

MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Report No. 25/2014

DOI: 10.4171/OWR/2014/25

## Interactions between Algebraic Geometry and Noncommutative Algebra

Organised by

Markus Reineke, Wuppertal

J. Toby Stafford, Manchester

Catharina Stroppel, Bonn

Michel Van den Bergh, Diepenbeek

18 May – 24 May 2014

ABSTRACT. The workshop presented the current developments in the field of noncommutative algebra geometry and its interactions with algebraic geometry and representation theory.

*Mathematics Subject Classification (2010):* 14A22, 16K20, 16S38, 16P90.

### Introduction by the Organisers

This meeting had 53 participants from 12 countries (Belgium, Canada, Czechoslovakia, China, France, Germany, Japan, Norway, Russia, Switzerland, UK and the US) and 24 lectures were presented during the five day period. The sponsorship of the European Union and other organizations allowed the organizers to invite and secure the participation of a number of young investigators, some of whom presented thirty-minute lectures. As always, there was a substantial amount of mathematical activity outside the formal lecture sessions. This meeting explored the applications of ideas and techniques from algebraic geometry to noncommutative algebra and vice-versa. A number of lectures presented open problems. Areas covered include

- noncommutative projective algebraic geometry,
- (quantized) quiver varieties/quiver representations,
- deformation theory,
- representation theory of Cherednik and related Hecke algebras
- $\mathcal{D}$ -module theory,

A number of advances in the above areas were presented and possible starting points for further research proposed. The breadth of the conference is illustrated by the abstracts.

*Acknowledgement:* The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, “US Junior Oberwolfach Fellows”. Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Ragnar-Olaf Buchweitz and Thomas A. Nevins in the “Simons Visiting Professors” program at the MFO.

**Workshop: Interactions between Algebraic Geometry and Non-commutative Algebra****Table of Contents**

Jason Bell (joint with Stéphane Launois, Omar Leon-Sanchez, Rahim Moosa)	
<i>A counterexample to the Poisson Dixmier-Moeglin equivalence</i> . . . . .	1369
Gwyn Bellamy	
<i>Caloger-Moser Spaces and Grothendieck's Ring of Varieties</i> . . . . .	1369
Ragnar-Olaf Buchweitz	
<i>Geometric Tilting Objects</i> . . . . .	1370
Christopher Dodd	
<i>Construction of <math>\mathcal{D}</math>-Modules with prescribed <math>p</math>-support</i> . . . . .	1370
Hans Franzen	
<i>Cohomology of Non-Commutative Hilbert Schemes as Modules over the Cohomological Hall Algebra</i> . . . . .	1371
K. R. Goodearl (joint with M.T. Yakimov)	
<i>Quantum cluster algebras and quantum nilpotent algebras</i> . . . . .	1374
Henning Krause	
<i>Highest weight and monoidal structure for strict polynomial functors</i> . . .	1374
Birge Huisgen-Zimmermann	
<i>Irreducible components of varieties of representations</i> . . . . .	1375
Dmitry Kaledin	
<i>Cyclotomic Mackey Functors</i> . . . . .	1376
Ivan Losev	
<i>Quantized Gieseker moduli spaces</i> . . . . .	1376
Wendy Lowen	
<i>Hochschild cohomology with support</i> . . . . .	1377
Andrew Morrison (joint with Brent Pym and Balázs Szendrői)	
<i>Motivic Donaldson–Thomas invariants of quantum <math>\mathbb{C}^3</math></i> . . . . .	1382
Sergey Mozgovoy (joint with Markus Reineke)	
<i>Higgs Bundles and Quiver Representations</i> . . . . .	1384
Thomas Nevins (joint with Gwyn Bellamy, Chris Dodd, Kevin McGerty)	
<i>Morse theory of <math>\mathcal{D}</math>-module categories</i> . . . . .	1385
François Petit	
<i>The codimension-three conjecture for holonomic DQ-modules</i> . . . . .	1385

Daniel Rogalski (joint with Manuel Reyes, James Zhang)	
<i>Frobenius Ext-algebras arising from Artin-Schelter Gorenstein Algebras</i>	.1387
Antonio Sartori	
<i>Categorification of <math>\mathfrak{gl}(1 1)</math>-representations</i>	.....1388
Travis Schedler (joint with Pavel Etingof)	
<i>Poisson and Hochschild-de Rham Homology and Symplectic Resolutions</i>	1389
Olaf M. Schnürer (joint with Valery A. Lunts)	
<i>Some enhancements of derived categories of coherent sheaves and applications</i>	.....1389
Susan J. Sierra (joint with Dan Rogalski, Toby Stafford)	
<i>Ring-theoretic blowing down</i>	.....1391
Adam-Christiaan van Roosmalen (joint with Donald Stanley)	
<i>Serre functors and derived equivalences for hereditary Artin algebras</i>	...1393
Chelsea Walton	
<i>On Quantum Symmetry</i>	.....1395
Ben Webster	
<i>Category <math>\mathcal{O}</math> and KLR algebras</i>	.....1396

## Abstracts

### A counterexample to the Poisson Dixmier-Moeglin equivalence

JASON BELL

(joint work with Stéphane Launois, Omar Leon-Sanchez, Rahim Moosa)

Let  $A$  be a finitely generated complex algebra that is equipped with a Poisson bracket  $\{, , , \}: A \times A \rightarrow A$ ; that is,  $\{, , , \}$  is a Lie bracket and for every  $a \in A$ , we have  $\{a, -\}$  is a derivation of  $A$ . An ideal  $I$  of  $A$  is called a *Poisson ideal* if  $\{I, A\} \subseteq I$ . Then we say that  $A$  is Poisson algebra. Brown and Gordon asked whether the Poisson Dixmier-Moeglin equivalence holds for prime Poisson ideals in any complex affine Poisson algebra; that is, whether the sets of Poisson rational ideals, Poisson primitive ideals, and Poisson locally closed ideals coincide. Here, we recall that a prime ideal  $P$  is Poisson locally closed if it is locally closed in the space of Poisson prime ideals (with topology given by the induced topology by the Zariski topology);  $P$  is Poisson rational if the induced Poisson bracket on the field of fractions of  $A/P$  has the property that if  $f \in \text{Frac}(A/P)$  has the property that  $\{f, x\} = 0$  for all  $x \in A/P$  then  $f$  is in  $\mathbb{C}$ ; finally  $P$  is Poisson primitive if there is a maximal ideal  $M$  that contains  $P$  such that  $P$  is the largest Poisson ideal contained in  $M$ .

We give a complete answer is given to this question using techniques from differential-algebraic geometry and model theory. In particular, it is shown that while the sets of Poisson rational and Poisson primitive ideals coincide, in every Krull dimension at least four there are complex affine Poisson algebras with Poisson rational ideals that are not Poisson locally closed. A weaker version of the Poisson Dixmier-Moeglin equivalence is proven for all complex affine Poisson algebras, from which it follows that the full equivalence holds in Krull dimension three or less. Finally, we show that everything, except possibly that rationality implies primitivity, can be done over an arbitrary base field of characteristic zero.

#### REFERENCES

- [1] J. Bell, S. Launois, O. Leon-Sanchez, R. Moosa, *Poisson algebras via model theory and differential algebra*. [arxiv.org.1406.0867](https://arxiv.org/abs/1406.0867).

### Caloger-Moser Spaces and Grothendieck's Ring of Varieties

GWYN BELLAMY

Associated to a smooth curve  $C$ , finite group  $G$  of automorphisms of  $C$ , and a positive integer  $n$  is the *Calogero-Moser deformation*  $\text{CM}_n(C, G) \rightarrow \mathfrak{c}_n$ . It is a flat Poisson deformation of  $T^*(C^n/\mathfrak{S}_n \wr G)$  over the affine base  $\mathfrak{c}_n$ , where  $\mathfrak{S}_n$  is the symmetric group. The deformation can be constructed as a certain coarse moduli space. We would like to understand these spaces better. A first step in doing so is

to consider, for each  $\mathbf{c} \in \mathfrak{c}_n$ , the class of the fiber  $\mathrm{CM}_{n,\mathbf{c}}(C, G)$  in Grothendieck's ring of varieties  $K_0(\mathrm{Var}_{\mathbb{C}})$ . We show that, for  $\mathbf{c}$  Weil generic,

$$\sum_{n \geq 0} [\mathrm{CM}_{n,\mathbf{c}}(C, G)] t^n = \left( \prod_{n \geq 1} \frac{1}{1 - \mathbb{L}^n t^n} \right)^{[C/G]}$$

where  $\mathbb{L}$  is the Lefschetz motive. This implies that  $[\mathrm{CM}_{n,\mathbf{c}}(C, G)]$  equals the class of the Hilbert scheme  $\mathrm{Hilb}^n(T^*(C/G))$  of  $n$  points in  $T^*(C/G)$  in  $K_0(\mathrm{Var}_{\mathbb{C}})$ . For arbitrary parameters  $\mathbf{c}$ , we present a conjectural formula for the power series  $\sum_{n \geq 0} [\mathrm{CM}_{n,\mathbf{c}}(C, G)] t^n$  in terms of certain generalized characters of higher level Fock spaces for the quantum group  $U_q(\widehat{\mathfrak{sl}}_{\infty})$ . This talk is based on joint work in progress with Oleg Chalykh.

### Geometric Tilting Objects

RAGNAR-OLAF BUCHWEITZ

After reviewing the definitions and structure theorem for  $d$ -representation infinite and  $(d + 1)$ -higher preprojective algebras (due to Keller, Herschend-Iyama-Oppermann, Amiot-Iyama-Reiten, Minamoto-Mori) we discussed the notion of geometric objects in *Ext*-finite triangulated categories with Serre functor starting from a suggestion by Bondal.

We say that an object  $F$  in such a category has Serre dimension  $d$  if there exists an  $a \geq 0$  such that for each  $i \in \mathbb{Z}$ ,

$$\mathrm{Ext}^j(S_d^i F, F) = 0 \text{ for } j \notin [-a, a + d]$$

where  $S_d := S \circ [-d]$  is the translated Serre functor. The object is sheaf-like if one can take  $a = 0$ . If  $T$  is a tilting object with endomorphism algebra  $\Lambda$ , then  $T$  has Serre dimension  $d = \mathrm{gldim}(\Lambda)$  and is sheaf-like if and only if  $\Lambda$  is  $d$ -representation infinite and then the associated  $(d + 1)$ -preprojective algebra is  $\bigoplus_{i \geq 0} \mathrm{Hom}(S_d^i T, T)$  which one can interpret as the pullback of  $T$  to the (virtual) canonical bundle still being a tilting object. If the  $(d + 1)$ -preprojective algebra is graded coherent (as is conjectured to always hold), then the work of Miramoto implies that  $D^b(\Lambda)$  carries a geometric  $t$ -structure of dimension  $d$  and thus, satisfies Bondal's requirement

### Construction of $\mathcal{D}$ -Modules with prescribed $p$ -support

CHRISTOPHER DODD

In this talk, I discussed the construction of (a part of) the *quantization correspondence* for algebraic  $\mathcal{D}$ -modules motivated by conjectures of Kontsevich and Belov-Kanel. We explained how, to any smooth Lagrangian  $\mathcal{L} \subset T^*X$  (where  $X$  is a smooth complex affine variety) with  $H^1(\mathcal{L}, \mathbb{C}) = 0$ , one can associate in a natural way a holonomic  $\mathcal{D}_X$ -module  $\mathcal{M}$ . This  $\mathcal{D}_X$ -module is irreducible and it has the property that after reduction mod  $p$  its " $p$ -support" is equal to  $\mathcal{L}_k^{(1)} \subset T^*X_k^{(1)}$  where  $k$  is a perfect field of positive characteristic for all  $p \gg 0$ . If  $H^1(\mathcal{L}, \mathbb{Z}) = 0$ , this

module is unique. As a consequence, we explained that this implies the existence of an isomorphism

$$\text{Aut}_{\text{Mor}}(\mathcal{D}_n) \xrightarrow{\sim} \text{Aut}_{\text{Sym}}(T^*\mathbb{A}^n)$$

which is a weak version of a very surprising conjecture if Kotsevich and Belov-Kanel

## Cohomology of Non-Commutative Hilbert Schemes as Modules over the Cohomological Hall Algebra

HANS FRANZEN

### 1. NON-COMMUTATIVE HILBERT SCHEMES

Non-commutative Hilbert schemes generalize classical Hilbert schemes in the following way: The Hilbert scheme  $\text{Hilb}_d(\mathbb{A}^m)$  is a variety whose (complex valued) points parametrize ideals  $I$  of the polynomial algebra  $\mathbb{C}[x_1, \dots, x_m]$  such that  $\mathbb{C}[x_1, \dots, x_m]/I$  has dimension  $d$  as a complex vector space.

In contrast, the *non-commutative* Hilbert scheme  $\text{Hilb}_d^{(m)}$  is a variety whose points are in bijection to *left* ideals  $I$  of the free non-commutative algebra  $A := \mathbb{C}\langle x_1, \dots, x_m \rangle$  in  $m$  variables having codimension  $d$  (as a complex vector space).

A result of Nakajima [4] asserts that the direct sum  $\bigoplus_d H^*(\text{Hilb}_d(\mathbb{A}^2))$  of the (singular) cohomology groups (with rational coefficients) carries the structure of a representation of the infinite-dimensional Heisenberg algebra; in fact, it is isomorphic to the bosonic Fock space. Using this structure, Lehn and Sorger (cf. [3]) were able to identify the cohomology *rings*  $H^*(\text{Hilb}_d(\mathbb{A}^2))$  with the ring of class functions of the symmetric group  $S_d$ . We would like to find a suitable analog of these results for the cohomology of the non-commutative Hilbert schemes  $\text{Hilb}_d^{(m)}$ .

