Mathematisches Forschungsinstitut Oberwolfach

Report No. 52/2013

DOI: 10.4171/OWR/2013/52

Large Scale Stochastic Dynamics

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27 October – 2 November 2013

Abstract. In focus are interacting stochastic systems with many components, ranging from stochastic partial differential equations to discrete systems as interacting particles on a lattice moving through random jumps. More specifically one wants to understand the large scale behavior, large in spatial extent but also over long time spans, as entailed by the characterization of stationary measures, effective macroscopic evolution laws, transport of conserved fields, homogenization, self-similar structure and scaling, critical dynamics, dynamical phase transitions, metastability, large deviations, to mention only a few key items.

Mathematics Subject Classification (2000): 60Gxx, 82-XX, 35-XX.

Introduction by the Organisers

"Large Scale Stochastic Dynamics" is at the crossroad of probability theory and statistical physics, the central theme being the stochastic evolution of a system with many interacting components. A prototypical example is the stochastic Ising model: at the sites of a regular lattice one has spins which take values ± 1. A specified spin flips at random times with a rate depending on the current neighboring spin configuration. Such a seemingly simple model has a very rich phenomenology. For example, let us impose that at the initial time the spin values are random according to a Bernoulli measure, whereas the dynamics runs at low temperatures forcing spins to align. Which laws govern the resulting spatial coarsening process? One may modify the dynamics by requiring the number of up spins (= particles) to be conserved, which is implemented by exchanging the spin values for a neighboring pair of spins. This model leads to a system of interacting symmetric random

walks. One can drive the system by a uniform force field making the random walks asymmetric. The variations are without bound. Mathematically one has to focus on a few central issues. In fact, at the conference interesting advance was reported on zero temperature dynamics of lozenges and on cuve shortening for zero temperature Glauber dynamics.

Our workshop is a snap-shot of the current activities. A partial list of topics reads

- random walks in random environments, including tree graphs
- low temperature Ising dynamics
- stochastic conservation laws with several components
- transport processes, mixing times, spectral gaps
- metastable systems
- condensation and coarsening phenomena
- hydrodynamic limits

We had 46 participants from 13 countries, mostly probabilists, but also experts from partial differential equations, numerical analysis, and statistical physics. They all enjoyed tremendously the unique and stimulating atmosphere at the Mathematische Forschungsinstitut Oberwolfach and hope to return some day.

Claudio Landim,
Stefano Olla,
Herbert Spohn

Workshop: Large Scale Stochastic Dynamics

Table of Contents

Abstracts

Dynamical Gibbs-non-Gibbs transitions for interacting particle systems

Frank den Hollander
(joint work with Roberto Fernández and Julian Martínez)

In [2] the following phenomenon was discovered. A spin system is started from a Gibbs state μ_0 and is subjected to a high-temperature spin-flip dynamics. Let μ_t be the state of the system evolved at time $t > 0$. Then μ_t need not be Gibbs. Three scenarios were found:

(1) μ_t is Gibbs for all t.
(2) μ_t is Gibbs for small t and non-Gibbs for large t.
(3) μ_t is Gibbs for small t and for large t, but non-Gibbs for intermediate t.

For the Ising model on $\mathbb{Z}^d$, $d \geq 2$, with a ferromagnetic pair potential $J > 0$ and an external magnetic field $h \in \mathbb{R}$, and subjected to a high-temperature Glauber dynamics, it was found that (1) occurs for J small and h arbitrary, (2) occurs for J large and $h = 0$, while (3) occurs for J large and $h \neq 0$.

The fact that Gibbsianness may be lost over time did not come as a surprise. In the late 1980's and early 1990's many examples of systems were found where Gibbsianness is lost under *renormalisation transformations* (like decimation, clumping or projection). Stochastic dynamics may be seen as some sort of time-dependent renormalisation transformation, which places it in the same context. What did come as a surprise, however, was that loss of Gibbsianness under stochastic dynamics is *the rule rather than the exception*. In the past 10 years many examples have been discovered, and it was found that the Gibbs-non-Gibbs phenomenon is not at all restricted to particular choices of the interaction parameters. For an overview, see [1].

Lack of Gibbsianness can be detected by finding *bad configurations*, i.e. configurations that are the discontinuity points of the conditional probability distribution for the spin at the origin given the spins outside the origin. For a bad configuration this conditional probability varies by some fixed $\epsilon > 0$ in total variation norm when the configuration is altered outside a large box Λ containing the origin *irrespectively* of how large Λ is. The presence of bad configurations indicates that the system is not quasi-local, i.e., the system *cannot be described by a locally summable interaction Hamiltonian*. Thus, lack of Gibssianness is equivalent to the existence of bad configurations, signaling the presence of a *hidden phase transition* in the behavior of the spin at the origin as the spins "infinitely far away" are varied (these spins play the role of a boundary condition). For the Ising model with J large and $h = 0$, it was found in [2] that no bad configuration exists for μ_t for small t, while the alternating configuration is bad for μ_t for large t (scenario (2)).

The above way of detecting lack of Gibbsianness is *static*: for fixed t, bad configurations at time t are identified that correspond to a phase transition of the

system at time 0. In [3] a new view on the Gibbs-non-Gibbs phenomenon was put forward, namely, a *dynamic* rather than a static point of view. It was conjectured that lack of Gibbsianness of the state μ_t at time t is equivalent to a certain large deviation rate function having more than one global minimizer. More precisely, consider the *trajectory of the empirical distribution* $(\pi_s)_{s\in[0,t]}$ of the system during the time interval $[0,t]$. Look at the conditional probability distribution for this trajectory given that at time t it ends in a certain value, say $\pi_t = \nu$ for some ν in the space of probability measures on the set of spin configurations. Let $I_{t,\nu}$ denote the large deviation rate function for this conditional probability distribution (in the infinite-volume limit). The conjecture says: *the state μ_t at time t is not Gibbs if and only if there is some ν for which $I_{t,\nu}$ has more than one global minimizer.* We may think of ν as a *bad empirical distribution*, taking over the role of the bad configuration in the static point of view.

The above conjecture remains open in general. For one, it is generally very hard to compute $I_{t,\nu}$ explicitly for a given interacting particle systems. However, progress has been made by looking at *mean-field models*. For mean-field models the Hamiltonian depends on the size n of the system and therefore the single state μ_t at time t for lattice systems must be replaced by a sequence $(\mu_{t,n})_{n\in\mathbb{N}}$ of states at time t labelled by the system size n. The notion of what Gibbs means needs to be modified accordingly. As was suggested in [6], the sequence $(\mu_{t,n})_{n\in\mathbb{N}}$ is to be called Gibbs if for all $\alpha \in \mathbb{R}$ the conditional probability distribution under $\mu_{t,n}$ of the first spin given that the *magnetization* of the $n-1$ other spins equals α_{n-1} converges as $n \to \infty$ to a limiting distribution whenever $\lim_{n\to\infty} \alpha_{n-1} = \alpha$, and this limiting distribution is continuous in α.

In [4] the Curie-Weiss model subject to an independent spin-flip dynamics was analyzed, and in [5] a Kac-type version of this model. It was shown that the conjecture raised in [3] is true for all values of the interaction parameters. Moreover, a complete classification of the optimal trajectories was obtained, and it was found that the optimal trajectories may exhibit interesting properties, like avoiding certain *forbidden regions* and being *non-monotone in time*.

Future work will focus on mean-field and Kac-type models of $\mathbb{R}$-valued spins subject to a dynamics of weakly interacting diffusions. In the continuum-spin setting the powerful machinery of stochastic analysis can be brought into play. An additional scenario shows up: (4) μ_t is non-Gibbs for all $t > 0$.

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Macroscopic diffusion for some energy models with mechanical origin

Makiko Sasada

We report recent studies on the derivation of heat transport from mechanical dynamics via a so-called two-step approach. In the context of the derivation of heat transport, the two-step approach is the strategy consists of (i) a derivation of a mesoscopic stochastic dynamics for energies from a microscopic mechanical dynamics via a rare (or weak) interaction limit and (ii) a derivation of a time evolution equation describing the macroscopic energy transport for the mesoscopic stochastic dynamics via a proper space-time scaling limit (more precisely, the hydrodynamic limit). There are two typical examples of the dynamics studied by this approach: these are "localized hard balls with elastic collisions" studied by Gaspard and Gilbert [2, 3] and "energy transfer in a fast-slow Hamiltonian system" studied by Dolgopyat and Liverani [1]. From these dynamics, they derived respective stochastic dynamics for energies.

In this talk, we consider the generalized stochastic processes of these examples, called stochastic energy exchange models (SEE) introduced by Grigo et al. [4] and energy conserving stochastic Ginzburg-Landau models (ECGL) which has been studied by Varadhan [8]. We remark that a two-particle model in the class of ECGL was also obtained from a microscopic dynamics in [5].

SSE and ECGL are both Markov processes on the state space $\mathbb{R}_+^N := (0,\infty)^N = \{(x_i)_{i=1}^N; x_i > 0\}$ where N represents the number of particles and x_i represents the energy of i-the particle.

SSE is a pure jump process with model parameters (Λ, P) where a continuous function $\Lambda : \mathbb{R}_+^2 \to \mathbb{R}_+$ represents the collision rate between particles and a probability measure-valued continuous function $P : \mathbb{R}_+^2 \to \mathcal{P}((0,1))$ represents the collision kernel. The infinitesimal generator $\mathcal{L}$ of SSE acting on bounded functions $f : \mathbb{R}_+^N \to \mathbb{R}$ is

$$\mathcal{L}f(x) \;=\; \sum_{i=1}^{N-1} \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, \mathrm{d}\alpha) \big(f(T_{i,i+1,\alpha}x) - f(x)\big)$$

where

$$(T_{i,i+1,\alpha}x)_k \;=\; \begin{cases} \alpha(x_i + x_{i+1}), & \text{if } k = i, \\ (1-\alpha)(x_i + x_{i+1}), & \text{if } k = i+1, \\ x_k, & \text{if } k \neq i, i+1. \end{cases}$$

ECGL is a multi-dimensional diffusion process with model parameters (κ, σ) where $\kappa : \mathbb{R}_+^2 \to \mathbb{R}$ and $\sigma : \mathbb{R}_+^2 \to \mathbb{R}_+$ are smooth functions satisfying a condition to

guarantee the process remains in the state space $\mathbb{R}_+^N$ forever. The infinitesimal generator $\mathcal{L}$ of ECGL acting on smooth bounded functions $f\colon \mathbb{R}_+^N \to \mathbb{R}$ is

$$\mathcal{L}f(x) \;=\; \sum_{i=1}^{N-1} \kappa(x_i, x_{i+1})\big(\partial_{x_{i+1}} - \partial_{x_i}\big) f \;+\; \frac{1}{2}\sigma^2(x_i, x_{i+1})\,\big(\partial_{x_{i+1}} - \partial_{x_i}\big)^2 f.$$

The process is also defined by the following SDEs:

$$\begin{cases} \mathrm{d}x_i &= \mathrm{d}J_{i-1,i} - \mathrm{d}J_{i,i+1} \\ \mathrm{d}J_{i,i+1} &= \kappa(x_i, x_{i+1})\,\mathrm{d}t + \sigma(x_i, x_{i+1}\,\mathrm{d}B_{i,i+1}. \end{cases}$$

Our goal is to derive the hydrodynamic limit for SSE or ECGL under "good" condition on model parameters (Λ, P) or (κ, σ). Here, "good" means the class satisfying the condition is general enough as it includes all the examples mentioned above, but not so general as we can show the hydrodynamic limit rigorously for the models in the class. Unfortunately, we do not achieve the goal so far since both of the models are of non-gradient type, but we have some results on the keys of the proof of hydrodynamic limit for non-gradient systems. Precisely, they are the characterization of reversible measures and the spectral gap estimate.

To study a sufficient condition for SSE or ECGL to be reversible with respect to some product measure, we introduce the notion of mechanical form. Actually, the notion of mechanical form for SSE was already introduced in [4]. Note that it might be a natural condition for models with mechanical origin.

Definition 1. *A pair* (Λ, P) *is said to be of* mechanical form *if*

- $\Lambda(E_1, E_2) = \Lambda_{\mathrm{sum}}(E_1 + E_2)\,\Lambda_{\mathrm{ratio}}(\frac{E_1}{E_1+E_2})$,
- $P(E_1, E_2, \mathrm{d}\alpha) = P_{\mathrm{ratio}}(\frac{E_1}{E_1+E_2}, \mathrm{d}\alpha)$.

Definition 2. *A pair* (κ, σ) *is said to be of* mechanical form *if*

- $\kappa(E_1, E_2) = \kappa_{\mathrm{sum}}(E_1 + E_2)\,\kappa_{\mathrm{ratio}}(\frac{E_1}{E_1+E_2})$,
- $\sigma(E_1, E_2) = \sigma_{\mathrm{sum}}(E_1 + E_2)\,\sigma_{\mathrm{ratio}}(\frac{E_1}{E_1+E_2})$.

With these notions, a sufficient condition for SSE or ECGL to be reversible with respect to an identical product measure is obtained.

Proposition 1 ([4])**.** *Assume* (Λ, P) *is of mechanical form and* $P_{\mathrm{ratio}}(\beta, \mathrm{d}\alpha)$ *has a unique invariant distribution* $p(\mathrm{d}\beta)$ *as a transition probability kernel on* $(0, 1)$*. Then, SSE with* (Λ, P) *is reversible with respect to the product Gamma-distribution with parameter* γ *if and only if* $p(\mathrm{d}\beta) = \frac{1}{Z}(\beta(1-\beta))^{\gamma-1}\Lambda_{\mathrm{ratio}}(\beta)\,\mathrm{d}\beta$*. Moreover, if SSE with* (Λ, P) *is reversible with respect to some identical product probability measure, then it must be a Dirac measure or a product Gamma-distribution.*

Proposition 2 ([7])**.** *Assume* (κ, σ) *is of mechanical form and the process has a unique invariant distribution with positive density. Then, ECGL with* (κ, σ) *is reversible with respect to some identical product probability measure if and only if* $\kappa(a, b) = \frac{1}{2}p_\gamma(a)^{-1}p_\gamma(b)^{-1}(\partial_b - \partial_a)p_\gamma(a)p_\gamma(b)\sigma^2(a, b)$ *for some* $\gamma > 0$ *where* p_γ *is the density of the Gamma-distribution with parameter* γ*, and if it holds then the measure must be the product Gamma-distribution with parameter* γ*.*

We remark that the models obtained in [2, 3, 1] are all of mechanical form and reversible with respect to a product Gamma-distribution.

For the processes reversible with respect to a product Gamma-distribution, we consider the spectral gap for SSE or ECGL on the canonical state space $\mathcal{S}_{e,N} := \{x \in \mathbb{R}_+^N \ ; \ \frac{1}{N}\sum_{i=1}^N x_i = e\}$ denoted by $\lambda(e,N)$. We have the following spectral gap estimate.

Theorem 1 ([6]). *Assume (Λ, P) is of mechanical form and SSE with (Λ, P) is reversible with respect to a product Gamma-distribution with parameter γ. We also assume that $\Lambda_{\rm sum}(s) \geq As^m$ for some $m \geq 0$ and $A > 0$. Then, $\exists C = C(m,\gamma) > 0$ s.t. $\forall N \geq 2$ and $\forall e > 0$,*

$$\lambda(e,N) \ \geq \ C\lambda(1,2)\,\frac{\mathrm{e}^m}{N^2}.$$

Theorem 2 ([7]). *Assume (κ, σ) is of mechanical form and ECGL with (κ, σ) is reversible with respect to a product Gamma-distribution with parameter γ. We also assume that $\sigma_{\rm sum}(s) \geq As^{m+1}$ for some $m \geq 0$ and $A > 0$. Then, $\exists C = C(m,\gamma) > 0$ s.t. $\forall N \geq 2$ and $\forall e > 0$,*

$$\lambda(e,N) \ \geq \ C\lambda(1,2)\,\frac{\mathrm{e}^m}{N^2}.$$

We remark that the models obtained in [2, 3] satisfy the assumption in Theorem 1 as $\Lambda_{sum}(s) = \sqrt{s}$, but the model obtained in [1] does not satisfy the assumption in Theorem 2 as $\sigma_{sum}(s) = s^{\frac{1}{4}}$.

In this talk, we also discuss a formal description of the macroscopic diffusion coefficient for the hydrodynamics equation. With this description, we can show that under the assumptions in Theorems 1,2 and also assuming $\Lambda_{sum}(s) = s^m$ or $\sigma_{sum}(s) = s^{m+1}$ for some m, the diffusion coefficient $D(E)$ should be $D(E) = cE^m$ for some constant c. Surprisingly, it is true even if the process is of non-gradient type. But note that the constant c is not explicit for the non-gradient models though it is explicit for the gradient models.

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Expansion of the Green-Kubo formula in the weak coupling limit

Cédric Bernardin

(joint work with François Huveneers, Joel L. Lebowitz, Carlangelo Liverani and Stefano Olla)

Energy transport in nonequilibrium macroscopic systems is described phenomenologically by Fourier's law. This relates the energy flux J, at the position r in the system, to the temperature gradient at r, via $J = -\kappa\nabla T$. The computation of the thermal conductivity κ, which depends on the temperature and the constitution of the system, from the underlying microscopic dynamics is one of the central mathematical problems in nonequilibrium statistical mechanics (see [4, 12, 7] and references therein).

The Green-Kubo (GK) formula gives a linear response expression for the thermal conductivity. It is defined as the asymptotic space-time variance for the energy currents in an infinite system in equilibrium at temperature $T = \beta^{-1}$, evolving according to the appropriate dynamics. For purely Hamiltonian (or quantum) dynamics, there is no proof of convergence of the GK formula (and consequently no proof of Fourier law). One way to overcome this problem is to add a dash of randomness (noise) to the dynamics [3]. In the present work we explore the resulting GK formula and start an investigation of what happens when the strength of the noise, ς, goes to zero.

Our basic setup is a chain of coupled systems. Each uncoupled system (to which we will refer as a *cell*) evolves according to Hamiltonian dynamics (like a billiard, a geodesic flow on a manifold of negative curvature, or an anharmonic oscillator...) perturbed by a dynamical energy preserving noise, with intensity ς. We will consider cases where the only conserved quantity for the dynamics with $\varsigma > 0$, is the energy. The cells are coupled by a smooth nearest neighbor potential εV. We assume that the resulting infinite volume Gibbs measure has a convergent expansion in ε for small ε. We are interested in the behaviour of the resulting GK formula for $\kappa(\varepsilon, \varsigma)$ for small ς and ε keeping the temperature β^{-1} and other parameters fixed.

We start by noting that for $\varsigma > 0$, the GK formula is well defined and has a finite upper bound [3]. We do not however have a strictly positive lower bound on $\kappa(\varepsilon, \varsigma)$ except in some special cases [3]. We believe however that $\kappa(\varepsilon, \varsigma) > 0$ whenever $\varepsilon > 0, \varsigma > 0$, i.e. there are no (stable) heat insulators. The fact that $\kappa(\varepsilon, \varsigma) \geq 0$ follows from the definition of the GK formula. The situation is different when we let $\varsigma \to 0$. In that case we have examples where $\kappa(\varepsilon, \varsigma) \to 0$ (disordered harmonic chains [2]), and where $\kappa(\varepsilon, \varsigma) \to \infty$ (periodic harmonic systems).

To make progress in elucidating the properties of $\kappa(\varepsilon, \varsigma)$, when $\varsigma \to 0$, we carry out a purely formal expansion of $\kappa(\varepsilon, \varsigma)$ in powers of ε: $\kappa(\varepsilon, \varsigma) = \sum_{n \geq 2} \kappa_n(\varsigma)\varepsilon^n$.

This is formal because space-time correlations entering in the GK formula are non-local function and depends themselves on ε.

We then investigate the structure of the term $\kappa_2(\varsigma)$, which we believe, but do not prove, coincides with the $\lim_{\varepsilon\to 0}\kappa(\varepsilon,\varsigma)/\varepsilon^2$. We show that $\kappa_2(\varsigma)$ is finite and strictly positive for $\varsigma > 0$ by proving that it is equal to the conductivity obtained from a weak coupling limit in which there is a rescaling of time as $\varepsilon^{-2}t$ (cf. [13, 14]). We argue further that the $\lim_{\varsigma\to 0}\kappa_2(\varsigma)$ exists and is closely related to the weak coupling macroscopic conductivity obtained for the purely Hamiltonian dynamics $\varsigma = 0$ from the beginning. The latter is computed for a geodesic flow on a surface of negative curvature, and is strictly positive [10]. A proof would require the extension to random perturbations for the theory developed for deterministic perturbations in [6, 5]. This should be possible by arguing as in the discrete time case [11].

Nevertheless the identification of $\kappa_2(\varsigma)$ with the weak coupling limit conductivity (suggested by H. Spohn [16]) gives some hope that the higher order terms, can also be shown to be well defined and studied in the limit $\varsigma \to 0$. This could then lead (if nature and mathematics are kind) to a proof of the convergence and positivity of the GK formula for a Hamiltonian system.

