ARBEITSGEMEINSCHAFT ON SUPERRIGIDTY
MARCH 30 - APRIL 5, 2014

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1. General Plan

This is a (tentative) plan for the talks at the Spring 2014 Arbeitsgemeinschaft at Oberwolfach, March 30 - April 5, 2014. The general title of Superrigidity will span the following themes:

(S) Margulis-Zimmer superrigidity for higher rank Lie groups and their lattices.
(C) Character superrigidity.
(D) Deformation/rigidity techniques.
(A) From superrigidity to Arithemeticity of lattices.

Some background notions and ideas that are common to these and will appear throughout the talks include:

(B) Amenability of groups and actions, boundaries for groups.
(B) Kazhdan's property (T), cohomology of unitary representations.
(B) Measured equivalence relations, von Neumann algebras, traces and characters.
(B) Howe-Moore property, commensurators.
(B) Measurable cocycles.

The list of tentative talks and their content is given below. We hope to have more detailed discussions on the blog at

http://aboutrigidity.wordpress.com

Tentative titles of the talks.

1. Backgound talks.

(B1) Amenability and boundaries
(B2) Property (T) and cohomological formulations
(B3) Von Neumann algebras
(B4) Measured equivalence relations

2. Character superrigidity.

(C1) Thompson's groups
(C2) Special linear groups
(C3) Commensurators
(C4) Commensurators

3. Higher rank superrigidity and Normal Subgroup Theorems.
   - (S1) Algebraic representations of
   - (S2) Bi-actions and superrigidity for lattices in $\mathrm{SL}_3(k)$.
   - (S3) Super-rigidity.
   - (S4) Irreducible Products (Bader-Shalom)
4. Deformation/rigidity.
   - (D1) Cocycle superrigidity for (T) over Bernoulli and compact actions
   - (D2) Applications to equivalence relations
   - (D3) Unique group-measure space decomposition
   - (D4) Unique group-measure space decomposition
5. Arithmeticity and commensurators.
   - (A1) Arithmeticity from superrigidity
   - (A2) Commensurator superrigidity
   - (A3) More details

2. Details

Amenability of groups and group actions (B1).

Goal: Discuss fixed point property of amenable groups, and define amenable actions. Show that if $\Gamma < G$ is a lattice, $\Gamma \curvearrowright M$ action on a compact metrizable space, then there is a meaurable Γ-map $\phi : G/P \to \mathrm{Prob}(M)$.

Suggested plan: Amenability of Abelian and compact groups, amenability is closed under extensions; deduce amenability of compact extensions of soluable groups. Deduce amenability of a lattice Γ acting on G/P where P is a compact extension of soluable; or at least the existance of a Γ-map ϕ as above.

Property (T) and cohomological formulations (B2).

Goal and plan: Define proeprty (T), give the main classes of examples, discuss property (FH), and the equivalence, time permitting (FH) $\implies$ (T).

Sources: for example Bekka, Valette, de la Harpe [6].

Von Neumann Algberas (B3).

Goal: A brief overview of von Neumann algebras and completely positive maps; including a discussion on injective von Neumann algebras, as well as a discussion of finite von Neumann algebras and the introduction of fundamental tools from Popa's deformation/rigidity theory.

Suggested plan: Given the amount of material to cover, this should be more of a survey lecture, presenting details of proofs only as time permits, and with emphasis placed on the more recent results.

General von Neumann algebras [37]: Von Neumann's double commutant theorem, states, traces, types of factors.

Completely positive maps (pp. 9-13 from [7]): Stinespring's representation theorem and consequences, conditional expectations, injective von Neumann algebras, Umegaki's theorem on conditional expectations on finite von Neumann algebras.

Finite von Neumann algebras: Standard representation, correspondences ([27], or Sections 1.1, and 1.2 from [28]), amenability and property (T) (Section 2 from [9], and [10]), Popa's intertwining theorem for Cartan subalgebras (Appendix C from [34], Appendix F from [7], or pp. 17-22 from [17]).

Measured equivalence relations (B4).

Goal: Introduction of von Neumann algebras and equivalence relations associated with group actions. A discussion of properties which are preserved under these constructions.

