# Arbeitsgemeinschaft mit aktuellem Thema:

## Totally disconnected groups

## Mathematisches Forschungsinstitut Oberwolfach

## October 5 – 10, 2014

## Organizers:


| Pierre-Emmanuel Caprace | Nicolas Monod |
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| UCLouvain, Belgium | EPFL, Switzerland |

**Contact organizers:** locally.compact@epfl.ch


## Introduction:

*Locally compact groups* arise as the symmetry groups of all sorts of structures across many areas of mathematics. This includes Lie groups, $p$-adic and adélic groups, isometry groups of general proper metric spaces. Even discrete structures such as locally finite graphs give rise to very interesting locally compact automorphism groups. Besides the groups themselves, one of the most important motivations to study locally compact groups is that they frequently appear as the "envelope" in which abstract groups of interest appear as lattices. This is notably the case for arithmetic groups and Kac–Moody groups. It has often happened that the most interesting theorems about those abstract groups are proved by transfering the problem to the ambient locally compact group and solving it there.

In the study of locally compact groups, it is usually understood that the focus is on *non-discrete* groups since otherwise it remains within "abstract" group theory. The case of Lie groups has extensively studied for well over a century and largely classified in the early twentieth century. The next significant period of research culminated in the 1950s with the solution to Hilbert's Fifth Problem, giving a satisfactory picture of the conected case.

Therefore, the main locus of modern research on locally compact groups is the study of non-discrete *totally disconnected* locally compact groups, since a general locally compact group decomposes as an extension of a connected group by a totally disconnected group.

The revival of this topic can arguably be dated to the work of G. Willis starting two decades ago. This gave a new impetus to the study of the *local structure* of totally disconnected groups. More recently, there has been progress both on the global and local structure. In addition, the compact case (i.e. profinite groups) has also witnessed important recent progress on the algebraic side.

The goals of the Arbeitsgemeinschaft are: to learn the necessary prerequisites, to study substantial parts of the recent developments and to reach the point where open problems can be discussed. The detailed roadmap is exposed below.

## Overview:

| | A: Introductory | B: Examples |
|---|---|---|
| 1 | Basics | $p$-adic Lie groups |
| 2 | Transformation groups | Tree groups |
| 3 | Metric viewpoint | Prescribed local actions |
| 4 | | Neretin group |

| | C: Decomposing | D: Willis' scale |
|---|---|---|
| 1 | Minimal normal subgroups | The scale function |
| 2 | Elementary groups | Tidy subgroups |
| 3 | | Commensurated subgroups |
| 4 | | Contraction groups |

| | E: (Tree) Lattices | F: Locally normal | G: Profinite groups |
|---|---|---|---|
| 1 | Margulis' NSP | Structure lattice | Nikolov–Segal |
| 2 | Bader–Shalom' NSP | Centraliser lattice | |
| 3 | Burger–Mozes' simple lattices | | |

# Talks:

## A Introductory

### Talk A.1 Basics

- Basic decomposition into connected and totally disconnected; van Dantzig's theorem. Reference: Chapter II, §7 from [HR79].
- Hilbert's 5th problem; brief discussion of the the Gleason–Yamabe Theorem, see Theorem 1.1.17 in [Tao12].
- The amenable radical Ramen($G$) and the decomposition of the quotient $G$/Ramen($G$). Reference e.g. §11.3 in [Mon01].

### Talk A.2 Transformation groups and permutation actions

- Automorphism groups of connected locally finite graphs are tdlc.
- Cayley–Abels graphs; references: e.g. §11 in [Mon01] and §2 in [CCMT11].
- Link between tdlc groups and abstract groups with commensurated subgroups and (relative) Schlichting completions. Reference: Section 3 in [SW13].

### Talk A.3 The metric viewpoint

- Overview of the metric geometry of locally compact groups; special emphasis on compact presentations. Reference: [CH].

## B Examples of tdlc groups

### Talk B.1 $p$-adic Lie groups

- Introduction to the basic features of $p$-adic Lie groups. Reference: [Glö].
- Existence of a Lie algebra over $\mathbf{Q}_p$; adjoint representation.
- Closed subgroups of $\mathrm{GL}_n(\mathbf{Q}_p)$ are $p$-adic Lie groups.

### Talk B.2 Automorphism groups of trees: generalities

- Basic features; reference: Chapter 1 from [FTN91].
- Dynamics on the boundary.
- Tits' simplicity theorem from [Tit70]

### Talk B.3 Automorphism groups of trees: prescribed local actions

- Burger–Mozes universal groups; reference: §3.2 from [BM00a].
- Variations by Banks–Elder–Willis [BEW].

