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Mini-Workshop: Constructive Homological Algebra with Applications to Coherent Sheaves and Control Theory

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Abstract. The main objective of this mini-workshop is to bring together recent developments in constructive homological algebra. There, the current state already reached a level of generality which allows simultaneous application to diverse fields of applied and theoretical mathematics. In this workshop, we want to focus on simultaneous applications to system theory on the one side and to coherent sheaves and their cohomology on the other side. Surprisingly, these apparently remote fields share a considerable amount of common constructive methods. Bringing category theory and homological algebra to the computer leads to questions in logic and type theory. One goal of this workshop is to promote and enlarge this overlap.

Mathematics Subject Classification (2000): 14Qxx, 18Gxx, 13Dxx, 18E35, 18A40.

Introduction by the Organisers

The quest for constructivity in mathematics is as old as mathematics itself. Thanks to the emergence of powerful computers, constructivity is now regaining increased interest in several fields of applied and theoretical mathematics. System theory and algebraic geometry are apparently two remote representatives of such fields. However, as homological algebra is invading system theory, both fields now share a common powerful tool which turns out to be one of the keys to constructivity questions in both theories. A unification of the existing computational approaches is becoming necessary to extend the scope of applicability of constructive homological algebra developed by researchers in the different fields. The study of such a unification also suggests a comparison between the different type systems underlying the computational models of existing computer implementations.

Modeling Abelian categories of coherent sheaves on non-affine schemes is more involved than modeling those of modules. The talks addressed recent results allowing a constructive description of Abelian categories of coherent sheaves on spaces with a finitely generated Cox ring S as a Serre quotient category of the category of finitely presented graded S-modules. This includes the constructive treatment of (local and) global Ext's of which sheaf cohomology is a particular case.

Introducing modules as an intrinsic description of linear functional systems in system theory opened the door to apply homological techniques stemming from algebraic analysis. This provides a unified framework for system theory, which expresses itself in terms of common concepts, techniques, results, algorithms, and even implementations. There are numerous module-theoretic properties with a system-theoretic interpretation, e.g., the module being torsion, torsion-free, pure, reflexive, projective, stably free, free. All these properties have system-theoretic counterparts, e.g., the existence of autonomous elements, of minimal (resp. injective or Monge) parametrizations, of Bézout identities. This makes their study crucial for applications in control theory and mathematical physics, e.g., motion planning, quadratic optimal control, solving variational problems, and searching for potentials or conservation laws, to name a few. Luckily, these module-theoretic properties can be described in terms of homological algebra using resolutions, extension and torsion functors, projective dimension, and the purity filtration.

The talks focused on new developments in constructive homological algebra with applications to

(1) coherent sheaves and their cohomology; equivariant vector bundles;
(2) system and control theory based on homological algebra techniques;
(3) discrete vector fields in algebraic topology.

One of the discussion sessions was devoted to the "univalent foundation of mathematics". Thierry Coquand, who attended the special year at the IAS, gave an informal talk about Voevodsky's new interpretation of dependent types where a type is thought of as a homotopy type. This interpretation suggests a new way to represent constructive mathematics in type theory. One remarkable feature of this formalism is that two algebraic structures that are isomorphic are equal which allows to transport properties from one structure to an isomorphic one.

Many participants profited from further stimulating discussion with participants of the two parallel mini-workshops "Spherical Varieties and Automorphic Representations" and "Localising and Tilting in Abelian and Triangulated Categories". For example, we would like to mention discussions with Stefan Schwede and Bernhard Keller about Morita theory, tilting theory, and (algebraically) triangulated categories.

This interdisciplinary mini-workshop was attended by 17 participants from different areas of mathematics: algebraic geometry, algebraic topology, constructive algebra, differential algebra, system and control theory, logic and type theory. The participants had enough time for several informal discussions about various topics including Boij-Söderberg theory, equivariant vector bundles, symmetries of algebraic surfaces, discrete vector fields, and comparisons between the programing paradigms and implementations of homological algorithms in GAP4 and Coq. The organizers and the participants would like to thank the MFO for the excellent organization and the great hospitality. Christine Berkesch was funded by the "US Junior Oberwolfach Fellows" program of the NSF.

Mini-Workshop: Constructive Homological Algebra with Applications to Coherent Sheaves and Control Theory

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Abstracts

Computing Global Extension Modules

Gregory G. Smith

Let X be a smooth projective toric variety over a (computable) field $\Bbbk$, and consider two coherent $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$. Arguably, the most important examples of sheaves are simply the structure sheaves of closed subschemes such as those arising from a presentation of a Mori dream space, the tropical compactification of a very affine variety, numerous models in algebraic statistics, or complete intersection Calabi-Yau manifolds in string theory. We write $\operatorname{Ext}^m_X(\mathcal{M},\mathcal{N})$ for the m-th global extension module; these modules play a central role in duality theory. Although there are various equivalent ways to define $\operatorname{Ext}^m_X(\mathcal{M},\mathcal{N})$ including right derived functors, hypercohomology, and equivalence classes of extensions (i.e. Yoneda Ext), all of the classical approaches are ostensibly non-constructive. In contrast, this extended abstract outlines an effective method for computing $\operatorname{Ext}^m_X(\mathcal{M},\mathcal{N})$. Both our presentation and the ensuing procedure constitute a relatively straightforward generalization of the algorithm described in [5] for the case $X = \mathbb{P}^d$.

Working over the smooth toric variety X provides a convenient means for encoding the sheaves $\mathcal{M}$ and $\mathcal{N}$. To be more explicit, identify $\operatorname{Pic}(X)$ with $\mathbb{Z}^r$ and write $\mathcal{O}_X(\mathbf{v})$ for the line bundle associated to $\mathbf{v} \in \mathbb{Z}^r$. Let $S := \Bbbk[x_1, \dots, x_n]$ be the $\mathbb{Z}^r$-graded Cox ring (a.k.a. total coordinate ring), and let B denote the irrelevant ideal. From a geometric perspective, S is determined by torus-invariant divisors on X and the reduced monomial ideal B is determined by the torus-fixed points on X. In terms of combinatorics, S is given by the rays (i.e. one-dimensional cones) in the fan corresponding to X and B is given by the maximal cones in the same fan. Since the precise definitions will not be needed, we refer to §5.2 in [1] for the details. With this notation, the toric variety X is the geometric quotient $\big(\mathbb{A}^n - \mathbf{V}(B)\big)/(\Bbbk^\times)^r$ and $d := \dim(X) = n - r$; see Theorem 5.1.11 in [1]. Moreover, the category of coherent $\mathcal{O}_X$-modules is equivalent to the quotient category of finitely generated $\mathbb{Z}^r$-graded S-modules by the thick subcategory of all B-torsion modules; see §5.3 in [1]. In this context, our problem becomes the following: given two finitely generated $\mathbb{Z}^r$-graded S-modules M and N, calculate the module $\operatorname{Ext}^m_X(\mathcal{M},\mathcal{N})$ where $\mathcal{M}$ and $\mathcal{N}$ are the coherent sheaves corresponding to M and N respectively.

Our projectivity assumption on X is also valuable. Specifically, this hypothesis guarantees that there exists an r-dimensional polyhedral cone $\mathbf{C} \subset \mathbb{Z}^r$ such that $\operatorname{Ext}^m_X\big(\mathcal{O}_X, \mathcal{O}_X(\mathbf{v})\big) = H^m\big(X, \mathcal{O}_X(\mathbf{v})\big) = 0$ for all $m > 0$ and $\mathbf{v} \in \mathbf{C}$. In particular, the cone of numerically effective line bundles $\operatorname{Nef}(X)$ is contained in $\mathbf{C}$; see Theorem 9.2.3 in [1]. The maximal vanishing cone $\mathbf{C} \subset \mathbb{Z}^r$ is obtained from the reduced homology of the induced subfans of X. Alternatively, the cone $\mathbf{C}$ is determined by the twists in the minimal $\mathbb{Z}^r$-graded free resolution of the irrelevant ideal B; see §3 in [4]. Roughly speaking, we use the line bundles corresponding to elements in $\mathbf{C}$ to form an acyclic resolution of the coherent sheaf $\mathcal{M}$.

The solution to our problem involves relating $\operatorname{Ext}^m_X(\mathcal{M},\mathcal{N})$ to $\big(\operatorname{Ext}^m_X(M',N)\big)_{\mathbf{0}}$ where M' is another $\mathbb{Z}^r$-graded S-module corresponding to $\mathcal{M}$. To express our main result, let $F_\bullet(M)$ denote a minimal $\mathbb{Z}^r$-graded free resolution of M. It follows that there is an exact sequence

$$0 \leftarrow M \leftarrow F_0(M) \leftarrow F_1(M) \leftarrow \cdots \leftarrow F_i(M) \leftarrow \cdots$$

where each $\mathbb{Z}^r$-graded S-module $F_i(M)$ is a direct sum of twists of S:

$$F_i(M) = \bigoplus_{j=1}^{b_i(M)} S\big(-\mathbf{a}_{i,j}(M)\big) \qquad \text{for some } \mathbf{a}_{i,j}(M) \in \mathbb{Z}^r.$$

The maps in this resolution are given by matrices of homogeneous forms and minimality is equivalent to saying that none of the entries are nonzero constants. Under this condition, all of the numerical invariants $b_i(M)$ and $\mathbf{a}_{i,j}(M)$ are independent of the choice of minimal free resolution.

Given these preliminaries, our main result is the following.

Theorem. *Let M and N be finitely generated $\mathbb{Z}^r$-graded S-modules, let m be an integer, and let $\mathbf{v},\mathbf{u} \in \mathbb{Z}^r$ be two vectors such that $\mathcal{O}_X(\mathbf{u})$ is ample. If $e \in \mathbb{N}$ satisfies $\mathbf{v} + \mathbf{a}_{m-k,\ell}(S_{e\mathbf{u}}M) - \mathbf{a}_{i,j}(N) \in \mathbf{C}$ for all $0 \leqslant k \leqslant m$, $1 \leqslant \ell \leqslant b_{m-k}(M)$, $0 \leqslant i \leqslant d-m$ and $1 \leqslant j \leqslant b_i(N)$, then there is a canonical isomorphism $\operatorname{Ext}^m_X\big(\mathcal{M},\mathcal{N}(\mathbf{v})\big) = \operatorname{Ext}^m_S(S_{e\mathbf{u}}M,N)_{\mathbf{v}}$.*

The condition $\mathbf{v}+\mathbf{a}_{m-k,\ell}(S_{e\mathbf{u}}M)-\mathbf{a}_{i,j}(N) \in \mathbf{C}$ holds for all $e \gg 0$, because $\mathcal{O}_X(\mathbf{u})$ is ample. When $\mathcal{M} = \mathcal{O}_X$, we essentially recover Theorem 0.2 in [2].