The variety  $\text{Hilb}_d^{(m)}$  can be constructed using geometric invariant theory. Let

$$\hat{R}_d := \mathbb{C}^d \times R_d$$

where  $R_d := M_{d \times d}(\mathbb{C})^m$ . The vector space  $\hat{R}_d$  is equipped with an action of the general linear group  $\text{Gl}_d$  by left multiplication on the vector and simultaneous conjugation on the matrices. We take the stability condition induced by the determinantal character of  $\text{Gl}_d$ . This means that a point  $(v, \phi_*)$  is (semi-)stable if and only if  $v$  generates the  $d$ -dimensional left- $A$ -module which arises by the  $m$ -tuple of matrices  $\phi_*$ . The geometric quotient

$$\hat{R}_d^{\text{st}} / \text{Gl}_d,$$

which is even a principal  $\text{Gl}_d$ -fiber bundle, is isomorphic to  $\text{Hilb}_d^{(m)}$  by identifying the orbit of a stable point  $(v, \phi)$  with the left-ideal  $\text{Ann}_A(v)$ . Note that  $\text{Hilb}_d^{(m)}$  is a smooth variety.

## 2. COHOMOLOGICAL HALL ALGEBRA

The Cohomological Hall algebra, which we will call CoHa in the following, was invented by Kontsevich and Soibelman (cf. [2]). It was constructed in order to provide a categorification of (motivic) Donaldson-Thomas invariants. Their construction works in vast generality; we are considering the CoHa  $\mathcal{H}$  of the  $m$ -loop quiver (and trivial potential). As a graded vector space, it is defined as direct sum

$$\mathcal{H} = \bigoplus_{d \geq 0} H_{\mathrm{Gl}_d}^*(R_d)$$

of equivariant singular cohomology groups with rational coefficients. The multiplication is induced by suitable ‘‘Hecke correspondences’’

$$R_p \times R_q \leftarrow R_{p,q} \rightarrow R_{p+q},$$

where  $R_{p,q}$  is the vector space of  $m$ -tuples of  $(d \times d)$ -matrices having a block upper triangular shape  $\begin{pmatrix} * & * \\ & * \end{pmatrix}$ . Kontsevich and Soibelman give an algebraic description of the thus obtained multiplication by identifying

$$H_{\mathrm{Gl}_d}^*(R_d) \cong H_{\mathrm{Gl}_d}^*(\mathrm{pt}) \cong H_{T_d}^*(\mathrm{pt})^{S_d} \cong \mathrm{Sym}(X(T_d))^{S_d} \cong \mathbb{Q}[x_1, \dots, x_d]^{S_d}.$$

Using Bott localization, they prove:

**Theorem 1** (Kontsevich-Soibelman). *For  $f \in \mathcal{H}_p$  and  $g \in \mathcal{H}_q$ , the product  $f * g$ , regarded as a symmetric polynomial in  $d = p + q$  variables, equals*

$$\sum_{\substack{1 \leq i_1 < \dots < i_p \leq d \\ 1 \leq j_1 < \dots < j_q \leq d \\ \text{complementary}}} f(x_{i_1}, \dots, x_{i_p}) g(x_{j_1}, \dots, x_{j_q}) \prod_{\mu=1}^p \prod_{\nu=1}^q (x_{j_\nu} - x_{i_\mu})^{m-1}.$$

## 3. A COHA-MODULE

We are going to define a left- $\mathcal{H}$ -module structure on the direct sum

$$\bigoplus_{d \geq 0} H^*(\mathrm{Hilb}_d^{(m)})$$

using the fact that  $H^*(\mathrm{Hilb}_d^{(m)}) = H_{\mathrm{Gl}_d}^*(\hat{R}_d^{\mathrm{st}})$ . We work with a modified version of the Hecke correspondences

$$R_p \times \hat{R}_q^{\mathrm{st}} \leftarrow \hat{R}_{p,q}^{\mathrm{st}} \rightarrow \hat{R}_{p+q}^{\mathrm{st}},$$

where  $\hat{R}_{p,q}^{\mathrm{st}}$  is by definition  $(\mathbb{C}^d \times R_{p,q}) \cap \hat{R}_d^{\mathrm{st}}$ . These types of CoHa-modules arising from stable framed objects are also considered in [6] in greater generality. The equivariant maps  $R_d \leftarrow \hat{R}_d \leftrightarrow \hat{R}_d^{\mathrm{st}}$  induce a map

$$j^* : \mathcal{H} \rightarrow \bigoplus_{d \geq 0} H^*(\mathrm{Hilb}_d^{(m)})$$

which can easily be seen to be  $\mathcal{H}$ -linear (with  $\mathcal{H}$  considered as a left-module over itself) and surjective. In [1], we show:

**Theorem 2.** *The kernel of  $j^*$  equals*

$$\bigoplus_{p \geq 0, q > 0} \mathcal{H}_p * (e_q \cup \mathcal{H}_q),$$

where  $*$  is the CoHa-multiplication,  $e_q(x_1, \dots, x_q) = x_1 \dots x_q$ , and  $\cup$  is the cup product on  $\mathcal{H}_q = H_{\text{Gl}_q}^*(R_q)$  which identifies with the multiplication of symmetric polynomials.

The proof of the above theorem relies on Harder-Narasimhan methods and computations with fiber bundles. It can be generalized to modules over the CoHa of an arbitrary quiver arising from arbitrary framing data. As a consequence of Theorem 2, we obtain a description of the cohomology ring  $H^*(\text{Hilb}_d^{(m)})$  in terms of generators and relations which can be regarded as an analog of Lehn-Sorger’s result. Choose a basis of  $\mathcal{H}_p \otimes \mathcal{H}_q$  over  $\mathcal{H}_d$  whose elements we may assume to be of the form  $f_{\lambda,p} \otimes g_{\lambda,q}$ .

**Corollary 3.** *The kernel of the map  $j^* : \mathbb{Q}[e_1, \dots, e_d] \rightarrow H^*(\text{Hilb}_d^{(m)})$  is generated by expressions of the form*

$$f_{\lambda,p} * (e_q \cup g_{\lambda,q})$$

with  $p = 0, \dots, p - 1$ ,  $q = d - p$  and  $\lambda = 1, \dots, \binom{d}{p}$ .

The non-commutative Hilbert scheme  $\text{Hilb}_d^{(m)}$  is a fine moduli space which means it possesses a universal bundle  $\mathcal{U}$  of rank  $d$ . Under the map  $j^*$ , the generator  $e_i$  is sent to  $c_i(\mathcal{U})$ . Using a cell decomposition of Reineke (cf. [5]), we obtain a monomial basis in Chern classes of  $\mathcal{U}$  for the cohomology with integral coefficients.

**Proposition 4.** *The cohomology  $H^*(\text{Hilb}_d^{(m)}; \mathbb{Z})$  with integral coefficients is a free abelian group with a basis given by the monomials*

$$c_1(\mathcal{U})^{b_{d-1}} \dots c_{d-1}(\mathcal{U})^{b_1}$$

where  $b = (b_1, \dots, b_d)$  is a tuple of non-negative integers satisfying  $b_1 + \dots + b_i \leq (m - 1)d$  for every  $i = 1, \dots, d - 1$ .

#### REFERENCES

- [1] H. Franzen, *On Cohomology Rings of Non-Commutative Hilbert Schemes and CoHa-Modules*, Preprint (2013), arXiv:1312.1499
- [2] M. Kontsevich, Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants*, Commun. Number Theory Phys. **5** (2011), 231–352
- [3] M. Lehn, C. Sorger, *Symmetric groups and the cup product on the cohomology of Hilbert schemes*, Duke Math. J. **110** (2001), 345–357
- [4] H. Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) **145** (1997), 379–388
- [5] M. Reineke, *Cohomology of noncommutative Hilbert schemes*, Algebr. Represent. Theory **8** (2005), 541–561
- [6] Y. Soibelman, *Remarks on Cohomological Hall algebras and their representations*, Preprint (2014), arXiv:1404.1606

## Quantum cluster algebras and quantum nilpotent algebras

K. R. GOODEARL

(joint work with M.T. Yakimov)

The theme of this talk was general quantum cluster algebra structures and their construction within the large, axiomatically defined class of quantum nilpotent algebras. The approach was developed in [1, 2]. Many important families of quantized coordinate rings are subsumed in the class covered by these methods, such as that of quantum Schubert cell algebras. In particular, the results extend the theorem of Geiss, Leclerc and Schröer for the case of symmetric Kac–Moody groups. A consequence is the verification of the Berenstein–Zelevinsky conjecture postulating the existence of quantum cluster algebra structures on all quantized coordinate rings of double Bruhat cells in simple Lie groups.

### REFERENCES

- [1] K.R. Goodearl and M.T. Yakimov, *From quantum Ore extensions to quantum tori via noncommutative UFDs*, arXiv:1208.6267.
- [2] K.R. Goodearl and M.T. Yakimov, *Quantum cluster algebra structures on quantum nilpotent algebras*, arXiv:1309.7869.

## Highest weight and monoidal structure for strict polynomial functors

HENNING KRAUSE

In representation theory of artin algebras, it is an old idea of Auslander [3] to ‘resolve’ an algebra  $A$  by finding an algebra  $B$  of finite global dimension together with an idempotent  $e$  such that  $eBe$  is isomorphic to  $A$ . Later, this idea was refined by asking for the algebra  $B$  to be quasi-hereditary. A prototypical example is the pair  $(A, B)$  consisting of the group algebra  $A$  of the symmetric group and  $B$  the corresponding Schur algebra. In fact, the right perspective seems to be the study of the category of strict polynomial functors (in the sense of Friedlander–Suslin [5]), which is equivalent to modules over a Schur algebra and contains the representation theory of the symmetric group.

In some recent work [7] the highest weight structure for the category of strict polynomial functors is explained in elementary terms, using classical facts about Schur functors [1] (including the Cauchy decomposition formula) and working over an arbitrary commutative ring. Of course, this implies the well-known fact that Schur algebras are quasi-hereditary and have finite global dimension. The point here is to offer an elementary approach which should help to understanding more interesting situations, as they arise for instance in recent work by Buchweitz–Leuschke–Van den Bergh [4].

The second aspect of this work concerns the monoidal structure. There is a tensor product for the category of strict polynomial functors which one can use to describe the Koszul–Ringel duality [6]. Recent work of Aquilino–Reischuk [2] shows that this tensor product is actually compatible with the tensor product for

representations of the symmetric group. Thus we have a resolution (in Auslander’s sense) which is in fact a monoidal functor.

## REFERENCES

- [1] K. Akin, D. A. Buchsbaum and J. Weyman, *Schur functors and Schur complexes*, Adv. in Math. **44** (1982), no. 3, 207–278.
- [2] C. Aquilino and R. Reischuk, *The monoidal structure of strict polynomial functors*, in preparation.
- [3] M. Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes, London, 1971.
- [4] R.-O. Buchweitz, G. J. Leuschke and M. Van den Bergh. *On the derived category of Grassmannians in arbitrary characteristic*, arXiv:1006.1633 [math.RT].
- [5] E. M. Friedlander and A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), no. 2, 209–270.
- [6] H. Krause, Koszul, *Ringel and Serre duality for strict polynomial functors*, Compos. Math. **149** (2013), no. 6, 996–1018.
- [7] H. Krause, *The highest weight structure for strict polynomial functors*, arXiv:1405.1691 [math.RT].

### Irreducible components of varieties of representations

BIRGE HUISGEN-ZIMMERMANN

Let  $A$  be a finite dimensional algebra over an algebraically closed field. In this general setting, we “corner” the irreducible components of the varieties  $\text{Rep}_{\mathbf{d}}A$  parametrizing the  $A$ -modules with fixed dimension vector  $\mathbf{d}$  through upper semi-continuous maps from  $\text{Rep}_{\mathbf{d}}A$  to suitable posets. This process leads to a finite collection of closed irreducible subvarieties of  $\text{Rep}_{\mathbf{d}}A$ , which includes the irreducible components. The subsequent process of sifting is, however, not expected to allow for a satisfying general solution, but needs to be customized for special classes of algebras. The first nontrivial instances of a full listing of the components were provided by Schröer (with different methods) for the class of Gelfand-Ponomarev algebras, a good illustration of the intricacy of the task.

We show how, for truncated path algebras of arbitrary quivers, the above strategy allows to detect and separate the components in terms of generic invariants of their modules. We are thus led to easily applicable criteria for determining the components from quiver and Loewy length of  $A$ ; the components are paired with an array of representation-theoretic properties of their modules. In the special case where  $A$  is local (that is, based on a quiver with a single vertex and a finite number  $r$  of loops), the component problem is particularly easily resolved. In particular, it is easy to count the number of components for any dimension  $d$  as a function of  $r$  and the Loewy length of  $A$ . The latter counts generalize existing formulas for  $r = 2$  and Loewy length 2, first obtained by Donald and Flanigan.

## Cyclotomic Mackey Functors

DMITRY KALEDIN

In noncommutative geometry, differential forms and de Rham cohomology classes appear in the form of Hochschild and cyclic homology. The corresponding natural linear algebraic structures such as the de Rham differential and the Hodge filtration are conveniently packaged using the category  $\Lambda$  introduced by A. Connes. However, for algebraic varieties over a finite field or a ring of  $p$ -adic integers, de Rham cohomology can be refined to the crystalline cohomology of A. Grothendieck and all parts of the story have their counterparts such as the de Rham-Witt complex of P. Deligne and L. Illusie and the filtered Dieudonné module structure on cohomology in the sense of J-M. Fontaine. If this refinement were to extend to the noncommutative setting, one would need a way to package additional linear algebra in a sufficiently simple and natural way, possibly extending Connes' approach. The notion of a cyclotomic Mackey functor is designed to give such a packaging. Its definition combines Connes' cyclic objects and the so-called Mackey functors for the group  $\mathbb{Z}$  and automatically provides all the linear algebra one needs.

