! There is no contradiction only because the local upper bound (2) is restricted to a fixed bounded subset $\Omega \subset M$.

Strikingly the estimate (4) shows that automorphic forms in higher rank must achieve their largest value in the cuspidal region (e.g. for all $n \geq 6$).

Connection with periods. To bound $\|f\|_\infty$ from below we establish a version of Hecke bound [6, p. 484]. It is convenient to use the modern formalism of period integrals that was a main theme of this Oberwolfach workshop. See [1, 13] for the details of the proof. We formulate the Hecke bound as follows:

$$\|f\|_\infty \gg \frac{\|f\|_2}{L(1,\pi,\mathrm{Ad})} \prod_p h(\pi_p),$$

where we introduce the local invariant $h(\pi_p)$ of the representation π_p as

$$h(\pi_p) := \max_{g \in G(\mathbb{Q}_p)} |W(g)| \,/\, \|W\|_2 \,.$$

Here W is a Whittaker newvector of the representation π_p. It is a matrix coefficient for the horospherical variety $(N\backslash G, \psi)$ in the sense that it comes from a smooth embedding $\pi_p^\infty \to \mathbb{C}^\infty(N(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p), \psi)$. It now remains to estimate $h(\pi_p)$ and we shall consider the two situations 1. and 2. separately.

The local estimate (p-adic). Our goal is to estimate $h(\pi_p)$ for a finite prime $p \geq 2$. Here $W = W_\circ$ is the newvector which we can normalize by $W(\mathbf{e}) = 1$. We shall focus on the case that $\pi_p = \mathbb{1} \boxplus \chi$ is a principal series representation with χ of level p^c with $c \geq 0$.

If $c = 0$ the Casselman-Shalika formula [2] implies that

$$W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = |y|\, \mathbb{1}_{\mathfrak{o}}(y) \sum_{0 \leq a \leq v(y)} \chi(p^a).$$

In particular $h(\pi_p) \asymp 1$. For general $c \geq 1$ the properties of the newvector as in the work of Casselman [3] imply that

$$W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = |y|^{\frac{1}{2}}\, \mathbb{1}_{\mathfrak{o}}(y). \tag{5}$$

The first step in computing $h(\pi_p)$ is to write the decomposition

$$\mathrm{GL}_2(\mathbb{Q}_p) = \bigsqcup_i NA\mathbf{k}_i K_0(p^c). \tag{6}$$

Here $K_0(p^c) \subset \mathrm{GL}_2(\mathbb{Z}_p)$ is the Hecke congruence subgroup of level c, which is the preimage of the Borel subgroup of upper-triangular matrices under the projection $\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{Z}/p^c\mathbb{Z})$. And for all $0 \leq i \leq c$, we let $\mathbf{k}_i := \begin{pmatrix} p^i & 0 \\ p^i & 0 \end{pmatrix}$ which is a convenient choice of representatives of the double cosets. We establish in [14] a formula for $W(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \mathbf{k}_i)$ generalizing (5).

We shall omit the details of the formula here. The direct consequence is that if $c = 2$ then $h(\pi_p) \asymp p^{\frac{1}{4}}$, which is the key ingredient in the proof of the lower bound (1). More generally we have shown [14] that $h(\pi_p) \asymp p^{\frac{1}{2}\lfloor \frac{c}{2} \rfloor}$ for all $c \geq 0$.

We view this decomposition (6) as the disjoint union of Borel orbits over the horospherical variety $N\backslash \mathrm{GL}_2$ if we take the points defined over the finite ring $\mathbb{Z}/p^c\mathbb{Z}$. The aforementioned computation of $h(\pi_p)$ could be interpreted in this framework and this is suggestive on how to proceed in greater generality.

We can mention that a general result of Casselman-Shalika [2] gives an expression for $W(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\mathbf{k}_i)$ as a finite sum of multiplicative characters of prescribed exponents weighted by Schwartz functions. See also the recent work of Lapid–Mao [10] and Sakellaridis–Venkatesh [12]. In this respect our contribution in [14] is to compute and estimate the Schwartz function in this very specific case.