We next show that we obtain the same $\kappa_2(\varsigma)$ for the thermal conductivity of an open system: N coupled cells in which cell 1 and cell N are in contact with Langevin reservoirs at different temperatures, when we let $N \to \infty$ and the two reservoir temperatures approach to β^{-1}.

We then study in details $\kappa_2(\varsigma)$ for 3 examples: the isolated cell hamiltonian is (1) a pinned anharmonic oscillator, (2) a rotor; (3) the system at $\varsigma = 0$ is a random (positively) pinned harmonic chain. In all cases we can prove that, generically, $\limsup_{\varsigma\to 0}\kappa_2(\varsigma) < +\infty$, as contrasted with the regular harmonic chain when $\kappa_2(\varsigma) \to \infty$ when $\varsigma \to 0$ [1, 8]. In case (1) and (2) we have no lower bound for this limit, but we believe that it will be strictly positive. In case (3) we prove that the limit is 0, as in the harmonic chain for $\varsigma \to 0$, with random pinning springs. Phase mixing, due to lack of resonances between frequencies of different cells at different energies, is the relevant ingredient for the finiteness of $\kappa_2(\varsigma)$ when $\varsigma \to 0$.

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Error estimates in the numerical computation of transport coefficients

Gabriel Stoltz
(joint work with Ben Leimkuhler and Charles Matthews)

Computational statistical physics aims at approximating the average properties of a physical system of interest. The microstate of the system of N particles is described by a configuration $(q,p) = (q_1, \dots, q_N, p_1, \dots, p_N) \in \mathcal{E}$ and a Hamiltonian energy function

$$H(q,p) \ = \ V(q) + \frac{1}{2}\, p^T\, M^{-1}\, p,$$

where the symmetric, definite, positive matrix M is the mass matrix. In most practical cases, the configuration part $\mathcal{M}$ of the phase space $\mathcal{E} = \mathcal{M} \times \mathbb{R}^{3N}$ is a periodic box $(L\mathbb{T})^{3N}$. Static properties can be written as the average of an appropriate observable $A(q,p)$ with respect to some probability measure defining the thermodynamic ensemble, typically the canonical measure

$$\mu(\mathrm{d}q\,\mathrm{d}p) \ = \ Z_{\mathrm{NVT}}^{-1}\, \mathrm{e}^{-\beta H(q,p)}\, \mathrm{d}q\,\mathrm{d}p, \qquad \beta \ = \ \frac{1}{k_{\mathrm{B}}T}.$$

The corresponding very high-dimensional integral is approximated in practice by time-averages of ergodic dynamics such as the Langevin dynamics

$$\begin{cases} \mathrm{d}q_t &= M^{-1} p_t\, \mathrm{d}t, \\ \mathrm{d}p_t &= -\nabla V(q_t)\, \mathrm{d}t - \gamma M^{-1} p_t\, \mathrm{d}t + \sqrt{\frac{2\gamma}{\beta}}\, \mathrm{d}W_t, \end{cases}$$

where $\gamma > 0$ is the friction parameter, and W_t is a standard $3N$-dimensional Brownian motion. More precisely, the following Law of Large Number holds:

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t A(q_s, p_s)\, \mathrm{d}s \ = \ \int_{\mathcal{E}} A(q,p)\, \mu(\mathrm{d}q\,\mathrm{d}p) \quad \text{a.s.} \tag{1}$$

Additional convergence results on the law of the process can also be stated, see for instance [1, 2, 7] for convergence results in the $H^1(\mu)$ norm, and [5] for Lyapunov techniques leading to a convergence in weighted L^∞ spaces. In fact, an important element for the proofs of our results are fine pointwise estimates on the resolvent of the generator $\mathcal{L}$ of the Langevin dynamics, which ensure that functions with derivatives growing at most polynomially are still at most of polynomial growth under the action of the resolvent (see [6] as well as the recent account [3]).

One aim of numerical analysis in the domain of computational statistical physics is to propose numerical methods to approximate (1) by numerically integrating the Langevin dynamics with a time step Δt, and to give error bounds as a function of the time step. More precisely, we look for a Markov chain (q^n, p^n), where (q^n, p^n) is an approximation of $(q_{n\Delta t}, p_{n\Delta t})$. For instance, it is possible to consider splitting schemes where the various parts of the dynamics are analytically integrated one after the other. To this end, we decompose the generator as $\mathcal{L} = A + B + C$ with

$$A \;=\; M^{-1}p\cdot\nabla_q, \qquad B \;=\; -\nabla V(q)\cdot\nabla_p, \qquad C \;=\; -M^{-1}p\cdot\nabla_p + \frac{1}{\beta}\,\Delta_p.$$

A possible scheme consists for instance in first integrating the elementary dynamics associated with A, then B, then C, leading to the scheme

$$\begin{cases} \widetilde{p}^{\,n+1} &= p^n - \Delta t\,\nabla V(q^n), \\ q^{n+1} &= q^n + \Delta t\, M^{-1}\widetilde{p}^{\,n+1}, \\ p^{n+1} &= \alpha_{\Delta t}\widetilde{p}^{\,n+1} + \sqrt{\frac{1-\alpha_{\Delta t}^2}{\beta}M}\,G^n \end{cases}$$

where G^n are i.i.d. Gaussian and $\alpha_{\Delta t} = \exp(-\gamma M^{-1}\Delta t)$. A numerical scheme is characterized by its evolution operator

$$P_{\Delta t}\,\psi(q,p) \;=\; \mathbb{E}\Big[\psi\big(q^{n+1},\,p^{n+1}\big)\;\Big|\;(q^n,p^n)=(q,p)\Big].$$

The idea is that $P_{\Delta t} \simeq \mathrm{e}^{\Delta t\,\mathcal{L}}$ in a sense to be made precise.

We prove in [4] the ergodicity of splitting schemes similar to the one above, in the sense that a Law of Large Number holds for a modified measure $\mu_{\gamma,\Delta t}(dq\,dp)$

$$\frac{1}{N_{\rm iter}}\sum_{n=1}^{N_{\rm iter}} A(q^n,p^n) \;\xrightarrow[N_{\rm iter}\to+\infty]{}\; \int A(q,p)\,\mu_{\gamma,\Delta t}(\mathrm{d}q\,\mathrm{d}p),$$

and a uniform-in-Δt convergence rate holds in some weighted L^∞ space:

$$\left\| P_{\Delta t}^n f - \int_{\mathcal{E}} f\,\mathrm{d}\mu_{\gamma,\Delta t}\right\|_{L^\infty_W} \;\le\; K\,\mathrm{e}^{-\lambda n\Delta t}\,\|f\|_{L^\infty_W}.$$

Error estimates on the computation of average properties are obtained as

$$\begin{aligned}\int_{\mathcal{E}}\psi(q,p)\,\mu_{\gamma,\Delta t}(dq\,dp) \;&=\; \int_{\mathcal{E}}\psi(q,p)\,\mu(\mathrm{d}q\,\mathrm{d}p) \\ &\quad+\;\Delta t^\alpha\int_{\mathcal{E}}\psi(q,p)f_{\alpha,\gamma}(q,p)\,\mu(\mathrm{d}q\,\mathrm{d}p)\;+\;\mathrm{O}(\Delta t^{\alpha+1}),\end{aligned}$$

where the correction function $f_{\alpha,\gamma}$ satisfies some Poisson equation $\mathcal{L}^* f_{\alpha,\gamma} = g_\gamma$ (with adjoints taken on $L^2(\mu)$ and g_γ depends on the scheme). It is then possible to rewrite the correction term as an integrated correlation function, and approximate it on-the-fly to reduce the bias on the invariant measure (see [4]). Uniform error estimates in the overdamped limit $\gamma \to +\infty$ can also be stated.

The numerical analysis quantifying the errors on the invariant measure can be extended to obtain error estimates on transport properties such as the mobility. In general, transport coefficients are expressed as integrated correlation functions (Green-Kubo formulas), or can be obtained as the linear response of some observable for an appropriately forced dynamics. We present in [4] error estimates on (i) integrated autocorrelation functions based on a modification of the observables appearing in the correlation product to increase the precision of the method; as well as on (ii) the linear response of nonequilibrium dynamics, by a double expansion with respect to the time step and the magnitude of the forcing parameter.

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Exit distributions for a class of anisotropic random walks in random environment

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(joint work with Erich Baur)

We consider the standard random walk in random environment (RWRE for short) on $\mathbb{Z}^d$: This is a Markov chain with "disordered " transition probabilities. The disorder is given by $\omega = \{p_x\}_{x\in\mathbb{Z}^d} \in \Omega \stackrel{\text{def}}{=} \mathcal{P}^{\mathbb{Z}^d}$, where

$$\mathcal{P} \stackrel{\text{def}}{=} \big\{p : \text{prob. distr. on } \{e : |e| = 1\}\big\}.$$

We exclusively use e to denote a lattice point in $\mathbb{Z}^d$ which is nearest to the origin.

For a fixed "environment "ω, and starting point $x \in \mathbb{Z}^d$ the Markov chain $\{X_n\}_{n\geq 0}$ has the "quenched "law

$$\mathrm{P}_{x,\omega}\big[X_0 = x\big] \;=\; 1 \quad \text{and} \quad \mathrm{P}_{x,\omega}\big[X_{n+1} = y + e \,\big|\, X_n = y\big] \;=\; p_y(e).$$

The basic assumptions we make are:

(I) The disorder is random and i.i.d.: $\mathbb{P}$ on Ω is a product measure $\mathbb{P} = \mu^{\mathbb{Z}^d}$, where μ is a probability distribution on $\mathcal{P}$

(II) Small disorder assumption: For some (small) $\varepsilon > 0$ thefollowing holds:

$$\mu\left[\left\{p : \left|p(e) - \frac{1}{2d}\right| \;\leq\; \varepsilon\right\}\right] \;=\; 1.$$

Example 1. *Choose a preferred $\hat{e}$ with probability $\frac{1}{2d}$, and then take $p(\hat{e}) = \frac{1}{2d} + \varepsilon$, $p(e) = \frac{1}{2d} - \frac{\varepsilon}{2d-1}$, $e \neq \hat{e}$.*

In contrast to the quenched law, there is the marginal measure on the paths contracted from the joint law $\mathbb{P} \otimes \mathrm{P}_{x,\cdot}$. Under this so called *averaged law*, $\{X_n\}$ is *not* Markovian.

Remark 1.

(1) *$P_{x,\omega}$ is not reversible for $d \geq 2$ (and not for $d = 1$ if one steps away from the nearest neighbor case)*

(2) *A lot is known under ballisticity conditions, as Sznitman's condition (T'). In that case*

$$\lim_{n\to\infty} \frac{X_n}{n} \;\neq\; 0.$$

The one-dimensional case is well studied in works of Solomon [7], [4], [6], and many others. In the higher-dimensional case, a lot is known under ballisticity conditions, like Sznitman's condition (T'), see e.g. [8]. The non-ballistic case in more than one dimension proved to be very delicate. For $d = 2$, nothing is known rigorously, except in special cases, like the balanced case investigated by Lawler [5]. For $d \geq 3$, under the assumption that the μ is invariant under lattice isometries, there is the celebrated work by Bricmont and Kupiainen [3], and more recently, for the continuous space setting, the work by Sznitman and Zeitouni [9], and in the lattice case, concentrating on exit distributions [2].

An important open problem is to achieve a complete perturbative picture at least for dimension $d \geq 3$:

Conjecture 1. *Given $d \geq 3$ there exists $\varepsilon(d) > 0$ such that whenever the above conditions (I) and (II) are satisfied, there exists $a(\mu) \in \mathbb{R}^d$, and a positive definite matrix $\Sigma(\mu)$ such that*

$$\mathcal{L}\left(\frac{X_n - na(\mu)}{\sqrt{n}}\right) \;\longrightarrow\; \mathcal{N}\big(0, \Sigma(\mu)\big), \qquad \mathbb{P}\text{-}a.s.$$

This appears to be beyond present days techniques. If true, it would prove the open 0-1-law in the perturbative regime. There is in fact no hope to have

an explicit computation of $a(\mu)$, $\Sigma(\mu)$, or an explicit characterization of non-ballisticity $\{\mu : a(\mu) = 0\}$.

The main new result we have is about exit distributions from Euclidean balls, under the assumption that μ is symmetric under reflections only, and not necessarily under rotations.

Consider Euclidean balls in $\mathbb{Z}^d$: $V_L \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^d : |x| \le L\}$, and define

$$\Pi_{L,\omega}(x,z) \stackrel{\text{def}}{=} \mathrm{P}_{x,\omega}\big[X_{\tau_L} = z\big], \quad x \in V_L,\ z \in \partial V_L,$$

where τ_L is the first exit time from $V_L(x) \stackrel{\text{def}}{=} x + V_L$.

Theorem 1 (E. Baur and E. Bolthausen, 2013)**.** [1] *Assume that μ is invariant under reflections only. (This rules out the possibility of $a(\mu) \neq 0$). If $\varepsilon > 0$ is small enough and the conditions (I) and (II) are satisfied, there is a sequence p_L of nearest neighbor (non-random, but L-dependent) transition probabilities with $p_L(e) = p_L(-e)$, $|e| = 1$, such that*

a)

$$p_\infty(e) \;=\; \lim_{L\to\infty} p_L(e) \qquad \textit{and} \qquad p_\infty(e) \;>\; 0, \quad \textit{for } |e| = 1.$$

b) *If π_L is the exit distributions from V_L by a random walk with transition probabilities p_L, one has that $\Pi_L \approx \pi_L$ on large scales as $L \to infty$. Precisely,*

$$\lim_{L\to\infty} \sum_y \Big| \sum\nolimits_x \Big(\Pi_L(0,x) - \pi_L(0,x)\Big)\, \phi_L(x,y)\Big| \;=\; 0,$$

where ϕ_L is a (rather arbitrary) smoothing operation on a scale $\gg 1$ but possibly $\ll L$.

Remark 2. (1) *A smoothing operation is necessary because there are effects of the random environment close to the boundary.*

(2) *An alternative phrasing is that Π_L is asymptotically the same as the exit distributions of a Brownian motion with covariance matrix*

$$\Sigma_\infty \;=\; \begin{pmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_d^2 \end{pmatrix},$$

with $\sigma_i^2 > 0$.

The main tool is the perturbation expansion for exit distributions as in [2]. Assume first invariance under all lattice isometries, and put define g_L^{ORW} to be the Green's function of ordinary random walk on V_L, killed at exiting the set. Put

$$\Delta_y(e) \;=\; \omega_y(e) - \frac{1}{2d}.$$

Then one has the expansion

$$\begin{aligned}\Pi_{L,\omega}(x,z) \;=\; & \pi_L(x,z) + \sum_{y\in V_L,e} g_L^{\mathrm{ORW}}(x,y)\,\Delta_y(e)\,\Big(\pi_L(y+e,z)-\pi_L(y,z)\Big)\\ & + \sum_{\substack{y,y'\in V_L\\ e,e'}} g_L^{\mathrm{ORW}}(x,y)\,\Delta_y(e)\,\Big(g_L^{\mathrm{ORW}}(y+e,y')-g_L^{\mathrm{ORW}}(y,y')\Big)\\ & \times\; \Delta_{y'}(e')\,\Big(\pi_L(y'+e',z)-\pi_L(y',z)\Big) \;+\; \cdots\end{aligned}$$

First the question: Why is $d \geq 3$ "easy "?

If we set $\xi(x) \stackrel{\mathrm{def}}{=} \mathrm{E}_{x,\omega}[X-x]$, $\xi_L \stackrel{\mathrm{def}}{=} L^{-1}\mathrm{E}_{0,\omega}[X_{\tau_L}]$, then

$$\begin{aligned}\xi_L \;=\; & \frac{1}{L}\sum_x g_L(0,x)\,\xi(x)\\ & + \frac{1}{L}\sum_{x,y} g_L(0,x)\,\Delta_x(e)\,\Big(g_L(x+e,y)-g_L(x,y)\Big)\,\xi(y) \;+\; \cdots\end{aligned}$$

For fixed L, $\varepsilon \to 0$ this leads to a valid expansion of $\mathcal{L}(\xi_L)$. If we set $\mathrm{cov}_{\mathbb{P}}[\xi] \stackrel{\mathrm{def}}{=} \delta\, I_d$, $\mathrm{cov}_{\mathbb{P}}[\xi_L] \stackrel{\mathrm{def}}{=} \delta_L\, I_d$, and taking into account only the first term gives

$$\delta_L \;\approx\; \Big(L^{-2}\sum\nolimits_{x\in V_L} g_L(0,x)^2\Big)\,\delta.$$

For $d = 1$, one has $g_L(0,x) \approx L$, and therefore $\delta_L \approx \mathrm{const}\times L\,\delta$. So, in one dimension, the disorder is enlarged, even taking into account only the leading term in the expansion. The two-dimensional case is the borderline: Here $g_L(0,x) \approx 1$, and in fact, a simple computation gives

$$\lim_{L\to\infty}\lim_{\delta\to 0}\frac{\delta_L}{\delta} \;=\; \frac{2}{\pi}.$$

The factor $2/\pi$ is irrelevant. One has to take into account that on scale L one has dependencies between $\xi_L(x)$ and $\xi_L(0)$ for $|x| < 2L$. Taking that properly into account leads to 1 instead of $2/\pi$. Therefore, in the two-dimensional case, the disorder is not contracting in leading order. This is the case for $d \geq 3$ where one gets

$$\delta_L \;\approx\; \begin{cases}\delta\, L^{-1}, & \text{for } d=3,\\ \delta\, L^{-2}\log L, & \text{for } d=4,\\ \delta\, L^{-2}, & \text{for } d\geq 5.\end{cases}$$

There are however clearly problems with this simple computation. In fact, we want to investigate the case with $\delta > 0$ (or $\varepsilon > 0$) fixed (small), and $L \to \infty$ and *not* L fixed (large), $\delta \to 0$. The computation above investigates only the latter, and it is clear that for δ fixed and $L \to \infty$, the behavior is not properly described by the first terms in the expansion. The way out is to use a multiscale procedure: Take a sequence of scales $1 = L_1 < L_2 < \cdots$ such that from scale L_k to L_{k+1} one can work with the expansion.

There is also the problem that $\Pi_{L,\omega}(0,\cdot) - \pi_L(0,\cdot)$ cannot go to 0 in total variation if $L \to \infty$: If $z \in \partial V_L$, $\Pi_{L,\omega}(0,z)$ "feels "the disorder near z. The way to handle this problem is to use some smoothing after exit.

The key recursion is to represent $\Pi_{L_{k+1}}$ through centered exits distributions from $V_{L_k} + x$, $x \in V_{L_{k+1}}$.

Here is a summary of the main problems and steps.

(1) For the induction step, one uses the (schematic) expansion

$$\begin{aligned}\Pi_{L_{k+1}} \;=\; & \pi_{L_{k+1}} \,+\, g_{L_{k+1}}^{(L_k)}\, \Delta^{(L_k)}\, \pi_{L_{k+1}} \\ & +\, g_{L_{k+1}}^{(L_k)}\, \Delta^{(L_k)}\, g_{L_{k+1}}^{(L_k)}\, \Delta^{(L_k)}\, \pi_{L_{k+1}} \,+\, \cdots\,,\end{aligned}$$

where

$$\Delta^{(L_k)}(x,\cdot) \;\stackrel{\text{def}}{=}\; \Pi_{V_{L_k}+x}(x,\cdot) - \pi_{V_{L_k}+x}(x,\cdot),$$

$g_{L_{k+1}}^{(L_k)}$ the Green's function of L_k-coarse grained ORW, killed when exiting $V_{L_{k+1}}$. The choice of the sequence is made in such a way that one uses sophisticated estimates only for the first term, and can estimates the others very crudely.

(2) One needs $g^{(L_k)}$ or $\pi_{L_{k+1}}$ as smoothing operations. There is the problem that g is not a very good smoother. The way to handle that is to split g into a part which has good smoothing properties, and another kernel which is "small ", and which needs some additional care.

(3) We propagated two properties: Estimates for the globally smoothed differences, and for the non-smoothed differences. The latter is needed for the part of $g^{(L_k)}$ which is not properly smoothing.

(4) Presence of bad L_k-balls inside $V_{L_{k+1}}$: The induction of course should tell us that the probability that a box is bad is decaying. "Badness "is measured in terms of total variations of (smoothed) exit distribution differences.

(5) A substantial technical improvement in [1] over [2] is that we prove that the RWRE-Green's functions is dominated with large probability by a deterministic kernel which behaves well under convolutions, and which itself is comparable with the RW-Green's function.

The key difficulty with dropping the rotational symmetry is that one does not know to what exit distribution one has to compare. Technically, the rotational symmetry is used for a crucial cancellation property using that the averaged exit distributions are automatically rotational invariant. It is therefore natural that for level L, one adapts p_L such that the exit distributions under RW have covariance which matches the covariance for the *averaged* RWRE. There are many problems, the main ones are:

- On every level, one has to work with different one step distributions. This creates problems with the induction step.
- One finally has to prove that the p_L form a Cauchy sequence.