The proof of this theorem, following §2 of [5], has three steps. One first proves a uniform vanishing result. The appropriate weak form of multigraded Castelnuovo-Mumford regularity (cf. §6 in [4]) states that, if $\mathbf{v}-\mathbf{a}_{i,j}(N) \in \mathbf{C}$ for all $0 \leqslant i \leqslant d-m$ and $1 \leqslant j \leqslant b_i(N)$, then we have $H^m\big(X,\mathcal{N}(\mathbf{v})\big) = 0$. Now, let $\mathcal{F}_\bullet(M)$ denote the locally free resolution of $\mathcal{M}$ corresponding to the sheafification of minimal free resolution $F_\bullet(M)$. Exploiting these vanishing conditions gives sufficient conditions to force the spectral sequence

$$E_2^{p,q} = \mathrm{H}^p\Big(H^q\big(X,\mathcal{H}om_X\big(\mathcal{F}_\bullet(M),\mathcal{N}(\mathbf{v})\big)\big)\Big) \Longrightarrow \operatorname{Ext}^{p+q}_X\big(\mathcal{M},\mathcal{N}(\mathbf{v})\big)$$

to degenerate. The second step shows that, if $\mathbf{v} + \mathbf{a}_{m-k,\ell}(M) - \mathbf{a}_{i,j}(N) \in \mathbf{C}$ for all $1 \leqslant k \leqslant m$, $1 \leqslant \ell \leqslant b_{m-k}(M)$, $0 \leqslant i \leqslant d-m$ and $1 \leqslant j \leqslant b_i(N)$, then we have $\operatorname{Ext}^m_X\big(\mathcal{M},\mathcal{N}(\mathbf{v})\big) = \mathrm{H}^m\big(\operatorname{Hom}_X\big(\mathcal{F}_\bullet(M),\mathcal{N}(\mathbf{v})\big)\big)$. Having reduced the problem to calculating the cochain complex $\operatorname{Hom}_X\big(\mathcal{F}_\bullet(M),\mathcal{N}(\mathbf{v})\big)$, one uses the exact sequence

$$0 \leftarrow H^1_B(N) \leftarrow \bigoplus\nolimits_{\mathbf{v}\in\mathbb{Z}^r} H^0\big(X,\mathcal{N}(\mathbf{v})\big) \leftarrow N \leftarrow H^0_B(N) \leftarrow 0$$

to obtain additional restrictions. Specifically, the third step establishes that, if $\mathbf{v}+\mathbf{a}_{m,\ell}(M)-\mathbf{a}_{i,j}(N) \in \mathbf{C}$ for all $1 \leqslant \ell \leqslant b_m(M)$, $0 \leqslant i \leqslant d+1$ and $1 \leqslant j \leqslant b_i(N)$, then we have $\mathrm{H}^m\big(\operatorname{Hom}_X\big(\mathcal{F}_\bullet(M),\mathcal{N}(\mathbf{v})\big)\big) = \operatorname{Ext}^m_S(M,N)_{\mathbf{v}}$. Combining all three steps yields a proof for the theorem.

To efficiently implement the algorithm arising from our theorem, one needs minimize the size of $F_\bullet(S_{e\mathbf{u}}M)$. To accomplish this, one should replace N with its saturation with respect to B in order to minimize the $\mathbf{a}_{i,j}(N)$. One should also use a more refined vanishing result which keeps track of distinct vanishing sets $\mathbf{C}_m$ for each cohomology group $H^m\big(X, \mathcal{O}_X(\mathbf{v})\big)$ where $0 \leqslant m \leqslant d$. Moreover, experimental evidence suggests that one should employ the e-th Frobenius power of the monomial ideal generated in degree $\mathbf{u}$ (i.e. $(S_{\mathbf{u}})^{[e]}$) rather than the e-th ordinary power (i.e. $S_{e\mathbf{u}}$). Building on the approach presented in this extended abstract, methods for computing the cohomology of sheaves on many normal toric varieties and the global extension modules for coherent sheaves on projective space are available in *Macaulay2* [3].

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Morphisms between Discrete Vector Fields

Francis Sergeraert

(joint work with Ana Romero)

The tool *Discrete Vector Fields* is now important in Constructive Algebraic Topology. It allowed us to find out a new understanding of the Eilenberg-Zilber theorem, powerful as well for the theoretical point of view as for concrete programming.

The Eilenberg-Zilber theorem is so understood as an obvious generalization of the standard triangulation of the square by two triangles (!). The *same* (!) vector field gives as well the *twisted* Eilenberg-Zilber theorem, and also which is obviously *the* right proof of the Eilenberg-MacLane conjecture (1953) about the correspondence between the topological and algebraic notions of *Classifying Spaces.*

More precisely, the algorithm implementing this correspondence, crucial when computing homotopy groups through the Whitehead tower, has been immediately implemented and gives striking results: for example the computing time of the homotopy group $\pi_5(\Omega S^3 \cup_2 D^3)$ has so been divided by 18: five minutes vs one hour and a half.

Numerous machine tests show an *experimental* evidence for the correctness of this new version of the Eilenberg-MacLane correspondence, but the proof is not so easy, not yet finished. The compatibility between the Eilenberg-Zilber vector field and the underlying algebraic structures in fact is not obvious at all.

The talk has been devoted to a central point of this subject, the *naturality* of the reduction canonically induced by a discrete vector field. The result is rather surprising. The natural (!) definition of morphisms between discrete vector fields is not at all the right one, and the right one is surprising: simply the image of a critical (resp. target) cell must be a critical (resp. target cell) but no condition at all is to be required for the source cells!

For the foreseen application to Eilenberg-Zilber vector fields, it so happens the behaviour of this vector field with respect to the source cells is totally anarchic, but fortunately this behaviour is the right one with the target and critical cells.

A didactic exposition of this subject is the heart of the talk, the proofs being quite elementary, a simple combinatorial game about some diagrams in finite lattices. The talk finishes with the application: the reduction given by the so-called Eilenberg-Zilber vector field is nothing but the old Eilenberg-Zilber reduction, as defined and described by Eilenberg, Zilber and MacLane more than sixty years ago. Allowing us to keep the benefit of the previous works in this area.

A detailed exposition of this work is available on the web site of the speaker [1].

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Discrete Vector Fields and Fundamental Algebraic Topology
`www-fourier.ujf-grenoble.fr/∼sergerar/Papers/Vector-Fields.pdf`

Boij–Söderberg theory and tensor complexes

Christine Berkesch
(joint work with Daniel Erman, Manoj Kummini, Steven V Sam)

The conjectures of M. Boij and J. Söderberg [3], proven by D. Eisenbud and F.-O. Schreyer [8] (see also [7, 4]), link the extremal properties of invariants of graded free resolutions of finitely generated modules over the polynomial ring $S = k[x_1, \dots, x_n]$ with the Herzog–Huneke–Srinivasan Multiplicity Conjectures. Here k is any field and S has the standard $\mathbb{Z}$-grading. In the course of their proof, Eisenbud and Schreyer introduce a groundbreaking relationship between the study of free resolutions over the S and the study of the cohomology of coherent sheaves on $\mathbb{P}_k^{n-1}$, via a nonnegative pairing of their associated numerics. This pairing has recently been categorified through work of Eisenbud and Erman [6], further extending the reach of Boij–Söderberg theory to larger classes of derived objects.

We now outline the main result of Boij–Söderberg theory for S. For simplicity, we restrict our attention to a graded S-module M that is of finite length; minor modifications yield the general situation. A minimal free resolution of M is an acyclic complex $(F_\bullet, \partial_\bullet)$ such that $H_0(F_\bullet) = M$, $\partial_i(F_i) \subseteq \langle x_1, \dots, x_n\rangle F_{i-1}$ for each i, and $F_i = \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{i,j}}$. The ranks $\beta_{i,j}$ of the free modules are independent of the choice of resolution $F_\bullet$ of M, and they are called the *Betti numbers* of M. We record these Betti numbers into a *Betti table* for M, denoted $\beta(M)$.

It appears to be a difficult question to classify which integer tables can be realized as the Betti table of a graded S-module. In a shift in perspective, Boij and Söderberg suggested that this task be approached *up to scalar multiple.* In other words, view $\beta(M) \in \bigoplus_{i=0}^{n} \bigoplus_{j\in\mathbb{Z}} \mathbb{Q} =: \mathbb{V}$ and describe instead the *cone of Betti tables*

$$B_{\mathbb{Q}}(S) := \mathbb{Q}_{\geq 0} \cdot \{\, \beta(M) \mid M \text{ graded } S\text{-module of finite length} \,\}.$$

In this direction, we say that $d = (d_0 < d_1 < \cdots < d_n) \in \mathbb{Z}^{n+1}$ is a *degree sequence* for S, and we define a partial order on these sequences via $d \leq d'$ if $d_i \leq d'_i$ for all i. Associated to a degree sequence d, we define the *pure diagram* $\pi_d \in \mathbb{V}$ by

$$\beta_{i,j}(\pi_d) = \begin{cases} \frac{1}{\prod_{\ell\neq i} |d_i - d_\ell|} & \text{if } j = d_i, \\ 0 & \text{otherwise.} \end{cases}$$

These are related to the Herzog–Kühl equations for *pure free resolutions*, see [13].

Theorem 1 ([8])**.** *The extremal rays of $B_{\mathbb{Q}}(S)$ are precisely those spanned by the π_d. Furthermore, if $D \in B_{\mathbb{Q}}(S)$, then there exist $a_i \in \mathbb{Q}_{\geq 0}$ and degree sequences $d^1 < d^2 < \cdots < d^\ell$ such that $D = \sum_{i=1}^{\ell} a_i \pi_{d^i}$.*

The decomposition of D in Theorem 1, which endows $B_{\mathbb{Q}}(S)$ with the structure of a simplicial fan, arises from a greedy algorithm. An important ingredient in the proof of the theorem is to show that each pure diagram π_d is realizable, up to scalar multiple, as $\beta(M)$ for some module M. Originally, Eisenbud and Schreyer applied a nonconstructive pushforward argument to show the existence of such M. A symmetrization of this argument produces *tensor complexes*, and the symmetry of these resolutions also allows them to be described explicitly.

