Arbeitsgemeinschaft mit aktuellem Thema:
Sofic Entropy
Mathematisches Forschungsinstitut Oberwolfach
Oct 6 - Oct 11

Organizers:

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Introduction:

The concept of entropy was introduced into ergodic theory by Kolmogorov in the late 1950s. Kolmogorov showed that, for a measure-preserving transformation of a probability space, the dynamical averages of the Shannon entropy have a common limiting value over all generating partitions. Sinai subsequently proposed the now standard definition that takes a supremum of such limits over all partitions to produce an invariant that does not depend on the existence of a generator. The Kolmogorov-Sinai theorem asserts that the supremum is attained on a generating partition, thereby making it possible to compute. This set-up extends most generally to actions of discrete amenable groups, for what is essential in the proof of the Kolmogorov-Sinai theorem is the ability to average within the group in an asymptotically invariant way. A corresponding theory of topological entropy was initiated by Adler, Konheim, and McAndrew in the early 1960s, and it is related to the Kolmogorov-Sinai theory by way of the variational principle.

The immediate significance of entropy is that it can be used to distinguish Bernoulli shifts. In fact Bernoulli shifts are classified by their entropy due to a celebrated result of Ornstein, and this classification was later extended by Ornstein and Weiss to Bernoulli actions of arbitrary infinite amenable groups. Entropy has greatly expanded its reach beyond its roots in Bernoulli structure to become a pervasive presence in dynamics with rich connections to a variety of other fields including number theory, diophantine approximation, Riemannian geometry, smooth dynamics, Banach spaces, and operator algebras.

Naturally the question arose as to whether entropy theory could be extended beyond the frontier of amenability, and an example of Ornstein and Weiss in [OW87] suggested that this might not be possible. However, about five years ago Bowen showed that one could produce an entropy invariant by taking the internal averaging in the Kolmogorov-Sinai definition and externalizing it to a finite set, where one counts the number of models for the dynamics [Bo10b]. To express this modeling one requires that the group approximately act on these finite sets. The existence of such quasi-actions for a group is a characterization of soficity, a concept introduced by Gromov in the late 1990s [Gr99]. Bowen was able to apply his notion of sofic measure entropy to classify Bernoulli actions of all countably infinite nontorsion sofic groups [Bo10b].

Later Kerr and Li took an operator algebra approach to sofic measure entropy which allowed them to remove a generator hypothesis in Bowen's definition, formulate a corresponding notion of sofic topological entropy, and establish a variational principle [KL11]. We know now that the general theory of sofic entropy can be entirely developed in the conventional language of partitions and points [KL13a, Ke13], and we will mostly adhere to this viewpoint in the lectures.

The lectures will address in a largely comprehensive way the developments in sofic entropy over the last five years. A central theme should be pointed out here that already appears in the amenable case but becomes more dramatic in the broader sofic context: once one passes from $\mathbb{Z}$ to other acting groups, the applicability of entropy to smooth dynamics and related geometric phenomena essentially vanishes (e.g., smooth $\mathbb{Z}^d$-actions for $d > 1$ always have zero entropy) and the theory becomes closely associated with probabilistic, analytic, and geometric structures more immediately tied to the group, which in the case of algebraic actions involves further connections to homological algebra, noncommutative harmonic analysis, and operator algebras.

Talks:

1. **Sofic groups**

   This lecture introduces the concept of a sofic group [Gr99, We00, Pe08]. Various formulations of soficity can be given (e.g., in the finitely generated case the Cayley graph is a Benjamini-Schramm limit of finite labeled graphs (this does not depend on the choice of generating set), and in the countable case the group is embeddable into a Hamming metric ultraproduct of symmetric groups [Pe08]) but should include the version used to define entropy (see [Ke13]). Surprisingly it is not known whether all groups are sofic. It should be shown that all amenable groups and residually finite groups are sofic. After this basic fact, the lecture could be a short survey on sofic groups (as in [Pe08]) or could focus on permanence properties (as in [ES06] and [CHR12]) or prove some interesting facts such as the Kaplansky conjecture and Lück approximation for sofic groups [ES04].

2. **Entropy for single automorphisms**

   This talk should serve as an introduction to (or refresher on) classical entropy theory for actions of a single map. There are two cases: topological entropy and measure entropy. For the purposes of extending the notions to sofic groups later on in the workshop, it is most suitable to define topological entropy in terms of ϵ-separated subsets as for example in [Do11, §6.1.1]. It is important to explain that this definition does not depend on the choice of metric [Do11, Theorem 6.1.8], and that one can even use a continuous pseudometric which is dynamically generating in a natural sense. To illustrate, one can compute the entropy of the shift map on $\{1, \ldots, n\}^{\mathbb{Z}}$. One could also indicate how to compute the entropy of a hyperbolic total automorphism.