## Quantized Gieseker moduli spaces

IVAN LOSEV

This talk describes my recent results on the representation theory of quantized Gieseker moduli spaces, [2]. Let us define those quantized spaces.

We start with a pair  $n, r$  of positive integers. Set  $V := \mathbb{C}^n, W := \mathbb{C}^r$ . We form the space  $R := \mathfrak{sl}(V) \oplus \text{Hom}(V, W)$ , this space comes equipped with a natural action of the group  $G = \text{GL}(V)$ . We have an induced action of  $G$  on  $T^*R = R \oplus R^* \cong \mathfrak{sl}(V)^{\oplus 2} \oplus \text{Hom}(V, W) \oplus \text{Hom}(W, V)$ . This action is Hamiltonian with moment map  $\mu : T^*R \rightarrow \mathfrak{g}^* \cong \mathfrak{g}$  given by  $\mu(A, B, i, j) := [A, B] + ji$ . The dual map  $\mu^* : \mathfrak{g} \rightarrow \mathbb{C}[T^*R]$  is given by  $x \mapsto x_R$ , where  $x_R$  denotes the vector field on  $R$  induced by  $x$ , this vector field is viewed as a function on  $T^*R$ .

Using the Hamiltonian action, we can form several different versions of a Hamiltonian reduction. First, set  $\mathcal{M}(n, r) := \mu^{-1}(0)//G$ , i.e.,  $\mathcal{M}(n, r)$  is an affine algebraic variety with  $\mathbb{C}[\mathcal{M}(n, r)] = (\mathbb{C}[T^*R]/\mathbb{C}[T^*R]\{x_R | x \in \mathfrak{g}\})^G$ . Next, pick a nonzero integer  $\theta$  that can be viewed as a character of  $G$  via the identification  $\mathbb{Z} \xrightarrow{\sim} \text{Hom}(G, \mathbb{C}^\times), \theta \mapsto [g \mapsto \det(g)^\theta]$ . So we can form the GIT quotient  $\mathcal{M}^\theta(n, r) := \mu^{-1}(0)^{\theta-ss}//G$ . The product of  $\mathcal{M}^\theta(n, r)$  with  $\mathbb{C}^2$  is the Gieseker moduli space parameterizing the torsion free rank  $r$  degree  $n$  sheaves on  $\mathbb{P}^2$  trivialized at infinity. Finally, we can form the quantum Hamiltonian reduction quantizing  $\mathcal{M}(n, r)$ . Namely, pick  $\lambda \in \mathbb{C}$  and view  $\lambda$  as the character  $x \mapsto \lambda \text{tr}(x)$  of  $\mathfrak{g}$ . The quantum Hamiltonian reduction  $\mathcal{A}_\lambda(n, r)$  is, by definition,  $(D(R)/D(R)\{x_R - \langle \lambda, x \rangle, x \in \mathfrak{g}\})^G$ , where  $D(R)$  stands for the algebra of linear differential operators on  $R$ . The algebra  $\mathcal{A}_\lambda(n, r)$  is filtered and the associated graded coincides with  $\mathbb{C}[\mathcal{M}(n, r)]$ . We could also form a quantization  $\mathcal{A}_\lambda^\theta(n, r)$  of

$\mathcal{M}^\theta(n, r)$  that is a sheaf of filtered algebras playing a crucial role in the theory but we are not going to do that in order to keep the exposition simple.

The goal of this project is to study the representation theory of  $\mathcal{A}_\lambda(n, r)$ . One motivation is an application to the representation theory of the quantizations of more general Nakajima quiver varieties. For example, in [2], results on  $\mathcal{A}_\lambda(n, r)$  were applied to extend the main theorem of [1] on the number of finite dimensional irreducible representations of quantized quiver varieties to the case of affine type quivers with arbitrary framing.

Let us list some results on the representation theory of  $\mathcal{A}_\lambda(n, r)$ .

**Theorem 1.** *The algebra  $\mathcal{A}_\lambda(n, r)$  has a finite dimensional irreducible representation if and only if  $\lambda$  is a rational number of the form  $\frac{q}{n}$  with  $\text{GCD}(q, n) = 1$  lying outside the interval  $(-r, 0)$ . If  $\lambda$  has this form, then there is a unique finite dimensional irreducible representation and every finite dimensional representation is completely reducible.*

**Theorem 2.** *The algebra  $\mathcal{A}_\lambda(n, r)$  has infinite homological dimension if and only if  $\lambda \in (-r, 0)$  is a rational number with denominator  $\leq n$ .*

**Theorem 3.** *Suppose  $\mathcal{A}_\lambda(n, r)$  has finite homological dimension and let  $n'$  denote the denominator of  $\lambda$ . Then there are precisely  $[n/n']$  proper two-sided ideals in  $\mathcal{A}_\lambda(n, r)$ . They form a chain and all of them are prime.*

#### REFERENCES

- [1] R. Bezrukavnikov, I. Losev, *Etingof conjecture for quantized quiver varieties*, arXiv:1309.1716.
- [2] I. Losev, *Etingof conjecture for quantized quiver varieties II: affine quivers*, arXiv:1405.4998.

### Hochschild cohomology with support

WENDY LOWEN

In this talk, we present our recent work [8], in which we propose a new point of view in order to treat and obtain certain decompositions for Hochschild cohomology. We illustrate this by revisiting several results on Hochschild cohomology of schemes which were obtained in joint work with Michel Van den Bergh [10],[11].

**0.1. Hochschild cohomology.** Hochschild cohomology is originally a cohomology theory for algebras. For a  $k$ -algebra  $A$  (with  $k$  a field), we have

$$HH^n(A) = \text{Ext}_{A-A}^n(A, A)$$

computed in the category of  $A$ -bimodules. It is the cohomology of the Hochschild complex  $\mathbf{C}(A)$ , which is a  $B_\infty$ -algebra, in particular it is endowed with all the algebraic structure relevant for higher order deformation theory. In general, the cohomology is notoriously hard to compute, although in some special cases, computations can be carried out:

- (1.a) For a commutative affine regular algebra  $A$ , the classical Hochschild-Kostant-Rosenberg theorem states that  $HH^n(A) = \Lambda_A^n \text{Der}(A)$ .

- (1.b) For an upper triangular matrix algebra  $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ , we have an associated Happel long exact sequence [4]

$$\dots \longrightarrow HH^i(R) \longrightarrow HH^i(A) \oplus HH^i(B) \longrightarrow Ext_{A-B}^i(M, M) \longrightarrow \dots$$

Hochschild cohomology has also been considered for a (quasi-compact, separated) scheme  $X$  by Kontsevich, Swan, Yekutieli and others, using the following expression:

$$HH^n(X) = Ext_{X \times X}^n(\mathcal{O}_X, \mathcal{O}_X).$$

From this definition, it is not clear whether there exists a  $B_\infty$ -algebra  $\mathbf{C}(X)$  which computes the cohomology. One possible approach to obtain such a  $B_\infty$ -algebra follows from the work of Gerstenhaber and Schack in two steps:

- (2.a) These authors define Hochschild cohomology for an arbitrary presheaf  $\mathcal{A}$  of algebras as  $HH^n(\mathcal{A}) = Ext_{\mathcal{A}-\mathcal{A}}^n(\mathcal{A}, \mathcal{A})$ , and for a scheme  $X$ , they propose  $HH^n(X)$ , where  $\mathcal{A}$  is the structure sheaf restricted to an affine basis, as the correct cohomology of  $X$  [2]. One can show that  $HH^n(X) = HH^n(\mathcal{A})$  [10].
- (2.b) For a presheaf of algebras  $\mathcal{A}$ , these authors construct a single algebra  $A$  with  $HH^n(\mathcal{A}) \cong HH^n(A)$  [3]. Thus, we arrive at the Hochschild complex  $\mathbf{C}(A)$  as a  $B_\infty$ -algebra computing  $HH^n(X)$ .

**0.2. Map-graded categories.** The aim of this talk is to present a general framework in which to understand results like (1.b) and (2.b), and in which to state some general decomposition techniques for Hochschild complexes. These techniques are of a different nature than the classical HKR type decompositions, and do not rely on assumptions like smoothness or characteristic zero.

The fundamental objects in our approach are *map-graded categories* [6]. A map-graded category  $\mathfrak{a}$  can be viewed as a monoid-graded algebra with several objects. It has an underlying grading category  $\mathcal{U}$ , object sets  $\mathfrak{a}_U$  for  $U \in \mathcal{U}$ , and morphism modules  $\mathfrak{a}_u(A, A')$  for  $u : U \rightarrow U'$  in  $\mathcal{U}$ ,  $A \in \mathfrak{a}_U$  and  $A' \in \mathfrak{a}_{U'}$ . These data should satisfy the obvious category-type axioms. The following are examples of map-graded categories:

- (3.a) If  $\mathcal{U}$  satisfies  $\mathcal{U}(U, U') = \{*\}$  or  $\mathcal{U}(U, U') = \emptyset$  for all  $U, U' \in \mathcal{U}$ , then  $\mathcal{U}$ -graded categories are in 1-1 correspondence with linear categories  $\mathfrak{a}$  with  $\text{Ob}(\mathfrak{a}) = \mathcal{U}$  and  $\mathfrak{a}(U, U') = 0$  if  $\mathcal{U}(U, U') = \emptyset$ . Thus, in this case a grading prescribes a certain “shape” for a linear category. This situation was described in [10] using so called censoring relations on a category.
- (3.b) For a presheaf of  $k$ -algebras  $\mathcal{A} : \mathcal{U} \rightarrow \text{Alg}(k)$ , there is an associated  $\mathcal{U}$ -graded category  $\mathfrak{a}$  obtained as a kind of  $k$ -linear Grothendieck construction from  $\mathcal{A}$  in the spirit of [1] (where the input is a pseudofunctor  $\mathcal{U} \rightarrow \text{Cat}$  and the output is a category fibered over  $\mathcal{U}$ ). The category  $\mathfrak{a}$  is an intermediate object between  $\mathcal{A}$  and the algebra  $A$  from (2.b).
- (3.c) For two linear categories  $\mathfrak{a}$  and  $\mathfrak{b}$  and an  $\mathfrak{a}$ - $\mathfrak{b}$ -bimodule  $M$ , Keller’s *arrow category* ( $\mathfrak{b} \rightarrow_M \mathfrak{a}$ ) [5] is naturally graded over the path category of  $\bullet \rightarrow \bullet$ . It is the natural categorical version of the algebra  $R$  from (1.b).

Small map-graded categories (over a fixed  $k$ ) can be organized into a category **Map**. For a functor  $\varphi : \mathcal{V} \rightarrow \mathcal{U}$  between small categories and a  $\mathcal{U}$ -graded category  $\mathfrak{a}$ , we obtain a naturally induced  $\mathcal{V}$ -graded category  $\mathfrak{a}^\varphi$  with  $\mathfrak{a}_V^\varphi = \mathfrak{a}_{\varphi(V)}$  for  $V \in \mathcal{V}$  and similarly for morphisms. This way, **Map** becomes fibered over the category **Cat** of small categories through the functor

$$\mathbf{Map} \longrightarrow \mathbf{Cat} : (\mathcal{U}, \mathfrak{a}) \longmapsto \mathcal{U}.$$

**0.3. Limited functoriality.** An important shortcoming of Hochschild cohomology of algebras is its lack of functoriality, which is known to be remedied in part by turning to linear categories [5]. Our point of view is that limited functoriality is in fact fundamentally determined by grading categories.

Let  $\mathfrak{a}$  be a  $\mathcal{U}$ -graded category. Let the nerve  $\mathcal{N}(\mathfrak{a})$  of  $\mathfrak{a}$  be the simplicial set with  $n$ -simplices  $\sigma = (u, A)$  given by data

$$\begin{array}{ccccccc} A_0 & & A_1 & & \dots & & A_n \\ \\ U_0 & \xrightarrow{u_0} & U_1 & \xrightarrow{u_1} & \dots & \xrightarrow{u_{n-1}} & U_n \end{array}$$

with  $u_i \in \mathcal{U}, A_i \in \mathfrak{a}_{U_i}$ .

The *Hochschild complex* of  $\mathfrak{a}$  is the complex  $\mathbf{C}_{\mathcal{U}}(\mathfrak{a})$  with

$$\mathbf{C}_{\mathcal{U}}^n(\mathfrak{a}) = \prod_{(u, A) \in \mathcal{N}(\mathfrak{a})} \text{Hom}_k(\mathfrak{a}_{u_{n-1}}(A_{n-1}, A_n) \otimes \dots \otimes \mathfrak{a}_{u_0}(A_0, A_1), \mathfrak{a}_{u_{n-1} \dots u_1 u_0}(A_0, A_n))$$

with the simplicial Hochschild differential. This complex is in fact a  $B_\infty$ -algebra [6].

Let  $\mathbf{Map}_c \subseteq \mathbf{Map}$  denote the full subcategory of cartesian morphisms with respect to the fibered category  $\mathbf{Map} \rightarrow \mathbf{Cat}$ . It is easily observed that taking Hochschild complexes defines a functor

$$\mathbf{C} : \mathbf{Map}_c \longrightarrow B_\infty : (\mathcal{U}, \mathfrak{a}) \longmapsto \mathbf{C}_{\mathcal{U}}(\mathfrak{a}).$$

**Example 1.** Consider an arrow category  $(\mathfrak{b} \rightarrow_M \mathfrak{a})$  as in (3.c). The functor  $(\bullet \rightarrow \bullet) \rightarrow (\bullet \rightarrow \bullet)$  underlies a cartesian morphism  $\mathfrak{b} \amalg \mathfrak{a} \rightarrow \mathfrak{b} \rightarrow_M \mathfrak{a}$ . Functoriality yields an exact sequence

$$0 \longrightarrow \mathbf{C}(\mathfrak{b}; \mathfrak{a}; M) \longrightarrow \mathbf{C}(\mathfrak{b} \rightarrow_M \mathfrak{a}) \longrightarrow \mathbf{C}(\mathfrak{b}) \oplus \mathbf{C}(\mathfrak{a}) \longrightarrow 0$$

in which the kernel  $\mathbf{C}(\mathfrak{b}; \mathfrak{a}; M)$  is seen to compute  $\text{Ext}_{\mathfrak{b}-\mathfrak{a}}(M, M[-1])$ . The resulting long exact cohomology sequence generalizes Happel’s sequence from (1.b).