Fix $(b_0, \ldots, b_n) \in \mathbb{N}^{n+1}$, let $R = \mathbb{Z}[x_{i,J}]$, where $1 \leq i \leq b_0$, $J = (j_1, \ldots, j_n)$, $1 \leq j_\ell \leq b_\ell$, and let $\phi = (x_{i,J}) \in R^{b_0} \otimes (R^{b_1})^* \otimes \cdots \otimes (R^{b_n})^*$ be the universal tensor. In [1], we construct, from the tensor ϕ and a choice $w \in \mathbb{Z}^{n+1}$ from an infinite family of appropriate weight vectors, a *tensor complex* $F(\phi, w)_\bullet$ with the following properties.

Theorem 2 ([1])**.** *A tensor complex $F(\phi, w)_\bullet$ satisfies the following:*

(i) *It is a graded pure free resolution of a Cohen–Macaulay module $M(\phi, w)$.*
(ii) *It is uniformly minimal over $\mathbb{Z}$, i.e., $F(\phi, w)_\bullet \otimes_R k[x_{i,J}]$ is a minimal free resolution for any field k.*
(iii) *It respects the multilinearity of ϕ, i.e., it is $\mathrm{GL}_{b_0} \times \cdots \times \mathrm{GL}_{b_n}$-equivariant.*
(iv) *Its differentials can by made explicit after fixing appropriate bases for the free modules $F(\phi, w)_i$.*

The construction of tensor complexes provides detailed new examples of minimal free resolutions, as well as a unifying view on a wide variety of complexes, including the Eagon–Northcott, Buchsbaum–Rim, and similar complexes, called Buchsbaum–Eisenbud matrix complexes [5, §A2.6], as well as the complexes used by Gelfand–Kapranov–Zelevinsky and Weyman to compute hyperdeterminants in [11, §14] and [14, §9.4]. In addition, tensor complexes have applications in Boij–Söderberg theory, as they provide infinitely many new families of pure resolutions, as well as the first explicit description of the differentials of the Eisenbud–Schreyer pure resolutions.

Corollary 3 ([1]). *There is a map* $R \to S = k[x_1, \dots, x_n]$ *such that* $S \otimes_R F(\phi, w)_\bullet$ *is the Eisenbud–Schreyer pure resolution constructed in* [8]. *In particular, these resolutions can be made explicit.*

The `BoijSoederberg` and `TensorComplexes` packages of the computer algebra software `Macaulay2` contain implementations of the work discussed in this talk [12]. For surveys on Boij–Söderberg theory, see [2, 9, 10].

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A Constructive Setup for Coherent Sheaves

Markus Lange-Hegermann
(joint work with Mohamed Barakat)

Our motivation is to establish a constructive setup for homological algebra in the category of coherent sheaves on several classes of schemes X. In the following we only consider the projective n-space $X = \mathbb{P}^n$ over a computable field k. However, the methods discussed also work for projective schemes relative to an abstract scheme and toric varieties [12]. Our approach is (partially) implemented in the homalg project.

Our goal is homological algebra in the context of Abelian categories. We proceed in a foundational manner: to implement all constructive proofs we need to provide algorithms for all existential quantifiers and disjunctions in the defining axioms of Abelian categories. We call any Abelian category *computable*, when this is possible. For example the category $S\mathrm{grmod}$ of finitely presented graded modules over the graded ring $S := k[x_0, \ldots, x_n]$ with $\deg(x_i) = 1$ is computable due to Gröbner basis methods [5]. Such categories allow all standard homological constructions and thus to work with them in tangible terms on a computer. This includes the computation of spectral sequences, in particular the filtrations they induce on their limits [2]. The goal of this talk is to show that the category $\mathrm{Coh}\,\mathbb{P}^n$ of coherent sheaves on projective n-space is computable.

Usually, one represents coherent sheaves by finitely generated graded modules, as the sheafification functor $\widetilde{\ } : S\mathrm{grmod} \to \mathrm{Coh}\,\mathbb{P}^n$ is essentially surjective and exact. However, used naively, this approach has the following problem. The inclusion $\iota : 0 \hookrightarrow S/\langle x_0, \ldots, x_n\rangle$ sheafifies to the isomorphism $\widetilde{\iota} : 0 \xrightarrow{\sim} 0$, as $S/\langle x_0, \ldots, x_n\rangle$ is Artinian and thus in the kernel of the sheafification functor. The isomorphism $\widetilde{\iota}$ has an inverse, which we want to compute. However, ι has no inverse; this is due to the sheafification not being an equivalence of categories.

The correct framework for $\mathrm{Coh}\,\mathbb{P}^n$ is that of Serre quotient categories. It turns out that $\mathrm{Coh}\,\mathbb{P}^n$ is equivalent to the Serre quotient $S\mathrm{grmod}/S\mathrm{grmod}^0$, where $S\mathrm{grmod}^0$ is the thick subcategory of finitely generated Artinian graded S-modules. A similar statement has long been known for a quotient of quasi-finitely generated modules [10] and has only recently been extended to finitely generated modules [9]. The following theorem provides the computability for Serre quotient categories.

Theorem ([4]). *Let $\mathcal{A}$ be a computable Abelian category and $\mathcal{C}$ a thick subcategory. Assume that we can decide whether an object of $\mathcal{A}$ lies in $\mathcal{C}$. Then the category $\mathcal{A}/\mathcal{C}$ is Abelian and as such computable.*

An easy corollary is that $\mathrm{Coh}\,\mathbb{P}^n$ is computable, as $S\mathrm{grmod}$ is computable and a module is Artinian if and only if its Hilbert polynomial is zero. In these categories graded modules are used to model sheaves but morphisms of sheaves are modeled in a more complicated way. More precisely, the proof uses a description of Serre quotient categories by a 3-arrow formalism, which we call Gabriel morphisms, to replace the direct limit description of $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}$ by a constructive dynamical process.

As a special case this implies that the computation of spectral sequences of filtered complexes is possible for the category of coherent sheaves on projective space. It remains to describe the construction of interesting filtered complexes. Usually, these arise from resolutions and application of functors. Now, we describe the computation of some functors. The computation of sheaf Hom and the tensor product of $\mathrm{Coh}\,\mathbb{P}^n$ is just a sheafification of the graded Hom and the tensor product in the category $S\mathrm{grmod}$. Both their derivations $\mathcal{E}xt$ and Tor can be computed algorithmically using locally free resolutions, which can be constructed by sheafifying graded free resolutions. Assume that the canonical quotient functor $Q : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ has a right adjoint S, the so-called section functor. Under this assumption we have $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(Q(M), Q(N)) \cong \mathrm{Hom}_{\mathcal{A}}(M, (S \circ Q)(N))$. There are algorithms to compute $S \circ Q$ [7, 3] and thus the Hom-functors is also algorithmical. The computability of Ext in this context is established in [8, 1].

The computation of these functors and the computability of the Abelian category of coherent sheaves is demonstrated by an example using the implementation in the homalg project [11, 6]. We compute of the equidimensional filtration on a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ using the spectral sequence

$$\mathcal{E}xt^p(\mathcal{E}xt^{-q}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^n}), \mathcal{O}_{\mathbb{P}^n}) \Rightarrow \begin{cases} \mathcal{F}, & p+q=0 \\ 0, & \text{otherwise.} \end{cases}$$

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Computing Ext in Serre quotient categories

Mohamed Barakat

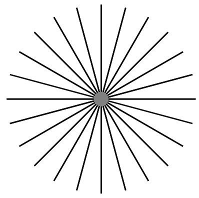
(joint work with Markus Lange-Hegermann)

The ability to compute Ext groups is of importance in algebra and geometry. For example, the higher derived functors $H^i(X, -)$ of the global section functor $\Gamma(X, -) = \mathrm{Hom}(\mathcal{O}_X, -)$ on a scheme X can be defined in terms of Ext:

$$H^i(X, -) = \mathrm{Ext}^i(\mathcal{O}_X, -),$$

where $\mathcal{O}_X$ is the structure sheaf of X. Categories of coherent sheaves $\mathfrak{Coh}\, X$ can often be described as a SERRE quotient of some category $\mathcal{A}$ of finitely generated graded modules over a computable graded ring S (where Ext is computable) modulo a thick subcategory $\mathcal{C}$. This is the case for $X = \mathbb{P}^n$, $X = X_\Sigma$ a normal toric variety, or X a MORI dream space, where S is the COX ring graded by the divisor class group $\operatorname{Cl} X$. The case $X = \mathbb{P}^n$ was already described in [4]: Let $S = k[x_0, \dots, x_n]$ denote the polynomial ring with its standard $\mathbb{Z}$-grading. Graded S-modules descend to modules on the orbit space

$$\mathbb{A}^{n+1}/k^* = \mathbb{P}^n \,\dot{\cup}\, \underbrace{\{0\}}_{\substack{\text{irrelevant}\\ \text{locus}}}$$



Hence, $\mathfrak{Coh}\,\mathbb{P}^n$ is equivalent to the category $S-\operatorname{grmod}$ of finitely presented graded S-modules but where the thick subcategory $S-\operatorname{grmod}^0$ of modules supported on the irrelevant locus (here, the ARTINians) is treated as the subcategory of zero objects, i.e.,

$$\mathfrak{Coh}\,\mathbb{P}^n \simeq S-\operatorname{grmod}/S-\operatorname{grmod}^0.$$

Given a general SERRE quotient category $\mathcal{A}/\mathcal{C}$ of an ABELian category $\mathcal{A}$, we describe how to compute $\operatorname{Ext}^i_{\mathcal{A}/\mathcal{C}}$ by relating it to a limit of $\operatorname{Ext}^i_{\mathcal{A}}$'s in the ambient category $\mathcal{A}$, under certain conditions depending on $i \geq 0$ (cf. [2]). Our approach differs from the one used in GREG SMITH's talk [6] as it is formulated in the context of abstract ABELian categories without enough projectives or injectives[1] (and without an internal $\mathcal{H}om$). Our excuse for not making a convergence analysis like in [5, 6] is that it does not make much sense in this generality. We thereby only generalize the first step in his proof.

From now on let $\mathcal{A}$ denote an ABELian category and $\mathcal{C} \subset \mathcal{A}$ a thick (or SERRE) subcategory. Recall, the SERRE quotient category $\mathcal{A}/\mathcal{C}$ is defined by setting

- $\operatorname{Obj} \mathcal{A}/\mathcal{C} = \operatorname{Obj} \mathcal{A}$;
- $\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(M, N) = \varinjlim_{\left\{ \substack{M' \leq M, N' \leq N \\ M/M', N' \in \mathcal{C}} \right\}} \operatorname{Hom}_{\mathcal{A}}(M', N/N')$.