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Anomalous shock fluctuations in TASEP

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(joint work with Peter Nejjar)

We considers the simplest non-reversible interacting stochastic particle system, namely the totally asymmetric simple exclusion process (TASEP) on $\mathbb{Z}$. Despite its simplicity, this model is full of interesting features. In TASEP, particles independently try to jump to their right neighbor site at a constant rate and jumps occur if the exclusion constraint is satisfied: no site can be occupied by more than one particle. Under hydrodynamic scaling, the particle density solves the deterministic Burgers equation (see e.g. [18, 1]). This model belongs to the Kardar-Parisi-Zhang (KPZ) universality class [16] (see [5] for a recent review).

We are interested in the fluctuations around the macroscopic behavior given in terms of the solution of the Burgers equation and we focus on the fluctuations of particles' positions. Depending on the initial condition, the deterministic solution may have parts of constant and decreasing density, as well as a discontinuity, also referred to as shock. The fluctuations of the shock location have attracted a lot of attention.

For TASEP product Bernoulli measures are the only translation invariant stationary measures [17]. In the first works one considered initial configurations to have a shock at the origin, with Bernoulli measures with density ρ (resp. λ) at its left (resp. right), with $\rho < \lambda$. The shock location is often identified by the position of a second class particle. In this case, the shock fluctuations are Gaussian in the scale $t^{1/2}$ [9, 10, 14]. Microscopic information on the shock are available too [7, 11, 8, 3]. The origin of the $t^{1/2}$ fluctuations lies in the randomness of the initial conditions, since fluctuations coming from the dynamics grow only as $t^{1/3}$. If the initial randomness is only at one side of the shock, a similar picture still

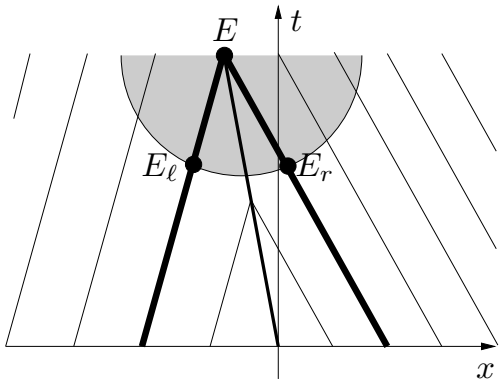

FIGURE 1. Illustration of the characteristics for TASEP. E is the shock location, where two characteristics merges (the thick lines). The gray region is of order t^ν for some $2/3 < \nu < 1$. Due to the slow decorrelation along characteritics, at large time t the fluctuations at E originates from the ones at E_ℓ and E_r.

holds. For example, in [4] one considers the initial condition is Bernoulli-ρ to the right and periodic with density $1/2$ to the left of the origin. When $\rho > 1/2$ there is a shock with Gaussian fluctuations in the scale $t^{1/2}$. In that work, the fluctuations of the shock position are derived from the ones of the particle positions. The result fits in with the heuristic argument in [19] (Section 5). The Gaussian form of the distribution function is not robust (see for instance Remark 17 in [4]).

In the paper [12] we study the fluctuation laws around a shock occurring without initial randomness are analyzed. In that case, one heuristically expects that the shock fluctuations, but also tagged particles fluctuations, live only on a scale of order $t^{1/3}$, see [2] for a physical argument. We find that the distribution function of a particle position (and also of tagged particles) is a product of two other distribution functions. The reason of the product form of the distribution function is that (1) at the shock two characteristics merge and (2) along the characteristics decorrelation is slow [13, 6].

More precisely, if we look at the history of a particle close to the shock at time t, it has non-trivial correlations with a region of width $\mathcal{O}(t^{2/3})$ around the characteristics, see Figure . At the shock the two characteristics come together with a positive angle so that at time $t - t^\nu$, $2/3 < \nu < 1$, their distance will be farther away than $\mathcal{O}(t^{2/3})$ (as proven for the step-initial condition situation by Johansson in [15]). This implies that the fluctuations built up along the two characteristics before time $t - t^\nu$ will be (asymptotically) independent. But if we stay on a characteristic, then the dynamical fluctuations created between time $t - t^\nu$ and time t are only $o(t^{1/3})$, which are irrelevant with respect to the total fluctuations present at time $t - t^\nu$ that are of order $t^{1/3}$ (this is also known as the slow-decorrelation phenomenon [13, 6]).

To generate a shock between two regions of constant density, we consider the initial condition where $2\mathbb{Z}$ is fully occupied and where the jump rates of particles

starting to the left (resp. right) of the origin is equal to 1 (resp. $\alpha < 1$). We prove in Corollary 1.5 of [12] the following result.

Theorem 1. *Let $x_n(0) = -2n$ for $n \in \mathbb{Z}$. For $\alpha < 1$ let $\mu = \frac{4}{2-\alpha}$ and $v = -\frac{1-\alpha}{2}$. Then it holds*

$$\lim_{t\to\infty} \mathbb{P}\Big[x_{t/\mu+\xi t^{1/3}}(t) \geq vt - st^{1/3}\Big] = F_1\Big(\frac{s-2\xi}{\sigma_1}\Big)\, F_1\Big(\frac{s-2\xi/(2-\alpha)}{\sigma_2}\Big), \tag{1}$$

with $\sigma_1 = \frac{1}{2}$ and $\sigma_2 = \frac{\alpha^{1/3}(2-2\alpha+\alpha^2)^{1/3}}{2(2-\alpha)^{2/3}}$. F_1 is the GOE Tracy-Widom distribution function [20].

As one can see from (1) the shock moves with speed v. When ξ is very large we are in the region before the shock, where the density of particle is $1/2$. Indeed, by replacing $s \to s + 2\xi$ and taking the $\xi \to \infty$ limit, then (1) converges to $F_1(s/\sigma_1)$. Similarly, when $-\xi$ is very large we are already in the shock, where the density of particles in $(2-\alpha)/2$. Indeed, by replacing $s \to s+2\xi/(2-\alpha)$ and taking $\xi \to -\infty$, then (1) converges to $F_1(s/\sigma_2)$. This is the reason why we call this situation a F_1–F_1 shock.

Actually, in [12] we describe also other shock situations (see Corollaries 1.6 and 1.7 therein). Further, for the proof it is convenient (not strictly necessary) to look at the problem from a last passage percolation point of view. In [12] we first determine the analogue results for that model and in a second time relate this to the TASEP picture.

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Mixing time and cutoff for the adjacent transposition shuffle and the simple exclusion

Hubert Lacoin

Let us consider the following way of shuffling a deck of N cards. At each step, with probability $1/2$ we interchange the position of a pair of adjacent cards chosen uniformly at random (among the $N-1$ possible choices) and with probability $1/2$ we do nothing. How many steps do we need to perform until the deck has been shuffled?

Even though this shuffling method may be of very little practical use for card players (indeed the usual rifle-shuffles allow a much faster mixing of the deck if executed properly, see [2]), this question has raised a considerable interest in the domain of Markov chain for a number of years since Aldous in [1, Section 4] proved that $O(N^3 \log N)$ steps where sufficient to mix the deck and that $\Omega(N^3)$ steps where necessary. It appears in [6, Chapter 23] in a short list of open problem concerning Markov chains mixing times.

A first reason that can be given for this interest is that it is that allowing only local moves (i.e adjacent transpositions) adds a constraint which makes the problem more challenging than the usual transposition shuffle (see [5] for a computation of the mixing time and [3] for a recent more probabilistic proof and some additional related results).

A second one is that shuffling with a geometrical constraint is a reasonable toy-model to describe relaxation of a low density gas. Consider N (labeled) particles in a box with erratic moves and local interactions, we can ask ourselves the difficult question: how much time is needed for the system to forget all the information about its initial configuration? Of course the adjacent-transposition is an over-simplification of the problem because it is one dimensional and the only motion that particles (or card) can make is exchanging the position with a neighbor, but a solution to the toy-problem might gives an idea of the qualitative behavior of the system. This connection with particle system becomes more obvious when the

Simple Exclusion process (which corresponds to the case of unlabeled particles) is introduced in the next Section.

The last substantial progress toward a solution before the writing of this paper was by Wilson [10], who proved that $\frac{1}{\pi^2}N^3 \log N$ steps where necessary and that $\frac{2}{\pi^2}N^3 \log N$ where sufficient, and conjectured that the first was the correct answer. In our talk we explain how we solved this conjecture by showing that the pack is mixed after $\frac{1}{\pi^2}N^3 \log N(1 + o(1))$ steps.

The exclusion process. A part of our work as also been devoted to the study of the mixing of the exclusion process, which is a projection of the adjacent transposition shuffle. The simplest way to describe it is the following: consider a segment with N sites, and place $k \in \{1, \dots, N-1\}$ particle on this segment, with *at most* one particle per site.

We consider the following dynamics: When a particle is situated next to an empty site, it jumps on it with rate one, and in the case of two empty site in the neighborhood it jumps on each one with rate one. A more formal description is given in the next section. We want to know how long we must wait to reach the equilibrium state of the particle system, for which all configurations are equally likely.

This model too has a long history and can be considered in a more general setup, with an $N \times N$ grid instead of a segment (or an higher dimensional cube, or a more general graph), we refer to [7, Section VIII] for a classical introduction. The problem of computing the mixing time of the exclusion process has also been well developed both in the case of $\mathbb{Z}^d$ grid or torus and of general graphs (see [8] and [9] and reference therein).

We show that for the mixing time is of the exclusion process with k particles is asymptotically equivalent to

$$\frac{1}{2\pi^2}N^2 \log k(1 + o(1))$$

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Random Walks in Divergence-Free Random Drift Field: H_{-1} suffices

Balint Tóth
(joint work with Illés Horváth, Bálint Vető and Gady Kozma)

I prove central limit theorem under diffusive scaling for the displacement of a random walk on $\mathbb{Z}^d$ in stationary divergence-free random drift field, under the H_{-1}-condition imposed on the drift field. The condition is equivalent to assuming that the stream tensor be stationary and square integrable. This improves the best existing result in Komorowski, Landim, Olla (2012), see [2], where it is assumed that the stream tensor is in $\mathcal{L}^{\max\{2+\delta,d\}}$, with $\delta > 0$. The proof relies on the Relaxed Sector Condition of Horváth, Tóth, Vető [1] and is technically considerably simpler than the proofs in Komorowski, Landim, Olla [2], or Oelschläger [3] (where a similar result is proved for diffusion in divergence-free random drift field).

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Biased random walks on Galton-Watson trees

Alexander Fribergh
(joint work with Gérard Ben Arous and Vladas Sidoravicius)

We will present different results related to the speed of biased random walks in random environments. Our main focus will be on a recent paper (by Ben Arous, Fribergh and Sidoravicius, see [1]) proving that the speed of the biased random walk on a Galton-Watson tree without leaves is increasing for high biases. This partially addresses a question asked by Lyons, Pemantle and Peres.

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Local limit theorem for the random conductance model in a degenerate ergodic environment

Martin Slowik
(joint work with Sebastian Andres and Jean-Dominique Deuschel)

One of the fundamental theorems in probability theory is the *central limit theorem*. Its *functional* version, first proven by Donsker [9], describes how to rescale a simple random walk in space and time in order to obtain a Brownian motion in the limit. *Local limit theorems* however provide much finer results. The classical local limit theorem (see e.g. [10, 15, 12]) states that the transition probabilities of a simple random walk properly rescaled converge to the Gaussian transition densities of the limiting Brownian motion on.

We are interested in establishing a local limit theorem for the *random conductance model* on the d-dimensional Euclidean lattice $(\mathbb{Z}^d, E^d)$. The conductance model is a reversible Markov process $\{X_t : t \geq 0\}$ on $\mathbb{Z}^d$ in continuous time with generator, $\mathcal{L}^\omega$, which acts on bounded functions $f\colon \mathbb{Z}^d \to \mathbb{R}$ as

$$\big(\mathcal{L}^\omega f\big)(x) \;=\; \sum_{y\sim x} \frac{\omega(\{x,y\})}{\mu^\omega(x)}\,\big(f(y)-f(x)\big),$$

where $\omega = \{\omega(e) \in [0,\infty] : e \in E^d\}$ is a family of non-negative weights (also called *conductances*) and $\mu^\omega(x) = \sum_{y\sim x}\omega(\{x,y\})$. We denote by $q^\omega(t,x,y)$ for $x,y \in \mathbb{Z}^d$ and $t \geq 0$ the transition density (or heat kernel associated with $\mathcal{L}^\omega$) of the Markov process $\{X_t\}$ with respect to the reversible measure μ^ω, i.e.

$$q^\omega(t,x,y) \;:=\; \frac{\mathrm{P}^\omega_x\big[X_t = y\big]}{\mu^\omega(y)}.$$

Of particular interest is the case when the conductances are itself random variables with law $\mathbb{P}$. For the *random* conductances model, a local limit theorem has been proven by Barlow and Hambly under the assumption that the conductances are i.i.d. random variables, i.e. $\mathbb{P}$ is a product measure. In addition, they assumed that either $\omega(e) \in \{0,1\}$ with $\mathbb{P}[\omega(e) > 0] > p_c$ for all $e \in E^d$ [4, Theorem 5.2] or that the conductances are uniformly elliptic [4, Theorem 5.7], that is there exists $c_1, c_2 \in (0,\infty)$ such that $c_1 \leq \omega(e) \leq c_2$ for all $e \in E^d$. The former model describes a random walk on a supercritical percolation cluster. In the later the additional assumption has been relaxed for i.i.d. conductances in [3, Theorem 5.14] by assuming that the conductances are only uniformly bounded away from zero.

Our main objective is to study this model under the following assumption on the law of the conductances.

Assumption 1. *Assume that the law $\mathbb{P}$ of the conductances satisfies:*

(i) $\mathbb{P}\big[0 < \omega(e) < \infty\big] = 1$ *for all* $e \in E^d$ *and* $\mathbb{E}[\mu^\omega(0)] < \infty$.

(ii) $\mathbb{P}$ *is ergodic with respect to translations of* $\mathbb{Z}^d$.

As a first result, we proved in [1] a quenched functional central limit theorem (QFCLT)

Theorem 1. *Suppose that $d \geq 2$ and the Assumptions 1 holds. Further, assume that that there exists $p, q \in (1, \infty]$ satisfying $1/p + 1/q < 2/d$ such that*

$$\mathbb{E}\big[\omega(e)^p\big] \,<\, \infty \qquad \textit{and} \qquad \mathbb{E}\big[1/\omega(e)^q\big] \,<\, \infty.$$

Then, $\mathbb{P}$-a.s., the rescaled process $\big\{\frac{1}{n}\, X_{n^2 t}\big\}$ converges under P_0^ω in law to a Brownian motion on $\mathbb{R}^d$ with a deterministic non-degenerate covariance matrix Σ^2.

Let us stress the fact that in dimensions $d = 1, 2$, it was shown in [5] that the QFCLT holds provided that $\mathbb{E}[\omega(e)] < \infty$ and $\mathbb{E}[1/\omega(s)] < \infty$. Therefore, the moment conditions we assumed are obviously not optimal. It is believed that the following conjecture should be true.

Conjecture 1. *Suppose that $d \geq 1$ and the Assumption 1 holds. If for all $e \in E^d$*

$$\mathbb{E}\big[\omega(e)\big] \,<\, \infty \qquad \textit{and} \qquad \mathbb{E}\big[1/\omega(e)\big] \,<\, \infty.$$

the the quenched functional central limit theorem holds with a deterministic and non-degenerate covariance matrix Σ^2.

It is well known, see e.g. [2], that in situations where the conductances are i.i.d. but the law $\mathbb{P}$ has a fat tail at zero, trapping phenomena may occur. As a consequence, the heat kernel decays only sub-diffusive so that the transition density does not have enough regularity for a local limit theorem to hold true, [2, Theorem 2.2]. Hence, it is clear that some sufficiently large moment conditions are needed.

Theorem 2. *Suppose that $d \geq 2$ and the conductances satisfy the conditions as specified in the Theorem above. Then, for $T_1, T_2 \in (0, \infty)$ with $T_1 < T_2$ and $K \in (0, \infty)$*

$$\lim_{n \to \infty} \sup_{|x| \leq K} \sup_{t \in [T_1, T_2]} \Big| n^d\, q^\omega\big(n^2 t, 0, \lfloor nx \rfloor\big) \,-\, a\, k_t(0, x)\Big| \,=\, 0, \qquad \mathbb{P}\textit{-a.s.}$$

with $a := 1/\mathbb{E}[\mu^\omega(0)]$ and k is the heat kernel of the limiting Brownian motion.

Fontes and Mathieu studied in [11] the random conductance model with a particular choice of an ergodic law $\mathbb{P}$ which admits a sub-Gaussian heat kernel decay. More precisely, in their example the conductances are uniformly bounded from above, i.e. $p = \infty$, and have a polynomial tail at zero ensuring that $\mathbb{E}[1/\omega(e)^q] < \infty$ for some q. They proved that the Gaussian heat kernel behaviour fails once $q < d/2$. This makes us believe that in contrast to the QFCLT at least the moment condition on $1/\omega(e)$ is optimal.

The Method. The proof of the local limit theorem is based on the approach in [4] and [6]. The two main ingredients are

1. a quenched functional central limit theorem (QFCLT) and
2. a Hölder-continuity estimate on the heat kernel,

which enables us to replace the weak convergence given by the QFCLT by the pointwise convergence in Theorem 2. As discussed above, the QFCLT has been established in [1]. In order to derive the Hölder-continuity estimate, we prove

a *parabolic Harnack principle* (PHI). Since the pioneering works [13, 14] Moser's iteration technique is by far the best-established tool in order to prove both elliptic and parabolic Harnack inequalities. Moser's iteration is based on two main ideas: a Sobolev-type inequality which allows to control the ℓ^r norm with $r = r(d) = d/(d-2) > 1$ in terms of the Dirichlet form, and a control of the Dirichlet form for a given harmonic and caloric function u, respectively. In the uniformly elliptic case this is rather standard [7, 8]. In our case where the conductances are unbounded from above and below, we need to work with a dimension dependent weighted Sobolev inequality, which we obtain from Hölder's inequality.

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Stochastic Ising model at zero temperature and curve-shortening flow

François Simenhaus
(joint work with Hubert Lacoin and Fabio L.Toninelli)

This talk is based on main results of [2, 3].

Short abstract. Let $\mathcal{D}$ be a bounded, smooth enough domain of $\mathbb{R}^2$. For $L > 0$ consider the continuous time, zero-temperature heat bath stochastic dynamics for the nearest-neighbor Ising model on $(\mathbb{Z}/L)^2$ (the square lattice with lattice spacing $1/L$) with initial condition such that $\sigma_x = -1$ if $x \in \mathcal{D}$ and $\sigma_x = +1$ otherwise. We prove the following classical conjecture due to H. Spohn [1]: In the diffusive limit where time is rescaled by L^2 and $L \to \infty$, the boundary of the droplet of "$-$" spins follows a *deterministic* anisotropic curve-shortening flow, such that the normal velocity is given by the local curvature times an explicit function of the local slope (see (1)). Locally, in a suitable reference frame, the evolution of the droplet boundary follows the one-dimensional heat equation.

Given $L \in \mathbb{N}$ we consider the zero-temperature stochastic Ising model on $(\mathbb{Z}/L)^2$ (the square lattice with lattice spacing $1/L$). The state space is the set $\Omega = \{-1, +1\}^{(\mathbb{Z}/L)^2}$ of spin configurations $\sigma = (\sigma_x)_{x\in(\mathbb{Z}/L)^2}$ with $\sigma_x = \pm 1$. The dynamics is a Markov process $(\sigma(t))_{t\geq 0}$, with $\sigma(t) = (\sigma_x(t))_{x\in(\mathbb{Z}/L)^2} \in \Omega$. Each spin σ_x is updated with unit rate: when the update occurs, σ_x takes the value of the majority of its four neighbors, or takes values ± 1 with equal probabilities if exactly two neighbors are $+1$ and two neighbors are -1.

We consider a compact, simply connected subset $\mathcal{D} \subset [-1, 1]^2$ whose boundary $\partial\mathcal{D}$ is a "nice" Jordan curve. The initial condition of the stochastic dynamics will be set to be "$-$" inside $\mathcal{D}$ and "$+$" outside:

$$\sigma_x(0) \;=\; \begin{cases} -1 & \text{if } x \in (\mathbb{Z}/L)^2 \cap \mathcal{D}, \\ +1 & \text{otherwise.} \end{cases}$$

We want to compute the scaling limit of the set $\mathcal{M}_L(t)$ of "$-$" spins (considered as a subset of $\mathbb{R}^2$ by adding a square of size $1/L$ centered at each "$-$" spin) at positive times, when $L \to \infty$.