**Example 2.** Let  $X$  be a quasi-compact separated scheme with an affine open basis  $\mathcal{B}$ . By (2.a), we have  $HH^n(X) = \text{Ext}_{\mathcal{O}_{\mathcal{B}}-\mathcal{O}_{\mathcal{B}}}^n(\mathcal{O}_{\mathcal{B}}, \mathcal{O}_{\mathcal{B}})$ . By the Cohomology Comparison Theorem from [11], this cohomology is computed by the complex  $\mathbf{C}_{\mathcal{B}}(\mathfrak{o}_{\mathcal{B}})$  where  $\mathfrak{o}_{\mathcal{B}}$  is the Grothendieck construction of  $\mathcal{O}_{\mathcal{B}}$  as in (3.b). We thus interpret this complex as the Hochschild complex  $\mathbf{C}_{\mathcal{B}}(X)$  of  $X$  with respect to the basis  $\mathcal{B}$ . For an inclusion  $V \subseteq U$  between open subsets  $U, V \subseteq X$ , let  $\mathcal{B}_U$  and  $\mathcal{B}_V$  be the

restrictions of  $\mathcal{B}$  to subsets within  $U$  and  $V$  respectively. By functoriality, the inclusion  $\mathcal{B}_V \subseteq \mathcal{B}_U$  yields  $\mathbf{C}_{\mathcal{B}_U}(U) \rightarrow \mathbf{C}_{\mathcal{B}_V}(V)$ . This approach remedies the lack of natural maps  $\mathbf{C}(\mathcal{O}(U)) \rightarrow \mathbf{C}(\mathcal{O}(V))$ .

**0.4. Sheaves and Mayer Vietoris sequences.** In order to make maximal use of grading categories, we now introduce a family of Grothendieck pretopologies on  $\mathbf{Cat}$ .

**Definition 1.** Let  $n \in \mathbb{N} \cup \{\infty\}$ . A collection of functors  $(\varphi_i : \mathcal{V}_i \rightarrow \mathcal{U})_{i \in I}$  is an  $n$ -cover provided that it induces jointly surjective collections of maps  $(\mathcal{N}(\varphi_i) : \mathcal{N}_k(\mathcal{V}_i) \rightarrow \mathcal{N}_k(\mathcal{U}))_{i \in I}$  between  $k$ -simplices of the simplicial nerves for all  $k \leq n$ .

For each  $n \in \mathbb{N} \cup \{\infty\}$ , we thus obtain a pretopology of  $n$ -covers on  $\mathbf{Cat}$ , and an induced pretopology of  $n$ -covers on  $\mathbf{Map}_c$ . We have:

- Theorem 4.**
- (1) The category  $\mathbf{Map}$  constitutes a stack on  $\mathbf{Cat}$  for each pretopology of  $n$ -covers with  $n \geq 3$ .
  - (2) The functor  $\mathbf{C} : \mathbf{Map}_c \rightarrow B_\infty$  is a sheaf for the pretopology of  $\infty$ -covers on  $\mathbf{Map}$ .

**Example 3.** Consider a  $\mathcal{U}$ -graded category  $\mathfrak{a}$  and two subcategories  $\varphi_i : \mathcal{V}_i \subseteq \mathcal{U}$  for  $i \in \{1, 2\}$  that constitute an  $\infty$ -cover of  $\mathcal{U}$ . Let  $(\mathcal{V}_i, \mathfrak{a}^{\varphi_i}) \rightarrow (\mathcal{U}, \mathfrak{a})$  and  $(\mathcal{V}_1 \cap \mathcal{V}_2, \mathfrak{a}^\varphi) \rightarrow (\mathcal{U}, \mathfrak{a})$  be corresponding cartesian morphisms. By the sheaf property of  $\mathbf{C}$ , we obtain a Mayer-Vietoris sequence of Hochschild complexes:

$$0 \rightarrow \mathbf{C}_{\mathcal{U}}(\mathfrak{a}) \rightarrow \mathbf{C}_{\mathcal{V}_1}(\mathfrak{a}^{\varphi_1}) \oplus \mathbf{C}_{\mathcal{V}_2}(\mathfrak{a}^{\varphi_2}) \rightarrow \mathbf{C}_{\mathcal{V}_1 \cap \mathcal{V}_2}(\mathfrak{a}^\varphi) \rightarrow 0.$$

**Example 4.** Let  $X$  and  $\mathcal{B}$  be as in Example 2. Let  $X = U_1 \cup U_2$  be an open cover and put  $U_{12} = U_1 \cap U_2$ . Let  $\mathcal{B}_1, \mathcal{B}_2$  and  $\mathcal{B}_{12}$  be the restrictions of  $\mathcal{B}$  to  $U_1, U_2$  and  $U_{12}$  respectively. Put  $\mathcal{B}_0 = \mathcal{B}_1 \cup \mathcal{B}_2 \subset \mathcal{B}$ . As categories,  $\mathcal{B}_1$  and  $\mathcal{B}_2$  together constitute an  $\infty$ -cover of  $\mathcal{B}_0$ , whence we obtain the Mayer-Vietoris sequence:

$$0 \rightarrow \mathbf{C}_{\mathcal{B}_0}(X) \rightarrow \mathbf{C}_{\mathcal{B}_1}(U_1) \oplus \mathbf{C}_{\mathcal{B}_2}(U_2) \rightarrow \mathbf{C}_{\mathcal{B}_{12}}(U_{12}) \rightarrow 0.$$

A more refined use of bases allows to obtain a sheaf of Hochschild complexes on quasi-compact opens, as was shown in [7]. On the other hand, the existence of Mayer-Vietoris long exact sequences for arbitrary ringed spaces was shown more generally in [10].

**Example 5.** Let  $\mathcal{U}$  be a finite poset and  $\mathfrak{a}$  a  $\mathcal{U}$ -graded category. Looking at a Hasse diagram of  $\mathcal{U}$ , we can  $\infty$ -cover  $\mathcal{U}$  “horizontally” by its “vertical” maximal simplices. This covering is such that both the involved simplices and their intersections are path categories of the following type:

$$\mathcal{V} = \langle \bullet \rightarrow \bullet \rightarrow \dots \rightarrow \dots \rightarrow \bullet \rightarrow \bullet \rangle$$

The Hochschild complex of each induced  $\mathcal{V}$ -graded category can be decomposed by treating it as an “iterated arrow category”. This illustrates the fact that the exact sequences obtained in Example 1 and Example 3 respectively should be seen as complementary tools for deconstructing Hochschild complexes.

*Acknowledgement.* The author is very grateful to Michel Van den Bergh for the original idea of introducing Hochschild complexes of map-graded categories, as a tool for obtaining a Hochschild cohomology local-to-global spectral sequence for arbitrary ringed spaces, dating back to one of her research stays at the Mittag-Leffler institute in 2004. A write up of the envisaged approach, which involves the theory of hypercoverings, remains a joint work in progress [9].

## REFERENCES

- [1] *Revêtements étales et groupe fondamental (SGA 1)*, Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3, Société Mathématique de France, Paris, 2003, Séminaire de géométrie algébrique du Bois Marie 1960–61. [Geometric Algebra Seminar of Bois Marie 1960–61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)]. MR MR2017446 (2004g:14017)
- [2] M. Gerstenhaber and S. D. Schack, *Algebraic cohomology and deformation theory*, Deformation theory of algebras and structures and applications (Il Ciocco, 1986), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 247, Kluwer Acad. Publ., Dordrecht, 1988, pp. 11–264. MR MR981619 (90c:16016)
- [3] ———, *The cohomology of presheaves of algebras. I. Presheaves over a partially ordered set*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 135–165. MR MR965749 (89k:16052)
- [4] D. Happel, *Hochschild cohomology of finite-dimensional algebras*, Séminaire d’Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), Lecture Notes in Math., vol. 1404, Springer, Berlin, 1989, pp. 108–126. MR 1035222 (91b:16012)
- [5] B. Keller, *Derived invariance of higher structures on the Hochschild complex*, preprint <http://www.math.jussieu.fr/~keller/publ/dih.dvi>.
- [6] W. Lowen, *Hochschild cohomology of presheaves as map-graded categories*, Int. Math. Res. Not. IMRN (2008), Art. ID rnn118, 32. MR 2449052 (2009i:18015)
- [7] ———, *A sheaf of Hochschild complexes on quasi compact opens*, Proc. Amer. Math. Soc. **136** (2008), no. 9, 3045–3050, preprint arXiv:0707.2605.
- [8] ———, *Hochschild Cohomology with Support*, Internat. Math. Res. Not. (2014), doi: 10.1093/imrn/rnu079.
- [9] W. Lowen and M. Van den Bergh, *A local-to-global spectral sequence for Hochschild cohomology*, in preparation.
- [10] ———, *Hochschild cohomology of abelian categories and ringed spaces*, Advances in Math. **198** (2005), no. 1, 172–221.
- [11] ———, *A Hochschild cohomology comparison theorem for prestacks*, Trans. Amer. Math. Soc. **363** (2011), no. 2, 969–986. MR 2728592 (2012c:16033)
- [12] B. Mitchell, *Rings with several objects*, Advances in Math. **8** (1972), 1–161. MR MR0294454 (45 #3524)

## Motivic Donaldson–Thomas invariants of quantum $\mathbb{C}^3$ .

ANDREW MORRISON

(joint work with Brent Pym and Balázs Szendrői)

*To the memory of Kentaro Nagao.*

Donaldson–Thomas invariants of a projective Calabi–Yau threefold  $(X, H)$  were defined in [11] as (virtual) counts of the number of stable sheaves on  $X$ . For rank one sheaves with vanishing second Chern character they give one of the ways of counting curves in  $X$ .

Motivic DT invariants [4] refine these deformation invariant numbers by associating to each moduli space of stable sheaves its *virtual motive*: an element of  $K_0^{\hat{\mu}}(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-\frac{1}{2}}]$  whose Euler characteristic is the classical DT invariant<sup>1</sup>. The refined virtual motive is no longer deformation invariant. It depends on the polarization and on the complex structure. This gives two possible paths into the non-commutative world.

In [8] Nagao and Nakajima discovered Szendrői’s non-commutative DT invariants of the conifold [10]. By enlarging the perspective to consider moduli of complexes in the derived category and varying the stability condition one can move from moduli spaces of sheaves on the commutative resolution (giving the classical DT invariants) to moduli spaces of modules for the algebra of the non-commutative resolution (giving Szendrői’s non-commutative DT invariants). Moreover, there is a wall and chamber decomposition of the space of stability conditions with explicit wall crossing formulas for the invariants. The picture was enhanced to motivic invariants in [6]. Indeed Kontsevich and Soibelman define a *polarization* of a non-commutative variety to be this extra data of a stability condition [4].

The goal of the talk is to take the second path into non-commutative geometry by making a non-commutative deformation of the three dimensional algebra. We present the simple case of quantum  $\mathbb{C}^3$ .

To begin consider the more general example of three dimensional Calabi–Yau algebras  $S_{a,b,c}$  defined as the quotient of  $\mathbb{C}\langle x, y, z \rangle$  by the cyclic derivatives of  $axyz + bxzy + c/3(x^3 + y^3 + z^3)$ . In particular the algebra  $S_{1,-q,0}$  defines quantum  $\mathbb{C}^3$  and in the case  $q = 1$  we recover  $\mathbb{C}[x, y, z]$ .

Motivated by considerations in physics one expects that the space of all modules over the algebra  $S_{a,b,c}$  should be described as the critical locus of a potential function on a bigger space of quantum fields. Indeed there exists a regular function  $f_{a,b,c} : \text{Hilb}(\mathbb{C}\langle x, y, z \rangle) \rightarrow \mathbb{C}$  from the smooth quasi-projective non-commutative Hilbert scheme whose critical locus  $\{df_{a,b,c} = 0\}$  equals  $\text{Hilb}(S_{a,b,c})$ , i.e. the scheme parameterizing finite dimensional cyclic left  $S_{a,b,c}$  modules. We denote by  $\text{Hilb}^n(S_{a,b,c})$  the component parametrizing length  $n$  modules.

The virtual motive  $[\text{Hilb}^n(S_{a,b,c})]_{\text{vir}} := \mathbb{L}^{\dim(\text{Hilb}^n(\mathbb{C}\langle x, y, z \rangle))/2} [\phi_{f_{a,b,c}}]$  is defined in terms of the *motivic class of vanishing cycles*  $[\phi_{f_{a,b,c}}]$ : a motivic class in the

---

<sup>1</sup>Elsewhere the motivic invariants are seen to have representation theoretic meaning with relation to co-homological Hall algebras and in string theory to BPS states [5].

Grothendieck ring of varieties with the same underlying monodromic mixed Hodge structure as the usual sheaf of vanishing cycles [3]. We collect all the virtual motives in a partition function  $Z_{a,b,c}(y) := \sum_{n \geq 0} [\text{Hilb}^n(S_{a,b,c})]_{\text{vir}} y^n$ .