### Talk B.4 Neretin's group of tree spheromorphisms

- Construction of the tdlc group topology.
- Link with Thompson's groups. Reference: [Kap99].
- Simplicity.

## C Global structure and decomposition theorems

### Talk C.1 Minimal normal subgroups

- Existence of minimal normal subgroups for certain compactly generated tdlc groups.
- Proof of Theorem B from [CM11].

### Talk C.2 Elementary groups

- Definition and permanence properties of the class of Wesolek's elementary groups; reference: Theorem 1.3 from [Wes14].
- Existence of an elementary radical in any second countable tdlc group; reference: Theorem 1.5 from [Wes14].
- Proof of the decomposition theorem for compactly generated tdlc groups; reference: Corollary 1.7 from [Wes14].

## D Willis' theory

The goal of this section is to cover part of the theory initiated by G. Willis in his seminal paper [Wil94].

### Talk D.1 The scale function

- First properties of the scale
- Proof that the set of periodic elements (= the union of all compact subgroups) is closed; reference: Theorem 3.5 in [Wes13].

### Talk D.2 Tidy subgroups

- Continuity of the scale (Theorem 1.1 from [Wes13]) and characterization of tidy subgroups as minimizing subgroups (Theorems 3.15 from [Wes13]).
- Application to the solution to Halmos' problem (following [PW03]): a l.c. group with an ergodic automorphism must be compact.

### Talk D.3 Commensurated subgroups of arithmetic groups

- Illustration of the use of the scale by Shalom–Willis' paper [SW13]. Suggestion: a discussion of *flat subgroups* can be shortcut by following the approach from Section 6.3 in [SW13].

### Talk D.4 Contraction groups and the scale

- Non-triviality of the contraction group for non-uniscalar elements, following Baumgartner–Willis [BW04].
- The nub and the closure of contraction groups; references: [BW04] and [Wil12].
- If time permits, discuss the Tits core, following [CRW14].

# E (Tree) Lattices

### Talk E.1 Margulis' Normal Subgroup Theorem

- Illustration of the use of contraction groups in the proof of Margulis' theorem. Reference: in addition to [Mar91, Chapter VIII], we especially recommend [Ben09, Lecture 4].

### Talk E.2 Bader–Shalom' Normal Subgroup Theorem

- Focus on the amenability part. Reference: [BS06]

- Mention the Burger–Mozes version for trees. Reference: Theorem 4.1 in [BM00b].

### Talk E.3 Burger–Mozes' simple lattices

- Explain how the normal subgroup theorem can be used to construct concrete examples of irreducible lattices in products of two trees, that are virtually simple. Reference: §5 and §6 in [BM00b]. See also [Rat04].

# F Locally normal subgroups

### Talk F.1 The structure lattice

- Basics on locally normal subgroups, following §2 and §3 from [CRWa].

- Compactly generated simple tdlc groups have no solvable locally normal subgroups; reference: Theorem 5.3 from [CRWb].

### Talk F.2 The centraliser lattice

- The centraliser lattice as a Boolean algebra; reference: Theorem I from [CRWa].

- Dynamical properties of the action; reference: Theorem F from [CRWb].

## G Abstract quotients of profinite groups, following Nikolov–Segal

### Talk G.1 Selected topics from [NS12]

- Context and motivation; maybe mention the old result (R. Alperin and folklore?) that any homomorphism to $\mathbf{Z}$ is automatically continuous.
- Dense normal subgroups following [NS12]. (Suggestion: the pro-$p$ case can be given a self-contained treatment, see Proposition 5.13 in [CRWb].)

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# Participation:

The idea of the Arbeitsgemeinschaft is to learn by giving one of the lectures in the program.

If you intend to participate, please send your full name and full postal address to

`locally.compact@epfl.ch`

by August 15, 2014 at the latest.

You should also indicate which talk you are willing to give:
First choice: talk no. ...
Second choice: talk no. ...
Third choice: talk no. ...

You will be informed shortly after the deadline if your participation is possible and whether you have been chosen to give one of the lectures.

The Arbeitsgemeinschaft will take place at Mathematisches Forschungsinstitut Oberwolfach, Lorenzenhof, 77709 Oberwolfach-Walke, Germany. The institute offers accomodation free of charge to the participants. Travel expenses cannot be covered. Further information will be given to the participants after the deadline.