The definition of the Hom-group leads to the 3-arrow formalism presented in MARKUS LANGE-HEGERMANN's talk [3]. Denote by $\mathscr{Q} : \mathcal{A} \to \mathcal{A}/\mathcal{C}, M \mapsto M, \phi \mapsto [\phi]$ the canonical functor. It is *exact* but in general neither full nor faithful.

The computability of $\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}$ reduces to that of $\operatorname{Hom}_{\mathcal{A}}$ once the thick subcategory $\mathcal{C} \subset \mathcal{A}$ is localizing, i.e., once the canonical functor $\mathscr{Q} : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ admits a right adjoint $\mathscr{S} : \mathcal{A}/\mathcal{C} \to \mathcal{A}$. $\mathscr{S}$ is called a section functor of $\mathscr{Q}$ since the counit $\mathscr{Q} \circ \mathscr{S} \to \operatorname{Id}_{\mathcal{A}/\mathcal{C}}$ is an isomorphism. The section functor $\mathscr{S}$, or rather the adjunction monad $\mathscr{S} \circ \mathscr{Q}$, allows getting rid of the limit in the definition of $\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}$:

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{C}}(\mathscr{Q}M, \mathscr{Q}N) \cong \operatorname{Hom}_{\mathcal{A}}(M, (\mathscr{S} \circ \mathscr{Q})(N)).$$

[1] Only the nonconstructive $\mathfrak{qCoh}\,\mathbb{P}^n \supset \mathfrak{Coh}\,\mathbb{P}^n$ has enough injectives.

Corollary. *If $\mathcal{A}$ is* Hom-*computable, $\mathcal{C}$ is constructively localizing (i.e., the monad $\mathscr{S} \circ \mathscr{Q}$ is computable) then $\mathcal{A}/\mathcal{C}$ is* Hom-*computable.*

In [1] we prove a theorem characterizing endofunctors $\mathscr{W} : \mathcal{A} \to \mathcal{A}$ equivalent to the adjunction monad $\mathscr{S} \circ \mathscr{Q}$.

Due to the lack of enough projectives or injectives we stick to YONEDA's description of

$$\operatorname{Ext}^c_{\mathcal{A}}(M, N) = \{[e : \underbrace{0 \leftarrow M \leftarrow G_c \leftarrow \cdots \leftarrow G_1 \leftarrow N \leftarrow 0}_{\text{exact}}]\}$$

as the ABELian group of certain equivalence classes of c-extensions. Applying the exact $\mathscr{Q}$ to a YONEDA cocycle $e \in \operatorname{Ext}^c_{\mathcal{A}}(M, N)$ yields a cocycle $\mathscr{Q}(e) \in \operatorname{Ext}^c_{\mathcal{A}/\mathcal{C}}(M, N)$. Since $\mathscr{Q}$ vanishes on $\mathcal{C}$ we get a binatural transformation

$$\mathscr{Q}^{\operatorname{Ext}} : \varinjlim_{\substack{M' \leq M, N' \leq N \\ M/M', N' \in \mathcal{C}}} \operatorname{Ext}^c_{\mathcal{A}}(M', N/N') \to \operatorname{Ext}^c_{\mathcal{A}/\mathcal{C}}(M, N).$$

It is now natural to ask for conditions sufficient for $\mathscr{Q}^{\operatorname{Ext}}$ to be an isomorphism. For $c = 1$ it turns out that assuming $\mathcal{C} \subset \mathcal{A}$ localizing is already sufficient for $\mathscr{Q}^{\operatorname{Ext}}$ to be an isomorphism.

Theorem ([2])**.** *If $\mathcal{C} \subset \mathcal{A}$ is a localizing subcategory then $\mathscr{Q}^{\operatorname{Ext}}$ is an isomorphism for $c = 1$.*

For the proof we need the following considerations. The adjunction monad yields an equivalence of categories

$$\mathscr{S} \circ \mathscr{Q} : \mathcal{A}/\mathcal{C} \xrightarrow{\sim} \operatorname{Sat}_{\mathcal{C}}(\mathcal{A}) \subset \mathcal{A},$$

where $\operatorname{Sat}_{\mathcal{C}}(\mathcal{A})$ is the full subcategory of $\mathcal{C}$-saturated objects, i.e., those objects $A \in \mathcal{A}$ satisfying $\operatorname{Ext}^i(C, A) = 0$ for $i = 0, 1$ and all $C \in \mathcal{C}$. So we are allowed to replace $\mathcal{A}/\mathcal{C}$ by $\operatorname{Sat}_{\mathcal{C}}(\mathcal{A})$ and $\mathscr{Q} : \mathcal{A} \to \mathcal{A}/\mathcal{C}$ by the corestriction $\widehat{\mathscr{Q}} = \operatorname{cores}_{|\operatorname{Sat}_{\mathcal{C}}(\mathcal{A})}(\mathscr{S} \circ \mathscr{Q}) : \mathcal{A} \to \operatorname{Sat}_{\mathcal{C}}(\mathcal{A})$. The rest follows easily from the left exactness of the embedding $\iota : \operatorname{Sat}_{\mathcal{C}}(\mathcal{A}) \hookrightarrow \mathcal{A}$.

For $c \geq 2$ we need further conditions on the categories $\mathcal{A}$ and $\mathcal{C}$. Let $H_{\mathcal{C}}(A)$ be the maximal $\mathcal{C}$-subobject of A defined as $\ker(A \to (\mathscr{S} \circ \mathscr{Q})(A))$. For an object $A \in \mathcal{A}$ we call a subobject $A^{\perp} \leq A$ an almost $\mathcal{C}$-complement if $A^{\perp} \cap H_{\mathcal{C}}(A) = 0$ and $A/\left(H_{\mathcal{C}}(A) + A^{\perp}\right) \in \mathcal{C}$. We call $\mathcal{C}$ an almost split localizing subcategory if for each $A \in \mathcal{A}$ there exists an almost $\mathcal{C}$-complement $A^{\perp}$. If every object $A \in \mathcal{A}$ has a maximal almost $\mathcal{C}$-complement then we call $\mathcal{C}$ a maximally almost split localizing subcategory of $\mathcal{A}$.

Theorem ([2])**.** *If $\mathcal{C}$ is a maximally almost split localizing subcategory of the* ABEL*ian category $\mathcal{A}$ then*

$$\widehat{\mathscr{Q}}^{\operatorname{Ext}} : \varinjlim_{\substack{M' \leq M, \\ M/M' \in \mathcal{C}}} \operatorname{Ext}^c_{\mathcal{A}}(M', N) \to \operatorname{Ext}^c_{\operatorname{Sat}_{\mathcal{C}}(\mathcal{A})}(M, N)$$

is an isomorphism (of ABEL*ian groups) for all* $\mathcal{C}$*-saturated* $M, N \in \mathcal{A}$*.*

For smooth projective toric varieties we saw the limit analysis in GREG SMITH's talk [6], in which he generalized his argument for the projective space which was based on the CASTELNUOVO-MUMFORD regularity [5]. The YONEDA approach presented here shows that

- the case $c = 1$ is special and should be treated separately in applications;
- a better understanding of maximal almost $\mathcal{C}$-complements might refine the convergence analysis.

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Isomorphisms and equivalences of linear functional systems

THOMAS CLUZEAU

(joint work with Alban Quadrat)

A linear functional system (e.g., a linear system of ordinary differential (OD) equations, partial differential (PD) equations, OD time-delay equations, difference equations) can generally be written as $R\,\eta = 0$, where R is a $q \times p$ matrix with entries in a noncommutative polynomial ring D of functional operators (e.g., OD or PD operators, time-delay operators, shift operators, difference operators) and η is a vector of unknown functions. More precisely, if $\mathcal{F}$ is a left D-module, then we consider the *linear system* $\ker_{\mathcal{F}}(R.) = \{\eta \in \mathcal{F}^p \mid R\,\eta = 0\}$. The algebraic analysis approach to mathematical system theory (see, e.g., [1, 3, 6, 7, 9, 10]) is based on the fact that the linear system $\ker_{\mathcal{F}}(R.)$ can be studied by means of the left D-module $M = D^{1\times p}/(D^{1\times q}\,R)$ *finitely presented by the matrix* R. Indeed, a remark of Malgrange [6] asserts that $\ker_{\mathcal{F}}(R.) \cong \hom_D(M, \mathcal{F})$, Hence, systemic properties of $\ker_{\mathcal{F}}(R.)$ can be studied by means of module properties of M and $\mathcal{F}$. Algorithms for checking certain module properties of M were recently developed based on constructive homological algebra for noncommutative polynomial rings D admitting Gröbner bases for admissible term orders. These algorithms were implemented in the packages and the computer algebra systems OreModules, OreMorphisms, Plural and `homalg`.

The first purpose of this talk is to develop a constructive version of Fitting's result [4] which asserts that two matrices presenting isomorphic left D-modules can be enlarged by blocks of 0 and I (identity matrix) to get equivalent matrices. This important result in module theory explains the relations between the key concepts of isomorphism of modules and equivalence of matrices, and has many applications in linear system theory. The results are gathered in the following theorem:

Theorem 1. *Let $R \in D^{q\times p}$, $R' \in D^{q'\times p'}$ and*

$$f : M = D^{1\times p}/(D^{1\times q}\, R) \quad \longrightarrow \quad M' = D^{1\times p'}/(D^{1\times q'}\, R')$$

be a left D-isomorphism, where $P \in D^{p\times p'}$ is a matrix such that $R\,P = Q\,R'$ for a certain matrix $Q \in D^{q\times q'}$. Moreover, let $R_2 \in D^{r\times q}$ (resp., $R'_2 \in D^{r'\times q'}$) be a matrix such that $\ker_D(.R) = D^{1\times r}\, R_2$ (resp., $\ker_D(.R') = D^{1\times r'}\, R'_2$). Then, there exist 6 matrices $P' \in D^{p'\times p}$, $Q' \in D^{q'\times q}$, $Z \in D^{p\times q}$, $Z' \in D^{p'\times q'}$, $Z_2 \in D^{q\times r}$ and $Z'_2 \in D^{q'\times r'}$ satisfying

$$\begin{cases} R'\,P' = Q'\,R, \\ P\,P' + Z\,R = I_p, \\ P'\,P + Z'\,R' = I_{p'}. \end{cases} \qquad \begin{cases} Q\,Q' + R\,Z + Z_2\,R_2 = I_q, \\ Q'\,Q + R'\,Z' + Z'_2\,R'_2 = I_{q'}, \end{cases}$$

and such that the following results hold:

(1) With the notation $s = q + p' + p + q'$, we have:

$$X = \begin{pmatrix} I_p & P \\ -P' & I_{p'} - P'\,P \end{pmatrix} \in \mathrm{GL}_{p+p'}(D), \quad X^{-1} = \begin{pmatrix} I_p - P\,P' & -P \\ P' & I_{p'} \end{pmatrix},$$

$$Y = \begin{pmatrix} I_q & 0 & R & Q \\ 0 & I_{p'} & -P' & Z' \\ -Z & P & 0 & P\,Z' - Z\,Q \\ -Q' & -R' & 0 & Z'_2\,R'_2 \end{pmatrix} \in \mathrm{GL}_s(D), \quad Y^{-1} = \begin{pmatrix} Z_2\,R_2 & 0 & -R & -Q \\ P'\,Z - Z'\,Q' & 0 & P' & -Z' \\ Z & -P & I_p & 0 \\ Q' & R' & 0 & I_{q'} \end{pmatrix}.$$

(2) The following commutative exact diagram holds

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & D^{1\times s} & \xrightarrow{.L} & D^{1\times(p+p')} & \xrightarrow{\pi\,\oplus\,0_{p'}} & M \longrightarrow 0 \\ & & \downarrow .Y & & \downarrow .X & & \downarrow f \\ & & D^{1\times s} & \xrightarrow{.L'} & D^{1\times(p+p')} & \xrightarrow{0_p\,\oplus\,\pi'} & M' \longrightarrow 0, \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where $\pi \oplus 0_{p'}$ *and* $0_p \oplus \pi'$ *are defined by*

$$\begin{array}{ccc} D^{1\times(p+p')} & \xrightarrow{\pi \oplus 0_{p'}} & M \\ (\lambda \quad \lambda') & \longmapsto & \pi(\lambda), \end{array} \qquad \begin{array}{ccc} D^{1\times(p'+p)} & \xrightarrow{0_p \oplus \pi'} & M' \\ (\lambda \quad \lambda') & \longmapsto & \pi'(\lambda'), \end{array}$$

and

$$L = \begin{pmatrix} R & 0 \\ 0 & I_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in D^{s\times(p+p')}, \quad L' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I_p & 0 \\ 0 & R' \end{pmatrix} \in D^{s\times(p+p')},$$

i.e., we have $L\,X = Y\,L'$*, and thus:* $L' = Y^{-1}\,L\,X \Longleftrightarrow L = Y\,L'\,X^{-1}$.

The consequences of this theorem on the Auslander transposes and adjoints of the finitely presented left modules are given. Then in the second part of the talk, we give an explicit characterization of isomorphic finitely presented modules in terms of certain inflations of their presentation matrices.

Theorem 2. *Let* $M_1 = D^{1\times p}/(D^{1\times q}\,R_1)$ *(resp.,* $M_2 = D^{1\times t}/(D^{1\times s}\,Q_2)$*) be a left* D*-module finitely presented by* $R_1 \in D^{q\times p}$ *(resp.,* $Q_2 \in D^{s\times t}$*) such that* $M_1 \cong M_2$*. Then, there exist matrices* $R_2 \in D^{q\times s}$*,* $Q_1 \in D^{p\times t}$*,* $Q_2 \in D^{s\times t}$*,* $S_1 \in D^{p\times q}$*,* $S_2 \in D^{s\times q}$*,* $T_1 \in D^{t\times p}$*,* $T_2 \in D^{t\times s}$*,* $V_1 \in D^{q\times l}$*,* $V_2 \in D^{t\times l}$*,* $W_1 \in D^{p\times m}$*, and* $W_2 \in D^{s\times m}$ *such that*

$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_{q+t} + \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} (P_1 \quad 0),$$

$$\begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} = I_{p+s} + \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} (0 \quad P_2),$$

where $P_1 \in D^{l\times q}$ *and* $P_2 \in D^{m\times s}$ *are defined by:*

$$\ker_D(.R_1) = D^{1\times l}\,P_1, \quad \ker_D(.Q_2) = D^{1\times m}\,P_2.$$

Fitting's theorem on the syzygy modules of these modules can then be found again. If the base ring is stably finite and one of the modules admits a full row rank presentation, this result yields a characterization of isomorphic modules as the completion problem characterizing Serre's reduction, i.e., the possibility to find a presentation of the module defined by less generators and less relations. This completion problem is shown to induce different isomorphisms between the modules finitely presented by the matrices defining the inflations which is then used to study Serre's reduction problem and completes results of [2].

Corollary 1. *With the notations and the assumptions of Theorem 2, let* D *be a stably finite ring (e.g., a noetherian domain* D*) and assume that* $q+t = p+s$*.*

(1) Then, we have:

$$(1) \quad \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_{q+t} \Leftrightarrow \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} = I_{p+s}.$$

(2) If either R_1 or Q_2 has full row rank, i.e., $\ker_D(.R_1) = 0$ *or* $\ker_D(.Q_2) = 0$*, then (1) holds. Equivalently, if R_1 or Q_2 has full row rank, then $M_1 \cong M_2$ is equivalent to the existence of $R_2 \in D^{q\times s}$, $Q_1 \in D^{p\times t}$, $Q_2 \in D^{s\times t}$, $S_1 \in D^{p\times q}$, $S_2 \in D^{s\times q}$, $T_1 \in D^{t\times p}$, and $T_2 \in D^{t\times s}$ such that:*

$$\begin{pmatrix} R_1 & R_2 \\ T_1 & T_2 \end{pmatrix} \begin{pmatrix} S_1 & Q_1 \\ S_2 & Q_2 \end{pmatrix} = I_{q+t}.$$

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A constructive approach to schemes and coherent sheaves

Henri Lombardi

We propose a constructive approach for the theory of Grothendieck' "spectral" schemes (these ones having as basis a spectral space). In particular, all Nœtherian schemes are spectral.

Our constructions neither use the affine schemes as sheaves of local rings nor the points (prime ideals) of the spectrum of a commutive ring. We "glue" finitely many commutative rings in a purely formal way. The same kind of construction works for the so called "sheaves of modules" over a spectral scheme. Our constructions are made possible by the existence of gluing theorems which work in the case of affine schemes, considering the category of affine schemes as the dual of the category of commutative rings.

First thing, gluing comaximal localizations of a commutative ring is fair. Consider a ring R with comaximal elements s_i (i.e. $\exists u_i$'s, $\sum_i u_i s_i = 1$)

$$R_i = R[1/s_i],\ R_{ij} = R[1/s_i s_j] \text{ and } \alpha_i : R \to R_i,\ \alpha_{ij} : R_i \to R_{ij}$$

Then in the following diagram R is the (inverse) limit of the R_i's and R_{ij}'s.

$$\begin{tikzcd}
 & & R_i \arrow[r, "\alpha_{ij}"] \arrow[ddr, "\alpha_{ik}"'] & R_{ij} \\
S \arrow[urr, "\psi_i"] \arrow[r, dashed, "\psi!"] \arrow[drr, "\psi_j"'] \arrow[ddrr, "\psi_k"'] & R \arrow[ur, "\alpha_i"'] \arrow[dr, "\alpha_j"] \arrow[ddr, "\alpha_k"'] & & \\
 & & R_j \arrow[uur, "\alpha_{ji}"'] \arrow[dr] & R_{ik} \\
 & & R_k \arrow[ur] \arrow[r, "\alpha_{kj}"'] & R_{jk}
\end{tikzcd}$$

Two views for this gluing.
1 – Elements of the ring can be given locally, and a morphism between commutative rings can be given locally.
2 – Good properties of the ring can be tested locally (e.g., to be a coherent ring, a pp-ring, a pf-ring, a normal ring, a Prüfer ring, a ring with a divisor theory, a Nœtherian ring), and in case of A-agebras, to be finitely generated, finitely presented, smooth, unramified, étale, flat or quasi finite.
NB. *The localized rings are not local rings. But they are finitely many.* Localizing at all primes is not a very good idea!

Second thing, the topological spectral space $\mathsf{Spec}\,R$ is to be replaced by its dual, the Zariski lattice $\mathsf{Zar}\,R$. There is fortunately a gluing theorem for distributive lattices which allows us to construct the base of the scheme obtained by gluing affine schemes.

This can be explained in two steps.

If we give a distributive lattice T, and $s_i \in T$ are "comaximal" (i.e. $\bigvee_i s_i = 1_T$), denoting $T_i = T/\!\uparrow s_i$, $T_{ij} = T/\!\uparrow(s_i s_j)$, $\pi_i : T \to T/\!\uparrow s_i$, $\ldots$, we get T as the limit of the T_i's and T_{ij}'s in the following diagram

$$\begin{tikzcd}
 & T \arrow[dl, "\pi_i"'] \arrow[d, "\pi_j"] \arrow[dr, "\pi_k"] & \\
T_i \arrow[d, "\pi_{ij}"'] \arrow[dr, "\pi_{ik}"] & T_j \arrow[dl, "\pi_{ji}"'] \arrow[dr, "\pi_{jk}"] & T_k \arrow[dl, "\pi_{ki}"'] \arrow[d, "\pi_{kj}"] \\
T_{ij} & T_{ik} & T_{jk}
\end{tikzcd}$$

When T is not given, we give T_i's, T_{ij}'s, T_{ijk}'s and s_{ij}'s in T_i for $i \neq j$, if we have compatible quotients ($\pi_{ij}(s_{ik}) = \pi_{ik}(s_{jk})$ and the diagram commutes)

$$
\begin{tikzcd}
T_i \arrow[d, "\pi_{ij}"'] \arrow[drr, "\pi_{ik}"] & & T_j \arrow[dll, "\pi_{ji}"'] \arrow[drr, "\pi_{jk}"] & & T_k \arrow[dll, "\pi_{ki}"'] \arrow[d, "\pi_{kj}"] \\
T_{ij} \arrow[drr, "\pi_{ijk}"'] & & T_{ik} \arrow[d, "\pi_{ikj}"] & & T_{jk} \arrow[dll, "\pi_{jki}"] \\
& & T_{ijk} & &
\end{tikzcd}
$$

the limit T of the diagram is good: we recover s_i''s in T, with $T_i = T/\!\uparrow s_i$. Moreover "good properties" of distributive lattices can be tested "locally" (i.e. tested on the T_i's).

Third thing, a similar situation works for gluing modules from their localizations. Here again there are two steps.