Our goal is to prove that, as $L \to \infty$, $\mathcal{M}_L(L^2 t)$ converges to the compact set $\mathcal{D}_t$ whose boundary $\gamma(t) = \partial\mathcal{D}_t$ is the solution of the anisotropic *curve shortening flow* described by the following p.d.e.

$$\partial_t \gamma \;=\; a(\theta)\, k\, \mathbf{N} \tag{1}$$

with initial condition $\gamma(0) := \partial\mathcal{D}$. This equation has to be read as follows. The normal velocity at a point $p \in \gamma(t)$ is given by the curvature k at point p times $a(\theta(p))$, with

$$a(\theta) \;=\; \frac{1}{2\left(|\cos(\theta)| + |\sin(\theta)|\right)^2}, \qquad 0 \leq \theta \leq 2\pi$$

and $\theta(p)$ the tangent angle to $\gamma(t)$ at p. The normal vector $\mathbf{N}$ at point p points inward and the curvature is positive (resp. negative) at points of local convexity

(resp. concavity) of $\gamma(t)$. The anisotropy function a is inherited from the geometry of the square lattice.

Since $a(\cdot)$ is not differentiable for θ multiple of $\pi/2$, the existence of a solution for (1) does not follow from the standard literature so that our first result concerns existence, uniqueness and regularity theorem for the solution of (1).

Theorem 1. *There exists a unique regular solution $(\gamma(t))_{t\leq T}$ of (1) that is a Jordan curve for $t < T$ (where T denotes the time where $(\gamma(t))_{t\geq 0}$ shrinks to a point).*

The corresponding theorem given in [3] is more precise. In particular it precises in what sense the solution is regular.

Now that we have defined the limit flow we can state the main result that gives convergence of the stochastic droplet $\mathcal{M}_L(L^2t)$ to the deterministic flow $\mathcal{D}_t$, that is the compact domain enclosed by $\gamma(t)$.

For $\eta > 0$ let $\mathcal{B}(x,\eta)$ denote the ball of radius η centered at $x \in \mathbb{R}^2$ and for any compact set $\mathcal{C} \subset \mathbb{R}^2$ we define

$$\mathcal{C}^{(\eta)} := \bigcup_{x\in\mathcal{C}} \mathcal{B}(x,\eta), \qquad \mathcal{C}^{(-\eta)} := \Big(\bigcup_{x\notin\mathcal{C}} \mathcal{B}(x,\eta)\Big)^c.$$

Theorem 2. *For any $\eta > 0$ the following holds:*

$$\lim_{L\to+\infty} \mathbb{P}\Big[\mathcal{D}_t^{(-\eta)} \subset \mathcal{M}_L(L^2t) \subset \mathcal{D}_t^{(\eta)},\ \forall t \geq 0\Big] = 1.$$

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Lozenge tilings, Glauber dynamics and macroscopic shape

Fabio Toninelli

(joint work with Benoit Laslier)

We consider uniform random tilings of a finite domain U_L of the plane, with lozenges of side $1/L$. There is a canonical way to associate a discrete height function to a lozenge tiling. It is known [5] that, if U_L tends when $L \to \infty$ to a smooth domain U and the height function on the boundary of U_L tends to a limit height function φ on ∂U, then under the uniform measure π_L the height function concentrates (for $L \to \infty$) around a non-random deterministic height function $\bar{\phi}$, called the *macroscopic shape*. The function $\bar{\phi}$ minimizes a certain surface tension functional. It is known also that, according to the boundary height

φ, the macroscopic shape can be either C^∞ smooth, or it can contain facets (frozen regions): this is the so-called arctic circle phenomenon [4].

In [1] we studied the Glauber dynamics of lozenge tilings: with rate one, triplets of lozenges with a common vertex are rotated by an angle π. The law at time t of the associated Markov process tends to the uniform measure π_L when time goes to infinity. We are interested in studying how long (as a function of L) the dynamics takes to approach equilibrium.

Let us note that the lozenge Glauber dynamics is equivalent to the Glauber dynamics of $+/-$ interfaces for the zero temperature, three dimensional Ising model.

Our main theorem is:

Theorem 1 ([1])**.** *Assume that the boundary height φ is such that the macroscopic shape $\bar{\phi}$ has no frozen region in U. Then, there exist sequences $T_L = L^{2+o(1)}$ and $\epsilon_L = o(1)$ such that at times $t > T_L$, and uniformly in the initial condition, with high probability the height function is within distance ϵ_L from the macroscopic shape.*

The scaling $L^{2+o(1)}$ is expected on general grounds [11] and actually one expects that, in the diffusive scaling, the height function evolution tends to a deterministic evolution described by a parabolic PDE.

Let us mention a couple of previous results on this question: (i) in [7], a non-local version of the Glauber dynamics was introduced, and it was shown that its mixing time is polynomial in L; (ii) in [6], the mixing time of the non-local dynamics was shown to be of order $L^2 \log L$; (iii) in [10], from [6] and comparison theorems for Markov chains it was deduced that the mixing time of the usual (local) Glauber dynamics (the one we are considering here) is $O(L^6 \log L)$; (iv) finally, in [3] it was proven that the mixing time of the local dynamics is $O(L^2 (\log L)^C)$, provided that the macroscopic shape is flat (an affine function).

A detailed proof of the above Theorem is given in [1]. An important point is to study equilibrium height fluctuations in mesoscopic domains of diameter $\approx L^{-1/2+o(1)}$, with rather arbitrary boundary conditions. In this respect, a central role in our work is played by the recent works [8, 9] by L. Petrov, who gives sharp asymptotics for the law of uniform lozenge tilings in special polygonal domains of the plane.

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Exponential extinction time for the contact process on finite graphs

Daniel Valesin
(joint work with Thomas Mountford, Jean-Christophe Mourrat, Qiang Yao)

The Contact Process $(\xi_t)_{t\geq 0}$ with infection rate $\lambda > 0$ on a locally finite graph $G = (V, E)$ is a Markov process with state space $\{0,1\}^V$. In the usual interpretation, vertices of the graph are individuals in a population, which can be either healthy (state 0) or infected (state 1). Infected individuals recover with rate 1, and healthy individuals become infected with rate λ times the number of infected neighbours. The configuration $\underline{0}$ in which all individuals are healthy is absorbing for the dynamics. A well-known fact about the Contact Process is that if the underlying graph is $\mathbb{Z}^d$, a phase transition occurs. Specifically, there exists $\lambda_c = \lambda_c(\mathbb{Z}^d) \in (0, \infty)$ such that, if $\lambda \leq \lambda_c$, then the process started with finitely many infections reaches $\underline{0}$ with probability one; if $\lambda > \lambda_c$, then this probability is less than one.

The Contact Process on finite graphs and large enough infection rate is known to exhibit metastable behaviour. This means that the process persists for a very long time in an equilibrium-like state with a positive density of infected sites, and eventually makes a quick passage to the real equilibrium $\underline{0}$. The most direct way to capture this behaviour is proving that the *extinction time* τ_G of the infection grows rapidly with the size of the graph. For the process defined on the finite graph G and started with all vertices infected, τ_G is simply the random time at which $\underline{0}$ is reached.

In [6], we obtain such results for the Contact Process on very general sequences of graphs. In order to state our main theorem, let us introduce some notation:

$$\Lambda(n,d) = \{\text{connected trees with } n \text{ vertices and degree bounded by } d\};$$
$$\mathcal{G}(n,d) = \{\text{graphs with } n \text{ vertices containing a spanning tree in } \Lambda(n,d)\}.$$

We prove:

Theorem 1. *Assume that $\lambda > \lambda_c(\mathbb{Z})$ and $d > 0$. Let (G_n) be a sequence of graphs so that, for each n, $G_n \in \mathcal{G}(n, d)$. Then, $\liminf_{n\to\infty} \log \mathbb{E}[\tau_{G_n}]/n > 0$. Additionally, $\tau_{G_n}/\mathbb{E}[\tau_{G_n}]$ converges in distribution to the exponential distribution with parameter 1.*

For the case of finite boxes of $\mathbb{Z}^d$, to which the above theorem is applicable, it is not difficult to replace the restriction $\lambda > \lambda_c(\mathbb{Z})$ by $\lambda > \lambda_c(\mathbb{Z}^d)$ and re-obtain previously existing results ([1], [8], [3], [4], [5]).

In addition, Theorem 1 can serve as the basic ingredient for establishing that, for the Contact Process on very general families of graphs, the extinction time grows exponentially with the number of vertices. Even when the condition $\lambda > \lambda_c(\mathbb{Z})$ is not satisfied, it is often possible to consider coarse grained versions of the process which have higher λ, and for which Theorem 1 can thus be applicable. One instance in which this program has been successfully carried out, also in [6], is now explained.

We take G_n as a random graph given as follows. The set of vertices is just a set with n points, $V_n = \{x_1, \ldots, x_n\}$. Let p be a probability measure on $\mathbb{N}$ satisfying $p(\{0, 1, 2\}) = 0$ and, for some $a > 2$,

$$0 \;<\; \liminf_{m\to\infty} m^a p(\{m\}) \;\leq\; \limsup_{m\to\infty} m^a p(\{m\}) \;<\; \infty$$

(in other words, p is supported on integers larger than 3 and is a power law with exponent larger than 2). We then take $d_1, \ldots, d_n$ independent, all with law p and conditioned on $\{\sum_{i=1}^n d_i \text{ is even}\}$. For $i = 1, \ldots, n$, we endow vertex x_i with d_i half-edges. We then pair up the half edges at random, uniformly among all possibilities. This produces the random graph G_n, which is often referred to as the configuration model. The Contact Process is then considered on this graph, typically for fixed λ and $n \to \infty$.

In [2], Chatterjee and Durrett obtained the surprising result that, no matter how small λ is, the extinction time τ_{G_n} grows quickly with n, and the process is in this sense "always supercritical". More precisely, they proved that for any $\lambda > 0$ and any $\beta > 0$, with probability that the extinction time for the Contact Process with parameter λ on G_n is larger than e^{n^β} tends to 1 as $n \to \infty$. We improved this result and showed

Theorem 2. *For any $\lambda > 0$, there exists $c > 0$ such that, as $n \to \infty$, $\tau_{G_n} > \mathrm{e}^{cn}$ with probability tending to 1.*

As explained earlier, since Theorem 2 holds for any positive $\lambda > 0$, and not only $\lambda > \lambda_c(\mathbb{Z})$, it does not follow directly from Theorem 1; rather, one has to obtained a coarse-grained version of the Contact Process on G_n for which the infection rate is larger than the original one. The building blocks in our coarse-grained process are vertices of very high degree that exist on G_n due to the fact that the degree distribution is a power law, and that have the property of sustaining the infection for a long time.

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Current symmetries for particle systems with several conservation laws

Gunter M. Schütz

(joint work with Rafael M. Grisi)

Valkó and Tóth have proved a symmetry relation for stationary currents of a specific family of lattice gas models with several conservation laws [1]. This relation is reminiscent of the Onsager reciprocity relations, but is, unlike the Onsager relations, valid arbitrarily far from equilibrium and it guarantees hyperbolicity of the associated system of conservation laws in the hydrodynamic limit. Following a different approach [2] we relax here for a more general class of models the restrictive assumption of [1] that the invariant measure of the process is a product measure. The central ingredients in our derivation are time-reversal and a requirement on the decay of correlations which is reminiscent of the absence of a phase transition. Thus our approach suggests that for very general Markovian stochastic dynamics only a fairly mild assumption on the correlations in the stationary distribution are required for these symmetry relations to be valid.

We restrict ourselves to one-dimensional conservative particle systems with finite local state space S on a finite torus $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$ with N sites. Microscopic configurations will be denoted by $\underline{\omega} = (\omega_k)_{k\in\mathbb{T}_N} \in \Omega := S^{\mathbb{T}_N}$ where ω_k denotes the state of lattice site k. A natural setting is a lattice gas of several conserved species of particles. All particles could be in some non-conserved internal state with a finite number of internal degrees of freedom such as a spin. The very general Markovian stochastic dynamics, defined below, then allows for jumps of particles to other lattice sites and possibly changes of the internal particle states, either independently of the jumps or simultaneously with the jumps.

More precisely, we introduce the switch function $\Theta_{k,m}^{\omega',\omega''}: \Omega \to \Omega$ by

$$\left(\Theta_{k,m}^{\omega',\omega''}\underline{\omega}\right)_n \;=\; \begin{cases} \omega_n; & \text{if } n \neq k, k+m \\ \omega'; & \text{if } n = k \\ \omega''; & \text{if } n = k+m. \end{cases}$$

The process evolves from $\underline{\omega}$ to $\Theta_{j,m}^{\omega',\omega''}\underline{\omega}$ with translation invariant transition rate $r_m\big(\sigma^j\underline{\omega};\omega',\omega''\big)$ where $m = 1,\dots,M$ and $\sigma:\Omega\to\Omega$ is the shift on Ω given by $(\sigma\underline{\omega})_k = \omega_{k+1}$, $\omega\in\Omega$. Then the infinitesimal generator of the process is given by

$$\big(\mathcal{L}f\big)(\underline{\omega}) \;=\; \sum_{m=1}^{M}\sum_{k=0}^{N-1}\sum_{\omega',\omega''\in S} r_m\big(\sigma^k\underline{\omega};\omega',\omega''\big)\,\Big(f\big(\Theta_{k,m}^{\omega',\omega''}\underline{\omega}\big) \;-\; f(\underline{\omega})\Big).$$

We make the following assumtions on the rates: (i) Only jumps between two different sites that are at a distance of at most M sites are allowed. (ii) The jump rate may not depend on the local configuration beyond a distance of M sites. (iii) One has locally conserved quantities, i.e. functions $\xi^\alpha: S\to\mathbb{R}$ such that

$$\xi^\alpha(\omega') - \xi^\alpha(\omega_k) \;=\; \xi^\alpha(\omega_{k+m}) - \xi^\alpha(\underline{\omega}'').$$

(Thus $\sum_k \xi_k^\alpha(\underline{\omega})$, where $\xi_k^\alpha(\underline{\omega}) := \xi^\alpha(\omega_k)$, is invariant under the dynamics.) (iv) For any fixed value of all the conserved quantities the dynamics is ergodic.

By a lattice version of the Noether theorem one has $\mathcal{L}\xi_k^\alpha(\underline{\omega}) = -j_k^\alpha(\underline{\omega}) + j_{k-1}^\alpha(\underline{\omega})$ where a lengthy, but straightforward computation yields the current

$$j_k^\alpha(\underline{\omega}) \;:=\; -\sum_{n=0}^{M-1}\sum_{m=n+1}^{M}\sum_{\omega',\omega''\in S} r_m\big(\sigma^{k-n}\underline{\omega};\omega',\omega''\big)\,\Big(\xi_{k-n}^\alpha(\Theta_{k-n,m}^{\omega',\omega''}\underline{\omega}) \;-\; \xi_{k-n}^\alpha(\underline{\omega})\Big),$$

of the conserved quantity ξ^α through the bond $(k,k+1)$. For the generator $\mathcal{L}^*$ of the time-reversed dynamics we denote the currents by $j_k^{\alpha *}(\underline{\omega})$.

Now let μ be an invariant measure for the dynamics. Denote by $\xi := (\xi^\alpha)_{\alpha\in I}$ the vector of conserved particle numbers, and also $\xi_k := (\xi_k^\alpha)_{\alpha\in I}$ for the vector of particle numbers at site k. For $\phi := (\phi^\alpha)_{\alpha\in I}$ we define the measure

$$\mu_\phi(\underline{\omega}) \;=\; \mu(\underline{\omega})\,\exp\Big(\sum\nolimits_{k\in\mathbb{T}_N}\phi\cdot\xi_k(\underline{\omega}) \;-\; G(\phi)\Big), \tag{1}$$

where the normalization

$$G(\phi) \;=\; \log\sum_{\underline{\omega}\in\Omega}\exp\Big(\sum\nolimits_{k\in\mathbb{T}_N}\phi\cdot\xi_k(\underline{\omega})\Big)\,\mu(\underline{\omega})$$

has the interpretation of the grand-canonical free energy. Since ξ^α are conserved quantities, also μ_ϕ is an invariant measure for every ϕ. By translation invariance and ergodicity of the process it follows that μ_ϕ is also translation invariant. We denote by $\langle\cdot\rangle_\phi$ the expectation according to the measure μ_ϕ. In particular, we define $\rho^\alpha := \langle\xi_k^\alpha\rangle_\phi$, which in our interpretation corresponds to the density of the particle species α, and the stationary currents $j^\alpha := \langle j_k^\alpha\rangle_\phi$. The main result is

Theorem 1 ([2]). *For the process constructed above one has for any finite N*

$$\frac{\partial j^\alpha}{\partial \phi^\beta} - \frac{\partial j^\beta}{\partial \phi^\alpha} = N\Big(\big\langle j^\alpha_{\frac{N}{2}}(\xi^\beta_1 - \rho^\beta)\big\rangle_\phi - \big\langle j^\beta_{\frac{N}{2}}(\xi^\alpha_0 - \rho^\alpha)\big\rangle_\phi\Big). \tag{2}$$

Proof. From the construction (1) of the invariant measure it follows

$$\frac{\partial \mu_\phi(\underline{\omega})}{\partial \phi^\alpha} = \mu_\phi(\underline{\omega}) \sum_{k\in\mathbb{T}_N} \big(\xi^\alpha_k(\underline{\omega}) - \rho^\alpha\big),$$

and therefore

$$\frac{\partial j^\beta}{\partial \phi^\alpha} = \sum_{k\in\mathbb{T}_N} \big\langle (\xi^\alpha_k - \rho^\alpha) j^\beta_0 \big\rangle_\phi.$$

On the other hand, we use the conservation law and time reversal to find that

$$\begin{aligned}\big\langle (\xi^\beta_k - \rho^\beta)(j^\alpha_{-1} - j^\alpha_0)\big\rangle_\phi &= \big\langle (\xi^\beta_k - \rho^\beta)\mathcal{L}(\xi^\alpha_0 - \rho^\alpha)\big\rangle_\phi \\ &= \big\langle (\xi^\alpha_0 - \rho^\alpha)\mathcal{L}^*(\xi^\beta_k - \rho^\beta)\big\rangle_\phi = \big\langle (\xi^\alpha_0 - \rho^\alpha)(j^{\beta *}_{k-1} - j^{\beta *}_k)\big\rangle_\phi.\end{aligned}$$

In the second and third equality we have also used translation invariance. With the telescopic property of the discrete lattice difference, a partial summation yields

$$\begin{aligned}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} k\,\big\langle (\xi^\beta_k - \rho^\beta)(j^\alpha_{-1} - j^\alpha_0)\big\rangle_\phi &= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} k\,\big\langle (\xi^\beta_0 - \rho^\beta)(j^\alpha_{-k-1} - j^\alpha_{-k})\big\rangle_\phi \\ &= -\frac{\partial j^\alpha}{\partial \phi^\beta} + N\,\big\langle (\xi^\beta_1 - \rho^\beta) j^\alpha_{\frac{N}{2}}\big\rangle_\phi.\end{aligned}$$

and similarly

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} k\,\big\langle (\xi^\alpha_0 - \rho^\alpha)(j^{\beta *}_{k-1} - j^{\beta *}_k)\big\rangle_\phi = -\frac{\partial j^\beta}{\partial \phi^\alpha} + N\,\big\langle (\xi^\alpha_0 - \rho^\alpha) j^\beta_{\frac{N}{2}}\big\rangle_\phi.$$

Finally one observes that for every $\alpha, \beta \in I$ and $N > 2M$ one has $j^{\alpha *} = -j^\alpha$ and $\langle (\xi^\alpha_{\frac{N}{2}} - \rho^\alpha) j^{\beta *}_0\rangle_\phi = -\langle (\xi^\alpha_{\frac{N}{2}} - \rho^\alpha) j^\beta_0\rangle_\phi$, which is ensured by $\xi^\alpha_{\frac{N}{2}}(\Theta^{\omega',\omega''}_{-n,m}\underline{\omega}) = \xi^\alpha_{\frac{N}{2}}(\underline{\omega})$ for every $0 \le n < m \le M$ when $N > 2M$. The restriction $N > 2M$ comes from the requirement that the microscopic currents must not be a function a of the conserved quantities at site $N/2$. This change is possible if site $N/2$ is out of this range, meaning $N > 2M$. These are the main ideas of the proof given in [2]. □

For special forms of the invariant measure we point out the corollary

Corollary 1. *For sufficiently fast decaying correlations where for all distinct pairs of indices α, β, $\langle (\xi^\beta_1 - \rho^\beta) j^\alpha_{\frac{N}{2}}\rangle_\phi = o(1/N)$ one has the current symmetry*

$$\frac{\partial j^\alpha}{\partial \phi^\beta} = \frac{\partial j^\beta}{\partial \phi^\alpha} \tag{3}$$

in the thermodynamic limit $N \to \infty$.