For the calculation we need the following description of our potential function  $f_{a,b,c}$ . The smooth moduli space  $\text{Hilb}^n(\mathbb{C}\langle x, y, z \rangle)$  is a free  $\text{GL}(\mathbb{C}^n)$  quotient of the set of stable points in the linear space  $V_n = \mathbb{C}^n \times \text{End}(\mathbb{C}^n)^3$  [9]. Here the vector gives the cyclic generator and the three endomorphisms describe the action of the co-ordinate functions. Now we can lift  $f_{a,b,c}$  to a  $\text{GL}(\mathbb{C}^n)$  invariant cubic polynomial  $F_{a,b,c}$  on  $V_n$  given by

$$F_{a,b,c}(v, X, Y, Z) = \text{Tr} (aXYZ + bXZY + c/3(X^3 + Y^3 + Z^3)).$$

The motivic class of vanishing cycles for such a quasi-homogenous polynomial is much simpler and equals  $[\phi_{F_{a,b,c}}] = [F_{a,b,c}^{-1}(1)] - [F_{a,b,c}^{-1}(0)]$  with a cyclic  $\mu_3$  action on the fiber over one. In this case it is shown [1] that the virtual motives  $[\text{Hilb}^n(S_{a,b,c})]$  can be recovered from those in the pre-quotient. Precisely it is shown that by setting

$$Z_{a,b,c}(y) := \sum_{n \geq 0} \frac{[\phi_{F_{a,b,c}}] \cdot \mathbb{L}^{-n^2}}{[\text{GL}(\mathbb{C}^n)]} y^n$$

we can recover  $Z_{a,b,c}(y) = Z_{a,b,c}(\mathbb{L}^{\frac{1}{2}}y) / Z_{a,b,c}(\mathbb{L}^{-\frac{1}{2}}y)$ . When  $c = 0$  the monodromy action is trivial and we can go further. Via a stratification of vanishing cycles we are left to compute the motivic classes of the moduli stacks of length  $n$  torsion sheaves on quantum  $\mathbb{C}^2$

$$Z_{a,b,0}(y) = \sum_{n \geq 0} \frac{[C_{-b/a}(n)]}{[\text{GL}(\mathbb{C}^n)]} y^n$$

where  $C_q(n) = \{(A, B) \in \text{End}(\mathbb{C}^n)^2 : AB = qBA\}$  is the deformed commuting variety. To state the theorem we use the plethystic exponential  $\text{Exp}$  with the basic properties that  $\text{Exp}(u + v) = \text{Exp}(u) \cdot \text{Exp}(v)$  and  $\text{Exp}(\mathbb{L}^i y^j) = (1 - \mathbb{L}^i y^j)^{-1}$ .

**Theorem.** *Let  $q \in \mathbb{C}^*$ , then for  $q$  a primitive  $r$ th root of unity we have*

$$Z_{1,-q,0}(y) = \text{Exp} \left( \frac{2\mathbb{L} - 1}{\mathbb{L} - 1} \frac{y}{1 - y} + (\mathbb{L} - 1) \frac{y^r}{1 - y^r} \right),$$

*otherwise*

$$Z_{1,-q,0}(y) = \text{Exp} \left( \frac{2\mathbb{L} - 1}{\mathbb{L} - 1} \frac{y}{1 - y} \right).$$

To prove the result we make a stratification of the deformed commuting variety. In the generic case all torsion sheaves must be supported on the union of the two coordinate axis. The exponent  $(2\mathbb{L} - 1)/(\mathbb{L} - 1)$  corresponds geometrically to the motivic class of the stack of such simple modules. In the  $r$ th root of unity case there can be  $r$ -dimensional simple modules supported away from the coordinate axis.

In forthcoming work [7] we will consider other non-commutative deformations of crepant resolutions.

## REFERENCES

- [1] J. Bryan, K. Behrend and B. Szendrői, *Motivic degree zero Donaldson–Thomas invariants*, *Inventiones mathematicae*, April 2013, Volume 192, Issue 1, pp 111-160.
- [2] J. Bryan and A. Morrison, *Motivic classes of commuting varieties via power structures*, To appear in *Journal of Algebraic Geometry*, arXiv:math/1206.5864.
- [3] J. Denef and F. Loeser, *Geometry on arc spaces of algebraic varieties*, *European Congress of Mathematics*, Vol. I (Barcelona, 2000) *Progr. Math.*, vol. 201, Birkhäuser, 2001, pp. 327-348.
- [4] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, 2008, arXiv:math/0811.2435.
- [5] M. Kontsevich and Y. Soibelman, *Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants*, *Com. Num. Th. and Phy.*, Vol. 5, 231-252.
- [6] A. Morrison, S. Mozgovoy, K. Nagao and B. Szendrői, *Motivic Donaldson–Thomas invariants of the conifold and the refined topological vertex*, *Advances in Mathematics*, 230 (4-6), 2065-2093.
- [7] A. Morrison, B. Pym, B. Szendrői, *Work in progress*.
- [8] K. Nagao and H. Nakajima, *Counting invariant of perverse coherent sheaves and its wall-crossing*, *Int. Math. Res. Not.*, 3855-3938.
- [9] M. Reineke, *Cohomology of noncommutative Hilbert schemes*, *Algebr. Represent. Theory*. 8 (2005), 541-561.
- [10] B. Szendrői, *Non-commutative Donaldson–Thomas theory and the conifold*, *Geometry & Topology* 12 (2008) 1171-1202.
- [11] R. Thomas, *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on fibrations*, *Journal of Differential Geometry* 54 (2000), no. 2, 367-438.

**Higgs Bundles and Quiver Representations**

SERGEY MOZGOVOY

(joint work with Markus Reineke)

We discussed the problem of counting semistable Higgs bundles (and twisted Higgs bundles) over a curve. So far this counting was done only for rank 2 (Hitchin) and 3 (Gothen) non-twisted Higgs bundles and for ranks 2 – 5 co-Higgs bundles on  $\mathbb{P}^1$  (twist of degree 2, Rayan). There exists a conjecture for the invariants of moduli spaces of arbitrary rank non-twisted Higgs bundles due to Hausel and Rodriguez-Villegas (generalized for twisted Higgs bundles by myself). The goal of the talk was to explain the general strategy of counting semistable Higgs bundles, give an explicit formula for the invariants of moduli spaces of twisted Higgs bundles on  $\mathbb{P}^1$  and to relate them to the invariants of moduli spaces of representations of some infinite symmetric quiver. We can verify on a computer for any rank and any twist that our formula gives the same result as a Hausel-Rodriguez-Villegas conjecture but could not prove that the formulas are equivalent

**Morse theory of  $\mathcal{D}$ -module categories**

THOMAS NEVINS

(joint work with Gwyn Bellamy, Chris Dodd, Kevin McGerty)

Suppose  $X$  is a smooth complex algebraic variety with the action of a reductive algebraic group  $G$ . In many instances, the Morse theory of  $X$  implies strong consequences for the ordinary or equivariant cohomology of  $X$ . I explained joint work with Bellamy, Dodd, and McGerty that develops parallel structure “one categorical level higher,” for categories of equivariant  $\mathcal{D}$ -modules on  $X$ , with consequences for representation theory and topology.

Associated to  $X$ ,  $G$ , and an additional choice of  $G$ -equivariant line bundle on  $T^*X$ , one typically gets an associated *Kirwan-Ness stratification* of  $T^*X$ . Such a stratification gives a filtration of the category  $\mathcal{D}(X/G)$  of  $G$ -equivariant  $\mathcal{D}$ -modules on  $X$  (really, an appropriate dg enhancement of a suitable derived category of such modules) by singular support, or microsupport, in  $T^*X$ . In joint work with McGerty, we prove that this filtration realizes  $\mathcal{D}(X/G)$  as glued from the categories microsupported on the strata: for each closed inclusion of strata there is a corresponding *recollement* of categories. In particular, we establish the existence and good properties of certain non-obvious adjoint functors. As a consequence, we prove a version of “hyperkaehler Kirwan surjectivity” for cotangent bundles.

Restricting attention to the top Kirwan-Ness stratum, i.e., the GIT-semistable locus, the resulting category is frequently identified with a category of deformation-quantization (DQ) modules on a smooth symplectic algebraic variety with a  $\mathbb{G}_m$ -action that rescales the symplectic form. In joint work with Bellamy, Dodd, and McGerty, we show that such a category satisfies an analogue of Kashiwara’s equivalence for  $\mathcal{D}$ -modules: modules with certain support conditions are equivalent to modules over a smaller variety. This provides a “categorical cell decomposition” of such categories in many examples, leading to concrete applications to  $K$ -theory and Hochschild and cyclic homology of the categories, consequences for compact generation, and more. The resulting applications for linear invariants, specialized to particular examples, imply numerical and linear-algebraic assertions about, for example, categories of representations of wreath product symplectic reflection algebras.

**The codimension-three conjecture for holonomic DQ-modules**

FRANÇOIS PETIT

This talk is concerned with the codimension-three conjecture for holonomic Deformation Quantization modules (DQ-modules). This is an analogue of the codimension-three conjecture for microdifferential modules formulated by M. Kashiwara at the end of the 1970’s and recently proved by M. Kashiwara and K. Vilonen (see [4]). The codimension-three conjecture is concerned with the extension of a holonomic microdifferential system over an analytic subset of the cotangent bundle of a complex manifold. Since DQ-modules provide a generalization of microdifferential

modules to arbitrary symplectic manifolds (see [3]), it is natural to try to extend the codimension-three conjecture to holonomic DQ-modules. More precisely, we have obtained, in the case of DQ-modules, the following results.

**Theorem 1.** *Let  $X$  be a complex manifold endowed with a DQ-algebroid stack  $\mathcal{A}_X$  such that the associated Poisson structure is symplectic. Let  $\Lambda$  be a closed Lagrangian analytic subset of  $X$  and  $Y$  be a closed analytic subset of  $\Lambda$  such that  $\text{codim}_\Lambda Y \geq 3$ . Let  $\mathcal{M}$  be a holonomic  $(\mathcal{A}_X^{\text{loc}}|_{X \setminus Y})$ -module, whose support is contained in  $\Lambda \setminus Y$ . Assume that  $\mathcal{M}$  has an  $\mathcal{A}_X|_{X \setminus Y}$ -lattice. Then,  $\mathcal{M}$  extends uniquely to a holonomic module defined on  $X$  whose support is contained in  $\Lambda$ .*

and

**Theorem 2.** *Let  $X$  be a complex manifold endowed with a DQ-algebroid stack  $\mathcal{A}_X$  such that the associated Poisson structure is symplectic. Let  $\Lambda$  be a closed Lagrangian analytic subset of  $X$  and  $Y$  be a closed analytic subset of  $\Lambda$  such that  $\text{codim}_\Lambda Y \geq 2$ . Let  $\mathcal{M}$  be a holonomic  $\mathcal{A}_X^{\text{loc}}$ -module whose support is contained in  $\Lambda$  and let  $\mathcal{M}_1$  be an  $\mathcal{A}_X^{\text{loc}}|_{X \setminus Y}$ -submodule of  $\mathcal{M}|_{X \setminus Y}$ . Then,  $\mathcal{M}_1$  extends uniquely to a holonomic  $\mathcal{A}_X^{\text{loc}}$ -submodule of  $\mathcal{M}$ .*

We hope that the submodule version of the codimension-three conjecture for holonomic DQ-modules will find application to the representation theory of Cherednik algebras via the localization results due to M. Kashiwara and R. Rouquier [2].

Though the codimension-three conjecture for holonomic DQ-modules is strictly more general than the version for microdifferential modules, it relies deeply on the tools and ideas elaborated by M. Kashiwara and K. Vilonen. Indeed, our proof follows their general strategy. One of the main differences between the two problems is that we are working in a non-conical setting and thus, we have to adapt their strategy to such a framework.

We briefly describe the proof of the above statements. We keep the notation of Theorem 1 and 2 and denote by  $j : X \setminus Y \rightarrow X$  the open embedding of  $X \setminus Y$  into  $X$ . Then, the problem amounts essentially to show that  $j_*\mathcal{M}$  and  $j_*\mathcal{M}_1$  are coherent. To prove this, we first notice this is a local problem. Thus, we can adapt a standard technique in complex analysis in several variables due to [5] and show that the coherence of  $j_*\mathcal{M}$  and  $j_*\mathcal{M}_1$  is equivalent to the coherence of the pushforward of these modules by a certain projection, the restriction of which to the support of the module is a finite map. Then, the coherence of the pushforward of  $j_*\mathcal{M}$  and  $j_*\mathcal{M}_1$  by the aforementioned projection follows from the below result due to M. Kashiwara and K. Vilonen. This is a difficult extension to coherent sheaves on  $\mathcal{O}_X[[\hbar]]$  of a classical theorem due to Frisch-Guenot, Trautmann and Siu (see [1, 6, 7]).

**Theorem 3** ([4, Theorem 1.6]). *Let  $X$  be a complex manifold and  $Y$  be a subvariety of  $X$  and  $j : X \setminus Y \rightarrow X$  the open embedding of  $X \setminus Y$  into  $X$ . If  $\mathcal{N}$  is a coherent reflexive  $\mathcal{O}_{X \setminus Y}[[\hbar]]$ -module and  $\text{codim} Y \geq 3$  then  $j_*\mathcal{N}$  is a coherent  $\mathcal{O}_X[[\hbar]]$ -module.*

## REFERENCES

- [1] J. Frisch and J. Guenot. Prolongement de faisceaux analytiques cohérents. *Invent. Math.*, 7:321–343, 1969.
- [2] M. Kashiwara and R. Rouquier. Microlocalization of rational Cherednik algebras. *Duke Math. J.*, 144(3):525–573, 2008.
- [3] M. Kashiwara and P. Schapira. *Deformation quantization modules*, volume 345 of *Astérisque*. Soc. Math. France, 2012.
- [4] M. Kashiwara and K. Vilonen. Microdifferential systems and the codimension-three conjecture. *ArXiv e-prints*, arXiv:1209.5124, 2012.
- [5] M. Sato, T. Kawai, and M. Kashiwara. Microfunctions and pseudo-differential equations. In *Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau)*, pages 265–529. Lecture Notes in Math., Vol. 287. Springer, Berlin, 1973.
- [6] Y. Siu. Extending coherent analytic sheaves. *Ann. of Math. (2)*, 90:108–143, 1969.
- [7] G. Trautmann. Ein Kontinuitätssatz für die Fortsetzung kohärenter analytischer Garben. *Arch. Math. (Basel)*, 18:188–196, 1967.