   Measure entropy is traditionally defined via partitions as in [Do11, Definition 4.1.1]. One should explain (and perhaps prove if there is time) the Kolmogorov-Sinai theorem [Do11, Theorem 4.2.2] which states that the entropy of an automorphism can be computed from the entropy with respect to any generating partition. Measure entropy can also be defined in a manner similar to topological entropy [Ka80, §1] (we will do something similar in defining sofic entropy). To illustrate, one can compute the measure entropy of a Bernoulli shift. The variational principle [Do11, Theorem 6.8.1] relates measure entropy with topological entropy: the topological entropy is the supremum of the measure entropies over all invariant probability measures. It would be helpful to sketch a proof of this fact if there is time. There are, of course, many other references one may wish to use to prepare this lecture (e.g., [Gl03, Pe89, Wa00]).

3. **Factors of Bernoulli shifts**

   A straightforward consequence of the definition of entropy and the Kolmogorov-Sinai theorem is that the full 2-shift over $\mathbb{Z}$ cannot factor onto the full 4-shift (in general, the full n-shift over a group G is the action of G on $([n]^G, u_n^G)$ where $[n] = \{1, \ldots, n\}$, u_n is the uniform probability measure on $[n]$, and the action $G \curvearrowright [n]^G$ is given by $(g \cdot x)_f = x_{g^{-1}f}$). In [OW87, Page 138, 1b] it is shown that, by contrast, the full 2-shift over the rank 2 free group does factor onto the full 4-shift. This example convinced many people that there could not exist an entropy theory for actions of non-amenable groups (see [El99] for example). Building on this example, in [Bo11] it is shown that if G is any group which contains a non-amenable free group then every Bernoulli shift over G factors onto every Bernoulli shift over G. It is an open question whether this result extends to all nonamenable groups. Even more strikingly, it [Ba05] it is shown that if G is any nonamenable group then there is some n such that the full n-shift over G factors onto $G \curvearrowright ([0,1]^G, \lambda^G)$ where λ is Lebesgue measure on $[0,1]$. From there it is easy to see that the full n-shift over G factors onto every Bernoulli shift over G. The proof uses a rudimentary form of a result of Gaboriau and Lyons [GL09]. A shorter proof can be given using [GL09]. This talk should present the results of [Bo11, Ba05].

4. **Sofic topological entropy**

   This lecture will introduce topological entropy for actions of sofic groups on compact metrizable spaces following the approach of Section 2 of [KL13b]. Special care should be given in presenting the proof of the fact the definition does not depend on the choice of dynamically generating continuous pseudometric, as this illustrates a fundamental idea that recurs in the Kolmogorov-Sinai theorem for sofic measure entropy, namely the playing off of two (sets of) parameters, with one expressing how good a sofic model is for the dynamics and the other representing the scale at which we can distinguish such models. A computation of the entropy of the shift action $G \curvearrowright \{1, \ldots, k\}^G$ should be included (see Section 2 of [KL13b]). It can also be explained why an action that fixes every point has zero entropy and why an invariant probability measure must exist if the entropy is not equal to $-\infty$.

5. **Gottschalk's surjunctivity problem**

   A group G is *surjunctive* if, for all finite sets A, every injective shift-equivariant continuous map $A^G \to A^G$ is surjective. Gottschalk asked whether all countable groups have this property [Go73], which can be viewed as an equivariant version of Dedekind finiteness. While Gottschalk's question remains unresolved, Gromov showed that all sofic groups are surjunctive, and in fact this was the motivation for his introduction of the concept of soficity [Gr99]. This lecture will demonstrate how topological entropy can be used to give a proof of surjunctivity as in Section 4 of [KL11]. This can first be done for amenable groups by using the classical definition of entropy and a relatively straightforward combinatorial argument involving Følner sets (the case of $\mathbb{Z}$ could also be treated separately as a warm-up). The sofic case is somewhat more complicated technically, and should be presented using the formulation of topological entropy in Definition 2.5 of [KL13b].

6. **Sofic measure entropy**

   This lecture introduces sofic entropy for probability-measure-preserving actions following the general approach of [Ke13]. The key basic result to be proved is the Kolmogorov-Sinai theorem (Theorem 2.6 of [Ke13]), which facilitates the problem of computation by a reduction to generating partitions or σ-subalgebras. The approach of [Ke13] should be related to the topological model definition in Section 3 of [KL13a] and the original definition of Bowen [Bo10b] (see Section 3 of [Ke13]). It should be pointed out that, unlike for classical entropy, computations always require a nontrivial argument independently of any use of the Kolmogorov-Sinai theorem, as can be illustrated in the case of an action which fixes every point and/or the standard action $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n$ with respect to Haar measure, both of which have zero entropy. Examples with positive entropy (notably Bernoulli actions) will further demonstrate this principle in later lectures.