Remark 1. *The assumptions of finite local state space and finite interaction range $N > 2M$ are basically technical. However, the presence of the finite-size term in* (2) *should be emphasized as it can lead to a breakdown of the infinite volume current symmetry* (3) *if correlations do not decay fast enough.*

The current symmetry has interesting consequences for the hydrodynamic limit behavior of the particle systems under Eulerian scaling. (i) One expects the conserved macroscopic densities $\rho^\alpha(x,t)$ to satisfy the system of conservation laws

$$\partial_t \rho^\alpha + \partial_x j^\alpha \;=\; 0 \tag{4}$$

where the macroscopic fluxes j^α as functions of the ρ^α are given by the expectations defined above in the thermodynamic limit $N \to \infty$. The current symmetry (3) then guarantees that the system is hyperbolic [1]. Thus we prove a result which one has long expected on general physical grounds. (ii) The quantity

$$S(\rho) \;=\; \sup_{\phi_\alpha \colon \alpha \in I} \big(\rho \cdot \phi - G(\phi)\big) \tag{5}$$

with the vector $\rho = (\rho^\alpha)_{\alpha \in I}$ is the usual thermodynamic entropy conjugate to the free energy $G(\phi)$. This quantity is a globally convex Lax entropy of the system (4) [3]. Indeed, defining the entropy flux $F(\rho)$ explicitly through the relation

$$\partial_\rho^\alpha F \;=\; \sum_\beta \phi^\beta \, \frac{\partial j^\beta}{\partial \rho^\alpha}$$

and observing that the construction (5) of the thermodynamic entropy implies that $\phi^\alpha = \partial_{\rho^\alpha} S(\rho)$ one obtains from (4) the scalar conservation law

$$\partial_t S(\rho) + \partial_x F(\rho) \;=\; 0.$$

The conservation of S is required for a proper estimate of the relative entropy of the time-evolved measure of the process with respect to the local equilibrium measure [1, 4].

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KPZ-behavior of hydrodynamics in one dimension

Henk van Beijeren

The dynamics of generic one-dimensional hamiltonian systems with translation invariant short-ranged interaction potentials are shown to be in the Kardar-Parisi-Zhang universality class. Scaling functions obtained by Prähofer and Spohn by solving the polynuclear growth model [1] can be used to obtain exact expressions for the long time behavior of the Green-Kubo integrands for heat diffusion and sound attenuation, as well as for system size dependent coefficients of heat conduction and sound damping. The Green-Kubo integrands decay with time as $t^{-2/3}$ respectively $t^{-3/5}$; the sound mode damping constant diverges with system size as $L^{1/2}$ and the heat conduction coefficient as $L^{1/3}$. Coefficients can be obtained exactly from the Prähofer-Spohn scaling functions combined with mode-coupling amplitudes as obtained by Ernst, Hauge and Van Leeuwen [2]. Due to the presence of three conserved densities (mass, momentum and energy), giving rise to three hydrodynamic modes with different propagation velocities (+ or $-c_0$, the adiabatic sound velocity for the sound modes and zero for the heat mode), there are important and still superdiffusive corrections to the asymptotic long time respectively large size behaviors. By using mode coupling techniques one can estimate these corrections as well. Simulations by Posch [3] on a square-shoulder model apparently confirm the predictions for the sound mode-sound mode correlation function. However, predictions for the wave number dependence of the decay rate of the heat mode correlation function are off by about a factor of 3. Correspondingly, for the tagged particle velocity autocorrelation function the simulation values are about a factor 2 smaller than the theoretical prediction. At present it is unclear what is causing these discrepancies.

More details can be found in [4].

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Anomalous diffusion in non-equilibrium, one-dimensional conservative systems

Milton Jara
(joint work with Cédric Bernardin and Patrícia Gonçalves)

Let us consider the following dynamics. Let $\{u_t(x); t \geq 0, x \in \mathbb{Z}\}$ be the solution of the system of ODE's given by

$$\tfrac{\mathrm{d}}{\mathrm{d}t} u_t(x) \;=\; u_t(x+1) - u_t(x-1); \qquad t \geq 0, x \in \mathbb{Z}.$$

Following [3], we call $u_t(x)$ the *volume* of site x at time t. Let us add a stochastic component to this dynamics. At each bond $\{x, x+1\}$, $x \in \mathbb{Z}$ we place a Poissonian clock of rate 1. Each time a Poissonian clock attached to a bond $\{x, x+1\}$, we switch the values of the volume at sites x and $x+1$. We call $\{\eta_t(x); t \geq 0, x \in \mathbb{Z}\}$ the Markov process obtained in this way. The generator of this process can be written as $L = S + A$, where

$$\begin{aligned} Af(\eta) \;&=\; \sum_{x \in \mathbb{Z}} \big(\eta(x+1) - \eta(x-1)\big)\, \frac{\partial f}{\partial \eta(x)}, \\ Sf(\eta) \;&=\; \sum_{x \in \mathbb{Z}} \big(f(\eta^{x,x+1}) - f(\eta)\big). \end{aligned}$$

Here $\eta = (\eta(x))_{x \in \mathbb{Z}}$ represents a volume configuration and $\eta^{x,x+1}$ represents the volume configuration obtained by switching the volume numbers at x and $x+1$. It turns out that the only conserved quantities on this model are the *volume* $\sum_x \eta(x)$ and the *energy* $\sum_x \eta(x)^2$. The extremal invariant measures are the product Gaussian measures

$$\mu_{\beta,\rho} \;=\; \prod_{x \in \mathbb{Z}} \sqrt{\frac{\beta}{2\pi}}\, \mathrm{e}^{-\frac{1}{2}\beta(\eta(x)-\rho)^2}\, \mathrm{d}\eta(x).$$

It is well-known that for diffusive systems the Green-Kubo formula relates the time evolution of stationary space-time correlations of conserved quantities to the current on stationary systems attached to heat reservoirs with different temperatures. In the case of one-dimensional Hamiltonian systems conserving both energy and momentum, violation of Fourier's law is expected. This has been theoretically confirmed for *stochastically* perturbed Hamiltonian systems [1], and also for the Bernardin-Stoltz model described above [3], [2]. This leaves open the question about what a good substitute for the Fourier's law (and consequently for the heat equation) is.

Our proposal is to study the time evolutions of the *energy fluctuations* of the Bernardin-Stoltz model in the harmonic case. Take $\rho = 0$ and let $\beta > 0$ be fixed. Consider the process $\{\eta_t; t \geq 0\}$ with initial distribution $\mu_{0,\beta}$. Let $n \in \mathbb{N}$ be a scaling parameter. For each test function $f \in \mathcal{C}_c^\infty(\mathbb{R})$ and for $t \geq 0$, define

$$\mathcal{E}_t^n(f) \;=\; \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} \big(\eta_{tn^{3/2}}(x) - \tfrac{1}{\beta}\big)\, f\big(\tfrac{x}{n}\big)$$

In this way, we have defined a distribution-valued process $\{\mathcal{E}_t^n; t \geq 0\}$ which we call the *energy fluctuation field*. Define $\mathcal{L} = -(-\Delta)^{3/4} + \nabla(-\Delta)^{1/4}$. We have the following result:

Theorem 1. *The process $\{\mathcal{E}_t^n; t \geq 0\}$ converges in distribution with respect to the J_1-Skorohod topology to the stationary solution of the infinite-dimensional Ornstein-Uhlenbeck equation*

$$\mathrm{d}\mathcal{E}_t \;=\; \mathcal{L}\,\mathcal{E}_t \mathrm{d}t + \mathrm{d}\mathcal{M}_t,$$

where $\mathcal{M}_t$ is a Gaussian martingale in $L^2(\mathbb{R})$ with correlation matrix

$$-\frac{2}{\beta} \int f(x)\,(-\Delta)^{3/4} g(x)\,\mathrm{d}x.$$

Let us explain what can we conclude out of this theorem. A first interesting feature is that the process $\mathcal{E}_t$ is space-time *self-similar* with self-similarity exponents $1:2:3$. This is a simple consequence of the $n^{3/2}$ time-scaling and of the $\frac{1}{n}$ space-scaling of the discrete model. Those exponents are exactly the same appearing in the KPZ universality class. Therefore, the process $\mathcal{E}_t$ provides an example of a Gaussian universality class with the same exponents of the KPZ universality class, and in particular different from the KPZ universality class. Another important feature of this model is the appearance of a *fractional heat equation* with driving operator $\mathcal{L}$. In fact, if we look at the space-time correlation functions

$$S(t,x) \;=\; \mathbb{E}\Big[\big(\eta_t(x)^2 - \tfrac{1}{\beta}\big)\,\big(\eta_0(0)^2 - \tfrac{1}{\beta}\big)\Big],$$

from Theorem 1 we conclude that

$$\lim_{n\to\infty} S\big(n^{3/2}t, nx\big) \;=\; P_t(x),$$

where $\{P_t(x); t \geq 0, x \in \mathbb{R}\}$ is the fundamental solution of the fractional heat equation $u_t = \mathcal{L}u$. Finally, since energy is conserved in a local sense, the model has a *fractional* pressure $p(u) = \big(-(-\Delta)^{-1/4} - \nabla(-\Delta)^{-3/4}\big)u$.

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KPZ equation, its renormalization and invariant measures

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(joint work with Jeremy Quastel)

The Kardar-Parisi-Zhang (KPZ) equation is a stochastic partial differential equation (SPDE), which describes the motion of growing interface with a random fluctuation. Denoting the height of the interface at time t and position $x \in \mathbb{R}$ by $h = h(t,x)$, it has the form

$$\partial_t h \;=\; \frac{1}{2}\,\partial_x^2 h \;+\; \frac{1}{2}\,\big(\partial_x h\big)^2 \;+\; \dot{W}(t,x), \tag{1}$$

where $\dot{W}(t,x)$ is the space-time Gaussian white noise, whose covariance structure is given by

$$\mathbb{E}\big[\dot{W}(t,x)\dot{W}(s,y)\big] \;=\; \delta(x-y)\delta(t-s). \tag{2}$$

We consider the equation (1) in one dimension. This equation is actually ill-posed because of inconsistency between the nonlinearity and the roughness of the noise. It is known that the solution $h(t,x)$ of the linear SPDE obtained from (1) by dropping the nonlinear term is $(\frac{1}{2}-\varepsilon)$-Hölder continuous in the space variable x for every $\varepsilon > 0$, so that the nonlinear term $\big(\partial_x h\big)^2$ would diverge. In fact, instead of (1), the renormalized equation

$$\partial_t h \;=\; \frac{1}{2}\,\partial_x^2 h \;+\; \frac{1}{2}\,\Big(\big(\partial_x h\big)^2 - \delta_x(x)\Big) \;+\; \dot{W}(t,x), \tag{3}$$

have the meaning in the following sense: Its Cole-Hopf solution defined as the logarithm of the solution of the linear stochastic heat equation (SHE) with a multiplicative noise:

$$\partial_t Z \;=\; \frac{1}{2}\,\partial_x^2 Z \;+\; Z\dot{W}(t,x), \tag{4}$$

i.e., $h(t,x) := \log Z(t,x)$ is a mathematically well-defined object and, by applying Itô's formula for this $h(t,x)$, we obtain (3) from (4) at least at a heuristic level. Note that, since $\big(\mathrm{d}W(t,x)\big)^2 = \delta_x(x)\mathrm{d}t$ from (2), the term $-\frac{1}{2}\delta_x(x)$ appears in (3) as an Itô correction term.

To give a meaning to (3) more precisely, we consider a simple approximation scheme for (3):

$$\partial_t h \;=\; \frac{1}{2}\,\partial_x^2 h \;+\; \frac{1}{2}\,\big((\partial_x h)^2 - \xi^\varepsilon\big) \;+\; \dot{W}^\varepsilon(t,x), \tag{5}$$

where $\dot{W}^\varepsilon(t,x) = \dot{W} * \eta^\varepsilon(t,x)$ is a smeared noise defined by applying a usual convolution kernel η^ε which tends to δ_0 as $\varepsilon \downarrow 0$, and $\xi^\varepsilon = \eta_2^\varepsilon(0)$ with $\eta_2^\varepsilon = \eta^\varepsilon * \eta^\varepsilon$. Then, by Itô's formula, we easily see that the solution $h = h^\varepsilon$ of (5) is given by the Cole-Hopf transform $h^\varepsilon = \log Z^\varepsilon$ of the solution $Z = Z^\varepsilon$ of the following SHE with the smeared noise:

$$\partial_t Z \;=\; \frac{1}{2}\,\partial_x^2 Z \;+\; Z\dot{W}^\varepsilon(t,x).$$

It is also easy to see that Z^ε converges to the solution Z of (4) as $\varepsilon \downarrow 0$. Thus, we can show that the solution h^ε of (5) converges to the Cole-Hopf solution of the KPZ equation. M. Hairer [1] has recently succeeded to give a meaning to (3), without bypassing the Cole-Hopf transform, and proved that the Cole-Hopf solution is the right solution of (3) under the periodic boundary condition.

In my talk, a different type of approximation scheme for the KPZ equation:

$$\partial_t h \;=\; \frac{1}{2}\,\partial_x^2 h \;+\; \frac{1}{2}\left((\partial_x h)^2 - \xi^\varepsilon\right) * \eta_2^\varepsilon \;+\; \dot{W}^\varepsilon(t,x),$$

was discussed. This type of approximation is appropriate from the view point to identify the invariant measures, since the applications of a certain operator A (in our case, the convolution operator) to the noise term and the same operator A twice to the drift term usually do not change the structure of the invariant measures; note that the second derivative ∂_x^2 and the convolution operator commute. The Cole-Hopf transform applied to this equation leads to an SHE with a smeared noise having an extra complex nonlinear term involving a certain renormalization structure:

$$\partial_t Z \;=\; \frac{1}{2}\,\partial_x^2 Z \;+\; \frac{1}{2}\,Z\left\{\left(\frac{\partial_x Z}{Z}\right)^2 * \eta_2^\varepsilon \;-\; \left(\frac{\partial_x Z}{Z}\right)^2\right\} \;+\; Z\dot{W}^\varepsilon(t,x). \tag{6}$$

It is shown that, under the situation that the corresponding tilt process is stationary, this complex term (the middle term in the RHS of (6)) can be replaced by a simple linear term divided by 24 in the limit, so that the limit equation is the linear SHE:

$$\partial_t Z \;=\; \frac{1}{2}\,\partial_x^2 Z \;+\; \frac{1}{24}\,Z \;+\; Z\dot{W}(t,x). \tag{7}$$

The constant $\frac{1}{24}$ is specific in KPZ world and frequently appears in related papers. The Wiener-Itô expansion and a similar method for establishing the so-called Boltzmann-Gibbs principle are effectively used to derive (7) from (6). As a result, it is shown that the distribution of a two-sided geometric Brownian motion with a height shift given by Lebesgue measure is invariant under the evolution determined by the SHE (4) on $\mathbb{R}$.

Multi-component KPZ equation for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d \in \mathbb{R}^d$:

$$\partial_t h^\alpha \;=\; \frac{1}{2}\,\partial_x^2 h^\alpha \;+\; \frac{1}{2}\,\Gamma_{\beta\gamma}^\alpha \partial_x h^\beta \partial_x h^\gamma \;+\; \dot{W}^\alpha(t,x),$$

was also discussed at approximating level:

$$\partial_t h^\alpha \;=\; \frac{1}{2}\,\partial_x^2 h^\alpha \;+\; \frac{1}{2}\,\Gamma_{\beta\gamma}^\alpha \left(\partial_x h^\beta \partial_x h^\gamma \;-\; \xi^\varepsilon \delta^{\beta\gamma}\right) * \eta_2^\varepsilon \;+\; \dot{W}^{\varepsilon,\alpha}(t,x),$$

and we gave its invariant measures, where $\Gamma_{\beta\gamma}^\alpha$ are constants which satisfy certain conditions.

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Scaling Limit for Brownian Motions with One-sided Collisions

Thomas Weiss

(joint work with Patrick L. Ferrari and Herbert Spohn)

We consider Brownian motions with one-sided collisions, meaning that each particle is reflected at its right neighbour. For a finite number of particles a Schütz-type formula is derived for the transition probability. We investigate an infinite system with periodic initial configuration, i.e. particles are located at the integer lattice at time zero. As a first main result, we provide an expression for the joint distribution at fixed time t.

Theorem 1. *The infinite system of interacting Brownian motions, $x_n(t)$, $n \in \mathbb{Z}$, starting at $x_n(0) = -n$ has a joint distribution function given by a Fredholm determinant with kernel*

$$K_t^{\mathrm{flat}}(x_1, n_1; x_2, n_2) \;=\; -\frac{(x_1 - x_2)^{n_2 - n_1 - 1}}{(n_2 - n_1 - 1)!}\, \mathbb{1}\{x_1 \geq x_2\}\, \mathbb{1}\{n_2 > n_1\} + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_-} \mathrm{d}z\, \frac{\mathrm{e}^{tz^2/2}\, \mathrm{e}^{-zx_1}\, (-z)^{n_1}}{\mathrm{e}^{t\varphi(z)^2/2}\, \mathrm{e}^{-\varphi(z)x_2}\, (-\varphi(z))^{n_2}}.$$

Here Γ_- is any path going from $\infty \mathrm{e}^{-\theta \mathrm{i}}$ to $\infty \mathrm{e}^{\theta \mathrm{i}}$ with $\theta \in [\pi/2, 3\pi/4)$, crossing the real axis to the left of -1, and

$$\varphi(z) \;=\; L_0\big(z\mathrm{e}^z\big)$$

with L_0 being the Lambert-W function, i.e. the principal solution for w in $z = w\mathrm{e}^w$.

Interesting and quite unexpected is the appearance of the Lambert function. It has a branch structure similar to the logarithm, but slightly more complicated.

The second main result of our contribution is a characterization of the law for the positions of the interacting Brownian motions in the large time limit. Due to the asymmetric reflections, the particles have an average velocity -1. For large time t the KPZ scaling theory suggests the positional fluctuations relative to the characteristic to be of order $t^{1/3}$. Nontrivial correlations between particles occur if the particle indices are of order $t^{2/3}$ apart from each other. Therefore, to describe the Brownian particles close to the origin at time t, we consider the scaling of the labels as

$$n(r,t) \;=\; \big\lfloor -t + 2^{5/3} t^{2/3} r \big\rfloor$$

and we define the rescaled process as

$$r \;\longmapsto\; X_t(r) \;=\; -\frac{x_{n(r,t)}(t) + 2^{5/3}\, t^{2/3}\, r}{(2t)^{1/3}}.$$

Theorem 2. *In the large time limit, X_t converges to the Airy$_1$ process,*

$$\lim_{t\to\infty} X_t(r) \;=\; \mathcal{A}_1(r),$$

in the sense of finite-dimensional distributions.

In particular for a single Brownian particle the limit is GOE Tracy-Widom distributed:

$$\lim_{t\to\infty} \frac{x_0(t)+t}{t^{1/3}} \;=\; \xi_{\mathrm{GOE}}.$$

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Hydrodynamic limit for a certain class of two-species zero-range processes

Kenkichi Tsunoda

Großkinsky and Spohn [1] studied several-species zero-range processes and gave a necessary and sufficient condition for translation invariant product measures to be invariant under such processes. Based on this result, they investigated the hydrodynamic limit. In this talk, we consider a certain class of two-species zero-range processes which are outside of the family treated by Großkinsky and Spohn. We prove a homogenization property for a tagged particle and apply it to derive the hydrodynamic limit under the diffusive scaling.

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Hydrodynamic limits for the velocity-flip model

Marielle Simon

I am interested in microscopic models of atoms whose time evolution is governed by a hybrid dynamics, namely a combination of deterministic and stochastic dynamics. A stochastic noise is added to the classical Newton's equations of motion, in such a way that the main features of the underlying Hamiltonian system are not destroyed. This stochastic noise provides good ergodic properties, and allows to derive the so-called hydrodynamic equations, which describe the macroscopic behavior of the system.

More precisely, in [5], I study the diffusive scaling limit for a chain of N harmonic coupled oscillators for which the Hamiltonian dynamics is perturbed with random flips of velocities. As a result, the total energy of the system is still conserved along the evolution. I look at the macroscopic behavior of this system as N goes to infinity, after rescaling space and time with N in the same way, with the diffusive

scaling. The system is considered under periodic boundary conditions. Let $p_x \in \mathbb{R}$ and $q_x \in \mathbb{R}$ denote the velocity and the position of the oscillator at site x. We define the deformations (r_x) by $r_x = q_{x+1} - q_x$. The Hamiltonian of the system is given by

$$\mathcal{H} \;=\; \sum_{x=0}^{N-1} \frac{p_x^2}{2} + \frac{r_x^2}{2}.$$

The stochastic noise can be easily described: each particle independently waits an exponentially distributed time interval and then flips the sign of its velocity. I derive the hydrodynamic equations by using the relative entropy method introduced for the first time by H. T. Yau [7] for a gradient[1] diffusive Ginzburg-Landau dynamics. In the context of diffusive systems, one need to establish the so-called *fluctuation-dissipation equations* in the mathematics literature (for example, in [3]). Moreover, since we observe the system on a diffusive scale and the system is non-gradient, second order approximations are needed. The good entropy estimate is obtained if the Gibbs local equilibrium state is corrected with a small term. This idea was first introduced in [2] and then used in [6] for interacting Ornstein-Uhlenbeck processes, and in [4] for the asymmetric exclusion process. However, it is the first time that this is applied for a system with several conservation laws in the diffusive scale.