## Frobenius Ext-algebras arising from Artin-Schelter Gorenstein Algebras

DANIEL ROGALSKI

(joint work with Manuel Reyes, James Zhang)

Let  $k$  be an algebraically closed field. An Artin-Schelter (AS) Gorenstein algebra is a finitely generated  $\mathbb{N}$ -graded  $k$ -algebra  $A = k \oplus A_1 \oplus A_2 \oplus \dots$  such that (i)  $A$  has finite injective dimension  $d$  as a right  $A$ -module, and (ii)  $\text{Ext}_A^i(k, A) = 0$  for  $i \neq d$ , and  $\text{Ext}_A^d(k, A) \cong k(\ell)$  (a graded shift of  $k$ ), where  $k = A/A_{\geq 1}$  is the trivial module. If (i) is replaced by the stronger condition that  $A$  has finite (graded) global dimension  $d$ , then  $A$  is called AS regular. Any AS Gorenstein algebra  $A$  has a graded automorphism  $\mu_A : A \rightarrow A$  associated to it called the *Nakayama automorphism*, which is a natural generalization to higher dimension of the usual Nakayama automorphism which is defined when  $d = 0$  and  $A$  is graded Frobenius. There is also the *homological determinant*  $\text{hdet} : \text{Aut}_{gr}(A) \rightarrow k$ , where  $\text{Aut}_{gr}(A)$  is the group of graded automorphisms of  $A$ , which is important in noncommutative invariant theory.

We conjecture that for any AS Gorenstein algebra  $A$ ,  $\text{hdet } \mu_A = 1$  always holds. In a previous paper [1] this conjecture was shown to have numerous applications; for example, if the conjecture is true, one obtains that any AS Gorenstein algebra  $A$  with  $\ell \neq 0$  is a graded twist of an AS Gorenstein algebra  $A'$  such that  $\mu_{A'}$  has finite order [1, Theorem 7.8].

In the current work we describe a method that can be used to prove the conjecture in quite wide generality. Let  $D_\epsilon(A)$  be the full triangulated subcategory of the derived category of graded left  $A$ -modules which consists of perfect complexes (bounded complexes of finitely generated projectives) with finite dimensional cohomologies. We show that this category has a Serre functor of the form  $\Sigma^d T^\ell \Phi$ , where  $\Sigma$  is the shift of complexes,  $T$  is induced by the shift of grading on modules,

and  $\Phi$  is induced by the action of the Nakayama automorphism  $\mu_A$  on modules. Then given any object  $X$  in  $D_\epsilon(A)$  which satisfies  $X \cong \Phi(X)$  we show that the associated graded Ext algebra  $E = \bigoplus_{i,j} \text{Hom}_{D_\epsilon(A)}(X, \Sigma^i T^j(X))$  is Frobenius, and we give a formula for its Nakayama automorphism  $\mu_E$ . As long as such an object  $X$  exists, by studying the relation between  $\mu_E$  and  $\mu_A$  and using that  $\text{hdet } \mu_E = 1$  (which is relatively easy to prove), we can prove the conjecture that  $\text{hdet } \mu_A = 1$ . The study of these Ext algebras  $E$  and their Nakayama automorphisms  $\mu_E$  seems interesting more generally, and we give other applications as well.

Ultimately, using this method we can prove the conjecture  $\text{hdet } \mu_A = 1$  for any AS Gorenstein algebra  $A$  such that  $D_\epsilon(A) \neq 0$ , by reducing to a case where a suitable object  $X$  exists. We show that the condition  $D_\epsilon(A) \neq 0$  holds in most important cases, for example whenever  $A$  is a factor ring of an AS regular algebra, and we conjecture that it holds for all AS Gorenstein algebras.

#### REFERENCES

- [1] M. Reyes, D. Rogalski, and J. J. Zhang, *Skew Calabi-Yau algebras and homological identities*, arXiv:1302.0437 (2013).

### Categorification of $\mathfrak{gl}(1|1)$ -representations

ANTONIO SARTORI

The representation theory of the complex general Lie superalgebra  $\mathfrak{gl}(1|1)$ , although quite elementary, presents interesting combinatorial properties. For example, the quantum invariant of links corresponding to its vector representation  $\mathbb{C}^{1|1}$  is the Alexander polynomial. This motivates our interest in computing intertwining operators between tensor products of representations and categorifying them.

In the talk, we describe in full detail the category of weight representations. Contrary to the case of usual semisimple Lie algebras, finite-dimensional representations of  $\mathfrak{gl}(1|1)$  need not be semisimple. In particular, the category of finite-dimensional weight representations decomposes into a semisimple and a non-semisimple summand.

We focus then on the semisimple monoidal subcategory generated by exterior powers of the natural representation: here we can define and compute an analog of Lusztig's canonical basis, and prove that it has integrality and positivity properties. We can then construct a categorification using subquotient categories of the BGG category  $\mathcal{O}(\mathfrak{gl}(n))$ .

Finally, we show how these categories have a natural geometric interpretation related to the geometry of the Springer fiber of hook type. In particular, the endomorphism rings of indecomposable projective modules in these categories are naturally isomorphic to the cohomology rings of attracting subvarieties for an action of a one-dimensional torus.

## REFERENCES

- [1] A. Sartori, *A diagram algebra for Soergel modules corresponding to smooth Schubert varieties*, ArXiv e-prints (2013), 1311.6968, To appear in Trans. Amer. Math. Soc.
- [2] A. Sartori, *Categorification of tensor powers of the vector representation of  $U_q(\mathfrak{gl}(1|1))$* , ArXiv e-prints (2013), 1305.6162.
- [3] A. Sartori, *The Alexander polynomial as quantum invariant of links*, ArXiv e-prints (2013), 1308.2047, To appear in Ark. Mat.

## Poisson and Hochschild-de Rham Homology and Symplectic Resolutions

TRAVIS SCHEDLER

(joint work with Pavel Etingof)

I defined the notion of Hochschild-de Rham homology of a deformation quantization of a Poisson variety. In previous work, we defined Poisson-de Rham homology of a Poisson variety, which recovers in the smooth symplectic case the de Rham cohomology of the variety; more generally, for varieties admitting a smooth symplectic resolution, this conjecturally recovers the de Rham cohomology of the symplectic resolution. For affine varieties the degree zero Poisson-de Rham homology coincides with the usual Poisson homology. Similarly, the Hochschild-de Rham homology recovers the de Rham cohomology when quantizing smooth symplectic varieties and conjecturally varieties admitting symplectic resolutions, and the degree zero Hochschild-de Rham homology of an affine variety is its degree zero Hochschild homology. In the case that the variety has finitely many leaves, the Hochschild and Poisson-de Rham homologies are finite-dimensional, and zero in degree of absolute value greater than the dimension.

## Some enhancements of derived categories of coherent sheaves and applications

OLAF M. SCHNÜRER

(joint work with Valery A. Lunts)

Let  $X$  be a quasi-projective scheme over a field  $k$ . The unbounded derived category  $D(\mathrm{Qcoh}(X))$  of quasi-coherent sheaves on  $X$  contains the bounded derived category  $D^b(\mathrm{Coh}(X))$  of coherent sheaves and the category of perfect complexes  $\mathfrak{P}erf(X)$  as full triangulated subcategories,

$$\mathfrak{P}erf(X) \subset D^b(\mathrm{Coh}(X)) \subset D(\mathrm{Qcoh}(X)).$$

The objects of  $\mathfrak{P}erf(X)$  can be characterized as the compact objects of  $D(\mathrm{Qcoh}(X))$ .

Recall that an enhancement of a triangulated ( $k$ -)category  $\mathcal{T}$  is a pretriangulated dg ( $k$ -)category  $\mathcal{A}$  together with an equivalence  $[\mathcal{A}] \cong \mathcal{T}$  where  $[\mathcal{A}]$  is the homotopy category of  $\mathcal{A}$ . For example, the full dg subcategory  $C(\mathrm{Qcoh}(X))_{\mathrm{h-inj}}$  of h-injective objects of the dg category  $C(\mathrm{Qcoh}(X))$  of complexes of quasi-coherent

sheaves forms an enhancement of  $D(\text{Qcoh}(X))$ . The obvious full dg subcategories of  $C(\text{Qcoh}(X))_{\text{h-inj}}$  define enhancements of  $D^b(\text{Coh}(X))$  and  $\mathfrak{P}erf(X)$ .

Fix a finite affine open covering  $(U_i)_{i=1}^n$  of  $X$  and consider for any complex  $P$  of vector bundles on  $X$  its  $*$ -Čech resolution

$$\mathcal{C}_*(P) = \left( \prod_i U_i P \rightarrow \prod_{i < j} U_{ij} P \rightarrow \dots \right)$$

where  $U P = u_* u^* P$  for  $u: U \hookrightarrow X$ . If  $P$  is a bounded complex of vector bundles the complex  $\mathcal{C}_*(P)$  is defined as the obvious totalization.

For integral  $X$  and bounded complexes  $P$  and  $Q$  of vector bundles the canonical map

$$\text{Hom}_{[C(\text{Qcoh}(X))]}(\mathcal{C}_*(P), \mathcal{C}_*(Q)) \rightarrow \text{Hom}_{D(\text{Qcoh}(X))}(\mathcal{C}_*(P), \mathcal{C}_*(Q))$$

is an isomorphism and hence the full dg subcategory of  $C(\text{Qcoh}(X))$  consisting of  $*$ -Čech resolutions  $\mathcal{C}_*(P)$  of bounded complexes  $P$  of vector bundles forms an enhancement of  $\mathfrak{P}erf(X)$ . For smooth projective  $X$  this enhancement was considered in [BLL04].

We found a modification of this construction which provides an enhancement of  $\mathfrak{P}erf(X)$  for any quasi-projective scheme  $X$  over  $k$ . The objects of this enhancement are bounded complexes of vector bundles and the morphism spaces are suitably defined dg submodules of  $\text{Hom}_{C(\text{Qcoh}(X))}(\mathcal{C}_*(P), \mathcal{C}_*(Q))$ . Using  $!$ -Čech resolutions we similarly define enhancements of  $\mathfrak{P}erf(X)$  and  $D^b(\text{Coh}(X))$ . We call these enhancements Čech enhancements.

Recall that a dg category  $\mathcal{A}$  is called  $k$ -smooth if the diagonal dg bimodule  $\mathcal{A}$  is a compact object of the derived category  $D(\mathcal{A} \otimes \mathcal{A}^{\text{op}})$  of dg  $\mathcal{A} \otimes \mathcal{A}^{\text{op}}$ -modules.

We say that  $\mathfrak{P}erf(X)$  (resp.  $D^b(\text{Coh}(X))$ ) is smooth over  $k$  if its h-injective enhancement is  $k$ -smooth as a dg category.

Using Čech enhancements we prove the following three theorems.

**Theorem 4** (Homological versus geometric smoothness). *Let  $\Delta: X \rightarrow X \times X$  be the diagonal immersion. The following three conditions are equivalent:*

- (1)  $\mathfrak{P}erf(X)$  is smooth over  $k$ ;
- (2)  $\Delta_*(\mathcal{O}_X) \in \mathfrak{P}erf(X \times X)$ ;
- (3)  $X$  is smooth over  $k$ .

**Theorem 5.** *If the field  $k$  is perfect then  $D^b(\text{Coh}(X))$  is smooth over  $k$ .*

**Theorem 6** (Fourier-Mukai kernels and dg bimodules). *Let  $X$  and  $Y$  be quasi-projective schemes over a field  $k$  and consider the projections  $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ . Then there are dg algebras  $A$  and  $B$  and equivalences of triangulated categories*

$$\begin{aligned} \theta_X &: D(\text{Qcoh}(X)) \xrightarrow{\sim} D(A), \\ \theta_Y &: D(\text{Qcoh}(Y)) \xrightarrow{\sim} D(B), \\ \theta_{X \times Y} &: D(\text{Qcoh}(X \times Y)) \xrightarrow{\sim} D(A^{\text{op}} \otimes B), \end{aligned}$$

such that for any  $K \in D(\text{Qcoh}(X \times Y))$  with corresponding  $M = \theta_{X \times Y}(K)$  the diagram

$$\begin{array}{ccc}
 D(\text{Qcoh}(X)) & \xrightarrow{\mathbf{R}q_*(p^*(-) \otimes^{\mathbf{L}} K)} & D(\text{Qcoh}(Y)) \\
 \theta_X \downarrow \sim & & \theta_Y \downarrow \sim \\
 D(A) & \xrightarrow{- \otimes_A^{\mathbf{L}} M} & D(B)
 \end{array}$$

commutes up to an isomorphism of triangulated functors.

Some variants of these results are claimed in the literature without proof or with gaps in the proofs, cf. the discussion in [LS14].

All three theorems admit short heuristic arguments making them plausible. However, turning these arguments into rigorous proofs seems to be hard. The different functors involved (inverse image, tensor product, direct image,  $\mathbf{R}\text{Hom}$ ) are usually computed via different types of replacements (h-flat, h-injective) which makes it difficult to treat them compatibly. It would be desirable to lift all these functors and the adjunctions among them from the derived level to the dg level of enhancements.