7. **Comparing amenable and sofic entropy**

   In both the topological and measurable cases, sofic entropy coincides with the classical entropy when the acting group is amenable [KL13a, Bo12b]. A complete argument in the measurable case is beyond the scope of a single lecture. One should instead concentrate on the proof in the case topological entropy (see Section 5 of [KL13a]) which illustrates some of the basic ideas relevant to both settings. The crucial tool is the Rokhlin lemma for sofic approximations (Lemma 4.5 of [KL13a]), to which some discussion should be devoted.

8. **Entropy of Bernoulli actions**

   Compute the entropy of a Bernoulli action $G \curvearrowright (Y, \nu)^G$ of a countable sofic group as in Section 4 of [Ke13]. The argument in the finite entropy case is due to Bowen [Bo10b]. The general case requires some basic properties of sofic entropy (Lemma 4.1 of [Ke13]) which should be established in the lecture. A consequence is that if the Shannon entropy of ν is infinite then the action $G \curvearrowright (Y, \nu)^G$ has no finite generating partition.

9. **Classification of Bernoulli actions**

   As shown in the previous lecture, the entropy of a Bernoulli action $G \curvearrowright (K, \kappa)^G$ of a sofic group is the Shannon entropy $H(K, \kappa)$, defined as $-\sum_{k \in K} \mu(\{k\}) \log(\mu(\{k\}))$ if K is countable and $+\infty$ if (K, κ) is not purely atomic. Thus if two Bernoulli shifts are measurably conjugate then the Shannon entropies of the base spaces agree. Ornstein famously proved the converse in the case $G = \mathbb{Z}$ [Or70a, Or70b]. This result was extended to arbitrary infinite amenable groups by Ornstein and Weiss [OW80]. See [Do11, DS12] for simplified proofs of Ornstein's result, and [BKS00, Gl03, Ru90] for proofs based on joinings.

   Stepin [St75] made the following definition: a group G is called *Ornstein* if two Bernoulli actions of G over standard bases are measurably conjugate whenever the Shannon entropies of the bases agree. Observe that no finite group is Ornstein. As mentioned above, all infinite amenable groups are Ornstein. Stepin observed that if G contains a Ornstein subgroup H then G itself is Ornstein. This is proved by building the isomorphism coset by coset. Details are available in [Bo11].

In [Bo12a] it is shown that if G is any countably infinite group then it is *almost Ornstein* in the sense that if $G \curvearrowright (K,\kappa)^G$ and $G \curvearrowright (L,\lambda)^G$ are Bernoulli actions of G over standard bases not merely consisting of two atoms, then they are measurably conjugate if the Shannon entropy of the bases agree.

The aim of the talk is to present the above results. Detailed arguments can be given where feasible.

10. **The variational principle**

The classical variational principle asserts that, for an action of a countable discrete amenable group on a compact metrizable space, the topological entropy is equal to the supremum of the measure entropies over all invariant probability measures. In [KL11] the same formula was shown to hold in the sofic case. The aim of the lecture is to present this sofic variational principle using the formulation of topological entropy in Definition 2.5 of [KL13b] and of measure entropy in Section 3 of [KL13a] (compare [Ch13]).

11. **Entropy of principal algebraic actions**

Following Section 7 of [KL11] but using instead the spatial definitions of topological and measure entropy from Section 2 of [KL13b] and Section 3 of [KL13a], apply the variational principle to compute the topological entropy of a principal algebraic action of a residually finite group with respect to a sofic approximation sequence built from finite quotients. The lecture should include a brief introduction to the Fuglede-Kadison determinant in group von Neumann algebras.

12. **Free group entropy**

A curious issue arising in sofic entropy theory is: how does the entropy depend on the choice of sofic approximation? Is there a "best" sofic approximation? In the special case when G is a free group, the best choice of sofic approximation appears to be the random choice. In other words, we choose a homomorphism from G to the symmetric group $\mathrm{Sym}(n)$ uniformly at random and use this to build a sofic approximation. There is an entropy associated to this random choice called the *f-invariant* [Bo10d]. It is not just the average of the sofic entropies! It was first introduced in [Bo10a]. This f-invariant is most interesting because unlike sofic entropy in general, the f-invariant satisfies a number of nice properties:

(a) it is additive under direct products (this is an easy exercise);

(b) it satisfies an ergodic decomposition formula [Se12b, Theorem 1.4];

(c) it satisfies an induced-subgroup-action formula [Se12a];

(d) there is a relative f-invariant and it satisfies the additivity property $f(\mathcal{P} \vee \mathcal{Q}) = f(\mathcal{P}) + f(\mathcal{Q}|\mathcal{P})$ [Bo10c];

(e) it is computable for Markov chains over the free group [Bo10c].