When M is a given R-module, defining $M_i = M[1/s_i]$ and so on ..., we get M as the limit of the M_i's and M_{ij}'s in the following diagram

$$
\begin{tikzcd}
& & M_i \arrow[r, "\varphi_{ij}"] \arrow[ddr, "\varphi_{ik}"'] & M_{ij} \\
N \arrow[urr, "\psi_i"] \arrow[r, dashed, "\psi!"] \arrow[drr, "\psi_j"'] \arrow[ddrr, "\psi_k"'] & M \arrow[ur, "\varphi_i"'] \arrow[dr, "\varphi_j"] \arrow[ddr, "\varphi_k"'] & & \\
& & M_j \arrow[uur, "\varphi_{ji}"'] \arrow[dr] & M_{ik} \\
& & M_k \arrow[ur] \arrow[r, "\varphi_{kj}"'] & M_{jk}
\end{tikzcd}
$$

If M is not given, if we give M_i's, M_{ij}'s, M_{ijk}'s and compatible localization morphisms

$$\big((M_i)_{i\in I}), (M_{ij})_{i<j\in I}, (M_{ijk})_{i<j<k\in I}; (\varphi_{ij})_{i\neq j}, (\varphi_{ijk})_{i<j, i\neq k, j\neq k}\big)$$

as in the following commutative diagram ($\varphi_{ij} : M_i \to M_{ij}$ is a localization morphism at s_j and $\varphi_{ijk} : M_{ij} \to M_{ijk}$ is a localization morphism at s_k), then the limit of this diagram reconstructs a good module.

$$
\begin{tikzcd}
M_i \arrow[d, "\varphi_{ij}"'] \arrow[drr, "\varphi_{ik}"] & & M_j \arrow[dll, "\varphi_{ji}"'] \arrow[drr, "\varphi_{jk}"] & & M_k \arrow[dll, "\varphi_{ki}"'] \arrow[d, "\varphi_{kj}"] \\
M_{ij} \arrow[drr, "\varphi_{ijk}"'] & & M_{ik} \arrow[d, "\varphi_{ikj}"] & & M_{jk} \arrow[dll, "\varphi_{jki}"] \\
& & M_{ijk} & &
\end{tikzcd}
$$

Morevover "good properties" of modules can be tested locally (e.g. finitely generated, finitely presented, flat, coherent, finitely generated projective, finitely generated projective of rank r, exactness of sequences).

Now we define **elementary schemes** as formal objects obtained by gluing finitely many affine schemes (i.e. commutative rings) along basic opens.

$$\begin{tikzcd}
R_i \arrow[d, "\alpha_{ij}"'] \arrow[dr, "\alpha_{ik}"] & R_j \arrow[dl, "\alpha_{ji}"'] \arrow[dr, "\alpha_{jk}"] & R_k \arrow[dl, "\alpha_{ki}"'] \arrow[d, "\alpha_{kj}"] \\
R_{ij} \arrow[dr, "\alpha_{ijk}"'] & R_{ik} \arrow[d, "\alpha_{ikj}"] & R_{jk} \arrow[dl, "\alpha_{jki}"] \\
 & R_{ijk} &
\end{tikzcd}$$

where $\alpha_{ij} : R_i \to R_{ij}$ are localization morphisms at $s_{ij} \in R_i$ and $\alpha_{ijk} : R_{ij} \to R_{ijk}$ are localization morphisms at $\alpha_{ij}(s_{ik}) \in R_{ij}$. Almost every usual scheme, including projective schemes and grassmannians, are elementary.
The "finite" data structure giving an elementary scheme is

$$\big((R_i)_{i\in I}), (R_{ij})_{i<j\in I}, (R_{ijk})_{i<j<k\in I}; (\alpha_{ij})_{i\neq j}, (\alpha_{ijk})_{i<j,j\neq k\neq i}, (s_{ij})_{i\neq j\in I}\big)$$

A **coherent sheaf of modules** over the previous elementary scheme is defined by formally gluing finitely presented coherent R_i-modules M_i with compatible localisations M_{ij} as R_{ij}-modules and M_{ijk} as R_{ijk}-modules. The "finite" data structure giving this "coherent sheaf" is

$$\big((M_i)_{i\in I}), (M_{ij})_{i<j\in I}, (M_{ijk})_{i<j<k\in I}; (\phi_{ij})_{i\neq j}, (\phi_{ijk})_{i<j,j\neq k\neq i}\big)$$

where each $\phi_{ij} : M_i \to M_{ij}$ is a localization morphism at s_{ij}, and each $\phi_{ijk} : M_{ij} \to M_{ijk}$ is a localization morphism at $\alpha_{ij}(s_{ik})$.

Once we have defined morphisms between elementary schemes, we can define a **spectral scheme** as obtained by gluing formally finitely many affine schemes (i.e. commutative rings) along finite unions of basic opens.

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The search for low rank vector bundles

Sebastian Posur

(joint work with Mohamed Barakat)

In this extended abstract we want to present a new computational approach to the search for indecomposable low rank vector bundles on the projective space $\mathbb{P}^n$ of dimension n over some field k. For this purpose our main tools are representation theory of finite groups and an equivariant version of the BGG equivalence [1].

Lots of problems concerning low rank vector bundles are still open and can be attacked by trying to construct explicit examples. So far the only known indecomposable $(n-2)$-bundles are essentially due to Horrocks and Mumford on $\mathbb{P}^4$ [3] and due to Horrocks on $\mathbb{P}^5$ [4]. No one knows if there exist examples in the case $n \geq 6$. Furthermore Hartshorne's famous conjecture implies the splitting of every 2-bundle on $\mathbb{P}^n$ for $n \geq 7$ [5].

In order to attack these problems, we first define an appropriate search space, i.e., a space which contains all vector bundles with some prescribed cohomology groups and which is computationally feasible. So every vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ can be regarded as an object in the category of coherent sheaves $\operatorname{Coh}\mathbb{P}^n$, which can in turn be regarded as a full subcategory of its bounded derived category $\mathrm{D}^{\mathrm{b}}(\operatorname{Coh}\mathbb{P}^n)$. For every coherent sheaf on $\mathbb{P}^n$ one can compute its so called Tate resolution [2], which gives rise to a triangulated equivalence between $\mathrm{D}^{\mathrm{b}}(\operatorname{Coh}\mathbb{P}^n)$ and the category of Tate sequences. A Tate sequence is a minimal complex of $\mathbb{Z}$-graded free modules over the exterior algebra $E := \wedge k^{n+1}$, where minimal means that the application of the functor $k \otimes_E -$ yields the trivial complex. A morphism between Tate sequences is a homotopy equivalence class of chain maps.

Let $\operatorname{Tate}(\mathcal{E})$ be the Tate resolution of a vector bundle and define ω_E as the k-dual of E. Then we have an isomorphism

$$\operatorname{Tate}(\mathcal{E})^d \cong \bigoplus_{i=0}^{n} \omega_E \otimes_k \mathrm{H}^i(\mathbb{P}^n, \mathcal{E}(d-i)),$$

i.e., the cohomology groups of $\mathcal{E}$ and all its twists are encoded in $\operatorname{Tate}(\mathcal{E})$ [2]. Thus we can apply Serre's criterion to read off from a Tate sequence if it corresponds to a vector bundle [6].

Every Tate sequence is uniquely determined by its zeroth syzygy-object, i.e, the kernel of the zeroth differential. In particular, it is uniquely determined by only considering the zeroth differential. Thus using the isomorphism given above, it is reasonable to define our desired search space as

$$\mathbb{P}\left(\operatorname{Hom}_E(\omega_E \otimes_k \bigoplus_{i=0}^{n} H^i_{-i}, \omega_E \otimes_k \bigoplus_{j=0}^{n} H^j_{-j+1})\right),$$

where H^i_{-i}, H^j_{-j+1} are some prescribed vector spaces. Every vector bundle $\mathcal{E}$ with cohomology groups $\mathrm{H}^i(\mathcal{E}(-i)) \cong H^i_{-i}$ and $\mathrm{H}^j(\mathcal{E}(-j+1)) \cong H^j_{-j}$ can be found in this search space. Its biggest drawback is its huge size. So for example if $n = 4$, $H^2_{-2} = k^2, H^1_0 = k^5$, then we know that the Horrocks Mumford bundle lies in $\mathbb{P}\left(\operatorname{Hom}_E(\omega_E \otimes_k k^2, \omega_E \otimes_k k^5)\right)$, a projective space of dimension 99 and as such it is not computationally feasible.

An effective strategy to decrease the dimension of the search space is the following: let a finite group G act on all the notions introduced above. The constraints given by this G-action will shrink the dimension of the search space dramatically. To be more precise, we start with an action of G on $\mathbb{P}^n$. Again, a G-equivariant vector bundle has now a G-equivariant Tate resolution which is uniquely determined by a G-morphism in $\operatorname{Hom}_E(\omega_E \otimes_k \bigoplus_{i=0}^{n} \mathrm{H}^i_{-i}(\mathcal{E}), \omega_E \otimes_k \bigoplus_{j=0}^{n} \mathrm{H}^j_{-j+1}(\mathcal{E}))$.

Our new search space is thus given by

$$\mathbb{P}\left(\mathrm{Hom}_E(\omega_E \otimes_k \bigoplus_{i=0}^{n} H^i_{-i}, \omega_E \otimes_k \bigoplus_{j=0}^{n} H^j_{-j+1})^G\right),$$

where H^i_{-i}, H^j_{-j+1} now are some prescribed G-modules. For example let G be the semidirect product of the Heisenberg group of order 125 and $\mathrm{SL}(2,5)$. Then there is a G-action on $\mathbb{P}^4$ and there are irreducible G-modules H^2_{-2}, H^1_0 of degree $2, 5$, respectively, such that the corresponding search space is a singleton only consisting of the Horrocks-Mumford bundle. This remains true if we choose Q_8 instead of $\mathrm{SL}(2,5)$.

A search algorithm constructed with the presented machinery above uses an exhaustive search strategy relying on some data base of finite groups. Roughly speaking, it proceeds as follows:

(1) Choose a group G and an action on $\mathbb{P}^n$.
(2) Choose some G-cohomology groups H^i_{-i}, H^j_{-j+1}.
(3) Construct the maps in the corresponding search space and use Serre's criterion to decide whether you have constructed a vector bundle.

This approach can be further improved by considering special constellations of prescribed cohomology groups which always yield singletons as search spaces. Furthermore, the Hilbert series of the zeroth syzygy object in a Tate sequence uniquely determines the rank and the Chern classes of a corresponding vector bundle and thus can also be used in an effective implementation of the search algorithm.