Up to present, the derivation of hydrodynamic equations for the harmonic oscillators perturbed by the velocity-flip noise was not rigorously achieved (see e.g. [1]), because of the control of large energies. Indeed, all the energy moments need to be controlled uniformly in time and with respect to the size of the system, and in [5], we get this uniform control. Let us notice that the harmonicity of the chain is crucial to prove the result (roughly speaking, it ensures that the Gaussian type of measures preserved along the dynamics).

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On the evolution of the ABC model in a strongly asymmetric regime

Ricardo Misturini

The *ABC model*, introduced by Evans et al. [4], is a stochastic conservative dynamics consisting of three species of particles, labeled A, B, C, on a discrete ring $\Lambda_N = \{-N, \dots, N\}$ (one particle per site). The system evolves by nearest neighbor transpositions: $AB \to BA$, $BC \to CB$, $CA \to AC$ with rate q and $BA \to AB$, $CB \to BC$, $AC \to CA$ with rate 1.

The asymptotic behavior of the process (and of some variations) has been widely studied in the *weakly asymmetric* regime $q = \mathrm{e}^{-\beta/N}$, introduced by Clincy et al. [3], when the system size N goes to infinity and β is a control parameter that plays the role of an inverse temperature. In this regime interesting phase transition phenomena arise as β is varied.

Our work goes in a different direction. We deal with a *strongly asymmetric* regime where $q = \mathrm{e}^{-\beta}$, $\beta \uparrow \infty$. We consider two types of asymptotics involving the process. As a first step, we examine the behavior of the process in the case where the number of particles of each species, N_A, N_B, and N_C, is fixed while $\beta \uparrow \infty$. And after that, we consider the large volume case, where N_A, N_B, and N_C depend on β.

We show that the particles almost always form three pure domains, one of each species, located clockwise in the cyclic-order ABC. For fixed volume, we show that, in the time scale $\mathrm{e}^{\min\{N_A,N_B,N_C\}\beta}$, the process evolves as a Markov process among these $2N + 1$ segregated configurations, which jumps from anywhere to anywhere at positive rates, expressible in terms of some absorption probabilities of a very simpler dynamics.

Analyzing the case where the system size N grows with β, with some restrictions on the speed of this growth, we prove that, if each type of particle preserves a positive proportion of the total mass, then this segregated shape evolves, in the time scale $N^2\mathrm{e}^{\min\{N_A,N_B,N_C\}\beta}$, as a Brownian motion on the circle. Removing the assumption that each type of particle has positive density, we have also found situations where the limit process is ballistic or, as an intermediate case, a Brownian motion with drift.

Results of the same nature (the finite volume and the Brownian motion cases) were obtained for the Kawasaki dynamics at low temperature in two dimensions by Beltrán, Gois and Landim [2, 5]. Many techniques used in our analysis of the ABC model come from these papers. In comparison with the Kawasaki dynamics, an important difference is that, with the exception of the case $N_A = N_B = N_C$, the ABC process is not reversible and its invariant measure is not even explicitly known.

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Convergence in higher mean of a random Schrödinger equation

Maximilian Butz

We consider the discrete random Schrödinger equation

$$\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\phi \;=\; H_\omega \phi$$

on $\ell^2\left(\mathbb{Z}^3\right)$ with Schrödinger operator

$$H_\omega \;=\; -\frac{\Delta}{2} + \lambda V_\omega,$$

with Δ the next-neighbor lattice laplacian and $\left(V_\omega(x)\right)_{x\in\mathbb{Z}^3}$ being independent centered Gaussian variables with $\mathbb{E}\big[V_\omega(x)^2\big] = 1$. The small positive parameter λ controls the strength of disorder. Due to this scaling of the potential, one expects to observe an effect of scattering only if one runs the dynamics up to space and time scales of order η^{-1}, with $\eta = \lambda^2$, so we use the macroscopic variables for position, momentum, time $(X, V, T) = (\eta x, v, \eta t)$. When taking $\lambda \to 0$ to obtain the weak-coupling, long-time *kinetic limit*, the η-scaled Wigner transform W^η of the (η-dependent, and, due to the dynamics, random) state $\phi_t^{(\eta)}$ has proven useful. W^η is a distribution acting on $J \in \mathcal{S}\left(\mathbb{R}^3 \times \mathbb{T}^3\right)$ ($\mathbb{T}^3$ the three-dimensional torus) by

$$\begin{aligned}\Big\langle J, W^\eta\big[\phi_t^{(\eta)}\big]\Big\rangle \;&=\; \sum_{X\in(\eta\mathbb{Z}/2)^3}\int_{\mathbb{T}^3}\mathrm{d}V\;\overline{J(X,V)}\;W^\eta\big[\phi_t^{(\eta)}\big](X,V)\\ &=\; \eta^{-3}\int_{\mathbb{R}^3\times\mathbb{T}^3}\mathrm{d}\xi\;\mathrm{d}v\;\widehat{J}(\xi/\eta, v)\;\overline{\widehat{\phi}_t^{(\eta)}(v-\xi/2)}\;\widehat{\phi}_t^{(\eta)}(v+\xi/2).\end{aligned}$$

Assuming that the initial states $\phi_0^{(\eta)}$ are bounded in $\ell^2\left(\mathbb{Z}^3\right)$, $W^\eta\big[\phi_0^{(\eta)}\big]$ weakly converges to a measure μ_0 on a subsequence of $\eta \to 0$, and we have for this subsequence by [2] (using the methods of [1]), a convergence result for the disorder-averaged Wigner transform.

Theorem 1. *For all macroscopic times $T > 0$, and all $J \in \mathcal{S}\big(\mathbb{R}^3 \times \mathbb{T}^3\big)$,*

$$\lim_{\eta\to 0}\mathbb{E}\Big[\,\Big\langle J, W^\eta\Big[\phi_{T/\eta}^{(\eta)}\Big]\Big\rangle\,\Big] \;=\; \big\langle J, \mu_T\big\rangle,$$

the measure μ_T given as the weak solution of the linear Boltzmann equation

$$\frac{\partial}{\partial T}\,\mu_T(X,V)$$
$$(1) \qquad = \; -\sin(2\pi V)\cdot\nabla_X\mu(X,V) \;+\; \int_{\mathbb{T}^3} \mathrm{d}U\; \sigma(U,V)\,\big(\mu_T(X,U)-\mu_T(X,V)\big).$$

In (1), $\sin(2\pi V)$ *is a vector with components* $\sin(2\pi V_j)$, $j=1,2,3$ *and the collision kernel* σ *equals*

$$\sigma(U,V) \;=\; 2\pi\,\delta\big(e(U)-e(V)\big).$$

It would be very desirable to not only understand how the disorder-averaged Wigner transform behaves in the kinetic limit, but to also have a control on how large the typical deviations from this average are. [3] shows that the random quantity $\big\langle J, W^\eta\big[\phi^{(\eta)}_{T/\eta}\big]\big\rangle$ converges to $\big\langle J,\mu_T\big\rangle$ in r-th mean for all $r>0$, (intuitively speaking, the variance of the Wigner transform goes to zero as $\eta\to 0$). Thus, the dynamics of the scaled Wigner transform approaches a deterministic behavior in the kinetic limit. However, *concentration of singularity* assumptions regarding the initial condition $\phi^{(\eta)}_0$ had to be made, and convergence in higher mean could only be established for $\phi^{(\eta)}_0$ allowing for a decomposition

$$(2) \qquad \widehat{\phi^{(\eta)}_0}(k) \;=\; f^{(\eta)}_\infty(k) + f^{(\eta)}_{\mathrm{sing}}(k)$$

such that

$$(3) \qquad \big\|f^{(\eta)}_\infty\big\|_{L^\infty(\mathbb{T}^3)} \;\le\; c$$

and

$$(4) \qquad \Big\|\,\big|f^{(\eta)}_{\mathrm{sing}}\big| * \big|f^{(\eta)}_{\mathrm{sing}}\big|\,\Big\|_{L^2(\mathbb{T}^3)} \;\le\; c'\eta^{4/5}$$

for c, c' constants independent of η. This result is somewhat unsatisfactory as the physical intuition or justification behind (2)–(4) is not immediately clear, and the "fate" of all other initial states remains open.

The main result I want to present is a result on convergence in higher mean that only uses the ℓ^2-boundedness of the initial states.

Theorem 2. *Let* $\big(\phi^{(\eta)}_0\big)_{\eta>0}$ *be a sequence of initial states bounded in* $\ell^2\big(\mathbb{Z}^3\big)$. *Then there is a constant* $c<\infty$ *such that for all macroscopic times* $T>0$, *and all test functions* $J\in\mathcal{S}\big(\mathbb{R}^3\times\mathbb{T}^3\big)$ *and* $r\ge 1$

$$\left(\mathbb{E}\left[\left|\left\langle J, W^\eta\left[\phi^{(\eta)}_{T/\eta}\right]\right\rangle \;-\; \mathbb{E}\left[\left\langle J, W^\eta\left[\phi^{(\eta)}_{T/\eta}\right]\right\rangle\right]\right|^r\right]\right)^{1/r} \le\; C(J,T)\;E_0\;\lambda^{\frac{1}{cr}}$$

for sufficiently small $\lambda>0$ *and* $C(J,T)<\infty$ *only depending on* J,T. *Together with Theorem 1 this yields*

$$\lim_{\lambda\to 0}\mathbb{E}\left[\left|\left\langle J, W^\eta\left[\phi^{(\eta)}_{T/\eta}\right]\right\rangle \;-\; \mathbb{E}\left[\big\langle J,\mu_T\big\rangle\right]\right|^r\right] \;=\; 0$$

for all $r,T>0$.

The proof starts just as in [3] with a Duhamel series, a subsequent graph expansion, and an expression of the unitary via resolvents, but provides a more refined classification of graphs and estimates for the resolvent integrals suitable for each class of graphs. This way we can avoid making ad-hoc assumptions like (2-4).

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Electric network for irreversible walks - but is it useful?

Márton Balázs
(joint work with Áron Folly)

There is a well-known analogy between reversible Markov chains and electric networks (see e.g. [1]): the probability of reaching a state a before another one b agrees with voltages in a corresponding network of resistances, and the electric current also has a probabilistic interpretation. Such analogies can be used to prove a variety of theorems regarding transience-recurrence, commute times, cover times. The electric counterpart is very simple, consists of resistors only. These simple components behave in a symmetric fashion, that's why the analogy only works for reversible chains.

We found the electric component using which together with resistances we can build a network that extends the above analogies from reversible Markov chains to the case of irreversible ones. The new part we use is a voltage amplifier with a rather simple characteristic: the ratio of the potentials on its left end and on its right end is a fixed constant, while inflowing current on the left agrees with the outgoing current on its right. (Of course realizing such an amplifier would require complicated engineering and external power source, but we just consider it a black box with the above characteristic.)

We explored some nice and less nice properties of this electric network as follows. First, not all networks correspond to Markov chains, only those which have arbitrary constant potential - no outer current source solutions. These networks have meaningful effective resistance between any two vertices, in connection to escape probabilities as in the reversible case. We have the transformations that correspond to the classical serial and parallel equivalences, but star-delta connections are only valid for special parameter cases. And, what is the most worrisome, monotonicity does not hold in the classical sense: we have an example where increasing some resistances decreases the effective resistance. In fact, we already could not find Dirichlet's or Thomson's energy minimum principles in our network.

As many of the known results in the field use the monotonicity property, we now wonder if our electric analogy can be used to prove some theorems analogous to the reversible case. This is very much work in progress, and any remarks in this direction are welcome!

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Metastability for condensing zero-range processes in the hydrodynamic limit

Stefan Grosskinsky

(joint work with Inés Armendáriz and Michail Loulakis)

The zero-range process (ZRP) is an interacting particle system where the jump rates of particles depend only on the occupation number of the departure site and are independent of the configuration at the target site. We consider a ZRP $(\eta_t : t \geq 0)$ on the one-dimensional discrete torus $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ with N particles and state space $X_{L,N} = \{\eta \in \mathbb{N}_0^{\Lambda_L} : \sum_{x\in\Lambda_L} \eta_x = N\}$. The dynamics is defined by the generator

$$\mathcal{L}f(\eta) \;=\; \sum_{x\in\Lambda_L} p(x,y)\, g(\eta_x)\big(f(\eta^{x,y}) - f(\eta)\big) \tag{1}$$

for suitable test functions $f : X_{L,N} \to \mathbb{R}$. We focus on symmetric, irreducible, translation invariant jump probabilities $p(x,y) \in [0,1]$ of finite range, and jump rates are of the form

$$g(n) \;=\; 1 + b/n^\gamma \qquad \text{with parameters } b, \gamma > 0.$$

The stationary measures of this process $\mu_{L,N}$ are

$$\mu_{L,N} \;=\; \nu^L\big[\,\cdot\,\big|S_L = N\big] \qquad \text{where} \quad S_L \;=\; \sum_{x\in\Lambda_L} \eta_x,$$

and can be written as conditional product measures with marginals $\nu^1(n) = 1/g(n)!$ [1, 2]. Denote by

$$M_L \;=\; \max_{x\in\Lambda_L} \eta_x$$

the maximum occupation number. For $\gamma = 1$, $b > 2$ or $\gamma \in (0,1)$, $b > 0$ there exists a finite critical density $\rho_c = \nu^1(\eta_x) \in (0,\infty)$, such that in the thermodynamic limit $L, N \to \infty$, $N/L \to \rho$

$$M_L/L \;\xrightarrow{P}\; \begin{cases} 0 & ,\ \rho \leq \rho_c \\ \rho - \rho_c & ,\ \rho > \rho_c \end{cases},$$

which has been shown in [2, 3]. The fluctuations are known to be Gaussian and of order $\sqrt{L}$ for $\gamma \in (0,1)$ or $\gamma = 1$ and $b > 3$, and to be of order $L^{1/(b-1)}$ with

a stable law for $\gamma = 1$ and $2 < b \leq 3$ [4]. Moreover, the distribution of the configuration outside the maximum is known to converge to the product measure ν with density ρ_c, so for supercritical densities configurations exhibit a unique extensive maximum with high probability, called the condensate.

Therefore, the location of the maximum $\psi_L(\eta) = \{x : \eta_x = M_L(\eta)\}$ is with high probability given by a single site in Λ, and by translation invariance, for $\rho > \rho_c$, $\frac{1}{L}\psi_L \to U(\mathbb{T})$ converges to a uniform random variable on the unit torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, the scaling limit of Λ_L. Due to ergodicity, the location $\psi_L(\eta_t)$ of the maximum is changing in time, but on a slow time scale. Our main result is that the zero-range process $(\eta_t : t \geq 0)$ exhibits metastability with respect to the observable ψ_L on the time scale θ_L in the following sense.

Theorem 1. *Assume $\gamma \in (0,1)$ or $\gamma = 1$ and $b > 8$. In the thermodynamic limit $N, L \to \infty$, $N/L \to \rho > \rho_c$ we have for the zero-range process (1) for typical initial conditions η_0 and scale $\theta_L = L^{1+b}$*

$$\Big(\frac{1}{L}\,\psi_L(\eta_{t\theta_L}) : t \geq 0\Big) \;\to\; (Y_t : t \geq 0).$$

$(Y_t : t \geq 0)$ is a Lévy-type process with stationary, independent increments, state space $\mathbb{T}$ and generator

$$\mathcal{L}^{\mathbb{T}} f(u) \;=\; \int_{\mathbb{T}\setminus\{0\}} K_{b,\rho}\, r(v)\big(f(u+v) - f(u)\big)\,\mathrm{d}v \;.$$

The constant $K_{b,\rho}$ can be computed explicitly and

$$r(v) \;=\; \lim_{L\to\infty} L^2 \mathrm{cap}_\Lambda\big(0, [vL]\big)$$

is the scaling limit of the capacities of a single particle on Λ, which is non-degenerate under our assumptions.

For example, for nearest-neighbour probabilities $p(x,y) = 1/2(\delta_{y,x+1} + \delta_{y,x-1})$ we have

$$r(v) \;=\; \frac{1}{|v|\big(1-|v|\big)}.$$

Since under our assumptions on $p(x,y)$ the single particle motion scales to Brownian motion, the capacities will have the same scaling in L and our result applies, whereas for long-range dynamics different scalings are possible.

The analogue of our result has been shown previously on a fixed lattice of size L in the limit $N \to \infty$ und more general symmetric rates [5], and has recently also been extended to asymmetric, non-reversible dynamics [6]. To prove our result in the thermodynamic limit we follow the same martingale approach adopted in this work, which has recently been summarized in [7]. As a first step, this includes the classical partition of the state space in metastable sets (called valleys), which have high stationary probability and are characterized by the location of the condensate. On finite lattices the number of valleys is fixed, and they contain attractors. These are particular configurations that are visited with high probability before the system exits a valley, which leads to a relatively simple renewal-type

argument to derive the exponential law for jumps between valleys, leading to a Markovian process in the limit. Matching upper and lower bounds for the rates between valleys can be computed using a potential theoretic approach that has been developed in [8], see also [9] and references therein. Recently, this has been used to obtain bounds on the capacities between individual valleys for the ZRP in the thermodynamic limit with diverging density, i.e. $L \to \infty$ and $N/L \to \infty$ [10].

Since we rescale and embed our lattice in the torus in the scaling limit, we can use a coarse-graining approach to get matching bounds between sets of valleys at macroscopic distances also for finite supercritical densities $\rho > \rho_c$. The main challenge in the thermodynamic limit is not the increasing number of metastable valleys, but the fact that the size of each valley grows exponentially with the system size L and there are no attractor configurations to prove an exponential exit law. Instead we show that the mixing time for the process restricted to a valley is much smaller than the typical time until the next jump to another valley under suitable, non-optimal conditions on the parameter b. A bound on the relaxation and mixing time is achieved by comparison of the ZRP in a valley with independent birth-death chains for which the spectral gap can be estimated. To control the system until mixing, we construct a coupling of the ZRP with $L-1$ copies of a two-site ZRP, which provides an upper bound for exit rate from any configuration in a valley.

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Random Walk in a one-dimensional Lévy random environment

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(joint work with Giampaolo Cristadoro, Marco Lenci and Marilena Ligabò)

Anomalous diffusions are stochastic processes $X(t)$, $t \in \mathbb{R}^+$, having an asymptotic variance which does not grow linearly in time, that is $\mathbb{E}\big[X^2(t)\big] \sim t^\delta$ with $\delta \neq 1$. This phenomenon is quite common in applications and it is especially related to the transport in inhomogeneous material, e.g., the motion of a light particle in an optical lattice [5, 6]. The basic mathematical models for anomalous diffusions are *Lévy flights*, which are random walks with step length provided by an i.i.d. sequence of Lévy α-stable random variables with $\alpha \in (0, 2)$ (see [9, 4]). In this simple case, the motion is indeed provided by an asymptotic super-diffusive behavior with $\delta = 2$, for $\alpha \in (0, 1]$, and $\delta = 3 - \alpha$, for $\alpha \in (1, 2)$ (Lévy scheme). To model the motion in inhomogeneous material, one would like to take also into account that steps are mutually correlated by their positions, which we may identify with the presence of scatterers in the media. To this aim, in [3] the so-called *Lévy-Lorentz gas* were introduced. This is a one-dimensional random walk in a Lévy-type random environment, where scatterers are placed as a renewal point process with inter-distances having a Lévy-type distribution, and jump probabilities depend on whether the position of the walker is on a scatterer or not.

We are then interested in providing a characterization of this walk under the quenched and annealed laws (recurrence/transience, LLN, scaling limits, large deviation of the empirical speed), and in determining whether (and under which law) the asymptotic behavior is super-diffusive. The theory of random walks in random environments have been intensively studied in the last forty years and many important results have been achieved, especially for one-dimensional systems that are generally quite well understood. Even so, classical results do not apply to this setting, mainly because of the non-ellipticity of the environment, and a different analysis is required. In [3] a first lower bound on the annealed second moment was derived, showing that for $\alpha \in (0, 1]$ the Lévy-Lorentz gas is super-diffusive. More recently, in [1, 7], some related annealed quantities were estimated and numerically simulated, also providing indications of the super-diffusivity of the annealed process for $\alpha \in (0, 1]$. The range of $\alpha \in (1, 2)$ is instead still under debate, as the results in [1, 7] are not completely in agreement and may lead to different conclusions.

In a work in progress with Cristadoro, Lenci and Ligabò, we studied the model for $\alpha \in (1, 2)$ and proved that the quenched law of the random walk satisfies a classical CLT and has moments converging to the moments of a diffusion.

We are now working to extend this result to the annealed law, showing that in spite of Lévy flights, the Lévy-Lorentz gas for $\alpha \in (1, 2)$ has a diffusive behavior.

The problems we are interested in are the following:

1. Analysis of the annealed law: CLT and moments convergence.
2. Study of the regime of $\alpha \in (0, 1)$: LLN, CLT and moments convergence w.r.t. the quenched and annealed law.

3. Construction and characterization of an analogous two-dimensional model, also following the physical analysis of *Lévy glasses* given in [5, 2, 8].