Suggested plan: Group-measure space construction, von Neumann's type classification (Proposition 8.6.4 from [19]), Orbit equivalence relations, full groups (e.g., Section 2 in [23]), Dye and Feldman-Moore's theorems on isomorphic equivalence relations (Theorem 1.2 in [17]), Zimmer's theorem on amenable actions and injective factors (Theorems 2.1 and 5.1 in [38]).

Thompson's groups (C1).

Goal: Give an introduction to characters on countable groups and present Dudko and Medynets' theorem classifying characters on Thompson's group F.

Suggested plan: The correspondence between extremal characters and factor representations (Theorem 5.1.10 in [25]). Character rigidity for the commutator subgroup of Thompson's group (Corollary 3.3 in [13]). Applications to non-free actions (Theorem 2.11 in [13] or Theorem 3.2 in [26]).

Special linear groups (C2).

Goal: Present the character rigidity results for special linear groups from [5] and [26].

Suggested plan: Character rigidity for $PSL_2(k)$ for k an infinite field. Discussion of how the proof changes for rings of integers (and their localizations) with infinitely many units (Theorem 2.6 in [26]). Comparison to Bekka's proof for $PSL_3(\mathbb{Z})$ [5]. Selected applications (e.g., Corollary 4.3 or Theorem 5.2 in [26]) depending on time.

Commensurators (C3), (C4).

Goal: To present the character rigidity theorem for an irreducible lattice $\Lambda < G \times H$, where G is a simple real Lie group with property (T), and trivial center, and when H is a product of p-adic Lie groups (Theorem D from [12]).

Suggested plan: Lecture 1: Probability measure-preserving action of G are free when restricted to Λ (Theorem 7.1 in [11] or Lemma 6 in [41]). Trace preserving actions of G are properly outer when restricted to Λ (Proposition 4.1 in [12]). Setting $\Gamma = \Lambda \cap (G \times U)$, proof of the theorem in the case when $\pi(\Gamma)$ is SOT-precompact (Propositions 6.1 and 6.2 in [12]). Proof of the theorem when $\pi(\Gamma)''$ is injective ([10], or Theorem B in [31]).

Lecture 2: From the first lecture what remains to be shown is that if $\pi : \Lambda \to \mathcal{U}(M)$ is a finite factor representation which is not the left-regular representation, then $N = \pi(\Gamma)''$ must be injective.

Let $G \curvearrowright (B, \eta)$ be the Poisson boundary of G, then

$$\mathcal{B} = \{\sigma^0_\gamma \otimes (J\pi_\gamma J) \mid \gamma \in \Gamma\}' \cap (L^\infty(B,\eta)\overline{\otimes}N)$$

is injective [38] and contains N, thus it is enough to show $\mathcal{B} = N$. (This is all background material, most of which will have been presented in earlier lectures).

Show that there are no non-trivial normal N-bimodular completely positive maps on $\mathcal{B}$ (Theorem 3.2 in [12]). Show that

$$\mathcal{B} = \{\sigma^0_\lambda \otimes (JE_N(\pi_\lambda)J) \mid \gamma \in \Gamma\}' \cap (L^\infty(B,\eta)\overline{\otimes}N)$$

(Proposition 5.1 in [12]). Show then that $\mathcal{B} = N$ (Theorem 4.4 in [12]).

Algebraic representations of ergodic actions (S1).

Goal: Present and discuss the notion of algebraic representations as defined in [4]. Prove Proposition 5.6 and Theorem 7.3.

Suggested plan: Discuss smoothness of algebraic group actions on algebraic varieties and spaces of measures (only for local fields). Define algebraic representations of T-ergodic action given a homomorphism $T \to G$ - don't touch cocycles! Follow §5 and §7 of [4] (convert all arguments and notations from a cocycle version to the simpler group representation version). Prove [4, Theorem 7.1] in a direct way: use the fact that G/P is compact to obtain a map $Y \to \mathrm{Prob}(G/P)$, use smoothness of the G action and amenability of stabilizers on this space.

Higher rank lattices super-rigidity (S2,S3).

Goal: Prove Margulis super-rigidity for lattices in $\mathrm{SL}_3(\mathbb{R})$ following [4].