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Constructive Stafford's theorems

Alban Quadrat
(joint work with Daniel Robertz)

The purpose of this work is to develop constructive versions of Stafford's theorems [10] on the module structure of Weyl algebras $A_n(k)$ (i.e., the rings of partial differential operators with polynomial coefficients) over a base field k of characteristic zero [1]. More generally, based on results of Stafford and Coutinho-Holland [4], we develop constructive versions of Stafford's theorems for the so-called very simple domains [4]. The algorithmization is based on the effective solvability of certain inhomogeneous quadratic equations over a very simple domain D.

We show how to explicitly compute a unimodular element of a finitely generated left D-module of rank at least two. This result is used to constructively decompose any finitely generated left D-module into a direct sum of a free left D-module and a left D-module of rank at most one. If the latter is torsion-free, then we explicitly show that it is isomorphic to a left ideal of D which can be generated by two elements. Then, we give an algorithm which reduces the number of generators of a finitely presented left D-module with module of relations of rank at least two. In particular, any finitely generated torsion left D-module can be generated by two elements and is the homomorphic image of a projective ideal whose construction is explicitly given. Moreover, a non-torsion but non-free left D-module of rank r can be generated by $r+1$ elements but no fewer.

These results are implemented in the Stafford package [7] for $D = A_n(k)$ and their system-theoretical interpretations [3, 6] are given within a D-module approach [1, 5].

Finally, we prove that the above results also hold for the ring of ordinary differential operators with either formal power series or locally convergent power series coefficients [8] and, using a result of Caro-Levcovitz [2], also for the ring of partial differential operators with coefficients in the field of fractions of the ring of formal power series or of the ring of locally convergent power series.

For more details and many illustrative examples, we refer to [9] and to the Stafford package [7].

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Applying Thomas decomposition and algebraic analysis to certain nonlinear PDE systems

Daniel Robertz
(joint work with Thomas Cluzeau, Alban Quadrat)

This talk is about work in progress in collaboration with Thomas Cluzeau (Université de Limoges) and Alban Quadrat (Inria Saclay). We report on first steps of a study of certain systems of nonlinear partial differential equations using a new algebraic analysis approach. By applying module-theoretic techniques to a new kind of linearization of the given equations, e.g., conservation laws of the given nonlinear system are computed. An effective version of this approach relies on methods of symbolic computation for both nonlinear and linear differential equations: a preparatory step applies a decomposition technique as proposed by J. M. Thomas in the 1930s, cf. [10], [1], [9]; the linearized system is dealt with using a version of Janet's algorithm performing normal form computations for the symbolic coefficients of the linearized system modulo the nonlinear system, cf. [5], [7], [8].

Given a system $R\,y = 0$ of *linear functional equations*, where R is a $q \times p$ matrix with entries in a (not necessarily commutative) ring D of functional operators and y is a vector of unknown functions, we associate to the system a left D-module. It is the cokernel of the homomorphism $D^{1\times q} \to D^{1\times p}$ which is induced by R. This module carries intrinsic information about the system, cf., e.g., [3]; in particular, equivalent systems give rise to isomorphic modules. Here the set $\mathcal{F}$ of candidates for the entries of y is also fixed to be a left D-module, the left action of D corresponding to the one used in $R\,y = 0$, and we assume that a faithful duality between equations and solutions is defined by these choices.

We study ways in which this approach can be applied to certain systems of nonlinear partial differential equations. One way is to allow the unknown functions and their derivatives to appear in the coefficients of the differential operators, i.e., in the matrix R. We assume that each (nonlinear) term in the given equations admits such a representation $d\,y$, where d is a differential operator. Simple examples are given by quasilinear PDE systems such as Burgers' equation, the Korteweg-de Vries equation or Euler's equations for an incompressible fluid.

The linearized system is a system of linear partial differential equations whose coefficients involve the unknown functions of the nonlinear system. A normal form computation for the linearized system must take the relations satisfied by these functions into account, cf. [8]. We obtain a (confluent and terminating) rewriting system for the coefficients by determining first a Thomas decomposition of the nonlinear system and choosing a so-called simple differential system in this decomposition. In such a simple differential system each equation is solved for (a power of) the highest derivative of an unknown function appearing in this equation.

With these techniques at hand we can now, for instance, compute homomorphisms from the left D-module defined by the adjoint operator of the differential operator given by the linearized system to the module associated with the linearized system, cf., e.g., [4], [2]. Using such a homomorphism to pull back solutions of the linearized system to solutions of the system given by the adjoint operator, the integration by parts formula yields a divergence expression which vanishes upon substitution of solutions of the nonlinear system, i.e., a conservation law, cf., e.g., [6].

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The Mayer-Vietoris exact sequence in Čech cohomology

Claire Tête

(joint work with joint work with Lionel Ducos and Claude Quitté)

This talk is about the foundation of the Mayer-Vietoris exact sequence in Čech cohomology, with respect to the augmented Čech complex, often called "the stable Koszul complex". Our treatment is elementary and uses neither the local cohomology of Grothendieck nor the well-known noetherian local cohomology. We use explicit objects giving us an elementary algebraic treatment in a general non-noetherian context, i.e. two finite sequences $\underline{a}$, $\underline{b}$ of a commutative ring and some Čech complexes built from these sequences. More precisely, our strategy consists in producing a short exact sequence of "type Čech" complexes having the expected cohomologies. Then arises a simplicial complex on the set ($\underline{a}$, $\underline{b}$, $\underline{ab}$) with a relatively surprising combinatory.

Some details about the local cohomology. In [5], Grothendieck defines the local cohomology groups of a sheaf $\mathcal{F}$ on a space X with support in a closed subset Y. If we particularize this to affine schemes, i.e. with an arbitrary ring $\mathbf{A}$ and a $\mathbf{A}$-module M :

$$X = \mathrm{Spec}(\mathbf{A}), \qquad Y = V(\mathfrak{a}) = V(\langle \underline{a} \rangle), \qquad \mathcal{F} = \widetilde{M}$$

then we find the Čech cohomology modules $\check{\mathrm{H}}^i_{\underline{a}}(M)$. But we don't find the local cohomology modules $H^i_{\mathfrak{a}}(M)$ of the noetherian algebraists (see [1], [3], [7]). This is the same thing when the ring is *noetherian.* In fact, the noetherian local cohomology modules are defined as the right derived functors of the functor

$$M \quad \longrightarrow \quad \Gamma_{\mathfrak{a}}(M) = \{x \in M \mid \forall\, a \in \mathbf{A},\ \exists\, e \in \mathbb{N},\ a^e x = 0\}$$

But this is not taken in the good category. In [4], Eisenbud says : "if $\mathbf{A}$ is non-noetherian, then the Čech complex does not always compute the derived functors in the category of $\mathbf{A}$-modules of $\Gamma_{\mathfrak{a}}()$, even for finitely generated $\mathfrak{a}$. Rather, it computes the derived functors in the category of (not necessarily quasi-coherent) sheaves of $\mathcal{O}_{\mathrm{Spec}(\mathbf{A})}$-modules. For this and other reasons, the general definition of the local cohomology modules should probably be made in this larger category. See [Hartshorne/Grothendieck 1967] for a treatment in this setting".

To sum up. The local cohomology of Grothendieck particularised with an affine scheme and a quasi-coherent sheaf is isomorphic to the Čech cohomology :

$$H^i_{V(\mathfrak{a})}(\mathrm{Spec}(\mathbf{A}), \widetilde{M}) \ \simeq\ \check{\mathrm{H}}^i_{\underline{a}}(M)$$

but is not isomorphic to the noetherian local cohomology defined by the functor $\Gamma_{\mathfrak{a}}()$ in the category of $\mathbf{A}$-modules :

$$H^i_{V(\mathfrak{a})}(\mathrm{Spec}(\mathbf{A}), \widetilde{M}) \ \not\simeq\ H^i_{\mathfrak{a}}(M)$$

With a *noetherian* ring $\mathbf{A}$, these 3 cohomologies are the same.

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Exceptional sequences and tilting in algebraic geometry (and computer algebra?)

Markus Perling

In this talk I gave an overview on semiorthogonal decompositions of derived categories together with some perspective towards constructive and computer algebraic aspects. More specifically, one would like to understand $D^b(X)$, the derived category of coherent sheaves on a smooth complete variety X (defined, say, over an algebraically closed field $\mathbb{K}$). Derived categories have been introduced by Grothendieck and Verdier [Ver77] as a general framework for homological algebra (for more recent introductions to the subject, see e.g. [Wei94] and [GM03]). It turns out that derived categories have a rich structure which to study is interesting in its own right. In particular, among the aspects which attract much attention in current research, are:

- The categorification of geometric invariants.
- The construction of interesting equivalences (e.g. in the form of non-commutative models for algebraic varieties).

An important tool in both cases is given by semi-orthogonal decompositions.

Definition: Two triangulated subcategories $\mathcal{P}, \mathcal{Q}$ of $D^b(X)$ represent a *semi-orthogonal decomposition* of $D^b(X)$ if $\mathcal{P}$ and $\mathcal{Q}$ together generate $D^b(X)$ and $\mathcal{P} \subseteq \mathcal{Q}^{\perp}$, i.e. $\mathrm{Hom}_{D^b(X)}(\mathcal{Q}, \mathcal{P}) = 0$.

This is equivalent to saying that for any object F in $D^b(X)$, there exist objects F' in $\mathcal{Q}$ and F'' in $\mathcal{P}$ such that there is a distinguished triangle

$$F' \longrightarrow F \longrightarrow F'' \longrightarrow F'[1] \tag{1}$$

(see [BK90]). It is an interesting aspect that the rather formal requirement of Hom-vanishing indeed can encode a lot of geometric information about X (though this is not entirely surprising, as it directly translates into Ext-vanishing, or, for many examples, into cohomology vanishing).

Example ([Beĭ78])**:** For a line bundle $\mathcal{O}(i)$ on $\mathbb{P}^n$, denote $\langle \mathcal{O}(i) \rangle$ the triangulated subcategory of $D^b(\mathbb{P}^n)$ generated by $\mathcal{O}(i)$. Then $\langle \mathcal{O}(-n) \rangle$, $\langle \mathcal{O}(-n+1) \rangle, \ldots, \langle \mathcal{O} \rangle$ form a semi-orthogonal decmposition of $D^b(\mathbb{P}^n)$.