This first of two talks on the f-invariant should present the definition (in terms of Shannon entropies, as in [Bo10a]) and its connection to sofic entropy as in [Bo10d]. As examples one can compute the f-invariant of a Bernoulli action [Bo10a], an action on a finite measure space, and actions on profinite completions of G.

13. **Markov chains over free groups**

This talk is partially a continuation of the previous talk. The purpose is to introduce Markov chains over free groups [Bo10c] and show that the f-invariant of a Markov chain is directly computable. Examples (as in [Bo10c, §8]) should be given to illustrate. The f^*-invariant can be presented as well as the relative f-invariant and the Abramov-Rohlin formula. Markov chains can be used to prove a variety of other formulas for the f-invariant including an ergodic decomposition formula [Se12b, Theorem 1.4], an induced-subgroup-action formula [Se12a] and Yuzvinskii's addition formula [BG12]. Be aware that there is a corrigendum to [Bo10c] available on the arXiv.

14. **Combinatorial independence and sofic entropy**

The main aim of this lecture is to explain how positive topological entropy can be characterized in terms of combinatorial independence via a positive density condition [KL13b]. This is to be done using IE-pairs (see Proposition 4.16(3) in [KL13b]). Basic properties of IE-pairs and IE-tuples can be described. Time permitting, IE-tuples can be applied to algebraic actions to answer to a question of Deninger on the Fuglede-Kadison determinant in the case of residually finite groups (Section 6 of [KL13b]).

15. **Entropy, Li-Yorke chaos, and distality**

    The goal of this lecture is to show that if the topological sofic entropy is positive for some sofic approximation sequence then the action is Li-Yorke chaotic (Section 8 of [KL13b]). One deduces from this that a distal action has entropy either zero or $-\infty$ if the group is sofic, and zero entropy if the group is amenable (Corollary 8.5 of [KL13b]). The role of distality in the structure theory of dynamical systems should be briefly explained (see [Gl03, GW05]).

16. **Sofic dimension**

    The sofic dimension of a group, and more generally of an equivalence relation, measure-preserving action, or discrete measured groupoid, measures the asymptotic growth of the number of its sofic approximations on larger and larger finite sets. This concept is analogous to Voiculescu's free entropy dimension. It was introduced in [DKP11] (see also [DKP12]). This lecture should give the definition of sofic dimension, present examples, explain that it is a measure-equivalence invariant, mention the connection to cost, and discuss the free product with amalgamation formula.

17. **Sofic mean dimension**

    Mean dimension was introduced by Gromov [Gr99b] about a decade ago as an analogue of dimension for dynamical systems, and was studied systematically by Lindenstrauss and Weiss [LW00] for continuous actions of countable amenable groups on compact metrizable spaces. Among other beautiful results, Lindenstrauss and Weiss used mean dimension to show that there exists a minimal action of $\mathbb{Z}$ on some compact metrizable space which can not be equivariantly embedded into $[0,1]^{\mathbb{Z}}$ equipped with the shift action. Mean dimension is further explored in [Co05, CK05, Gu11, Kr06, Kr09, Li99].

The goal of this lecture is to extend mean dimension to continuous actions of countable sofic groups G on compact metrizable spaces X following [Li12]. One should compute the sofic mean dimension for some examples, such as certain Bernoulli actions and actions with the small boundary property.

18. **Measured equivalence relations, groupoids and entropy theory**

In [RW00] Rudolph and Weiss proved that while entropy is not an orbit-equivalence invariant, relative entropy with respect to the orbit change σ–algebra is an invariant. From this fact it is possible to extend results concerning actions of the integers to actions of general amenable groups [RW00, Da01, DP02, DG02, Av05, Av10, Bo12a]. Moreover, it is possible to define the entropy of an amenable equivalence relation relative to a class-bijective factor [Da01].

Sofic measured equivalence relations were introduced in [EL10] and sofic groupoids in [DKP11]. This talk should present the Rudolph-Weiss result, show how it implies a definition for the entropy of an amenable equivalence relation relative to a class-bijective factor, explain the definition of a sofic measured equivalence relation, and show how sofic entropy theory can be extended to class-bijective extensions of sofic equivalence relations. The paper [Bo13] is written in the generality of groupoids. However, to keep it simple the lecture should probably focus only on equivalence relations.

Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`lpbowen@math.utexas.edu` or `kerr@math.tamu.edu`

by **10 August 2013** at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...

Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Schwarzwaldstrasse 9-11, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered.

Please note that departure is on 11 October, Friday afternoon because the capacity of the MFO is needed for a board meeting on Saturday. A stay overnight to Saturday is only possible in exceptional cases. Further information will be given to the participants after the deadline.

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