For simplicity we stated our results here for quasi-projective schemes over a field. This assumption can be weakened, see [LS14], where the above theorems and some more results characterizing properness are proved.

REFERENCES

[BLL04] Alexey I. Bondal, Michael Larsen, and Valery A. Lunts, *Grothendieck ring of pretriangulated categories*, Int. Math. Res. Not. (2004), no. 29, 1461–1495.  
 [LS14] Valery A. Lunts and Olaf M. Schnürer, *New enhancements of derived categories of coherent sheaves and applications*, arXiv:1406.7559 [math.AG] (2014).

Ring-theoretic blowing down

SUSAN J. SIERRA

(joint work with Dan Rogalski, Toby Stafford)

This work is part of an ongoing project to extend tools from the algebraic geometry of projective surfaces to noncommutative graded domains of GK-dimension 3, with the ultimate goal of classifying such algebras.

We first discuss the algebraic geometry we wish to generalise. Fix an algebraically closed field  $k$ ; all varieties and algebras will be defined over  $k$ . Let  $X$  be a smooth projective surface. Let  $x \in X$ , and let  $\pi : Bl_x(X) \rightarrow X$  be the blowup of  $X$  at  $x$ ; recall that  $\pi$  is a *monoidal transformation*. It is well-known that:

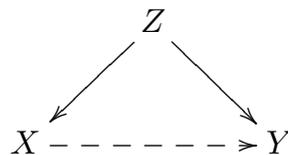
**Proposition 5.**  *$Bl_x(X)$  is a smooth projective surface. If  $L = \pi^{-1}(x)$  is the exceptional locus of  $\pi$ , then  $L \cong \mathbb{P}^1$  and  $L.L = -1$ .*

In the proposition above,  $\pi$  contracts  $L$  to  $x$  by construction. A celebrated theorem (“Castelnuovo’s contraction criterion”) says that the converse also holds: the properties of  $L$  characterise curves that can be contracted to smooth points.

**Theorem 7** (Castelnuovo). *Let  $Y$  be a smooth projective surface, and let  $L$  be a curve on  $Y$  so that  $L \cong \mathbb{P}^1$  and  $L.L = -1$ . Then there is a smooth projective surface  $X$  and a birational morphism  $\pi : Y \rightarrow X$  so that  $L$  is the exceptional locus of  $\pi$  and  $Y \cong Bl_x(X)$ , where  $x = \pi(L)$ .*

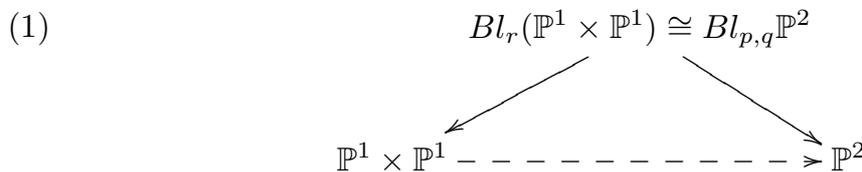
The other crucial theorem in the subject is the following consequence of Zariski’s Main Theorem, which shows that monoidal transformations are the building blocks of birational geometry of surfaces.

**Theorem 8** (Zariski). *Let  $X \dashrightarrow Y$  be a birational map of smooth projective surfaces. Then there are a smooth projective surface  $Z$  and compositions of monoidal transformations  $Z \rightarrow X, Z \rightarrow Y$  so that*



*commutes.*

The fundamental example of this theorem is the isomorphism between the blowup of  $\mathbb{P}^2$  at two points and the blowup of  $\mathbb{P}^1 \times \mathbb{P}^1$  at one point. The diagram above becomes:



To generalise this to the noncommutative setting, we first must extend some definitions. Let  $R, S$  be connected graded  $k$ -algebras that are domains. We say that  $R, S$  are *birational* if  $Q_{gr}(R)_0 \cong Q_{gr}(S)_0$ . (If  $R$  is a homogeneous coordinate ring of a projective variety  $X$ , then  $Q_{gr}(R)_0 \cong k(X)$ , so this generalises the definition from commutative algebraic geometry.) For example, let  $S$  be a quadratic Sklyanin algebra (a “noncommutative  $\mathbb{P}^2$ ”) and let  $S'$  be a cubic Sklyanin algebra (or more generally a smooth noncommutative quadric surface). Michel Van den Bergh [2, Theorem 13.4.1] has proved that  $S$  and  $S'$  are birational.

Our results apply to *elliptic algebras*: an elliptic algebra is a connected graded domain  $R$  containing a central  $g \in R_1$  so that  $R/(g)$  is isomorphic to a twisted homogeneous coordinate ring on an elliptic curve  $E$ . (We say that  $E$  is *associated to  $R$* .) For example, the 3-Veronese  $T := S^{(3)}$  of  $S$  is elliptic, as is  $T' := (S')^{(4)}$ .

For elliptic algebras, there is a good analogue of a monoidal transformation. Let  $R$  be an elliptic algebra with  $\dim R_1 \geq 4$ . Then  $R_1/(g)$  may be identified with global sections of an invertible sheaf  $\mathcal{L}$  on  $E$ . If  $p \in E$ , we define  $Bl_p(R)$  to be the subalgebra of  $R$  generated by the elements of  $R_1$  whose images mod  $g$  vanish at  $p$ . (The algebra  $Bl_p(R)$  also occurs in [3] as the coordinate ring of the more categorical blowups defined there.) A theorem of Rogalski [1, Lemma 9.1]

says that  $Bl_p(R)$  has a line module  $L$  so that  $R/Bl_p(R) \cong L^{\oplus \mathbb{Z}}$  as an (ungraded)  $Bl_p(R)$ -module. We refer to  $L$  as the *exceptional line module* for  $Bl_p(R)$ .

Let  $R$  be an elliptic algebra. A graded  $R$ -module  $L$  is a *line module* if  $L$  is cyclic and has Hilbert series  $1/(1-s)$ . We say that  $L$  has *self-intersection*  $(-1)$  if  $Ext_R^1(L, L) = 0$  (there is also an equivalent definition using noncommutative intersection theory). Then we prove noncommutative versions of Proposition 5 and Theorem 7:

**Proposition 6.** *Let  $R$  be an elliptic algebra with associated elliptic curve  $E$ , let  $p \in E$ , and let  $L$  be the exceptional line module for  $R' := Bl_p(R)$ . If the image of  $L$  has projective dimension 1 over  $R'/(g-1)$ , then  $L$  has self-intersection  $(-1)$ .*

(We note that it is possible for the condition on the projective dimension of  $L$  to fail and for  $Ext_R^1(L, L)$  to be nonzero, so the analogy with commutative geometry is not exact.)

**Theorem 9.** *Let  $R$  be an elliptic algebra with associated elliptic curve  $E$  and let  $L$  be a line module with self-intersection  $(-1)$ . Then there is an elliptic algebra  $R'$  associated to  $E$  and a point  $p \in E$  so that  $R \cong Bl_p(R')$ , with exceptional line  $L$ .*

As yet, there is no general analogue of Theorem 8. We do prove, however:

**Theorem 10.** *Let  $E$  be the elliptic curve associated to  $T'$ , defined above, and let  $r \in E$  be generic. Then there is a quadratic Sklyanin algebra  $S$  so that  $T = S^{(3)}$  is associated to  $E$  and points  $p, q \in E$  so that*

$$Bl_r(T') \cong Bl_{p,q}(T).$$

Our theorem also holds for smooth quadric surfaces, giving a noncommutative version of (1).

## REFERENCES

- [1] D. Rogalski, Blowup subalgebras of the Sklyanin algebra. *Adv. Math.* **226** (2011), no. 2, 1433–1473.
- [2] J.T. Stafford and M. van den Bergh, Noncommutative curves and noncommutative surfaces. *Bull. Amer. Math. Soc. (N.S.)* **38** (2001), no. 2, 171–216.
- [3] M. Van den Bergh, Blowing up of non-commutative smooth surfaces, *Mem. Amer. Math. Soc.* **154** (2001), no. 734.

## Serre functors and derived equivalences for hereditary Artin algebras

ADAM-CHRISTIAAN VAN ROOSMALEN

(joint work with Donald Stanley)

I reported on recent research by Donald Stanley and myself in which we studied a connection between Serre functors and derived equivalences (see [3]). Throughout, let  $k$  be a field. We will assume that all categories are  $k$ -linear. For a triangulated category  $\mathcal{C}$ , we will write  $\Sigma$  for the suspension.

### 1. SERRE DUALITY

Let  $\mathcal{C}$  be a Hom-finite triangulated category. A *Serre functor* ([2]) is a triangle autoequivalence  $\mathbb{S} : \mathcal{C} \rightarrow \mathcal{C}$  together with isomorphisms

$$\eta_{A,B} : \text{Hom}(A, B) \cong \text{Hom}(B, \mathbb{S}A)^*,$$

for any  $A, B \in \mathcal{C}$ , which are natural in  $A, B$  and where  $(-)^*$  is the vector space dual.

We say that an abelian category  $\mathcal{A}$  has *Serre duality* if  $D^b\mathcal{A}$  admits a Serre functor. It has been shown in [2] that the following categories have Serre duality:

- the category  $\text{mod } \Lambda$  of finite-dimensional modules over a finite-dimensional algebra  $\Lambda$  with finite global dimension, and
- the category  $\text{Coh } \mathbb{X}$  of coherent sheaves on a smooth projective variety  $\mathbb{X}$ .

### 2. $t$ -STRUCTURES

Let  $\mathcal{C}$  be a triangulated category. A  $t$ -structure ([1]) on  $\mathcal{C}$  is a pair  $(\mathcal{U}, \mathcal{V})$  of full subcategories of  $\mathcal{C}$  satisfying the following conditions:

- (1)  $\Sigma\mathcal{U} \subseteq \mathcal{U}$  and  $\mathcal{V} \subseteq \Sigma\mathcal{V}$ ,
- (2)  $\text{Hom}(\Sigma\mathcal{U}, \mathcal{V}) = 0$ ,
- (3)  $\forall C \in \mathcal{C}$ , there is a triangle  $U \rightarrow C \rightarrow V \rightarrow \Sigma U$  with  $U \in \mathcal{U}$  and  $V \in \Sigma^{-1}\mathcal{V}$ .

Furthermore, we will say that the  $t$ -structure  $(\mathcal{U}, \mathcal{V})$  is *bounded* if

$$\bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{U} = \bigcup_{n \in \mathbb{Z}} \Sigma^n \mathcal{V} = \mathcal{C}.$$

The *heart* of a  $t$ -structure  $(\mathcal{U}, \mathcal{V})$  is defined to be  $\mathcal{U} \cap \mathcal{V}$ . It has been shown in [1, Théorème 1.3.6] that the heart is an abelian category.

When  $\mathcal{C}$  is the bounded derived category  $D^b\mathcal{A}$  of an abelian category  $\mathcal{A}$ , the embedding  $\mathcal{H} \rightarrow D^b\mathcal{A}$  lifts to a triangle functor  $D^b\mathcal{H} \rightarrow D^b\mathcal{A}$ , called the *realization functor*. In general, the realization functor is not an equivalence. The following theorem is standard (see [1, Proposition 3.1.16]).

**Theorem 11.** *Let  $\mathcal{A}$  be an abelian category, and let  $(\mathcal{U}, \mathcal{V})$  be a  $t$ -structure in  $D^b\mathcal{A}$  with heart  $\mathcal{H}$ . The realization functor  $D^b\mathcal{H} \rightarrow D^b\mathcal{A}$  is an equivalence if and only if for all  $A, B \in \mathcal{H}$ , all  $n \geq 2$ , and every morphism  $f : A \rightarrow \Sigma^n B$ , there is a monomorphism  $B \rightarrow C$  in  $\mathcal{H}$  such that  $A \rightarrow \Sigma^n B \rightarrow \Sigma^n C$  is zero.*

### 3. MAIN RESULT

A first connection between Serre functors and derived equivalences is given in the following proposition.

**Proposition.** *Let  $\mathcal{A}$  be an abelian category, and let  $(\mathcal{U}, \mathcal{V})$  be a  $t$ -structure in  $D^b\mathcal{A}$  with heart  $\mathcal{H}$ . If the realization functor  $D^b\mathcal{H} \rightarrow D^b\mathcal{A}$  is an equivalence, then  $(\mathcal{U}, \mathcal{V})$  is bounded and  $\Sigma\mathcal{U} \subseteq \mathcal{U}$ .*

One can now wonder when the converse of the previous proposition holds.

**Question.** Let  $\mathcal{A}$  be an abelian category of which the bounded derived category  $D^b\mathcal{A}$  admits a Serre functor  $\mathbb{S} : D^b\mathcal{A} \rightarrow D^b\mathcal{A}$ , and let  $(\mathcal{U}, \mathcal{V})$  be a  $t$ -structure on  $D^b\mathcal{A}$  with heart  $\mathcal{H} = \mathcal{U} \cap \mathcal{V}$ . For which categories  $\mathcal{A}$  (and possibly for which restricted class of  $t$ -structures  $(\mathcal{U}, \mathcal{V})$  on  $D^b\mathcal{A}$ ) are the following statements equivalent:

- (1) the realization functor  $D^b\mathcal{H} \rightarrow D^b\mathcal{A}$  is an equivalence,
- (2) the  $t$ -structure  $(\mathcal{U}, \mathcal{V})$  is bounded and  $\mathbb{S}\mathcal{U} \subseteq \mathcal{U}$ ?

The main result I reported on, is that the answer to the previous question is positive when  $\mathcal{A}$  is the module category of a finite-dimensional hereditary algebra:

**Theorem 12.** Let  $\Lambda$  be a finite-dimensional hereditary algebra and write  $\text{mod } \Lambda$  for the category of finite-dimensional right  $\Lambda$ -modules. Let  $(\mathcal{U}, \mathcal{V})$  be a  $t$ -structure in  $D^b \text{mod } \Lambda$  with heart  $\mathcal{H}$ .