Example ([Orl93])**:** More generally, consider a projective bundle $\mathbb{P}(E) \overset{p}{\to} X$ of rank r with relative ample bundle $\mathcal{O}_E(1)$. For $k \in \mathbb{Z}$ we denote $D^b(X)_k \subset D^b(\mathbb{P}(E))$ the subcategory generated by objects of the form $p^*F \otimes \mathcal{O}_E(k)$. Then $D^b(X)_{-r}, D^b(X)_{-r+1}, \ldots, D^b(X)_0$ is a semi-orthogonal decomposition of $D^b(\mathbb{P}(E))$.

Example ([Orl93])**:** Consider the morphisms $\tilde{Z} \overset{j}{\hookrightarrow} Y \overset{b}{\to} X$, where b is a blow-up along some smooth subvariety $Z \subset X$ of codimension r and $\tilde{Z} \to Z$ the corresponding exceptional divisor. Then $\tilde{Z}$ is a projective bundle of rank $r-1$ over Z and categories $D^b(Z)_k$ are defined as before. Then there is a semi-orthogonal decomposition $j_* D^b(Z)_{-r+1}, \ldots, j_* D^b(Z)_{-1}, b^* D^b(X)$ of $D^b(Y)$.

Example: In [Kuz09], Kuznetsov shows that a cubic Fano threefold X admits a semi-orthogonal decomposition $\mathcal{A}_X, \langle \mathcal{O}(1) \rangle, \langle \mathcal{O}(2) \rangle$. In [BMMS12], it is shown that the component $\mathcal{A}_X$ is a birational invariant of X, i.e. two cubic Fano threefolds X, X' are isomorphic iff $\mathcal{A}_X$ and $\mathcal{A}_{X'}$ are equivalent as triangulated categories.

Above series of examples gives a taste of semi-orthogonal decompositions in increasing difficulty, i.e. the first example essentially represents the simplest type of decomposition. This kind of decomposition can be formalized as follows.

Definition ([Rud90])**:**
- (i) An object $E \in D^b(X)$ is called *exceptional* if $\mathbb{K} \cong \mathrm{Hom}_{D^b(X)}(E, E)$ and $\mathrm{Hom}_{D^b(X)}(E, E[k]) = 0$ for all $k \neq 0$.
- (ii) A sequence of exceptional objects $E_1, \ldots, E_n$ is called a *full exceptional sequence* if it induces a semiorthogonal decomposition $\langle E_1 \rangle, \ldots, \langle E_n \rangle$ of $D^b(X)$.
- (iii) An exceptional sequence is called *strongly* exceptional if $\mathrm{Hom}_{D^b(X)}(E_i, E_j[k]) = 0$ for all i, j and all $k \neq 0$.

It turns out that the construction of exceptional sequences is a surprisingly hard problem and no complete set of criteria, neither sufficient nor necessary, is known in general. Among the varieties which are known to admit full exceptional sequences are homogeneous spaces (e.g. [Kap88]), toric varieties [Kaw06], rational surfaces ([Orl93], see also [HP11a]) and the spaces $\mathcal{M}_{0,n}$ [MS12].

Definition: An object T in $D^b(X)$ is called *tilting object* if it generates $D^b(X)$ and $\mathrm{Hom}_{D^b(X)}(T, T[k]) = 0$ for all $k \neq 0$.

If a tilting object T exists, then we denote its endomorphism algebra $A := \mathrm{Hom}_{D^b(X)}(T, T)$, a finite-dimensional basic $\mathbb{K}$-algebra. By a result of Bondal [Bon90], the functor

$$\mathbf{R}\mathrm{Hom}(T, \,.\,) : D^b(X) \longrightarrow D^b(A - \mathrm{mod})$$

induces an equivalence of derived categories. Indeed, any full strongly exceptional sequence $E_1, \ldots, E_n$ gives rise to a tilting object $T := \bigoplus_{i=1}^n E_i$. However, in general the construction of tilting objects is even harder than the construction of exceptional sequences. So it is known that Kapranov's construction for Flag varieties yields strongly exceptional sequences, but even for toric varieties the existence of such objects is completely open (see also [HP11a] for a discussion).

Example: On $\mathbb{P}^2$, exceptional vector bundles have been classified by Drezet and Le Potier [DL85]. It was shown by Rudakov [Rud89] all full exceptional sequences $\mathcal{E}, \mathcal{F}, \mathcal{G}$ with ranks e, f, g, respectively, correspond precisely to solutions of the Markov equation

$$e^2 + f^2 + g^2 = 3efg,$$

i.e. up to dualizing and twisting, there is a one-to-one correspondence between exceptional sequences on $\mathbb{P}^2$ and solutions of the Markov equation. All such sequences are automatically strong and we can describe the correspoding endomorphism algebra as a path algebra with relations which is given by the quiver

$$\bullet \xrightarrow{3g} \bullet \xrightarrow{3e} \bullet,$$

where $\xrightarrow{t}$ denotes t parallel arrows. The relations can be characterized by the kernel of the following short exact sequence:

$$0 \to \mathrm{Hom}(\mathcal{E}, L_{\mathcal{F}}\mathcal{G}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{F}) \otimes_{\mathbb{K}} \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{E}, \mathcal{G}) \to 0.$$

Here, $L_{\mathcal{F}}\mathcal{G}$ denotes the left-mutation of $\mathcal{G}$ at $\mathcal{F}$ (see [Rud90]).

A general existence result is the following.

Theorem ([HP11b])**:** Any smooth, projective rational surface admits a tilting bundle.

Note however, that the construction of [HP11b] in general yields tilting bundles which do not decompose into strongly exceptional sequences.

Besides their conceptual relevance, exceptional sequences are also very interesting from computational perspective. As sequence (1) indicates, they can be used to decompose a given object into very simple parts. In the classical version of Beĭlinson, this has become a very important tool for the study of vector bundles on projective spaces. A variation of this theme, the so-called BGG correspondence [EFS03] is the basis for several implementations which facilitate the construction of vector bundles on projective spaces and the computation of their invariants ([DE02], [DGPS12], [B+]). One can hope that the construction of exceptional sequences and tilting correspondence in a similar way will become an effective tool to study sheaves on algebraic varieties and their derived categories. Among the challenges here is the algorithmic construction of exceptional sequences and the sufficiently high-level computer representation of tilting correspondence. The former can sometimes be done in a straightforward manner e.g. for certain types of sequences on toric varieties [Per04], the latter can be facilitated e.g. by packages such as `homalg` [B+].

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Deciding kernel membership of the sheafification functor on toric varieties

Sebastian Gutsche

The goal of this talk is to give a constructive description of the Abelian category of coherent sheaves on a toric variety. For this I use the notion of a Serre quotient of computable category. A category is called *computable* if all existential quantifiers are constructive. Given a computable category $\mathcal{A}$ and a thick subcategory $\mathcal{C} \subset \mathcal{A}$ the Serre quotient category $\mathcal{A}/\mathcal{C}$ is computable if for an $M \in \mathcal{A}$ the membership $M \in \mathcal{C}$ is decidable [1]. Given a normal toric variety X with Cox ring S we have $\mathfrak{Coh}\,(X) \cong S-\mathrm{grmod}/\ker \mathrm{Sh}$, where $\mathrm{Sh}: \ S-\mathrm{grmod} \to \mathfrak{Coh}\,(X)$ denotes the sheafification functor [2]. This means that $\mathfrak{Coh}\,(X)$ is computable if it is decidable whether the sheafification $\widetilde{M} := \mathrm{Sh}\,(M)$ of a module M is the zero sheaf. For smooth toric varieties there is a global description when a sheaf is zero.

Theorem. *Let X be a smooth toric variety with fan Σ and $B \leq S$ the irrelevant ideal of S, the ideal generated by the monomials $x^{\widehat{\sigma}} := \prod_{\rho\in\Sigma(1)-\sigma(1)} x_\rho$ with $\sigma \in \Sigma_{max}$. Then the sheafification of a $M \in S-grmod$ is 0 iff $B\,(X_\Sigma)^l\,M = 0$ for some $l \in \mathbb{Z}_{>0}$.*

For singular varieties this is wrong, for an example see [3, 5.3.11]. One way to see whether a module sheafifies to zero is looking at the modules of global sections on affine charts of an affine covering. A coherent sheaf is zero if and only if these modules vanish. For a toric variety with fan Σ there is a nice description of the modules of global sections on affine charts of the torus invariant affine covering induced by the maximal cones in Σ.

Proposition. *Let X be a toric variety with , and $M \in S-grmod$. Then, for every $\sigma \in \Sigma_{\max}$ it is*

$$\Gamma\left(U_\sigma, \widetilde{M}\right) = \left(M_{x^{\widehat{\sigma}}}\right)_0,$$

where U_σ denotes the affine subset of X associated to σ.

For a smooth cone σ this is easy to compute, one gets a description for $\left(S_{x^{\widehat{\sigma}}}\right)_0$ and $\left(M_{x^{\widehat{\sigma}}}\right)_0$ by setting the variables of S appearing in $x^{\widehat{\sigma}}$ to 1 [3, 5.2.10]. Again, this fails if σ is not smooth.
For an arbitrary normal toric variety X with fan Σ and a graded module M over

S, we can compute $(M_{x^{\hat{\sigma}}})_0$ as an $(M_{x^{\hat{\sigma}}})_0$- module as follows: Given a presentation map $\varphi: F_1 \to F_0$ for the S-module M, we can compute a presentation for the module $(M_{x^{\hat{\sigma}}})_0$ as an $(S_{x^{\hat{\sigma}}})_0$-module, without computing $M_{x^{\hat{\sigma}}}$. To do so, we first compute a HILBERT basis of the cone of monomials of degree 0 in $S_{x^{\hat{\sigma}}}$. Given this basis, we get a generating set for $(S_{x^{\hat{\sigma}}})_0$. Now we can present $(S_{x^{\hat{\sigma}}})_\alpha$ as an $(S_{x^{\hat{\sigma}}})_0$-module for every $\alpha \in \mathrm{Cl}(X)$, using MINKOWSKY-WEIL decomposition of the polyhedron of monomials of degree α in $S_{x^{\hat{\sigma}}}$. From that point, by looking at images of the generators of those modules, we can compute the degree zero part of $\varphi_{x^{\hat{\sigma}}}$, and get a presentation for $(M_{x^{\hat{\sigma}}})_0$.

The algorithm decides whether a module over the COX ring sheafifies to zero, and therefore lies in the kernel of Sh. This means the category of coherent sheaves on a normal toric variety is computable as a SERRE quotient.

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