The local estimate (archimedean). From now $G = \mathrm{PGL}_n(\mathbb{R})$. Let $K = PO_n(\mathbb{R})$ and A be the split torus of diagonal matrices with positive entries. The unramified Whittaker function $W \in \mathcal{C}^\infty(N\backslash G, \psi)$ attached to π_∞ of spectral parameter ν_π is right K-invariant. Its uniqueness is a result of Shalika and its existence follows from the analytic continuation of the Jacquet integral [9].

By the Iwasawa decomposition, the N, ψ-invariance from the left and right K-invariance we only need to study the extrema of $W(y)$ for $y \in A$.

We show [1] that these extrema are achieved at the critical values y of a certain covering map $f : \Sigma \to A$ that we now proceed to describe. The Lie algebra is decomposed according to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the subalgebra of symmetric matrices. We consider the subspace of tridiagonal symmetric matrices $\begin{pmatrix} * & y_1 & & & \\ y_1 & * & y_2 & & \\ & y_2 & * & \ddots & \\ & & \ddots & \ddots & y_{n-1} \\ & & & y_{n-1} & * \end{pmatrix} \in \mathfrak{p}^0$. Then we form the $n-1$-dimensional variety Σ as

$$\Sigma := \left\{ k \in K, \text{ s.t. } \mathrm{Ad}(k)\nu_\pi \in \mathfrak{p}^0 \right\}. \tag{7}$$

Let f be the projection map $k \mapsto (y_1, \ldots, y_{n-1})$.

The variety Σ is stable under left-multiplication by any element of $K \cap M$, the 2-group of diagonal matrices with ± 1 entries. Thus we may arrange without loss of generality that $y_1, \ldots, y_{n-1} > 0$. Then the map $f : \Sigma \to A$ with $f(k) = y$ is given by identifying $y = (y_1, \ldots, y_{n-1})$ with the simple roots of the diagonal matrix $y \in A$.

Now the image of f is the set of $y \in A$ such that $W(y)$ is not exponentially small. The critical values of f is the caustic set, which corresponds to certain Lagrangian singularities. The function W achieves its extrema on the caustic set.

For $n = 3$ there are A_2 singularities that correspond to fold singularities of f. And there is also an A_3 singularity that corresponds to a Whitney pleat of f. The nonsingular values of y (resp. the A_2 singularity, resp. the A_3 singularity) contribute an exponent $\lambda^{\frac{1}{2}}$ (resp. $\lambda^{\frac{1}{3}}$, resp. $\lambda^{\frac{1}{4}}$) to $h(\pi_\infty)$. Note that $\lambda = ||\nu_\pi||^2$.

By a stationary phase analysis we conclude an approximation the Whittaker function for $n = 3$ in the transition region. Indeed $W(y)$ is given by the Airy function near the fold singularities

$$\mathtt{Ai}(a) := \int_{-\infty}^{\infty} e(x^3 + ax)dx,$$

and the Pearcey function near the A_3 singularity

$$\mathtt{P}(a,b) := \int_{\infty}^{\infty} e(x^4 + ax^2 + bx)dx.$$

Example 3. In the case $n = 3$ suppose $\nu_\pi = \mathrm{diag}(\lambda^{\frac{1}{2}}, 0, -\lambda^{\frac{1}{2}})$, that is $\pi_\infty \simeq \pi_\infty^\vee$ is self-dual. Then the outer caustic of A_2 singularities is given up to scaling by the equation $y_1^2 + y_2^2 = 1$ (an arc of a circle!), while the inner caustic is given by

$$27y_1^4y_2^4 + 4y_1^2 + 4y_2^2 = 1 + 18y_1^2y_2^2.$$

The cusp with A_3 singularity is the point $y_1 = y_2 = \frac{1}{\sqrt{3}}$.

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