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Well distributed points in a generic graph

Alexandre Gaudillière

(joint work with Luca Avena)

For a given continuous time Markov process X on a finite state space $\mathcal{X}$ with generator L defined by

$$Lf(x) \;=\; \sum_{y\neq x} w(x,y)\big(f(y)-f(x)\big), \qquad f\colon \mathcal{X}\to\mathbb{R},$$

we look for a law $\mathbb{P}$ for some random subset R of $\mathcal{X}$ with a given size $|R| = m \leq |\mathcal{X}| = n$ and such that

$$\mathbb{E}\big[\mathrm{E}_x[T_R]\big] \quad \text{does not depend on } x \text{ in } \mathcal{X},$$

with $\mathrm{E}_x[T_R]$ the mean hitting time of the random set R for the process started from x in $\mathcal{X}$. To this end we consider for any positive parameter q the probability measure ν_q on rooted spanning forests ϕ of $\mathcal{X}$ that is defined by

$$\nu_q(\phi) \;=\; \frac{1}{Z(q)} \prod_{(x,y)\in\phi} w(x,y),$$

where "$(x,y) \in \phi$" means that (x,y) is an edge of the spanning forest ϕ with the convention that each branch of each tree is oriented towards the tree root, and where $Z(q)$ is the normalizing partition function.

Such a measure ν_q can be sampled through Wilson algorithm [2] and we prove the following results.

(1) $Z(q)$ is the characteristic polynomial of the generator L:

$$Z(q) \;=\; \det(q-L) \;=\; (q+\lambda_0)\cdots(q+\lambda_{n-1}) \;=\; a_1 q + \cdots + a_n q^n,$$

where $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of $(-L)$ ordered by non-decreasing real part.

(2) The number of trees in a forest sampled from ν_q is a sum of n independent Bernoulli variables with parameters (or mean values) $q/(q+\lambda_i)$ when L has only real eigenvalues. If L has some complex eigenvalues then, while we are not able to make sense of such Bernoulli variables, the same result holds by defining the law of such a sum through the same algebraic formula as in the real case. Such a formula with complex λ_i's does define a probability measure on integers between 1 and n.

(3) When Φ is a random forest with law ν_q the root set $\rho(\Phi)$ forms a determinantal process with kernel

$$K(x,y) \;=\; \mathrm{P}_x\big[X(T_q) = y\big],$$

where T_q stands for an exponential time with parameter q (or expected value $1/q$) that is independent of the process X. This means that for any $A \subset \mathcal{X}$

$$\mathbb{P}\big[A \subset \rho(\Phi)\big] \;=\; \det{}_A K \;=\; \det\Big(\big(K(x,y)\big)_{x,y\in A}\Big).$$

(4) This root set $\rho(\Phi)$ is a solution to our problem. Indeed, for all x in $\mathcal{X}$,

$$\mathbb{E}\big[\mathrm{E}_x[T_{\rho(\Phi)}]\big] \;=\; \frac{1}{q}\left(1 - \prod_{i=1}^{n-1} \frac{\lambda_i}{q+\lambda_i}\right)$$

is a quantity that does not depends on x, and so does the same quantity conditioned on having a given root number $m \le n$:

$$\mathbb{E}\Big[\mathrm{E}_x\big[T_{\rho(\Phi)}\big] \;\Big|\; \big|\rho(\Phi)\big| = m\Big] \;=\; \frac{a_{m+1}}{a_m},$$

where each a_i refers to the coefficient of degree i in $Z(q)$, characteristic polynomial of the generator L.

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Random walks on quasi one dimensional lattices and applications to molecular motors

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(joint work with Vittoria Silvestri)

Molecular motors have been and still are object of intensive study in biology and biophysics. Two fundamental paradigms have been proposed for their modelization. In the so called *Brownian ratchet* model [11, 19] the dynamics of the molecular motor is given by a one–dimensional diffusion on a periodic but typically asymmetric potential, which can switch to a different potential at random times (switching diffusion). The other paradigm, on which we concentrate here, is given by continuous time random walks (also with non exponential waiting times) on quasi linear graphs having a periodic structure [8, 9, 13, 14, 15, 16]. We call these graphs *quasi 1D lattices*, they are obtained by gluing together several copies of a fundamental cell in a linear fashion. The geometric complexity of the fundamental cell reflects the possible conformational transformations of the molecular motor in its mechanochemical cycle. The simplest example is given by a random walk on $\mathbb{Z}$ with periodic jump rates, while random walks on other classes of quasi 1D lattices (parallel–chain models and divided–chains models) have been studied motivated by experimental evidence of a richer structure [1, 2, 12].

Let us point out some selected results in the biophysical literature concerning random walks on quasi 1D lattices. Still before the study of molecular motors, both the asymptotic velocity and gaussian fluctuations for the random walk on $\mathbb{Z}$ with periodic jump rates have been obtained in [4] under a suitable *Ansatz*. Generalizing the same Ansatz, formulas have been given in [1, 2, 12] for the asymptotic velocity and gaussian fluctuations of parallel–chains models and divided–chains models. In [21] the authors consider a generic random walk on a quasi 1D lattice and, by first–passage time arguments, show that the asymptotic velocity can be computed by solving a suitable linear system. Results concerning the large deviations of the molecular motor position are rather limited. In [17] the authors have derived such a LDP for the random walk on $\mathbb{Z}$ with periodic jump rates with periodicity 2, showing in addition the presence of a Gallavotti–Cohen type symmetry. This kind of symmetry relations, often called *fluctuation theorems*, have received in the last decade much attention inside the non–equilibrium statistical physics of small systems and in particular for molecular motors (cf. [7, 17, 20] and references therein).

We now describe our contribution (cf. [5, 6]) and compare it with the previous results. We treat random walks on quasi 1d lattices in full generality and show their analysis reduces to the study of random time changes of sums of i.i.d. random variables. As a first consequence we prove a law of large number and a central limit theorem, giving explicit formulas for the asymptotic velocity and diffusion coefficient. We also show that for random walks in quasi 1d lattices the asymptotic velocity can be computed by means of suitable linear systems, in the same spirit of [21]. Although the techniques involved in this part are rather standard, we

arrive to some remarkable results. While applying our formulas to the period case we confirm Derrida's result [4], for the class of divided–chains and parallel–chains model we show that the formulas derived in [1, 2, 12] by extending Derrida's Ansatz are not correct and we give different formulas. Moreover, we discuss computability issues for computing the asymptotic velocity and the diffusion coefficient in the general case.

We explain our results concerning large deviations. The present results for random time changes of cumulative process do not cover completely our class of models. We therefore follow a different route since in our case all relevant information concerning the position of the random walk are encoded in an associated random walk on $\mathbb{Z}$ with nearest neighbor jumps and typically non–exponential holding time, that we called *reduced random walk*. Adapting to the latter the techniques developed in [3] we derive the LDP for the first–passage times as well for the position of the molecular motor. The tools developed in this part are fundamental to investigate the Gallavotti–Cohen type symmetry pointed out in [17]. We show indeed that it is not universal: it is satisfied for any choice of jump rates only by a suitable class of quasi 1d lattices, while it is unsatisfied for almost any choice of jump rates in the other quasi 1d lattices. Finally, we show that the validity of the Gallavotti–Cohen type symmetry for the above class of quasi 1d lattices is indeed a consequence of a universal symmetry for algebraic currents [7].

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Large deviations for degenerate jump processes

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(joint work with Lorenzo Bertini)

We consider a class of continuous time Markov chains on a compact metric space that admit an invariant measure strictly positive on open sets together with an absorbing state. We prove the joint large deviation principle for the empirical measure and flow. Due to the lack of uniform ergodicity, the zero level set of the rate function is not a singleton. As a corollary, we deduce the Donsker-Varadhan large deviation principle for the empirical measure.

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Phase transitions on the scaling limits of the symmetric slowed exclusion

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(joint work with Tertuliano Franco and Adriana Neumann)

The exclusion process is an interacting particle system widely studied in Probability and Statistical Mechanics. Roughly speaking, according to this model, particles perform continuous time random walks in a lattice, under the constraint that whenever a particle tries to jump to an already occupied site, the jump is suppressed. If that is the case, the particle has to wait a new random time, in order to move.

There is an intensive research on the behavior of exclusion processes, from many different aspects and varied view points. In this work, we consider the one-dimensional symmetric simple exclusion in the presence of a slow bond. To be precise, the dynamics of this model is defined as follows. The model is consider evolving on $\mathbb{Z}$ and at each bond $\{x, x+1\}$ we place a clock T_x which has exponential distribution with parameter $\xi^n_{x,x+1}$, where, for $\alpha > 0$ and $\beta \in [0, \infty]$, $\xi^n_{x,x+1} = \frac{\alpha}{n^\beta}$, if $x = -1$, otherwise, $\xi^n_{x,x+1} = 1$. We notice that for $\alpha = 1$ and $\beta = 0$, all the clocks have parameter equal to 1 and the process corresponds to the well known symmetric simple exclusion process. By increasing the vale of α and β, we are creating a microscopic barrier at the bond $\{-1, 0\}$, since all the bonds are crossed, by particles, at rate 1 except the slow bond whose rate of passage of particles is given by α/n^β. The questions we address here are: what is the macroscopic effect of increasing the value of these parameters, at the level of hydrodynamics and equilibrium fluctuations? Is there a phase transition? What is its dependence on α at the macroscopic level? Is there a critical β? Does the phase transition at the hydrodynamics and fluctuations level, have the same critical β and the same dependence on α? Here we give a precise answer to these questions. In order to state properly our results we need to introduce some notation.

The microscopic dynamics. The *symmetric slowed exclusion process* is the Markov process $\{\eta_t : t \geq 0\}$ with state space $\Omega := \{0, 1\}^{\mathbb{Z}}$ and with infinitesimal generator acting on local functions $f : \Omega \to \mathbb{R}$ as

$$(\mathcal{L}_n f)(\eta) = \sum_{x \in \mathbb{Z}} \xi^n_{x,x+1} \big(f(\eta^{x,x+1}) - f(\eta) \big),$$

where $\eta^{x,x+1}$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(x+1)$ and $\xi^n_{x,x+1}$ is given as above. The dynamics of the process can be informally described as follows. At each bond $\{x, x+1\}$ of $\mathbb{Z}$, there is an exponential clock of parameter $\xi^n_{x,x+1}$, all of them being independent. Suppose the configuration at the present is η. After a ring of the clock at the bond $\{x, x+1\}$, the occupation variables $\eta(x)$ and $\eta(x+1)$ are exchanged. It is well known that the Bernoulli product measures on Ω_n with parameter $\rho \in [0, 1]$, denoted by $\{\nu_\rho : 0 \leq \rho \leq 1\}$, are invariant and reversible for the dynamics introduced above.

Hydrodynamic Limit. For each configuration η we denote by $\pi^n(\eta; \mathrm{d}u)$ the empirical measure given by $\pi^n(\eta; \mathrm{d}u) = \frac{1}{n} \sum_{x \in \mathbb{Z}} \eta(x) \delta_{x/n}(\mathrm{d}u)$ and $\pi^n_t(\eta, \mathrm{d}u) := \pi^n(\eta_{tn^2}, \mathrm{d}u)$. The hydrodynamic limit is the following statement. Assume a Law of Large Numbers (L.L.N.) for $\{\pi^n_0\}_{n \in \mathbb{N}}$ to $\rho_0(u)\,\mathrm{d}u$, under an initial distribution of the system, then for any $t > 0$ the L.L.N. holds for $\{\pi^n_t\}_{n \in \mathbb{N}}$ to $\rho(t, u)\,\mathrm{d}u$ under the corresponding distribution of the system at time t, where $\rho(t, u)$ evolves according to the hydrodynamic equation of the process.

For this model, the hydrodynamic limit is stated as follows. Fix $\rho_0 : \mathbb{R} \to [0, 1]$ and a sequence of probability measures $\{\mu_n\}_{n \in \mathbb{N}}$ associated to ρ_0. For any $T \geq 0$, the sequence of measure valued processes $\{\pi^n_t(\mathrm{d}x);\ t \in [0, T]\}_{n \in \mathbb{N}}$ converges, as n tends to infinity, to some $\{u(t, x)\,\mathrm{d}x;\ t \in [0, T]\}$ in probability with respect to the

Skorohod topology of $\mathcal{D}([0,T],\mathcal{M}_+(\mathbb{R}))$, that is, the space of càdlàg paths taking values in $M_+(\mathbb{R})$) (the space of positive measures in $\mathbb{R}$ with total mass bounded by one). Moreover,

- for $\beta<1$, $\{u(t,x);t\geq 0,\,x\in\mathbb{R}\}$ is the unique weak solution of the heat equation $\partial_t u(t,x)=\ \partial^2_{xx}u(t,x), t\geq 0,\,x\in\mathbb{R}$.
- for $\beta=1$, $\{u(t,x);t\geq 0, x\in\mathbb{R}\}$ is the unique weak solution of the heat equation $\partial_t u(t,x)=\ \partial^2_{xx}u(t,x), t\geq 0,\,x\in\mathbb{R}\setminus\{0\}$ with a boundary condition of Robin's type at zero, namely $\partial_x u(t,0^+)=\partial_x u(t,0^-)=\alpha\big(u(t,0^+)-u(t,0^-)\big), t\geq 0$.
- for $\beta>1$, $\{u(t,x);t\geq 0,x\in\mathbb{R}\}$ is the weak solution of the heat equation $\partial_t u(t,x)=\ \partial^2_{xx}u(t,x), t\geq 0,\,x\in\mathbb{R}\setminus\{0\}$ with a boundary condition of Neumann's type at zero, namely, $\partial_x u(t,0^+)=\partial_x u(t,0^-)=0, t\geq 0$.

See [1, 2] for details on the proof of this statement.

Equilibrium fluctuations. We start by describing the equilibrium fluctuations for the density of particles, from where we obtain the equilibrium fluctuations for other observables of the Markov process as: the current, tagged particle and the occupation time.

Fix $\rho\in[0,1]$. We denote our space of test functions by $\mathcal{S}(\mathbb{R}\setminus\{0\})$, which consists in the space of functions $f\in C^\infty(\mathbb{R}\setminus\{0\})$, that are continuous from the right at $x=0$, with $\|f\|_{k,\ell}:=\sup_{x\in\mathbb{R}\setminus\{0\}}|(1+|x|^\ell)\,f^{(k)}(x)|<\infty$, for all integers $k,\ell\geq 0$, and $f^{(k)}(0^-)=f^{(k)}(0^+)$, for all k integer, $k\geq 1$. Let $\mathcal{S}_\beta(\mathbb{R})$ be the subset of $\mathcal{S}(\mathbb{R}\setminus\{0\})$ composed of functions f satisfying

- For $\beta<1$, $f(0^-)=f(0^+)$.
- For $\beta=1$, $f^{(1)}(0^+)=f^{(1)}(0^-)=\alpha\big(f(0^+)-f(0^-)\big)$.
- For $\beta>1$, $f^{(1)}(0^+)=f^{(1)}(0^-)=0$.

For $t\in[0,T]$, we define the density fluctuation field on functions $f\in\mathcal{S}_\beta(\mathbb{R})$ as $\mathcal{Y}^n_t(f):=\frac{1}{\sqrt{n}}\sum_{x\in\mathbb{Z}}f\big(\frac{x}{n}\big)\big(\eta_{tn^2}(x)-\rho\big)$. For this model, the equilibrium fluctuations for the density are stated as follows. The sequence of processes $\{\mathcal{Y}^n_t\}_{n\in\mathbb{N}}$ converges in distribution, as n tends to infinity, with respect to the Skorohod topology of $\mathcal{D}([0,T],\mathcal{S}'_\beta(\mathbb{R}))$ (the space of càdlàg paths taking values in $\mathcal{S}'_\beta(\mathbb{R})$) to a Gaussian process $\mathcal{Y}_t$ in $C([0,T],\mathcal{S}'_\beta(\mathbb{R}))$ (the space of continuous paths taking values in $\mathcal{S}'_\beta(\mathbb{R})$), which is the stationary solution of the Ornstein-Uhlenbeck equation

$$\mathrm{d}\mathcal{Y}_t\ =\ \Delta_\beta\mathcal{Y}_t\mathrm{d}t\ +\ \sqrt{2\chi(\rho)}\,\nabla_\beta\,\mathrm{d}\mathcal{B}_t,$$

where $\mathcal{B}_t$ is a $\mathcal{S}'_\beta(\mathbb{R})$-valued Brownian motion. This means that the trajectories of $\mathcal{Y}_t$ are in $C([0,T],\mathcal{S}'_\beta(\mathbb{R}))$, $\mathcal{Y}_0$ is a white noise of variance $\chi(\rho)$, namely if for any $f\in\mathcal{S}_\beta(\mathbb{R})$, the real-valued random variable $\mathcal{Y}_0(f)$ has a normal distribution of mean zero and variance $\chi(\rho)\int_{\mathbb{R}}(f(x))^2\mathrm{d}x$.

Now we look to other observables of our Markov Process. We start with the current. For $u\in\mathbb{R}$ define the current through the bond $\{\lfloor un\rfloor-1,\lfloor un\rfloor\}$, as $J^n_u(t)$, that is $J^n_u(t)$ counts the total number of jumps from the site $\lfloor un\rfloor-1$ to

the site $\lfloor un \rfloor$ minus the total number of jumps from the site $\lfloor un \rfloor$ to the site $\lfloor un \rfloor - 1$ in the time interval $[0, tn^2]$. Then, for every $t \geq 0$ and every $u \in \mathbb{R}$, $\{J_u^n(t)/\sqrt{n}\}_{n\in\mathbb{N}}$ converges, as n tends to infinity, to $J_u(t)$, in the sense of finite-dimensional distributions, where $J_u(t)$ is a Gaussian process with mean zero and variance given by

- for $\beta < 1$, $\mathbb{E}[(J_u(t))^2] = 2\chi(\rho)\sqrt{\frac{t}{\pi}}$;
- for $\beta = 1$, $\mathbb{E}[(J_u(t))^2] = 2\chi(\rho)\Big(\sqrt{\frac{t}{\pi}} + \frac{\Phi_{2t}(2u+4\alpha t)\, \mathrm{e}^{4\alpha u + 4\alpha^2 t} - \Phi_{2t}(2u)}{2\alpha}\Big)$;
- for $\beta > 1$, $\mathbb{E}[(J_u(t))^2] = 2\chi(\rho)\Big(\sqrt{\frac{t}{\pi}}\Big[1 - \mathrm{e}^{-u^2/t}\Big] + 2u\,\Phi_{2t}(2u)\Big)$, where $\Phi_{2t}(x) := \int_x^{+\infty} \frac{e^{-u^2/4t}}{\sqrt{4\pi t}} \mathrm{d}u$.

To observe the fluctuations for a tagged particle, we have to start the process from ν_ρ conditioned to have a particle at the site $\lfloor un \rfloor$. If $X_u^n(t)$ is the position at the time tn^2 of that tagged particle, then

$$\big\{X_u^n(t) \geq k\big\} \;=\; \Big\{J_u^n(t) \geq \sum\nolimits_{x=\lfloor un \rfloor}^{\lfloor un \rfloor + k - 1} \eta_{tn^2}(x)\Big\}.$$

Therefore, for all $\beta \geq 0$, every $u \in \mathbb{R}$ and $t \geq 0$ $\{X_u^n(t)/\sqrt{n}\}_{n\in\mathbb{N}}$ converges, as n tends to infinity, to $X_u(t)$, in the sense of finite-dimensional distributions, where $X_u(t) = J_u(t)/\rho$ in law. See [3] for details on the proofs of these results.

Finally, we define the occupation time of the origin as the process

$$\Gamma_n(t) \;:=\; \frac{1}{n^{3/2}} \int_0^{tn^2} \big(\eta_s(0) - \rho\big)\, \mathrm{d}s.$$

Then, for every $t \geq 0$, $\{\Gamma_n(t) : t \in [0,T]\}_{n\in\mathbb{N}}$ converges in distribution with respect to the uniform of $\mathcal{C}([0,T], \mathbb{R})$, as n tends to infinity, to a mean-zero Gaussian process $\{\Gamma(t) : t \in [0,T]\}$ with variance given by

- for $\beta < 1$, $\mathbb{E}\big[\big(\Gamma(t)\big)^2\big] = \frac{4}{3}\frac{\chi(\rho)}{\sqrt{\pi}} t^{3/2}$, that is, the limit process is (up to a constant) a fractional Brownian motion of Hurst exponent $3/4$.
- for $\beta = 1$, $\mathbb{E}\big[\big(\Gamma(t)\big)^2\big] = \frac{4}{3}\frac{\chi(\rho)}{\sqrt{\pi}} t^{3/2} + 2\chi(\rho) \int_0^t \int_0^s \frac{F_\alpha(s-r)}{\sqrt{4\pi(s-r)}} \mathrm{d}r\, \mathrm{d}s$, where
$$F_\alpha(t) \;=\; \frac{1}{2t} \int_0^{+\infty} z\, \mathrm{e}^{-z^2/4t - 2\alpha z}\, \mathrm{d}z.$$
Moreover, the limit process is not self-similar, hence it is not a fractional Brownian motion.
- for $\beta > 1$, $\mathbb{E}\big[\big(\Gamma(t)\big)^2\big] = \frac{8}{3}\frac{\chi(\rho)}{\sqrt{\pi}} t^{3/2}$, that is, the limit process is (up to a constant) a fractional Brownian motion of Hurst exponent $3/4$.