Suggested plan: Follow §9 of [4], but not in a general setting: focus on $Y = \mathrm{SL}_3(\mathbb{R})$ (as a measure space, endowed with the Haar measure), S being a lattice acting on the left, T being a unipotent subgroup (one of the standard six) acting on the right. Also replace the cocycle c by a (Zariski dense) representation $S \to G$ (G simple over a local field).

Use the startegy of the proof of theorem 10.1 to prove Margulis super-rigidity in this case.

Lattices in product of groups (S4).

Goal: Discuss and indicate the proof of normal subgroup theorem for irreducible lattices in products.

Suggested plan: State Margulis normal subgroup theorem and show how it follows from Margulis factor theorem and property (T) of higher rank groups, as explained for example in [40]. State [3, Theorem 1.1] and explain that it follows similarly from Theorem 1.9. Outline the proof of theorem 1.9.

Deformation/Rigidity methods for cocycles: lectures (D1) (D2).

Goal: to illustrate Popa's deformation vs. rigidity technique by proving cocycle superrigidity results of Popa [29] for Bernoulli actions, and Ioana [18] for profinite or descrete spectrum actions, following [14] and [15], and show basic applications to Orbit Equivalene rigidity.

Suggested plan: After statement of the cocyle superrigidity results (simplest cases of Bernoulli $\Gamma \curvearrowright [0,1]^{\Gamma}$ and $\Gamma \curvearrowright K$ for some dense imbedding $\Gamma \to K$ in a compact group) prove local rigidity for cocycles [14, Prop 5.14] followed by [14, Prop 5.16]. Time permitting prove some consequences, e.g. orbit equivalence superrigidity (some version of [15, Theorem 1.8]), and/or equivalence relation that cannot be generated by essentially free action of any group (e.g. follow [14, Section 4.3.1]).

Unique group-measure space decomposition (D3), (D4).

Goal: The papers [8,24,30,35] develop techniques to show that for a large class of groups, and for arbitrary free ergodic measure-preserving actions, the associated von Neumann crossed product has a unique (up to unitary conjugacy) group-measure space decomposition. The goal of these lectures is to prove this in the special, yet illustrative, case when Γ has an infinite subgroup with relative property (T), and yet has non-trivial cohomology $H^1(\Gamma, \pi) \neq 0$ for a mixing representation $\pi : \Gamma \to \mathcal{H}$.

Suggested plan: Lecture 1: Gaussian actions, the exponential of cocycles [32], and the associated s-malleable deformation [33]. Mixing properties (Lemmas 4.1 and 4.2 in [35]). Reduction of the problem to showing uniform convergence for the mystery Cartan subalgebra (Section 3.6 in [35]).

Lecture 2: Transfer of rigidity ([30], or Proposition 5.1 in [35]). From uniform convergence on the mystery "rigid subset" to uniform convergence on the mystery Cartan subalgebra (Theorem 4.1 in [35]).

Arithmeticity from Superrigidty (A1).

Goal: What is an arithmetic lattice? Show that superrigidity (with $\mathbf{C}$- and $\mathbf{Q}_p$- targets) implies arithmeticity. Time permitting state Margulis arithmeticity criterion.

Sources: Section 1 of the original paper of Margulis [20] (see also [21]), Witte Morris [36, Section 13C].

Commensurator Superrigidty (A2).

Goal: Discuss Margulis' Commensurator Superrigidity and how it implies arithmeticity criterion. Prove the CS assuming uniqueness of boundary maps.

Suggested plan: [1].

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Oberwolfach Arbeitsgemeinschaft:
Superrigidity

Date:

30 Mar - 5 Apr 2014 (ID: 1414)

Organizers:

Uri Bader, Haifa
Alex Furman, Chicago
Jesse Peterson, Nashville

Application for Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program. If you intend to participate, please send your full name and full postal address to

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by **10 February 2014** at the latest.

You should also indicate which talk you are willing to give:

- **First choice: talk no. X**
- **Second choice: talk no. Y**
- **Third choice: talk no. Z**

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures. The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Schwarzwaldstrasse 9-11, 77709 Oberwolfach-Walke, Germany.

The Institute covers board and lodging. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.