The realization functor  $D^b\mathcal{H} \rightarrow D^b\mathcal{A}$  is an equivalence if and only if  $(\mathcal{U}, \mathcal{V})$  is bounded and  $\mathbb{S}\mathcal{U} \subseteq \mathcal{U}$ .

#### REFERENCES

- [1] A. A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.
- [2] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337.
- [3] Donald Stanley and Adam-Christiaan van Roosmalen, *Derived equivalences for hereditary Artin algebras*, preprint.

## On Quantum Symmetry

CHELSEA WALTON

Let  $k$  be an algebraically closed field of characteristic zero. The purpose of this work is to study quantum analogues of group actions on commutative  $k$ -algebras. Here, we restrict our attention to the actions of finite quantum groups, i.e. finite dimensional Hopf algebras  $H$ . Two important subclasses of such  $H$  are those that are *semisimple* (that is to say, semisimple as an algebra; these Hopf algebras are always finite dimensional) and those that are *pointed* (that is to say, all simple  $H$ -comodules are 1-dimensional).

Naturally there are many choices on what (quantum)  $k$ -algebras we can act on, but we are first motivated from the viewpoint of classic invariant theory and algebraic geometry, where the examination of Hopf actions on commutative domains over  $k$  is of interest. The classification of semisimple Hopf actions on such  $k$ -algebras is completely understood by following result with Pavel Etingof.

**Theorem 1.** [2, Theorem 1.3] Any action of semisimple Hopf algebra  $H$  over  $k$  on a commutative domain over  $k$  factors through a finite group action.

The theorem fails if any of the hypotheses are omitted: see [2, Remark 4.3] for ‘domain’ being omitted, [2, Example 5.10] and [5, Examples 7.4–7.6] for ‘commutative’ being omitted, and Theorem 2 below for the omission of semisimplicity.

Next, Etingof and I investigate finite dimensional Hopf actions on quantizations of commutative domains. An important case of this task was established when the module algebra was the  $n$ -th Weyl algebra  $A_n(k)$  with standard filtration and the  $H$ -action preserves the filtration of  $A_n(k)$ . Here, we get that the  $H$ -action factors through a finite group action [2, Corollary 1.4]. Generalizing this result to actions on more general quantizations of commutative domains (including rings of differential operators, enveloping algebras of a finite dimensional Lie algebra, quantized quiver varieties, etc.) is a work in progress with Juan Cuadra [1]. Here, we will not assume that the  $H$ -action preserves the filtration of the module algebra.

In contrast to Theorem 1, we have that there are many actions of finite dimensional nonsemisimple Hopf algebras on commutative domains that do not factor through a group action. Etingof and I study the case when  $H$  is pointed of *finite Cartan type*. The study boils down to  $H$ -actions on fields (that do not factor through a group action), and we refer to such  $H$  as *Galois-theoretical*.

**Theorem 2.** [3, Theorem 1.2] There exist several examples and non-examples of Galois-theoretical Hopf algebras, including the Taft algebras,  $u_q(\mathfrak{sl}_2)$ , and some Drinfeld twists of other small quantum groups.

The classification of Galois-theoretical finite dimensional pointed Hopf algebras (of finite Cartan type) is a work in progress [4]. Please see the references below for several questions and conjectures related to this program.

#### REFERENCES

- [1] J. Cuadra, P. Etingof, and C. Walton, *Semisimple Hopf actions on quantizations* (preprint), in progress.
- [2] P. Etingof, C. Walton, *Semisimple Hopf actions on commutative domains*, Adv. Math **251** C (2014), 47–61.
- [3] P. Etingof and C. Walton, *Pointed Hopf actions on fields, I* (preprint), arXiv:1403.4673.
- [4] P. Etingof and C. Walton, *Pointed Hopf actions on fields, II* (preprint), in preparation.
- [5] E. Kirkman, J. Kuzmanovich, J.J. Zhang *Gorenstein subrings of invariants under Hopf algebra actions*, J. Algebra. **322** (2009) no. 10, 3640–3669.

### Category $\mathcal{O}$ and KLR algebras

BEN WEBSTER

We discuss a generalization of the BGG category  $\mathcal{O}$  which arises from other symplectic varieties, in particular from quiver varieties. This approach works by defining a category of D-modules that category  $\mathcal{O}$  is a quotient of. Using topological techniques, we can compute this category as the representations of a convolution algebra with a combinatorial description. In the quiver case, what arises is a generalization of KLR algebras, which we call weighted KLR algebras. We explain how these relations arise from the geometry of quivers.

*Reporter: Louis de Thanhoffer de Volcsey*

## Participants

**Hanno Becker**

Mathematisches Institut  
Universität Bonn  
Endenicher Allee 60  
53115 Bonn  
GERMANY

**Prof. Dr. Jason P. Bell**

Department of Pure Mathematics  
University of Waterloo  
Waterloo, Ont. N2L 3G1  
CANADA

**Dr. Gwyn Bellamy**

Department of Mathematics  
University of Glasgow  
University Gardens  
Glasgow G12 8QW  
UNITED KINGDOM

**Prof. Dr. Yuri Berest**

Department of Mathematics  
Cornell University  
Malott Hall  
Ithaca, NY 14853-4201  
UNITED STATES

**Prof. Dr. Rafael Bocklandt**

School of Mathematics and Statistics  
The University of Newcastle  
Newcastle upon Tyne NE1 7RU  
UNITED KINGDOM

**Prof. Dr. Ken A. Brown**

Department of Mathematics  
University of Glasgow  
University Gardens  
Glasgow G12 8QW  
UNITED KINGDOM

**Prof. Dr. Ragnar-Olaf Buchweitz**

Computer & Mathematical Sciences  
Dept.  
University of Toronto Scarborough  
1265 Military Trail  
Toronto Ont. M1C 1A4  
CANADA

**Prof. Dr. William Crawley-Boevey**

Department of Pure Mathematics  
University of Leeds  
Leeds LS2 9JT  
UNITED KINGDOM

**Louis de Thanhoffer de Volcsey**

Center for Statistics  
Hasselt University  
Agoralaan Building D  
3590 Diepenbeek  
BELGIUM

**Dr. Chris Dodd**

Department of Mathematics  
University of Toronto  
100 St. George Street  
Toronto, Ont. M5S 2E4  
CANADA

**Dr. Michael Ehrig**

Mathematisches Institut  
Universität Bonn  
Endenicher Allee 60  
53115 Bonn  
GERMANY

**Hans Franzen**

Fachbereich Mathematik  
Bergische Universität Wuppertal  
Gaußstr. 20  
42097 Wuppertal  
GERMANY

**Sian Fryer**

Department of Mathematics  
The University of Manchester  
Manchester M13 9PL  
UNITED KINGDOM

**Prof. Dr. Victor Ginzburg**

Department of Mathematics  
The University of Chicago  
5734 South University Avenue  
Chicago, IL 60637-1514  
UNITED STATES

**Prof. Dr. Kenneth R. Goodearl**

Department of Mathematics  
University of California at  
Santa Barbara  
South Hall  
Santa Barbara, CA 93106  
UNITED STATES

**Prof. Dr. Iain Gordon**

School of Mathematics  
University of Edinburgh  
James Clerk Maxwell Bldg.  
King's Buildings, Mayfield Road  
Edinburgh EH9 3JZ  
UNITED KINGDOM

**Prof. Dr. Birge**

**Huisgen-Zimmermann**  
Department of Mathematics  
University of California at  
Santa Barbara  
South Hall  
Santa Barbara, CA 93106  
UNITED STATES

**Prof. Dr. Colin Ingalls**

Department of Mathematics  
University of New Brunswick  
P.O. Box 4400  
Fredericton, N.B. E3B 5A3  
CANADA

**Prof. Dr. Osamu Iyama**

Graduate School of Mathematics  
Nagoya University  
Chikusa-ku, Furo-cho  
Nagoya 464-8602  
JAPAN

**Martin Kalck**

School of Mathematics  
University of Edinburgh  
James Clerk Maxwell Bldg.  
King's Buildings, Mayfield Road  
Edinburgh EH9 3JZ  
UNITED KINGDOM

**Dr. Dmitry Kaledin**

Department of Algebraic Geometry  
Steklov Mathematical Institute  
Gubkina 8  
Moscow 119 991  
RUSSIAN FEDERATION

**Prof. Dr. Henning Krause**

Fakultät für Mathematik  
Universität Bielefeld  
Postfach 100131  
33501 Bielefeld  
GERMANY

**Prof. Dr. Thomas H. Lenagan**

School of Mathematics  
University of Edinburgh  
James Clerk Maxwell Bldg.  
King's Buildings, Mayfield Road  
Edinburgh EH9 3JZ  
UNITED KINGDOM

**Prof. Dr. Thierry Levasseur**

Dept. de Mathématiques  
Université de Bretagne Occidentale  
6, Avenue Victor Le Gorgeu  
29238 Brest Cedex 3  
FRANCE

**Prof. Dr. Ivan Loseu**

Department of Mathematics  
Northeastern University  
519 LA, 360 Huntington Ave.  
Boston MA 02115  
UNITED STATES

**Dr. Wendy Tor Lowen**

Department of Mathematics  
University of Antwerpen  
Middelheimlaan 1  
2020 Antwerpen  
BELGIUM

**Prof. Dr. Valery A. Lunts**

Department of Mathematics  
Indiana University at Bloomington  
Swain Hall East  
Bloomington, IN 47405  
UNITED STATES

**Dr. Kevin R. McGerty**

Mathematical Institute  
Oxford University  
Woodstock Road  
Oxford OX2 6GG  
UNITED KINGDOM

**Dr. Andrew Morrison**

Departement Mathematik  
ETH Zürich; HG G 68.2  
Rämistr. 101  
8092 Zürich  
SWITZERLAND

**Prof. Dr. Sergey Mozgovoy**

Department of Mathematics  
Trinity College  
University of Dublin  
Dublin 2  
IRELAND

**Prof. Dr. Thomas A. Nevins**

Dept. of Mathematics, University of  
Illinois at Urbana-Champaign  
273 Altgeld Hall MC-382  
1409 West Green Street  
Urbana, IL 61801-2975  
UNITED STATES

**Dr. Francois Petit**

School of Mathematics  
University of Edinburgh  
James Clerk Maxwell Bldg.  
Edinburgh EH9 3JZ  
UNITED KINGDOM

**Prof. Dr. Markus Reineke**

Fachbereich Mathematik  
Bergische Universität Wuppertal  
42097 Wuppertal  
GERMANY

**Prof. Dr. Idun Reiten**

Department of Mathematical Sciences  
NTNU  
7491 Trondheim  
NORWAY

**Prof. Dr. Claus Michael Ringel**

Fakultät für Mathematik  
Universität Bielefeld  
Postfach 100131  
33501 Bielefeld  
GERMANY

**Dr. Daniel Rogalski**

Department of Mathematics  
University of California, San Diego  
9500 Gilman Drive  
La Jolla, CA 92093-0112  
UNITED STATES

**Antonio Sartori**

Department of Mathematics  
University of York  
York YO10 5DD  
UNITED KINGDOM

**Dr. Travis Schedler**  
Department of Mathematics  
The University of Texas at Austin  
1 University Station C1200  
Austin, TX 78712-0257  
UNITED STATES

**Olaf Schnürer**  
Mathematisches Institut  
Universität Bonn  
Endenicher Allee 60  
53115 Bonn  
GERMANY

**Dr. Susan J. Sierra**  
School of Mathematics  
University of Edinburgh  
James Clerk Maxwell Bldg.  
Edinburgh EH9 3JZ  
UNITED KINGDOM

**Prof. Dr. J. Toby Stafford**  
Department of Mathematics  
The University of Manchester  
Oxford Road  
Manchester M13 9PL  
UNITED KINGDOM

**Prof. Dr. Catharina Stroppel**  
Mathematisches Institut  
Universität Bonn  
Endenicher Allee 60  
53115 Bonn  
GERMANY

**Dr. Balazs Szendroi**  
Mathematical Institute  
Oxford University  
24-29 St. Giles  
Oxford OX1 3LB  
UNITED KINGDOM

**Prof. Dr. Michel van den Bergh**  
Department of Mathematics  
Limburgs Universitair Centrum  
Universitaire Campus  
3590 Diepenbeek  
BELGIUM

**Dr. Adam-Christiaan Van Roosmalen**  
Faculty of Mathematics and Physics  
Charles University  
Sokolovska 83  
18675 Praha 1  
CZECH REPUBLIC

**Dr. Chelsea Walton**  
Department of Mathematics  
Massachusetts Institute of  
Technology  
Cambridge, MA 02139-4307  
UNITED STATES

**Prof. Dr. Ben Webster**  
Department of Mathematics  
University of Virginia  
141 Cabell Drive, 324 Kerchof Hall  
P.O. Box 400137  
Charlottesville, VA 22904-4137  
UNITED STATES

**Dr. Thorsten Weist**  
Fachbereich Mathematik  
Bergische Universität Wuppertal  
Gaußstr. 20  
42097 Wuppertal  
GERMANY

**Prof. Dr. Michael Wemyss**  
School of Mathematics  
University of Edinburgh  
King's Buildings  
Mayfield Road  
Edinburgh EH9 3JZ  
UNITED KINGDOM

**Prof. Dr. Quan-Shui Wu**

Institute of Mathematics  
Fudan University  
Shanghai 200 433  
CHINA

**Prof. Dr. Amnon Yekutieli**

Department of Mathematics  
Ben-Gurion University of the Negev  
Beer Sheva 84 105  
ISRAEL

**Prof. Dr. James Zhang**

Department of Mathematics  
University of Washington  
Padelford Hall  
Box 354350  
Seattle, WA 98195-4350  
UNITED STATES