This is an example of a process for which the fluctuations of the current and of the occupation time are not of the same type. For details on the proof of this result, we refer the reader to [4].

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A quenched central limit theorem for random walks in random sceneries in two dimensions

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(joint work with Nadine Guillotin-Plantard and Julien Poisat)

Random walks in random sceneries (RWRS) are simple models of processes in disordered media. They were introduced at the end of the 70's independently by Kesten–Spitzer and Borodin, with the motivation of constructing new self-similar processes with stationary increments. They are defined from two sources of randomness: a field $\xi = (\xi(x))_{x\in\mathbb{Z}^d}$ of i.i.d. random variables, called the *random scenery*, and a random walk $S = (S_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}^d$, independent of the scenery. The RWRS $Z = (Z_n)_{n\in\mathbb{N}}$ is then defined as the scenery accumulated along the trajectory of the random walk, that is, $Z_n = \xi(S_1) + \ldots + \xi(S_n)$. The law of Z under the joint law $\mathbb{P}$ of ξ and S is called the *annealed* law, while its conditional law given ξ, i.e., under $\mathbb{P}[\,\cdot\,|\xi]$, is called the *quenched* law.

To describe the limit theorems under the annealed law, assume that the random walk increment and the scenery at the origin are taken in the normal domain of attraction of stable laws of indices a and b, respectively. In the case $d = 1 > a$, Kesten and Spitzer showed in [9] that the process $\left(n^{-\delta} Z_{\lfloor nt\rfloor}\right)_{t\geq 0}$, where $\delta = 1 - a^{-1} + (ab)^{-1}$, converges weakly under the annealed law to a self-similar process with stationary increments that is not stable. Later on, Bolthausen proved in [2] that, for $d = 2 = a = b$, a functional CLT holds for $(\sqrt{n\log n}^{-1} Z_{\lfloor nt\rfloor})_{t\geq 0}$, and his result also covers the case $d = 1 = a$, $b = 2$. The case $d = a \in \{1, 2\}$, $0 < b < 2$ was considered by Castell, Guillotin-Plantard and Pène in [6], who showed that in this case, the process rescaled by $n^{1/b}(\log n)^{1-1/b}$ converges to stable process of index b. The transient case, i.e., $a > d$, had already been considered by Borodin in [5]; rescaling by $n^{1/b}$ one obtains also as a limit a stable process of index b.

Limit theorems under the quenched law are more recent. One of the first results in this direction was obtained by Ben Arous and Černý in [1] for the Bouchaud trap model in dimensions $d \geq 2$. Recently, Guillotin-Plantard and Poisat proved in [7] in the case $b = 2$ that the functional central limit theorem holds under the quenched law for a class of transient random walks, including walks with finite covariance matrices in dimensions $d \geq 3$. In dimension $d = 2$, they were only

able to prove convergence along a subsequence, raising the question of whether the convergence takes place along the full subsequence.

Together with Nadine Guillotin-Plantard and Julien Poisat, we give in [8] a positive answer to this question when the scenery at the origin has slightly more than a second moment. More precisely, under the assumptions

(A1) The random walk increment S_1 has a finite non-singular covariance matrix Σ and is aperiodic in the sense of Spitzer [10];

(A2) $\mathbb{E}[\xi_0] = 0$, $\mathbb{E}[\xi_0^2] = 1$ and there exists a $\gamma > 0$ such that

$$\mathbb{E}\big[\xi_0^2(\log^+ |\xi_0|)^\gamma\big] \;<\; \infty,$$

where $\log^+ x := 0 \vee \log x$,

we are able to prove the following theorem:

Theorem 1. *If $d = 2$ and (A1)–(A2) hold, then, for $\mathbb{P}$-almost every ξ, the process*

$$\frac{Z_{\lfloor nt \rfloor}}{\sqrt{n \log n}}, \quad t \geq 0$$

converges weakly under $\mathbb{P}[\,\cdot\,|\xi]$ in the Skorohod topology to a Brownian motion with variance $\sigma^2 = (\pi\sqrt{\det \Sigma})^{-1}$.

The proof of the above theorem uses an approach developed by Bolthausen and Sznitman [3] in the context of random walks in random environments. The idea is to pass the known annealed functional CLT to the quenched law using concentration properties of the quenched expectations of Lipchitz functionals of the rescaled path. This concentration is in turn obtained via a martingale decomposition of the difference between the quenched and annealed expectations, and arises naturally as a consequence of the difference in the asymptotic growth of the so-called *self-intersection local time* of S and the *mutual intersection local time* of two independent copies of S, the former being much larger than the latter in dimensions $d \geq 2$.

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Occupation times in long-range asymmetric simple exclusion processes and KPZ exponents

Sunder Sethuraman

(joint work with Cedric Bernardin and Patricia Goncalves)

The main focus of the talk was to discuss the fluctuations of the occupation time at the origin with respect to asymmetric simple exclusion processes on $\mathbb{Z}^d$ when started from a Bernoulli invariant measure. When the jumps are short-range and $\rho = 1/2$, it is known that the fluctuations are superdiffusive in $d = 1, 2$ [3], [10]. Moreover, it can be inferred from the results in [1], [2], [6] that the exact variance order should be $t^{4/3}$ in $d = 1$ and $t(\log t)^{2/3}$ in $d = 2$. For the diffusivity, a related quantity, this has been already proved (cf. [4], [7], [11] and references therein). When the density $\rho \neq 1/2$, the fluctuations are in diffusive scale and convergence to Brownian motion is known [3], [8], [9].

Consider now long-range jumps, that is say when the jump law $p(x) = c\mathbb{1}\{x_i > 0, 1 \leq i \leq d\}\,|x|^{-d-\alpha}$ (more general asymmetries and relations between these asymmetries were also considered) for $\alpha > 0$ and density $\rho = 1/2$. We find in $d = 1$ a curious 'change-point' at $\alpha = 3/2$. That is, for $0 < \alpha \leq 3/2$, the variance orders (in Tauberian sense) are the same as under the associated symmetric process and equal t for $0 < \alpha < 1$ and $t^{2-1/\alpha}$ for $1 \leq \alpha \leq 3/2$. For $\alpha > 2$, we show that the variance order is the same as that under the short-range model. Also, for $3/2 < \alpha \leq 2$, we show superdiffusive behavior, that the variance is bounded below by $t^{1+(2\alpha)^{-1}}$.

One notices at $\alpha = 3/2$, the variance is $t^{4/3}$ and this is also the conjectured order when $\alpha > 2$. We propose therefore, since volatility should increase with α, that the variance is order $t^{4/3}$ for all $\alpha \geq 3/2$. Some preliminary 'explanation' of these results was given in terms of scaling limits of fluctuation fields of weakly-asymmetric long-range processes.

When $d = 2$, the variance is order t when $\alpha < 2$ and is order $t\log(\log(t))$ (in Tauberian sense) when $\alpha = 2$. When $\alpha > 2$, the variance is the same order as in the short-range case.

When $d \geq 3$, or when density $\rho \neq 1/2$ in $d = 1, 2$, the variance is always diffusive, that is order t no matter the α, as expected.

In addition, the scaling limits of occupation times in symmetric long-range systems were also explained. These, in $d = 1$ when $1 < \alpha < 2$, turn out to be fractional Brownian motions with Hurst parameter $H = 1 - (2\alpha)^{-1}$, and in all other cases and dimensions, the limits are Brownian motions. These results

complement those in symmetric finite-range models [5], [9]: In $d = 1$, the limit is fractional Brownian motion with $H = 3/4$ and Brownian motion in $d \geq 2$.

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Derivation of Euler and Nonlinear Sound Equations

József Fritz

Dedicated to the Memory of Hermann Rost

The Anharmonic Chain: It is perhaps the simplest microscopic model of one-dimensional elasticity. The Hamiltonian of coupled oscillators of unit mass on $\mathbb{Z}$ reads as

$$H(\omega) \;:=\; \sum_{k\in\mathbb{Z}} H_k(\omega)\,, \qquad H_k(\omega) \;:=\; \frac{p_k^2}{2} \;+\; V(q_{k+1}-q_k)\,,$$

where $\omega = \{(p_k, q_k) : k \in \mathbb{Z}\}$, $p_k, q_k \in \mathbb{R}$ denotes a configuration of the infinite system. In terms of the deformation (strain) variables $r_k := q_{k+1} - q_k$ the equations of motion read as

$$\dot{p_k} \;=\; V'(r_k) \;-\; V'(r_{k-1}) \qquad \text{and} \qquad \dot{r_k} \;=\; p_{k+1} \;-\; p_k \quad \text{for } k \in \mathbb{Z}\,;$$

in this formulation V needs not be symmetric. Total momentum $P := \sum p_k$, total deformation $R := \sum r_k$ and total energy H are preserved by the dynamics, thus we have a three-parameter family $\lambda_{\beta,\pi,\gamma}$ of translation invariant stationary product

measures, where $\beta > 0$ is the inverse temperature. Under $\lambda_{\beta,\pi,\gamma}$ the Lebesgue density of any couple $(p_k, r_k) \sim (y, z)$ reads as $\exp(\gamma z - \beta I(y, z|\pi) - F(\beta, \gamma))$, where $I(y, z|\pi) := (y - \pi)^2/2 + V(z)$; the normalization F is sometimes referred to as the free energy. Hyperbolic scaling means that we are interested in the behavior of the empirical processes $\pi_\varepsilon(t, x) := p_k(t/\varepsilon)$, $\rho_\varepsilon(t, x) := r_k(t/\varepsilon)$ and $\chi_\varepsilon(t, x) := H_k(\omega(t/\varepsilon))$ if $|\varepsilon k - x| < \varepsilon/2$, as $0 < \varepsilon \to 0$. A formal calculation results in the triplet of compressible Euler equations:

$$\partial_t \pi \;=\; \partial_x J(\tilde{I}, \rho), \qquad \partial_t \rho \;=\; \partial_x \pi \qquad \text{and} \qquad \partial_t \chi \;=\; \partial_x (\pi J(\tilde{I}, \rho)),$$

where $\tilde{I} = \chi - \pi^2/2$ is the internal energy and $J(\tilde{I}, \rho) = \gamma/\beta$ denotes the equilibrium mean of energy flux. To obtain a complete picture on stationary states, which is a first requirement of a correct derivation of hydrodynamic equations, random perturbations of the evolution law should be introduced.

Random exchange of momenta: This weak noise preserves the classical conservation laws, and it is sufficient for the strong ergodic hypothesis claiming that every translation invariant stationary measure is a superposition of our product measures $\lambda_{\beta,\pi,\gamma}$, see [2]. Therefore the relative entropy argument of H.-T. Yau [1] implies the above set of Euler equations in a smooth regime.

Ginzburg - Landau perturbation: The stochastic dynamics

$$\begin{aligned} \mathrm{d}p_k \;&=\; \big(V'(r_k) - V'(r_{k-1})\big)\,\mathrm{d}t \;+\; \sigma\,\big(p_{k+1} + p_{k-1} - 2p_k\big)\,\mathrm{d}t \\ &\quad +\; \sqrt{2\sigma}\,\big(\mathrm{d}w_k - \mathrm{d}w_{k-1}\big), \\ \mathrm{d}r_k \;&=\; \big(p_{k+1} - p_k\big)\,\mathrm{d}t, \quad k \in \mathbb{Z}, \end{aligned}$$

where $\sigma > 0$ is fixed and $\{w_k : k \in \mathbb{Z}\}$ are independent Wiener processes, violates the law of energy conservation, thus $\lambda_{\pi,\gamma} := \lambda_{1,\pi,\gamma}$ are the stationary product measures. In this case we get convergence to classical solutions of the nonlinear sound equation of elastodynamics:

$$\partial_t u \;=\; \partial_x S'(v) \qquad \text{and} \qquad \partial_t v \;=\; \partial_x u,$$

where $S(v)$ denotes the convex conjugate of $F(\gamma) := \log \int \exp(\gamma x - V(x))\,\mathrm{d}x$.

Artificial Viscosity: In a regime of shock waves the randomness must be very strong:

$$\begin{aligned} \mathrm{d}p_k \;&=\; \big(V'(r_k) - V'(r_{k-1})\big)\,\mathrm{d}t \;+\; \sigma(\varepsilon)\,\big(p_{k+1} + p_{k-1} - 2p_k\big)\,\mathrm{d}t \\ &\quad +\; \sqrt{2\sigma(\varepsilon)}\,\big(\mathrm{d}w_k - \mathrm{d}w_{k-1}\big), \qquad k \in \mathbb{Z}, \\ \mathrm{d}r_k \;&=\; (p_{k+1} - p_k)\,\mathrm{d}t \;+\; \sigma(\varepsilon)\,\big(V'(r_{k+1}) + V'(r_{k-1}) - 2V'(r_k)\big)\,\mathrm{d}t \\ &\quad +\; \sqrt{2\sigma(\varepsilon)}\,\big(\mathrm{d}\tilde{w}_{k+1} - \mathrm{d}\tilde{w}_k\big), \qquad k \in \mathbb{Z}, \end{aligned}$$

where $\{w_k\}$ and $\{\tilde{w}_k\}$ are independent families of independent Wiener processes. The macroscopic viscosity: $\varepsilon\sigma(\varepsilon) \to 0$, but $\varepsilon\sigma^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$. Conservation laws and stationary states are as before, thus again the sound equation is expected as the result of the hyperbolic scaling limit.

Conditions on V : The substitution of the microscopic current V' by its equilibrium expectation S' is done by means of a logarithmic Sobolev inequality, thus V must be strictly convex. Moreover, the genuine nonlinearity of its flux is a condition for existence of weak solutions to the sound equation, that is $S'''(v) = 0$ can not have more that one root. In terms of V this follows from the same property of V'''. For instance, $V(r) := r^2/2 - \log \mathrm{ch}(\kappa r)$, but there are many other examples, too. A technical condition of asymptotic normality is also needed: $V''(x)$ converges at an exponential rate as $x \to \pm\infty$.

Main Result: Since we are not able to prove the uniqueness of the hydrodynamic limit, our only hypothesis on the initial distribution is that its specific entropy (relative to $\lambda_{0,0}$) is finite. Then the distributions P_ε of the empirical process $(u_\varepsilon, v_\varepsilon)$ form a tight family with respect to the weak topology of the C space of trajectories, and its limit distributions are all concentrated on a set of weak solutions to the sound equation. The notion of weak convergence above changes from step to step of the argument. We start with the weak compactness of the family of Young measures of the block - averaged empirical processes, finally we get tightness in the strong local $L^p(\mathbb{R}^2_+)$ topology if $p < 2$, see [5] for details.

Compensated compactness: This is the most relevant keyword of the proofs, it is based on the evaluation of Lax entropy pairs (h, J). Although we have $\partial_t h(u, v) + \partial_x J(u, v) = 0$ along classical solutions, a nontrivial h is not preserved by the microscopic evolution because it can not admit extra conservation laws. Therefore h exhibits a non-gradient behavior, which can be controlled by a clever logarithmic Sobolev inequality, at least if $\varepsilon\sigma^2(\varepsilon) \to 0$ as $\varepsilon \to 0$. Finally, Tartar's celebrated factorization identity implies the Dirac property of the limiting Young measure by means of the delicate, slowly increasing Lax entropy pairs of J.W. Shearer [3] and Serre - Shearer [4].

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Couplings and attractiveness for interacting particle systems

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(joint work with Lucie Fajfrová and Thierry Gobron)

The simple exclusion process ([19]) is one of the most studied interacting particle systems: In spite of its simplicity, it is a toy model for most of the panorama of nowaday's issues. It is described as follows. Each site of $\mathbb{Z}^d$ is endowed with a rate 1 exponential clock (all clocks being mutually independent). The state space is $\Omega = X^{\mathbb{Z}^d}$, with $X = \{0, 1\}$: There is at most one particle per site, $\eta_t(x) = 1$ if there is a particle on site $x \in \mathbb{Z}^d$ at time $t \geq 0$, and $\eta_t(x) = 0$ if x is empty at t. If at time t the clock at x rings and $\eta_{t^-}(x) = 1$, the particle present at x attempts a jump to a target site y according to a probability transition $p(y - x)$, and the jump is performed only if y is empty, i.e. $\eta_{t^-}(y) = 0$ (this is the exclusion rule).

Attractiveness is a crucial property of simple exclusion, proved through basic coupling: In the Markovian coupled process $(\xi_t, \zeta_t)_{t\geq 0}$, particles of (ξ_t) and (ζ_t) attempt jumps together as much as possible, that is they obey to the same clocks and choose the same target site. This coupling satisfies:

(A) if $\xi_0 \leq \zeta_0$ (coordinate-wise), then for all $t \geq 0$, $\xi_t \leq \zeta_t$ a.s.

Attractiveness is a key tool to determine the extremal invariant and translation invariant measures of the process, and its hydrodynamic behavior under Euler scaling.

Our question is: What about attractiveness for other processes than simple exclusion, how to check it, how to use it as a tool to derive for instance invariant measures or hydrodynamics?

Our works focus on multiple particle jump models, which are conservative particle systems of state space $\Omega = X^{\mathbb{Z}}$ with $X \subset \mathbb{Z}$ or $X \subset \mathbb{N}$.

If $X \subset \mathbb{N}$, $\eta_t(z)$ is the number of particles at site z at time t; if $X \subset \mathbb{Z}$, $|\eta_t(z)|$ is the number of unit charges at site z (either positive for $\eta_t(z) > 0$ or negative for $\eta_t(z) < 0$). For such models, in a transition at time t, for a configuration η_t, and an integer $k \geq 1$, with a rate $\Gamma^k_{\eta_{t^-}(x),\eta_{t^-}(y)}(y - x)$, k particles (respectively k positive charges) 'attempt a jump' from a site x to a site y, that is, the values at sites x and y are changed to $\eta_{t^-}(x) - k$ and $\eta_{t^-}(y) + k$ as long as the resulting configuration belongs to the state space. The conserved quantity is $\eta_t(x) + \eta_t(y)$, the total number of particles (resp. the total charge) involved in the transition.

If k takes only the value 1, we recover simple exclusion when $\Gamma^1_{1,0}(y - x) = p(y-x)\times 1$, zero range process ([1]) when $\Gamma^1_{\alpha,\beta}(y-x) = p(y-x)g(\alpha)$, misanthropes process ([10]) when $\Gamma^1_{\alpha,\beta}(y - x) = p(y - x)b(\alpha, \beta)$ and target process ([16]) when $\Gamma^1_{\alpha,\beta}(y - x) = p(y - x)\mathbb{1}_{\{\alpha\geq 1\}}b(\beta)$. Two examples with $k \geq 1$ are a stick model related to the Hammersley process ([13, 21]), and a two species exclusion model with charge conservation ([11]).

It turns out that, if for some rates $k > 1$, then the basic coupling construction is not possible any more for such multiple particle jump models when they are

attractive. In [15], we have derived necessary and sufficient conditions for attractiveness of such models. Under these conditions, we have constructed an increasing coupling. The latter permits to determine the extremal invariant and translation invariant measures for the dynamics, under an *irreducibility condition* on the coupled transition rates. This latter condition ensures that for the coupled process, discrepancies of opposite signs in two coupled configurations will merge (as for simple exclusion, this part is the main step of the proof); we have to take into account a new feature appearing in those models, the partial (or total) *exchange of discrepancies* in a coupled transition. Finally, all this setup enables to derive hydrodynamics under Euler scaling for attractive multiple particle jump models, relying on the method in [2, 3, 4, 5]; we have for that to ascertain the *macroscopic stability* property ([9, 20]) for those models.

We are now working ([14]) on a class of examples that we call *mass migration processes*, for which $\Gamma^k_{\alpha,\beta} = p(y-x)g^k_{\alpha,\beta}$. This includes mass migration zero range process for which $\Gamma^k_{\alpha,\beta}(y-x) = p(y-x)g^k_\alpha$ (see [17] in finite volume), and mass migration target process for which $\Gamma^k_{\alpha,\beta}(y-x) = p(y-x)\mathbf{1}_{[k\leq\alpha]}\, g^k_{*,\beta}$.

We look for product invariant measures for a mass migration process $(\eta_t)_{t\geq 0}$. We exhibit relations between the transition rates and the single site marginal of a product probability measure $\bar\mu$ which is invariant and translation invariant for $(\eta_t)_{t\geq 0}$. These relations are exploited in two ways: (1) given the rates, check that a measure $\bar\mu$ is invariant; (2) given $\bar\mu$, define rates for which $\bar\mu$ is invariant.

This correspondence gives new examples of models, mass migration zero range processes, for which condensation ([12, 18, 6]) occurs, since the latter property concerns only stationary distributions: Having a stationary state $\bar\mu$ which produces condensation we can study different rates which lead to this $\bar\mu$.

We have also begun to work on exclusion processes with speed change. Davide Borrello has derived similar methods for attractiveness and couplings for non-conservative dynamics in [7], which enabled him to study biological models in [8].

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